

Gravitational waves in a de Sitter universe

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The construction of exact linearized solutions to the Einstein equations within the Bondi-Sachs formalism is extended to the case of linearization about de Sitter spacetime. The gravitational wave field measured by distant observers is constructed, leading to a determination of the energy measured by such observers. It is found that gravitational wave energy conservation does not normally apply to inertial observers but that it can be formulated for a class of accelerated observers, i.e., with worldlines that are timelike but not geodesic.

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I. INTRODUCTION

The standard formulation of gravitational wave (GW) theory assumes an asymptotically flat spacetime (for example, see Refs. [1,2]). However, astrophysical evidence has emerged over the last two decades that the Universe is undergoing an accelerated expansion which is well described by the Λ CDM model. Thus, it is important to investigate GW properties in such a spacetime, and further it is in a way urgent to do so since direct GW detection by facilities such as LIGO is regarded as imminent. In order to be able to model this problem, the first step is to decide on a background spacetime to be used. For ease of analysis, it should, in some way, be analytically simple, yet for any results to be astrophysically relevant, it should also be realistic. These two requirements are somewhat contradictory, and here we use de Sitter spacetime. Although the spacetime is unrealistic in that it has no matter content, it does include the key feature of an accelerated expansion. The key advantage for the present study of de Sitter spacetime is its simplicity.

In a series of recent papers, Ashtekar *et al.* [3–6] have investigated GWs in de Sitter spacetime. The focus of their work has been on the asymptotic structure and correct mathematical formulation of GWs. In addition, they have tackled the astrophysical issues of whether a de Sitter background would affect the generation or detection of GWs and concluded that the effect is negligible. There has also been recent related work by Date and Hoque [7]. In this paper, we study similar matters, although from a different perspective.

Exact linearized solutions about a background spacetime within the Bondi-Sachs formalism have also received recent attention [8,9]. The simplest background to consider is Minkowski, and Ref. [10] constructed such solutions for the purpose of providing a test bed for numerical relativity codes using the Bondi-Sachs formalism. Subsequently, the approach has been used to construct solutions about Schwarzschild [11], to investigate quasinormal modes [12], to find the spacetime geometry generated by binaries

in circular orbit [9,13], and to generalize previous results to the case of sourcing by arbitrary matter fields [8]. It is therefore natural to extend this approach further by seeking to linearize about de Sitter spacetime, particularly in light of the more general issues raised above.

In order to linearize about a given spacetime, its metric in Bondi-Sachs form must first be found. This is surprisingly difficult. It is of course easy to do so for both Minkowski and Schwarzschild. It can be done for Kerr [14], but the result involves elliptic integrals; the simplest cosmological model is Einstein-de Sitter, but the Bondi-Sachs form of its metric has to be constructed numerically. It is therefore somewhat noteworthy that the Bondi-Sachs metric for de Sitter spacetime is very simple and further that the solutions for linearized perturbations take a simple polynomial form. As such, one may hope to be able to use the solutions to gain physical insight into the perturbations, i.e., the GWs, in de Sitter spacetime.

The analytic solutions obtained are valid throughout the Bondi-Sachs domain external to the source. Conformal compactification is not introduced, and so issues involving the treatment of \mathcal{I}^+ are avoided. We do, however, consider the leading order term in an expression, and this is equivalent to taking an asymptotic limit. The GW analysis is from the physical viewpoint of the geodesic deviation that would be measured by an observer. In order for the usual concept of conservation of GW energy to apply, this quantity would be expected to have an asymptotic falloff of $1/r$. Perhaps surprisingly, it is found that this falloff does not apply to inertial observers (i.e., on a timelike geodesic) but only to a class of accelerated observers. Thus, we believe that the study of the solutions presented here can provide useful physical insight, which may be helpful toward a proper understanding of GW energy in de Sitter spacetime.

As well as investigating GWs from the viewpoint of a detector, we also investigate the effect of a de Sitter background on the generation of GWs. This is achieved by a straightforward adaptation of the procedure about Minkowski [13]. As reported in other work, we also find

that the consequence of the change in background is normally completely negligible. Certainly, this is the case for direct detection of GWs, and it is difficult to conceive of any measurable astrophysical process that would be indirectly affected.

In Sec. II, the coordinate transformation from usual de Sitter coordinates to the Bondi-Sachs form is determined and implemented. Then, the linearized Einstein equations are constructed and solved in Sec. III, with the properties of the associated GWs developed in Sec. IV. GWs from an equal mass binary in de Sitter spacetime are calculated (Sec. V), and the paper ends with the conclusion in Sec. VI. Throughout the paper, extensive use is made of computer algebra, and the associated scripts are described in the Appendix; further, the scripts are available in the Supplemental Material [15].

