Analytic self-force calculations in the post-Newtonian regime:
Eccentric orbits on a Schwarzschild background

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We present a method for solving the first-order Einstein field equations in a post-Newtonian (PN) expansion. Our calculations generalize the work of Bini and Damour and subsequently Kavanagh *et al.* to consider eccentric orbits on a Schwarzschild background. We derive expressions for the retarded metric perturbation at the location of the particle for all ℓ-modes. We find that, despite first appearances, the Regge-Wheeler gauge metric perturbation is C^0 at the particle for all ℓ. As a first use of our solutions, we compute the gauge-invariant quantity $\langle U \rangle$ through 4PN while simultaneously expanding in eccentricity through e^{10}. By anticipating the $e \to 1$ singular behavior at each PN order, we greatly improve the accuracy of our results for large e. We use $\langle U \rangle$ to find 4PN contributions to the effective one body potential $\hat{Q}$ through e^{10} and at linear order in the mass ratio.

I. INTRODUCTION

Recent years have seen a large amount of research in the regime of overlap between two complementary approaches to the general relativistic two body problem: gravitational self-force (GSF) and post-Newtonian (PN) theory. GSF calculations are made within the context of black hole perturbation theory, an expansion in the mass ratio $q \equiv \mu/M$ of the two bodies, which is valid for all speeds. PN theory, on the other hand, is an expansion in small velocities (or equivalently, large separations), but is valid for any q. In the area of parameter space where the two theories overlap, they can check each other and also be used to compute previously unknown parameters.

Much of the recent GSF/PN work has been performed with the eventual goal of detecting gravitational waves. With Advanced LIGO [1] now performing science runs, the need for accurate waveforms is immediate. One very effective framework for producing such waveforms is the effective one body (EOB) model [2]. Careful GSF calculations, when performed in the PN regime, can then be used to provide information on the PN behavior of EOB potentials.

This calibration is possible only because gauge-invariant quantities can be computed separately using the GSF and PN theory. The first such invariant was the "redshift invariant" u^t, originally suggested by Detweiler [3], who then computed u^t numerically with a GSF code and compared it to its PN value through third PN order [4]. Subsequently, a number of papers used numerical techniques to compute u^t to ever higher precision, and fit out previously unknown PN parameters at ever higher order [5–9]. Work by Shah *et al.* [7], in particular, opened a new

avenue to obtaining high-order PN parameters in the linear-in-q limit. Rather than solve the first-order field equations through numerical integration, they employed the function expansion method of Mano, Suzuki and Takasugi (MST) [10,11]. Combining MST with computer algebra software like Mathematica, one can solve the field equations to hundreds or even thousands of digits when considering large radii orbits.

In addition to u^t, other local, circular orbit gauge invariants have been calculated. Barack and Sago [12] computed the shift to the innermost stable circular orbit due to the GSF (which subsequently was used by Akcay *et al.* to inform EOB in Ref. [13]). Then, inspired by work of Harte [14], Dolan *et al.* [15] computed the so-called "spin-invariant" ψ, which measures the GSF effect on geodetic spin precession. Subsequently (as suggested in Ref. [16]), Dolan *et al.* [17] computed higher-order "tidal invariants," followed by Nolan *et al.* [18] computing "octupolar invariants."

All of those calculations were restricted to circular orbits, which are much simpler computationally than eccentric orbits. There are a number of reasons why. First, because circular orbits possess only one harmonic for each ℓm mode, the field equations at that mode reduce from a set of $1 + 1$ partial differential equations to ordinary differential equations. When considering eccentric orbits, one must either solve the $1 + 1$ equations directly in the time domain as an initial value problem, or stay in the frequency domain, wherein the number of harmonics goes from only one to a countably infinite set (although in practice, of course, this is truncated based on some accuracy criterion). Additionally, finding the particular solution in the frequency domain is

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no longer a simple matching, but rather requires an integration over one period of the source's libration.

Despite these difficulties, gauge invariants have been computed for eccentric orbits as well. As suggested first by Damour [19], Barack et al. [20] computed the GSF effect on the precession rate of slightly eccentric orbits around a Schwarzschild black hole. Then, Barack and Sago [21] generalized Detweiler's u^t to generic bound orbits on a Schwarzschild background by proper time averaging it over the perturbed orbit to form $\langle U \rangle$. In that same work, they computed the periapsis advance of eccentric orbits, another gauge invariant. The work in both Refs. [20] and [21] employed a time domain code in the strong field regime. Since then, $\langle U \rangle$ has been thoroughly examined in the weaker field by Akcay et al. [22] using a frequency domain code and direct analytic PN calculations through 3PN. In addition, van de Meent and Shah have computed $\langle U \rangle$ for equatorial orbits on a Kerr background [23] for the first time. Along with providing checks between PN and GSF, these calculations have served as important internal consistency checks for GSF, wherein $\langle U \rangle$ has been found in Lorenz, radiation, and Regge-Wheeler gauges.

To add to the numerical approaches merging PN and GSF, there has been ongoing analytic work. Indeed, the combined use of black hole perturbation theory and PN theory has an extensive history largely inspired by the original MST papers [10,11]. Sago, Nakano, Hikida, Fujita (and many others) [24–31] have used PN expansions to compute fluxes, waveforms and the GSF for a variety of orbits on both Schwarzschild and Kerr backgrounds. More recently, in a series of papers Bini and Damour [16,32–34] have used these methods to analytically find local gauge invariants. They used MST to compute u^t, ψ, as well as tidal invariants in the linear-in-q limit, with a focus on EOB calibration. Since then, Kavanagh et al. [35] have built upon the Bini and Damour approach and computed u^t, ψ, and the tidal invariants, to 21.5PN, the current state of the art.

In this work we present a method for computing GSF quantities sourced by eccentric orbits around a Schwarzschild black hole through use of an analytic PN expansion. Our method extends the circular orbit work of [32,35] by performing an expansion in small eccentricity at each PN order. Our calculations are performed in Regge-Wheeler [36] gauge for all $\ell \geq 2$ and make use of Zerilli's [37] analytic solutions for $\ell = 0, 1$ (although with a slight gauge transformation to the monopole). We collectively refer to this as Regge-Wheeler-Zerilli (RWZ) gauge. We obtain expressions for the retarded metric perturbation and its first derivatives for ℓ modes as a function of the particle's position. Significantly, in our final expressions, we identify poles in the small-e expansion. After factoring out singular-in-e terms at each PN order, we greatly improve the accuracy of our results for large eccentricities. As a use of our solutions, we compute $\langle U \rangle$, confirming the

numeric 4PN predictions of Ref. [23] and provide new PN parameters for 4PN coefficients through e^{10}. We note similar concurrent work [38] of Bini et al. [39], who worked to 6.5PN and e^2.

The paper is arranged as follows. In Sec. II we give an overview of eccentric orbits on a Schwarzschild background and our frequency domain method for solving the first-order field equations sourced by such orbits. In Sec. III we demonstrate our method for finding analytic expressions for the retarded metric perturbation in a double PN/small-eccentricity expansion. Section IV discusses the details of the gauge invariant we compute. Section V gives results for both the metric perturbation itself and $\langle U \rangle$, showing the merits of our resummation of the small-e series. We finish with a brief discussion in Sec. VI. Our Appendix provides details on our low-order modes and $\langle U \rangle$ written in a slightly different form, for comparison purposes.

Throughout, we use the $(-, +, +, +)$ metric signature and set $c = G = 1$ (although we briefly use $\eta = c^{-1}$ for PN power counting). Lowercase greek indices run over Schwarzschild coordinates, t, r, θ, φ. We make use of the Martel and Poisson [40] $\mathcal{M}^2 \times \mathcal{S}^2$ decomposition. Following their notation, lowercase latin indices indicate t or r while uppercase latin indices are either θ or φ.

II. INHOMOGENEOUS SOLUTIONS TO THE FIRST-ORDER FIELD EQUATIONS IN RWZ GAUGE

In this section we cover our method for solving the RWZ-gauge first-order field equations in the frequency domain, taking the source to be a point particle in eccentric orbit. With the exception of a brief discussion of the singular structure of RWZ gauge at the end of this section, we follow closely the more detailed presentation of Ref. [41]. We will work in Schwarzschild coordinates where the metric takes the standard form,

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \quad (2.1)$$

with $f \equiv 1 - 2M/r$.

A. Bound orbits on a Schwarzschild background

Let a small body, or particle, of mass μ orbit a static black hole of mass M, assuming $q \equiv \mu/M \ll 1$. We parametrize the particle's background geodesic by proper time τ, writing $x_p^\alpha(\tau) = [t_p(\tau), r_p(\tau), \pi/2, \varphi_p(\tau)]$. Here and subsequently we use the subscript p to indicate a quantity evaluated on the worldline. Note that by choosing $\theta_p = \pi/2$ we have confined the particle to the equatorial plane with no loss of generality. Differentiating x_p yields the four-velocity

$$u^\alpha = \left(\frac{\mathcal{E}}{f_p}, u^r, 0, \frac{\mathcal{L}}{r_p^2} \right), \quad (2.2)$$

where we have defined the two constants of motion, the specific energy $\mathcal{E}$ and specific angular momentum $\mathcal{L}$. The constraint $u^\alpha u_\alpha = -1$ implies the following relation,

$$\dot{r}_p^2(t) = f_p^2 - \frac{f_p^2}{\mathcal{E}^2}U_p^2, \quad U^2(r,\mathcal{L}^2) \equiv f\left(1+\frac{\mathcal{L}^2}{r^2}\right), \quad (2.3)$$

where a dot indicates a coordinate time derivative.