II. TRANSFORMATION OF DE SITTER SPACETIME FROM COMOVING TO BONDI-SACHS COORDINATES

De Sitter spacetime, in standard comoving coordinates (t, ρ, θ, ϕ) , is represented by the metric

$$ds^2 = -dt^2 + A^2 e^{2\alpha t} (d\rho^2 + \rho^2 d\Omega^2), \quad (1)$$

where A is a constant, $\alpha = \sqrt{\Lambda/3}$ (Λ is the cosmological constant), and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Now, make the coordinate transformation

$$(t, \rho) \rightarrow (u, r): \quad u = -\frac{\log(A\rho\alpha \exp(\alpha t) + 1) - \alpha t}{\alpha}, \quad (2)$$

$$r = A\rho e^{\alpha t},$$

with inverse

$$t = \frac{\log(1 + r\alpha) + u\alpha}{\alpha}, \quad \rho = \frac{r \exp(-u\alpha)}{A(1 + r\alpha)}. \quad (3)$$

The metric (1) is transformed to

$$ds^2 = -du^2(1 - r^2\alpha^2) - 2dudr + r^2 d\Omega^2, \quad (4)$$

which is in Bondi-Sachs form since it is clear that r is both a null and a surface area coordinate. Note that each null cone $u = u_0 = \text{constant}$ has a maximum radius in de Sitter coordinates $\rho_{\max} = \exp(-u_0\alpha)/(A\alpha)$ [from Eq. (3)], but r is unbounded.

We will also need the transformation for the 4-velocity when the motion is purely radial. V^a in Bondi-Sachs coordinates

$$V_{[BS]}^a = \left(V^0, \frac{1}{2V^0} - \frac{V^0(1 - r^2\alpha^2)}{2}, 0, 0 \right) \quad (5)$$

becomes in de Sitter coordinates

$$V_{[ds]}^a = \left(\frac{1}{2V^0(1+r\alpha)} + \frac{V^0(1+r\alpha)}{2}, \frac{1}{2V^0(1+r\alpha)^2} - \frac{V^0}{2}, 0, 0 \right). \quad (6)$$

In particular, we note that a ‘‘natural’’ de Sitter observer for which $V_{[ds]}^a = (1, 0, 0, 0)$ satisfies

$$V_{[BS]}^a = \left(\frac{1}{1+r\alpha}, r\alpha, 0, 0 \right). \quad (7)$$

III. LINEARIZED EINSTEIN EQUATIONS AND THEIR SOLUTION

The formalism for expressing Einstein’s equations as an evolution system based on characteristic, or null-cone, coordinates is based on work originally due to Bondi *et al.* [1,16] for axisymmetry and extended to the general case by Sachs [17]. The formalism is covered in the review by Winicour [18], and the conventions used here are as Ref. [19]. We start with coordinates based upon a family of outgoing null hypersurfaces. Let u label these hypersurfaces, x^A ($A = 2, 3$) label the null rays, and r be a surface area coordinate. In the resulting $x^\alpha = (u, r, x^A)$ coordinates, the metric takes the Bondi-Sachs form

$$ds^2 = -\left(e^{2\beta} \left(1 + \frac{W}{r} \right) - r^2 h_{AB} U^A U^B \right) du^2 - 2e^{2\beta} dudr - 2r^2 h_{AB} U^B dudx^A + r^2 h_{AB} dx^A dx^B, \quad (8)$$

where $h^{AB} h_{BC} = \delta_C^A$ and $\det(h_{AB}) = \det(q_{AB})$, with q_{AB} a metric representing a unit 2-sphere; for the computer algebra calculations presented later, it is necessary to have specific angular coordinates, and for that purpose stereographic coordinates are used with $x^A = (q, p)$ and

$$q_{AB} dx^A dx^B = \frac{4}{(1 + q^2 + p^2)^2} (dq^2 + dp^2). \quad (9)$$

W is related to the usual Bondi-Sachs variable V by $V = r + W$. It is convenient to describe angular quantities and derivatives by means of complex numbers using the spin-weighted formalism and the eth (\eth) calculus [20–22]. To this end, q_{AB} is represented by a complex dyad q_A with, for example, $q_A = (1, i)2/(1 + q^2 + p^2)$ in stereographic coordinates. Then, h_{AB} can be represented by its dyad component $J = h_{AB} q^A q^B/2$. We also introduce the field $U = U^A q_A$. (The field $K = \sqrt{1 + J\bar{J}}$ is needed in the nonlinear case but not here because linearization implies $K = 1$.)

The procedure for linearizing about de Sitter spacetime is very similar to that used for linearization about other fixed backgrounds. So, here we omit some detail and just outline the key steps highlighting the differences between the

current case and that of Minkowski spacetime as described in Ref. [10]. The Bondi-Sachs metric quantities

$$J, \quad \beta, \quad U, \quad w \quad (10)$$

(where $w = r^3\alpha^2 + W$) are regarded as being small [$\mathcal{O}(\epsilon)$], and all terms in the Einstein equations of order $\mathcal{O}(\epsilon^2)$ are set to zero. Most calculations are in vacuum, but if matter is present, it will be assumed that its density $\rho = \mathcal{O}(\epsilon)$. An important difference between this paper and previous work is that here the Einstein equations include the cosmological constant, so that the equations to be evaluated are

$$R_{ab} - 3\alpha^2 g_{ab} = 4\pi(2T_{ab} - Tg_{ab}). \quad (11)$$

We find

$$R_{11}: \frac{4}{r}\beta_{,r} = 8\pi T_{11} \quad (12)$$

$$q^A R_{1A}: \frac{1}{2r}(4\delta\beta - 2r\delta\beta_{,r} + r\bar{\delta}J_{,r} + r^3U_{,rr} + 4r^2U_{,r}) = 8\pi q^A T_{1A} \quad (13)$$

$$h^{AB}R_{AB}: (4(1 - 3r^2\alpha^2) - 2\delta\bar{\delta})\beta + \frac{1}{2}(\bar{\delta}^2 J + \delta^2 \bar{J}) + \frac{1}{2r^2}(r^4\delta\bar{U} + r^4\bar{\delta}U)_{,r} - 2w_{,r} = 8\pi(h^{AB}T_{AB} - r^2T) \quad (14)$$

$$q^A q^B R_{AB}: -2\delta^2\beta + (r^2\delta U)_{,r} - 2(r - 2\alpha^2 r^3)J_{,r} - (1 - \alpha^2 r^2)r^2 J_{,rr} + 2r(rJ)_{,ur} = 8\pi q^A q^B T_{AB}. \quad (15)$$

The remaining Einstein equations are needed only in the vacuum case and are

$$R_{00}: \frac{1}{2r^3}(r(r - \alpha^2 r^3)w_{,rr} + \delta\bar{\delta}w + 2(r - \alpha^2 r^3)\delta\bar{\delta}\beta + \alpha^2 r^4(\delta\bar{U} + \bar{\delta}U) + 12r^3\alpha^2(1 - r^2\alpha^2)\beta - 4r(r - \alpha^2 r^3)\beta_{,u} - r^3(\delta\bar{U} + \bar{\delta}U)_{,u} + 2rw_{,u}) = 0 \quad (16)$$

$$R_{01}: \frac{1}{4r^2}(2rw_{,rr} + 4\delta\bar{\delta}\beta + 24\beta\alpha^2 r^2 - (r^2\delta\bar{U} + r^2\bar{\delta}U)_{,r}) = 0 \quad (17)$$

$$q^A R_{0A}: \frac{1}{4r^2}(2r\delta w_{,r} - 2\delta w + 2r^2(r - r^3\alpha^2)(4U_{,r} + rU_{,rr}) + 4r^2U + r^2(\delta\bar{\delta}U - \delta^2\bar{U}) + 2r^2\bar{\delta}J_{,u} - 2r^4U_{,ur} - 4r^2\delta\beta_{,u}) = 0. \quad (18)$$

The solution to the above equations is constructed as described in Ref. [10]. We make a standard ‘‘separation of variables’’ ansatz,

$$\begin{aligned} \beta &= \beta_0(r)\Re(\exp(i\omega u))Z_{\ell m}, \\ w &= w_0(r)\Re(\exp(i\omega u))Z_{\ell m}, \\ U &= U_0(r)\Re(\exp(i\omega u))_1 Z_{\ell m}, \\ J &= J_0(r)\Re(\exp(i\omega u))_2 Z_{\ell m}, \end{aligned} \quad (19)$$

where the $Z_{\ell m}$ are real spherical harmonics and the ${}_s Z_{\ell m}$ are their spin-weighted extensions (see Refs. [10,23]). An important technical difference between this ansatz and that of Ref. [10] is that here we use ${}_s Z_{\ell m}$ as basis functions, instead of $\delta^s Z_{\ell m}$. The relation between the two sets of basis functions is

$$\delta^s Z_{\ell m} = \sqrt{\frac{(\ell - s)!}{(\ell + s)!}} {}_s Z_{\ell m}. \quad (20)$$