Any geodesic on a Schwarzschild background can be parametrized using $\mathcal{E}$ and $\mathcal{L}$. For bound, eccentric motion, however, it is convenient to instead use the (dimensionless) semilatus rectum p and the eccentricity e (see [42,43]). These two pairs of parameters are related by

$$\mathcal{E}^2 = \frac{(p-2)^2 - 4e^2}{p(p-3-e^2)}, \quad \mathcal{L}^2 = \frac{p^2 M^2}{p-3-e^2}. \quad (2.4)$$

Bound orbits satisfy the inequality $p > 6 + 2e$ [42].

Using the p, e parametrization, the radial position of the particle is given as a function of Darwin's [44] relativistic anomaly χ,

$$r_p(\chi) = \frac{pM}{1 + e\cos\chi}. \quad (2.5)$$

As χ runs from $0 \to 2\pi$, the particle travels one radial libration, starting at periapsis. The quantities τ_p, t_p, and φ_p are found by solving first-order differential equations in χ,

$$\frac{dt_p}{d\chi} = \frac{p^2 M}{(p-2-2e\cos\chi)(1+e\cos\chi)^2}$$
$$\times \left[\frac{(p-2)^2 - 4e^2}{p-6-2e\cos\chi}\right]^{1/2},$$
$$\frac{d\varphi_p}{d\chi} = \left[\frac{p}{p-6-2e\cos\chi}\right]^{1/2},$$
$$\frac{d\tau_p}{d\chi} = \frac{Mp^{3/2}}{(1+e\cos\chi)^2}\left[\frac{p-3-e^2}{p-6-2e\cos\chi}\right]^{1/2}. \quad (2.6)$$

There is an analytic solution for φ_p,

$$\varphi_p(\chi) = \left(\frac{4p}{p-6-2e}\right)^{1/2} F\left(\frac{\chi}{2}\bigg| -\frac{4e}{p-6-2e}\right), \quad (2.7)$$

where $F(x|m)$ is the incomplete elliptic integral of the first kind [45]. Note that there also exists a more extensive semianalytic solution for t_p [46], though we do not provide it here.

Eccentric orbits have two fundamental frequencies. The libration between periapsis and apoapsis is described by

$$\Omega_r \equiv \frac{2\pi}{T_r}, \quad T_r \equiv \int_0^{2\pi}\left(\frac{dt_p}{d\chi}\right)d\chi. \quad (2.8)$$

Meanwhile, the average rate of azimuthal advance over one radial period is

$$\Omega_\varphi \equiv \frac{\varphi_p(2\pi)}{T_r} = \frac{4}{T_r}\left(\frac{p}{p-6-2e}\right)^{1/2}\mathcal{K}\left(-\frac{4e}{p-6-2e}\right), \quad (2.9)$$

with $\mathcal{K}(m)$ being the complete elliptic integral of the first kind [45]. The two frequencies Ω_r and Ω_φ are only equal in the Newtonian limit.

B. Solutions to the time domain master equation

In RWZ gauge the field equations for the metric perturbation amplitudes can be reduced to a single wave equation for each ℓm mode. The equation is satisfied by a parity-dependent master function from which the metric perturbation amplitudes can be readily recovered. When $\ell + m$ is odd, we use the Cunningham-Price-Moncrief (CPM) function, $\Psi_{\ell m}^o$, and when $\ell + m$ is even, we use the Zerilli-Moncrief (ZM) function, $\Psi_{\ell m}^e$. In the remainder of this subsection we will use $\Psi_{\ell m}$ with no superscript to refer to either the ZM or CPM variable. In each case the master equation has the form