In the vacuum case, Eqs. (12) to (18) reduce to a system of ordinary differential equations in $\beta_0(r)$, $w_0(r)$, $U_0(r)$, $J_0(r)$. It is remarkable that, following the same procedure as Ref. [10] for the case of linearization about a Minkowski background, these equations can be solved exactly and that for the outgoing wave the solutions are simple polynomials in $1/r$. Equation (12) shows that $\beta_0 = \text{constant}$; then, Eqs. (13) and (15) may be combined to give (in the case $\ell = 2$)

$$x^2(x^2 - \alpha^2) \frac{d^2 J_2(x)}{dx^2} + 2x(2x^2 + i\omega x + \alpha^2) \frac{dJ_2(x)}{dx} - 2(2x^2 + i\omega x + \alpha^2)J_2(x) = 0, \quad (21)$$

where $x = 1/r$ and $J_2(x) = d^2 J(x)/dx^2$. Equation (21) has a solution of the form $J_2(x) = C_3 f_3(x) + C_4 f_4(x)$, but one of the functions, say $f_4(x)$, is of the form $\exp(2i\omega r)$ representing incoming radiation, so normally we set $C_4 = 0$. Integrating $J_2(x)$ to obtain $J_0(r)$ introduces two further integration constants C_1, C_2 . The procedure leading to Eq. (21) also leads to an expression for $U_0(r)$ in terms of $J_0(r)$ and so can now be evaluated. Next, Eq. (14) is integrated to give $w_0(r)$ in terms of an additional integration constant C_5 . The constraint equations Eqs. (16) to (18) impose two conditions on the constants of integration and are used to express C_2, C_5 in terms of β_0, C_1, C_3 .

We find in the leading order $\ell = 2$ case

$$\begin{aligned} \beta_0(r) &= \text{constant} \\ U_0(r) &= \sqrt{6} \left(\frac{-24i\omega\beta_0 + 3C_1(3\alpha^2 + \omega^2) - C_3\omega^2(4\alpha^2 + \omega^2)}{36} \right. \\ &\quad \left. + \frac{2\beta_0}{r} + \frac{C_1}{2r^2} + \frac{iC_3\omega}{3r^3} + \frac{C_3}{4r^4} \right) \\ w_0(r) &= -2\beta_0\alpha^2 r^3 + \frac{r^2}{6} (24i\omega\beta_0 - 3C_1(\omega^2 + 3\alpha^2) \\ &\quad + \omega^2 C_3(\omega^2 + 3\alpha^2)) \\ &\quad + \frac{r}{3} (-6\beta_0 + 3i\omega C_1 - C_3i\omega(\omega^2 + 4\alpha^2)) \\ &\quad - C_3(\omega^2 + \alpha^2) + \frac{i\omega C_3}{r} + \frac{C_3}{2r^2} \\ J_0(r) &= \sqrt{6} \left(\frac{24\beta_0 - i(4\omega\alpha^2 + \omega^3)C_3 + 3i\omega C_1}{18} + \frac{C_1}{2r} - \frac{C_3}{6r^3} \right). \end{aligned} \quad (22)$$

This solution has been checked by substituting it into all ten vacuum Einstein equations and confirming that $R_{ab} - 3\alpha^2 g_{ab} = 0$.

In the case of linearization about Minkowski, the modes with $\ell > 2$ also have solutions that are polynomial in $1/r$ but of higher order. The highest power of $1/r$ appears in $U_0(r)$ and is $1/r^{\ell+2}$. While it has not been explicitly checked, it can reasonably be expected that the same situation would apply to linearization about de Sitter.

IV. GRAVITATIONAL WAVES

In the asymptotically flat case, GWs are described in terms of the wave strain in the transverse traceless (TT) gauge ($h_+ + ih_\times$), the gravitational news (\mathcal{N}), or the Newman-Penrose quantity ψ_4 . These descriptors are related by time derivatives (and constant factors due to the use of different conventions in the original definitions), specifically

$$r\psi_4 = 2\partial_u \bar{\mathcal{N}} = r\partial_u^2 (h_+ - ih_\times). \quad (23)$$

In order to decide which descriptor is most convenient here, the issue of gauge freedom needs to be considered, which is whether results obtained would be affected if a small change $[\mathcal{O}(\epsilon)]$ were to be made to the coordinates. In such a case, calculations are more difficult because the coordinates have to be specified to satisfy geometrical conditions to $\mathcal{O}(\epsilon)$. Now, both $h_+ + ih_\times$ and \mathcal{N} are gauge dependent quantities, but ψ_4 is not. The Newman-Penrose quantity $\psi_4 = -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d$ is defined in terms of the Weyl tensor C_{abcd} and a null tetrad (specified below). It is tensorially a scalar, but gauge freedom can appear in the specification of the null tetrad. However, in the background

de Sitter metric, $C_{abcd} = 0$, so ψ_4 , and indeed all the ψ_i , are gauge independent quantities.

Suppose that an observer has 4-velocity V^a given by Eq. (5). Then, the null tetrad (n^a, ℓ^a, m^a) must be chosen so that

$$\sqrt{2}V^a = n^a + \ell^a \quad (24)$$

and for the unperturbed ($\epsilon = 0$) spacetime is

$$\begin{aligned} n^a &= \left(\sqrt{2}V^0, -\frac{V^0(1-r^2\alpha^2)}{\sqrt{2}}, 0, 0 \right), \\ \ell^a &= \left(0, \frac{1}{V^0\sqrt{2}}, 0, 0 \right), \quad m^a = \left(0, 0, \frac{q^A}{\sqrt{2}r} \right), \end{aligned} \quad (25)$$

where q^A is the dyad on the unit sphere. Evaluating ψ_4 , we obtain

$$\begin{aligned} \psi_4 &= (V^0)^2 {}_{-2}Z_{2m} \\ &\times \Re \left[\exp(i\omega u) C_3 \frac{\sqrt{6}}{12} \left(-\frac{2\omega^4 + 8\omega^2\alpha^2 + 3\alpha^4}{r} \right. \right. \\ &\quad \left. \left. + i \frac{4\omega^3 + 10\omega\alpha^2}{r^2} + 6 \frac{\omega^2 + \alpha^2}{r^3} - 6i \frac{\omega}{r^4} - \frac{3}{r^5} \right) \right]. \end{aligned} \quad (26)$$

Note further that the forms of ψ_0, \dots, ψ_3 have been checked, and they have the expected falloff behavior $C_3 r^{-5}, \dots, C_3 r^{-2}$. It is also important to note that the formula for ψ_4 involves only C_3 , and not β_0, C_1 . This is expected as the same applies in the asymptotically flat case to all the GW descriptors. Thus, as in the Minkowski case, the constants β_0, C_1 may be regarded as representing gauge freedoms.

The physical relevance of ψ_4 needs to be addressed, i.e., the relationship between ψ_4 and geodesic deviation (which is what is actually measurable by a detector such as LIGO). In a vacuum $\Lambda = 0$ spacetime, $R_{ab} = 0$ so that $C_{abcd} = R_{abcd}$, and the relevant geodesic deviation $R_{abcd}V^a m^b V^c m^d$ can be expressed, using Eq. (24), as $C_{abcd}(n^a + \ell^a)m^b(n^c + \ell^c)m^d/2$, which to leading order in the asymptotic falloff is $-\bar{\psi}_4/2$. In the de Sitter case, $R_{ab} = \Lambda g_{ab} \neq 0$, so the preceding argument cannot be applied. However, the result is still true since

$$\begin{aligned} &(C_{abcd} - R_{abcd})(n^a + \ell^a)m^b(n^c + \ell^c)m^d \\ &= \Lambda \left(-g_{a[c}g_{d]b} + g_{b[c}g_{d]a} + \frac{4g_{a[c}g_{d]b}}{3} \right) (n^a + \ell^a) \\ &\quad \times m^b(n^c + \ell^c)m^d, \end{aligned} \quad (27)$$

and every term on the right-hand side involves two inner products of null tetrad vectors, of which at least one inner product must vanish. [The only nonzero inner product in Eq. (27) is $g_{ab}n^a\ell^b = -1$]. Thus, there is an equivalence

between the gauge invariant quantity ψ_4 and geodesic deviation.