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_\ell(r)\right]\Psi_{\ell m}(t,r) = S_{\ell m}(t,r), \quad (2.10)$$

where both the potential V_ℓ and the source term $S_{\ell m}$ are parity dependent. The variable $r_* = r + 2M\log(r/2M - 1)$ is the standard tortoise coordinate. The source term $S_{\ell m}$ is of the form

$$S_{\ell m} = G_{\ell m}(t)\delta[r - r_p(t)] + F_{\ell m}(t)\delta'[r - r_p(t)], \quad (2.11)$$

where the coefficients $G_{\ell m}$ are $F_{\ell m}$ are smooth functions. While it is certainly possible to solve Eq. (2.10) directly in the time domain, at present we are interested in a frequency domain approach. As such, we decompose both $\Psi_{\ell m}$ and $S_{\ell m}$ in Fourier series as

$$\Psi_{\ell m}(t,r) = \sum_{n=-\infty}^{\infty} X_{\ell mn}(r)e^{-i\omega t},$$

$$S_{\ell m}(t,r) = \sum_{n=-\infty}^{\infty} Z_{\ell mn}(r)e^{-i\omega t}. \quad (2.12)$$

In these expressions, the frequency $\omega \equiv \omega_{mn} = m\Omega_\varphi + n\Omega_r$, which follows from the biperiodic source. Note that we use the notation $X_{\ell mn} \equiv X_{\ell m\omega_{mn}}$. The Fourier series coefficients are found as usual by

$$X_{\ell m n}(r) = \frac{1}{T_r} \int_0^{T_r} dt \, \Psi_{\ell m}(t, r) e^{i\omega t},$$

$$Z_{\ell m n}(r) = \frac{1}{T_r} \int_0^{T_r} dt \, S_{\ell m}(t, r) e^{i\omega t}. \qquad (2.13)$$

Combining Eqs. (2.10) and (2.12) yields the frequency domain master equation,

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_\ell(r) \right] X_{\ell m n}(r) = Z_{\ell m n}(r). \qquad (2.14)$$

This equation has two causal homogeneous solutions. At spatial infinity the "up" solution $\hat{X}_{\ell m n}^+$ trends to $e^{i\omega r_*}$. As $r_* \to -\infty$ at the horizon, the "in" solution $\hat{X}_{\ell m n}^-$ trends to $e^{-i\omega r_*}$. Here we use a hat to emphasize that these are un-normalized homogeneous solutions.

With a causal pair of linearly independent solutions, one would normally find the particular solution to Eq. (2.14) through the method of variation of parameters. However, in this case, the singular source (2.11) leads to a Gibbs phenomenon which spoils the exponential convergence when forming $\Psi_{\ell m}$ in Eq. (2.12). It is now standard to use the method of extended homogeneous solutions [41,47] to obtain exponential convergence of $\Psi_{\ell m}$ and all its derivatives at all locations, including the particle's. The method is covered extensively elsewhere, and so we simply recount the procedure here.

We start by performing a convolution integral between the homogeneous solutions and the frequency domain source, which yields normalization coefficients,

$$C_{\ell m n}^\pm = \frac{1}{W_{\ell m n}} \int_{r_{\min}}^{r_{\max}} dr \frac{\hat{X}_{\ell m n}^\mp(r) Z_{\ell m n}(r)}{f(r)}, \qquad (2.15)$$

where $W_{\ell m n}$ is the (constant in r) Wronskian

$$W_{\ell m n} = f(r) \left(\hat{X}_{\ell m n}^- \frac{d\hat{X}_{\ell m n}^+}{dr} - \hat{X}_{\ell m n}^+ \frac{d\hat{X}_{\ell m n}^-}{dr} \right). \qquad (2.16)$$

Note that the integral (2.15) is formally over all r, but we write it here as limited to the libration range $r_{\min} \leq r \leq r_{\max}$, since outside that region $Z_{\ell m n} = 0$. We next form the frequency domain extended homogeneous solutions,

$$X_{\ell m n}^\pm(r) \equiv C_{\ell m n}^\pm \hat{X}_{\ell m n}^\pm(r), \quad r > 2M, \qquad (2.17)$$

and subsequently define the time domain extended homogeneous solutions,

$$\Psi_{\ell m}^\pm(t, r) \equiv \sum_n X_{\ell m n}^\pm(r) e^{-i\omega t}, \quad r > 2M. \qquad (2.18)$$

The time domain extended homogeneous solutions are formed from a set of smooth functions and therefore the

sum (2.18) is exponentially convergent for all $r > 2M$, and all t. Finally, the particular solution to Eq. (2.10) is of the weak form

$$\Psi_{\ell m}(t, r) = \Psi_{\ell m}^+(t, r)\theta[r - r_p(t)] + \Psi_{\ell m}^-(t, r)\theta[r_p(t) - r], \qquad (2.19)$$

where θ is the Heaviside distribution.

C. Metric perturbation reconstruction

The even-parity metric perturbation amplitudes are reconstructed from the ZM master function via the relations

$$K^{\ell m, \pm}(t, r) = f\partial_r \Psi_{\ell m}^{e, \pm} + A\Psi_{\ell m}^{e, \pm},$$

$$h_{rr}^{\ell m, \pm}(t, r) = \frac{\Lambda}{f^2} \left[\frac{\lambda + 1}{r} \Psi_{\ell m}^{e, \pm} - K^{\ell m, \pm} \right] + \frac{r}{f} \partial_r K^{\ell m, \pm},$$

$$h_{tr}^{\ell m, \pm}(t, r) = r\partial_t \partial_r \Psi_{\ell m}^{e, \pm} + rB\partial_t \Psi_{\ell m}^{e, \pm},$$

$$h_{tt}^{\ell m, \pm}(t, r) = f^2 h_{rr}^{\ell m, \pm}, \qquad (2.20)$$

where $\Lambda(r) \equiv \lambda + 3M/r$, $\lambda \equiv (\ell + 2)(\ell - 1)/2$, and

$$A(r) \equiv \frac{1}{r\Lambda} \left[\lambda(\lambda + 1) + \frac{3M}{r} \left(\lambda + \frac{2M}{r} \right) \right],$$

$$B(r) \equiv \frac{1}{rf\Lambda} \left[\lambda \left(1 - \frac{3M}{r} \right) - \frac{3M^2}{r^2} \right]. \qquad (2.21)$$

The odd-parity metric perturbation amplitudes are reconstructed from the CPM variable,

$$h_t^{\ell m, \pm}(t, r) = \frac{f}{2} \partial_r (r\Psi_{\ell m}^{o, \pm}), \qquad h_r^{\ell m, \pm}(t, r) = \frac{r}{2f} \partial_t \Psi_{\ell m}^{o, \pm}. \qquad (2.22)$$

In these expressions we have included $\pm$ superscripts to indicate that metric perturbation amplitudes can be reconstructed on either the left or right side of the particle.

At last, the retarded metric perturbation (which we write as $p_{\mu\nu}$) can be synthesized by multiplying by spherical harmonics and summing over ℓm modes. Our particular harmonic decomposition is due to Martel and Poisson [40]. The spherical harmonics used below (even-parity scalar $Y^{\ell m}$ and odd-parity vector $X_B^{\ell m}$), along with the two-sphere metric (Ω_{AB}), can be found in that reference. Lowercase latin indices run over t and r while uppercase latin indices run over θ and φ. To emphasize a point about the singular nature of RWZ gauge, we perform the sum over spherical harmonics in two stages. First, we form the ℓm contribution to each metric perturbation component (see Ref. [41]),

$$p_{ab}^{\ell m}(x^\mu) = [h_{ab}^{\ell m,+}(t,r)\theta(z)$$
$$+ h_{ab}^{\ell m,-}(t,r)\theta(-z) + h_{ab}^{\ell m,S}(t)\delta(z)]Y^{\ell m}(\theta,\varphi),$$
$$p_{aB}^{\ell m}(x^\mu) = [h_a^{\ell m,+}(t,r)\theta(z)$$
$$+ h_a^{\ell m,-}(t,r)\theta(-z)]X_B^{\ell m}(\theta,\varphi),$$
$$p_{AB}^{\ell m}(x^\mu) = [K^{\ell m,+}(t,r)\theta(z)$$
$$+ K^{\ell m,-}(t,r)\theta(-z)]r^2\Omega_{AB}Y^{\ell m}(\theta,\varphi). \quad (2.23)$$

These expressions are written as weak solutions in terms of the Heaviside and Dirac distributions θ and δ, which depend on $z \equiv r - r_p(t)$. Note that in general the Martel-Poisson decomposition also includes scalar amplitudes $j_a^{\ell m}$, $G^{\ell m}$, and $h_2^{\ell m}$, but these are set to zero in RWZ gauge.

The expressions in Eq. (2.23) suggest that the RWZ gauge retarded metric perturbation is not only discontinuous at the location of the particle, but actually proportional to the Dirac delta function for certain components. Moreover, this singularity is spread over a two-sphere of radius $r = r_p(t)$. However, we find that, at least at the particle's location, this is in fact not true. Indeed, setting $\theta = \theta_p(t) = \pi/2$, $\varphi = \varphi_p(t)$ and summing the expressions (2.23) over m exactly cancels out the delta functions, and the remaining amplitudes are actually continuous at $r = r_p$ for all $\ell \geq 2$. Thus, we write

$$p_{ab}^\ell(x_p^\mu) = \sum_m h_{ab}^{\ell m,\pm}(t_p,r_p)Y^{\ell m}(\pi/2,\varphi_p),$$
$$p_{aB}^\ell(x_p^\mu) = \sum_m h_a^{\ell m,\pm}(t_p,r_p)X_B^{\ell m}(\pi/2,\varphi_p),$$
$$p_{AB}^\ell(x_p^\mu) = r_p^2\Omega_{AB}\sum_m K^{\ell m,\pm}(t_p,r_p)Y^{\ell m}(\pi/2,\varphi_p), \quad (2.24)$$