However, for a physical interpretation of detectable gravitational waves, we need to do rather more than just evaluate ψ_4 . Assuming that the detector is in free fall and so follows a timelike geodesic, allowance needs to be made for changes to the position and velocity of the detector, and further all results should be in terms of the detector's proper time τ rather than the Bondi-Sachs time coordinate u . Let $\partial_\tau \psi_4$ mean the rate of change of ψ_4 as observed by the detector; then,

$$\partial_\tau \psi_4 = V^0 \partial_u \psi_4 + V^1 \partial_r \psi_4 + V^0 \partial_{V^0} \psi_4. \quad (28)$$

The first term in Eq. (28) would appear in a calculation about Minkowski and represents the redshift factor, and the second term is unimportant because $\partial_r \psi_4 = \mathcal{O}(1/r^2)$, and so the term does not make a leading order contribution. The third term, however, is highly significant. From the geodesic equation, we find $\partial_\tau V^0 = -\alpha^2 r (V^0)^2$, so it is nonzero only in the de Sitter case; further, the multiplication by r means that the leading order part of the expression is affected. The expression found is rather long, and we present only the part to $\mathcal{O}(1/r)$:

$$\begin{aligned} \partial_\tau \psi_4 = & \frac{5(V^0)^3 \sqrt{6} C_3 \alpha^2 (8\alpha^2 \omega^2 + 2\omega^4 + 3\alpha^4)}{24} \\ & - i \frac{(V^0)^3 \sqrt{6} C_3 \omega (33\alpha^4 + 2\omega^4 + 20\alpha^2 \omega^2)}{12r}. \end{aligned} \quad (29)$$

In the $\Lambda = 0$ asymptotically flat case, the well-known ‘‘News = mass loss’’ theorem [1] applies. A key precursor to the result is that the magnitude of the GWs decays as r^{-1} so that

$$\int_S |\mathcal{N}|^2 \quad (30)$$

(where S is a spherical shell $u = r = \text{constant}$) is independent of r . For de Sitter spacetime, our objective is to investigate conditions under which energy conservation in the above sense applies. Using Eqs. (29) and (23), the total energy crossing a 2-surface $r = u = \text{constant}$ according to observers with 4-velocity V^a is

$$\begin{aligned} \int_S |\mathcal{N}|^2 = & \frac{(V^0)^2 C_3^2}{96\omega^4} [r^2 \alpha^4 (225\alpha^8 + 1200\alpha^6 \omega^2 + 1900\alpha^4 \omega^4 \\ & + 800\alpha^2 \omega^6 + 100\omega^8) + \omega^2 (16\omega^8 + 320\omega^6 \alpha^2 \\ & + 2128\omega^4 \alpha^4 + 5280\omega^2 \alpha^6 + 4356\alpha^8)]. \end{aligned} \quad (31)$$

The above can be made independent of r by the appropriate choice of V^0 . To leading order in α ,

$$\begin{aligned} V^a = & \left(D \left(1 - \frac{10}{\omega^2} \alpha^2 + \frac{668 - 25r^2 \omega^2}{8\omega^4} \alpha^4 \right), \right. \\ & \frac{1 - D^2}{2D} + \frac{10 + D^2(10 + r^2 \omega^2)}{2\omega^2 D} \alpha^2 \\ & \left. + \frac{132 + 25r^2 \omega^2 + D^2(668 + 55r^2 \omega^2)}{16\omega^4 D} \alpha^4, 0, 0 \right), \end{aligned} \quad (32)$$

where D is a constant.

A. Discussion

We now discuss the implications for GW energy conservation of the results above. For observers in free fall, energy conservation applies, provided Eq. (32) is satisfied. In order to make the implications more concrete, we consider the example where the constant D is fixed by the condition $V^0 \rightarrow 1$ as $r \rightarrow 0$; physically, this corresponds to observers close to the source being at rest relative to the source. In this case, we find

$$\begin{aligned} V^a = & \left(1 - 25 \frac{r^2 \alpha^4}{\omega^2}, \frac{r^2 \alpha^2}{2} \left(1 + 50 \frac{\alpha^2}{\omega^2} \right), 0, 0 \right) \\ & + \mathcal{O}(\alpha^6). \end{aligned} \quad (33)$$

Thus, while it is possible to find a set of observers for whom energy conservation holds, their velocities depend on both position and wave frequency. If the wave frequency changes, as happens during an inspiral, then the observer's velocity would need to be adjusted; i.e., the observer would need to be accelerated.

Abandoning the idea that the observers for whom energy conservation holds should be freely falling, we can, for example, set $V^0 = 1$ everywhere (so that physically observers near the source are at rest relative to the source). In order that such observers have $\partial_\tau V^0 = 0$, they must experience an acceleration $-r\alpha^2 + r^3 \alpha^4 / 2$.

We note that the natural de Sitter observers are freely falling and have V^a given by Eq. (7); they do not constitute a set of observers for whom energy conservation applies.