and a single-valued rhs becomes a check on all calculations. The full retarded metric perturbation is then a simple sum over all ℓ,

$$p_{ab}(x_p^\mu) = \sum_{\ell=0}^\infty p_{ab}^\ell(x_p^\mu),$$
$$p_{aB}(x_p^\mu) = \sum_{\ell=0}^\infty p_{aB}^\ell(x_p^\mu),$$
$$p_{AB}(x_p^\mu) = \sum_{\ell=0}^\infty p_{AB}^\ell(x_p^\mu). \quad (2.25)$$

Zerilli's analytic $\ell = 0$, 1 solutions are given in Appendix A.

III. POST-NEWTONIAN SOLUTIONS

We now present our method for finding the retarded metric perturbation induced by a particle in eccentric motion about a Schwarzschild black hole. We combine

the formalism of the previous section with PN expansions of all relevant quantities. Our final results, given in Sec. V, include expansions to 4PN (that is, four terms beyond leading order). At each PN order we expand to 10th order in eccentricity. For the purposes of pedagogy and space, however, in this section we keep only two PN orders and (where necessary) two terms in the small-e expansion. Our final solutions will give the metric perturbation at the location of the particle for each ℓ-mode, as in Eq. (2.24). Note that the presentation from Sec. III B onward is only relevant to modes $\ell \geq 2$.

A. Post-Newtonian expansions of orbit quantities

1. Position-independent orbit quantities

We take as our PN expansion parameter the inverse of the dimensionless semilatus rectum p. Assuming it to be small, we expand $dt_p/d\chi$ from Eq. (2.6). Inserting the resulting expansion in Eq. (2.8) we integrate order by order and find

$$\Omega_r = \frac{1}{M}\left(\frac{1-e^2}{p}\right)^{3/2}\left[1 - 3\frac{1-e^2}{p} + \mathcal{O}(p^{-2})\right]. \quad (3.1)$$

Expanding Eq. (2.9) in the same limit gives

$$\Omega_\varphi = \frac{1}{M}\left(\frac{1-e^2}{p}\right)^{3/2}\left[1 + 3\frac{e^2}{p} + \mathcal{O}(p^{-2})\right]. \quad (3.2)$$

As expected $\Omega_r = \Omega_\varphi$ at Newtonian order. We now introduce the dimensionless gauge-invariant PN parameter y,

$$y \equiv (M\Omega_\varphi)^{2/3}. \quad (3.3)$$

Combining Eqs. (3.2) and (3.3) we find a PN expansion for y in terms of p. We invert the expansion to get p in terms of y:

$$p = \frac{1-e^2}{y} + 2e^2 + \mathcal{O}(y^1). \quad (3.4)$$

This allows us to obtain expansions for both Ω_r and Ω_φ in powers of y,

$$\Omega_r = \frac{y^{3/2}}{M}\left[1 - \frac{3}{1-e^2}y + \mathcal{O}(y^2)\right], \quad \Omega_\varphi = \frac{y^{3/2}}{M}. \quad (3.5)$$

Note that the Ω_φ expression is exact due to the definition of y. With the above expressions, we are able to find PN expansions of all orbit quantities in terms of y. The specific energy and angular momentum follow from Eq. (2.4),

$$\mathcal{E} = 1 - \frac{y}{2} + \frac{3 + 5e^2}{8 - 8e^2} y^2 + \mathcal{O}(y^3),$$

$$\mathcal{L} = \sqrt{1 - e^2} M y^{-1/2} + \frac{3M(1 + e^2)}{2\sqrt{1 - e^2}} y^{1/2} + \mathcal{O}(y^{3/2}).$$

$$(3.6)$$

Later, we will also need the radial period as measured in coordinate time [Eq. (2.8)] and proper time (found by integrating $d\tau_p/d\chi$ from $0 - 2\pi$). They are, respectively,

$$T_r = \frac{2\pi M}{y^{3/2}} \left[1 + \frac{3}{1 - e^2} y + \mathcal{O}(y^2) \right],$$

$$\mathcal{T}_r = \frac{2\pi M}{y^{3/2}} \left[1 + \frac{3 + 3e^2}{2 - 2e^2} y + \mathcal{O}(y^2) \right]. \qquad (3.7)$$

2. Position-dependent orbit quantities

We now expand quantities that vary along the worldline as functions of the relativistic anomaly χ. We start with the radial position r_p, first expanding in the PN parameter y, and then at each PN order we expand in eccentricity e. The resulting double expansion (here keeping only the first two nonzero orders in each y and e) is

$$r_p(\chi) = M[1 - e\cos\chi + \mathcal{O}(e^2)]y^{-1} + M[2e^2$$
$$- 2e^3 \cos\chi + \mathcal{O}(e^4)]y^0 + \mathcal{O}(y^1). \qquad (3.8)$$

Next, consider the $\varphi_p(t)$ motion, which can be decomposed into two parts as [47]

$$\varphi_p(t) = \Omega_\varphi t + \Delta\varphi(t). \qquad (3.9)$$

The first term represents the mean azimuthal advance, while the second term is periodic in T_r. In our expansions we avoid ever using $\Omega_\varphi t$ explicitly, and work only with the $\Delta\varphi$. This is convenient because terms involving $\Omega_\varphi t$ can lead to linear-in-χ terms. For example, notice that after leading order in y, $e^{im\Omega_\varphi t}$ is not strictly oscillatory,

$$e^{im\Omega_\varphi t} = [e^{im\chi} - 2im\sin(\chi)e^{im\chi}e + \mathcal{O}(e^2)]y^0$$
$$+ [3im\chi e^{im\chi} + 3m(2m\chi - i)\sin(\chi)e^{im\chi}e$$
$$+ \mathcal{O}(e^2)]y + \mathcal{O}(y^2). \qquad (3.10)$$

On the other hand, $e^{in\Omega_r t}$ is oscillatory for all PN orders,

$$e^{in\Omega_r t} = [e^{in\chi} - 2in\sin(\chi)e^{in\chi}e + \mathcal{O}(e^2)]y^0$$
$$+ [3in\sin(\chi)e^{in\chi}e + \mathcal{O}(e^2)]y + \mathcal{O}(y^2). \qquad (3.11)$$

This qualitative difference between the fundamental frequencies can be traced back to the fact that Ω_φ is a "rotation-type" frequency describing average accumulation of phase while Ω_r is a "libration-type" frequency describing periodic motion in r. Since $\Delta\varphi$ is periodic in T_r we find that its expansion is free of linear-in-χ terms,

$$\Delta\varphi(\chi) = [2e\sin\chi + \mathcal{O}(e^2)]y^0$$
$$+ [4e\sin\chi + \mathcal{O}(e^2)]y + \mathcal{O}(y^2). \qquad (3.12)$$

Lastly, because they will be useful later, we also note the expansions

$$e^{-im\Delta\varphi(\chi)} = [1 - 2ime\sin\chi + \mathcal{O}(e^2)]y^0 + [-4ime\sin\chi + \mathcal{O}(e^2)]y + \mathcal{O}(y^2),$$

$$\frac{dt_p}{d\chi} = M[1 - 2e\cos\chi + \mathcal{O}(e^2)]y^{-3/2} + M[3 - 3e\cos\chi + \mathcal{O}(e^2)]y^{-1/2} + \mathcal{O}(y^{1/2}),$$

$$\frac{d\tau_p}{d\chi} = M[1 - 2e\cos\chi + \mathcal{O}(e^2)]y^{-3/2} + M\left[\frac{3}{2} - 2e\cos\chi + \mathcal{O}(e^2)\right]y^{-1/2} + \mathcal{O}(y^{1/2}). \qquad (3.13)$$

B. Frequency domain homogeneous solutions

We now derive expressions for the un-normalized frequency domain master function evaluated at the particle's location. We start with homogeneous solutions to the odd-parity master equation of the form derived in Refs. [32,35]. These solutions are written as a double expansion in small frequency and large r using the dimensionless expansion parameters

$$\mathcal{X}_1 \equiv \frac{M}{r}, \qquad \mathcal{X}_2 \equiv (\omega r)^2, \qquad (3.14)$$

which are assumed to be the same order of magnitude. Note that we choose the symbols $\mathcal{X}_1$ and $\mathcal{X}_2$ rather than the standard X_1 and X_2 to avoid confusion with the frequency domain master function. Written in terms of these variables, the $\ell = 2$ homogeneous solutions to the odd-parity master equation are

$$\hat{X}_2^{o,+} = \mathcal{X}_1^2 \eta^4 + \frac{1}{6} \mathcal{X}_1^2 (10\mathcal{X}_1 + \mathcal{X}_2)\eta^6 + \mathcal{O}(\eta^8),$$

$$\hat{X}_2^{o,-} = \frac{1}{\mathcal{X}_1^3} \eta^{-6} - \frac{\mathcal{X}_2}{14\mathcal{X}_1^3} \eta^{-4} + \mathcal{O}(\eta^{-2}), \qquad (3.15)$$