An important question is whether a positive cosmological constant will affect the interpretation of observations expected from detectors such as LIGO. The effects are ignorable, provided, from Eq. (29), $r\alpha^2 \ll \omega$. This is indeed the case, since the lowest ω can be is a few Hz, $\alpha \approx 1/(5 \text{ Gpc})$, and the largest expected value of r is of order 1 Gpc for a binary black hole merger. A similar conclusion was reported by Ashtekar *et al.* [3].

V. GRAVITATIONAL WAVES FROM AN EQUAL MASS BINARY

A procedure for calculating the gravitational field, linearized about Minkowski, for two equal mass M objects

in circular orbit radius r_0 was described in Ref. [13], and that result has recently been generalized [9]. The calculations about de Sitter and Minkowski proceed in the same way, and here we just provide an outline.

The objects are modeled as point particles, and so the matter density $\rho(u, r, x^A)$ is expressed in terms of δ -functions. It is then straightforward to decompose ρ into spherical harmonic components, i.e., $\rho = \sum \rho_{\ell m} Z_{\ell m}$. For the case $\ell = 2$, $\rho_{21} = \rho_{2,-1} = 0$, and ρ_{20} is constant in time and thus is not the source of any radiation. Only ρ_{22} and $\rho_{2,-2}$ are relevant. Turning now to the metric, there are two separate solutions, valid in $r < r_0$ and $r > r_0$, respectively, with each solution having its own set of integration constants. The exterior solution has three constants, and the interior solution has only one free constant (with the others fixed by the condition that spacetime must be regular at the origin). These constants are fixed by imposing four conditions at the interface $r = r_0$: continuity of J and U and jump conditions on β , w that follow from integrating the δ -functions in the right-hand sides of Eqs. (12) and (14) across $r = r_0$.

The result obtained for the metric in the region $r > r_0$ is

$$\begin{aligned}\beta &= \Re(\beta_0 \exp(i\omega u))Z_{22} + \Re(-i\beta_0 \exp(i\omega u))Z_{2,-2} \\ w &= \Re(w_0 \exp(i\omega u))Z_{22} + \Re(-w_0 \exp(i\omega u))Z_{2,-2} \\ J &= \Re(J_0 \exp(i\omega u))_2 Z_{22} + \Re(-iJ_0 \exp(i\omega u))_2 Z_{2,-2} \\ U &= \Re(U_0 \exp(i\omega u))_1 Z_{22} + \Re(-iU_0 \exp(i\omega u))_1 Z_{2,-2},\end{aligned}\tag{34}$$

where ω is the wave frequency which is twice the orbital frequency and where w_0 , J_0 , U_0 are given by Eqs. (22), with the integration constants taking the values

$$\begin{aligned}\beta_0 &= \frac{M}{r_0} \sqrt{15\pi}, \\ C_1 &= -\frac{2M}{3} \sqrt{15\pi} + \frac{M}{3} i\omega r_0 \sqrt{15\pi} + \frac{2M}{15} \omega^2 r_0^2 \sqrt{15\pi} \\ &\quad - \frac{8M}{15} \alpha^2 r_0^2 \sqrt{15\pi} + \mathcal{O}(r_0^3, \alpha^4), \\ C_3 &= \frac{4M}{5} r_0^2 \sqrt{15\pi} - \frac{4M}{5} i\omega r_0^3 \sqrt{15\pi} - \frac{64M}{105} \omega^2 r_0^4 \sqrt{15\pi} \\ &\quad - \frac{24M}{35} \alpha^2 r_0^4 \sqrt{15\pi} + \mathcal{O}(r_0^5, \alpha^4).\end{aligned}\tag{35}$$

As discussed earlier, the GW field is determined by the value of C_3 . From its expression in Eq. (35), it is clear that the effect of a de Sitter rather than Minkowski background is insignificant if $\alpha^2 r_0^2 \ll 1$. It is difficult to envisage any astrophysical scenario in which that would not be the case.

VI. CONCLUSION

Within the Bondi-Sachs formalism, exact solutions linearized about a de Sitter background have been constructed, and these have been used to investigate some properties of GWs in that spacetime. It was found that the most convenient GW descriptor to use is ψ_4 because of its gauge invariance, and it was also shown that ψ_4 is physically relevant because it is directly related to the geodesic deviation measured by an observer.

The paper investigated the gravitational wave energy determined by various observers. In particular, it was found to be difficult to define observers in free fall for which energy conservation of the GWs would apply. Such observers would need to have velocities that depend on position (which is not surprising) but also on the wave frequency. On the other hand, it was found to be straightforward to define accelerated observers for which energy conservation applies.

The paper determined the gravitational field around an equal mass binary, showing to leading order the additional terms introduced on using a de Sitter background.

Changing from a Minkowski to a de Sitter background introduced modifications to formulas for the generation, propagation, and detection of GWs. In realistic astrophysical scenarios, these modifications are so small that they are completely ignorable. However, the modifications may be significant for the mathematical understanding of the concept of GW energy in a de Sitter spacetime, since in particular the result that inertial observers are not appropriate for describing GW energy is somewhat counterintuitive. The development of a proper theory of mass loss and GW energy is a matter for future work.

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APPENDIX: COMPUTER ALGEBRA SCRIPTS

The Maple scripts used in this work are given in the Supplemental Material [15], and their purposes are summarized here:

- (i) The script *dS.map* evaluates the coordinate transformation from de Sitter to Bondi-Sachs coordinates to obtain Eqs. (4) and (6).
- (ii) The script *lin.map* uses ProcsRules.map and gamma.out and evaluates Eqs. (12) through (18). It also checks that the solution Eq. (22) satisfies all ten Einstein equations.

- (iii) The script *masterEqn.map* evaluates Eq. (21).
- (iv) The script *l2.map* solves Eqs. (12) through (18) to obtain the solution, Eq. (22).
- (v) The script *weyl.map* uses *ProcsRules.map* and *gamma.out* and evaluates the Weyl tensor C_{abcd} to find ψ_i and in particular ψ_4 in Eq. (26). It further evaluates Eqs. (29) through (32).
- (vi) The script *GWfromBinary.map* evaluates Eq. (35).
- (vii) The files *ProcsRules.map* and *gamma.out* are auxiliary files used by other scripts as stated above.

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- [1] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, *Proc. R. Soc. A* **269**, 21 (1962).
 - [2] R. Penrose, *Proc. R. Soc. A* **284**, 159 (1965).
 - [3] A. Ashtekar, B. Bonga, and A. Kesavan, [arXiv:1510.04990](https://arxiv.org/abs/1510.04990).
 - [4] A. Ashtekar, B. Bonga, and A. Kesavan, *Classical Quantum Gravity* **32**, 025004 (2015).
 - [5] A. Ashtekar, B. Bonga, and A. Kesavan, *Phys. Rev. D* **92**, 044011 (2015).
 - [6] A. Ashtekar, B. Bonga, and A. Kesavan, *Phys. Rev. D* **92**, 104032 (2015).
 - [7] G. Date and S. Hoque, [arXiv:1510.07856](https://arxiv.org/abs/1510.07856).
 - [8] C. E. Cedeño M. and J. C. N. de Araujo, *Phys. Rev. D* **92**, 124015 (2015).
 - [9] C. E. Cedeño M. and J. C. N. de Araujo, [arXiv:1503.07731](https://arxiv.org/abs/1503.07731).
 - [10] N. T. Bishop, *Classical Quantum Gravity* **22**, 2393 (2005).
 - [11] T. Mädler, *Phys. Rev. D* **87**, 104016 (2013).
 - [12] N. Bishop and A. Kubeka, *Phys. Rev. D* **80**, 064011 (2009).
 - [13] N. T. Bishop, D. Pollney, and C. Reisswig, *Classical Quantum Gravity* **28**, 155019 (2011).
 - [14] N. Bishop and L. Venter, *Phys. Rev. D* **73**, 084023 (2006).
 - [15] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevD.93.044025> for the Maple scripts.
 - [16] H. Bondi, *Nature (London)* **186**, 535 (1960).
 - [17] R. Sachs, *Proc. R. Soc. A* **270**, 103 (1962).
 - [18] J. Winicour, *Living Rev. Relativity* **8**, 10 (2005).
 - [19] N. T. Bishop and C. Reisswig, *Gen. Relativ. Gravit.* **46**, 1643 (2014).
 - [20] N. T. Bishop, R. Gómez, L. Lehner, and J. Winicour, *Phys. Rev. D* **54**, 6153 (1996).
 - [21] N. T. Bishop, R. Gómez, L. Lehner, M. Maharaj, and J. Winicour, *Phys. Rev. D* **56**, 6298 (1997).
 - [22] R. Gómez, L. Lehner, P. Papadopoulos, and J. Winicour, *Classical Quantum Gravity* **14**, 977 (1997).
 - [23] Y. Zlochower, R. Gómez, S. Husa, L. Lehner, and J. Winicour, *Phys. Rev. D* **68**, 084014 (2003).