with $+$ indicating the infinity-side (up) solution, and $-$ the horizon-side (in) solution. We use $\eta = c^{-1}$ to keep track of the PN order. The even-parity solutions are computed from the Chandrasekhar-Detweiler transformation [48]

$$\hat{X}_\ell^e = \left[\frac{(\ell-1)\ell(\ell+1)(\ell+2)}{24} + \frac{3(1-2\mathcal{X}_1)\mathcal{X}_1^2}{(\ell-1)(\ell+2)+6\mathcal{X}_1}\right]\hat{X}_\ell^o$$
$$+ \frac{M(1-2\mathcal{X}_1)}{2}\frac{d\hat{X}_\ell^o}{dr}, \tag{3.16}$$

which is true to all PN orders. We compute r derivatives with the relation

$$\frac{d}{dr} = \frac{\mathcal{X}_1}{M}\left(\sqrt{\mathcal{X}_2}\frac{d}{d\sqrt{\mathcal{X}_2}} - \mathcal{X}_1\frac{d}{d\mathcal{X}_1}\right). \tag{3.17}$$

Then, the even-parity $\ell = 2$ homogeneous solutions are

$$\hat{X}_2^{e,+} = \mathcal{X}_1^2\eta^4 + \left(\frac{2\mathcal{X}_1^3}{3} + \frac{\mathcal{X}_1^2\mathcal{X}_2}{6}\right)\eta^6 + \mathcal{O}(\eta^8),$$

$$\hat{X}_2^{e,-} = \frac{1}{\mathcal{X}_1^3}\eta^{-6} + \left(\frac{3}{2\mathcal{X}_1^2} - \frac{\mathcal{X}_2}{14\mathcal{X}_1^3}\right)\eta^{-4} + \mathcal{O}(\eta^{-2}). \tag{3.18}$$

For the remainder of this presentation we focus on the specific example of infinity side, odd parity. The even-parity and horizon-side calculations, as well as those for the r-derivatives of the master function, follow from an equivalent procedure.

We are interested in evaluating the homogeneous solutions at the location of the particle. Plugging in Eq. (3.14) with $r = r_p(t)$ we find

$$\hat{X}_2^{o,+} = \frac{M^2}{r_p^2}\eta^4 + \frac{M^2(10M + \omega^2 r_p^3)}{6r_p^3}\eta^6 + \mathcal{O}(\eta^8). \tag{3.19}$$

In this expression ω and r_p are valid to all PN orders. To make further progress, we now expand ω and r_p in both the PN parameter y and the eccentricity e. The double expansion of ω (again keeping only the first two nonzero orders in y and e) is

$$\omega = \omega_{mn} = \frac{(m+n)}{M}y^{3/2}$$
$$+ \left[-\frac{3n}{M} - \frac{3ne^2}{M} + \mathcal{O}(e^3)\right]y^{5/2} + \mathcal{O}(y^{7/2}). \tag{3.20}$$

Inserting this, along with the PN expression for r_p from Eq. (3.8) into Eq. (3.19) we find (dropping η in favor of y for counting PN orders)

$$\hat{X}_{2mn}^{o,+} = [1 + 2e\cos\chi + \mathcal{O}(e^2)]y^2$$
$$+ \left[\frac{1}{6}(m^2 + 2nm + n^2 + 10) + 5e\cos\chi + \mathcal{O}(e^2)\right]y^3$$
$$+ \mathcal{O}(y^4). \tag{3.21}$$

This expression, while a function of χ, which tracks the particle, is still just a homogeneous solution to the frequency domain master equation. Note that Eq. (3.21) is specific to $\ell = 2$, but is valid for $m = \pm 1$ and any n (though we will see that a finite e expansion limits the number of relevant harmonics n).

C. Frequency domain extended homogeneous solutions

Our next step is to find normalization coefficients so that we can form the frequency domain extended homogeneous solutions, as in Eq. (2.17). In Eq. (2.15) we wrote an expression for $C_{\ell mn}^{\pm}$ that depends on a generic source bounded between $r_{\min}$ and $r_{\max}$. We now specify to the form given in Eq. (2.11). It can be shown [41] that such a source, when combined with Eq. (2.13), leads to a normalization integral of the form

$$C_{\ell mn}^{\pm} = \frac{1}{W_{\ell mn}T_r}\int_0^{T_r}\left[\frac{1}{f_p}\hat{X}_{\ell mn}^{\mp}G_{\ell m} + \left(\frac{2M}{r_p^2 f_p^2}\hat{X}_{\ell mn}^{\mp}\right.\right.$$
$$\left.\left. - \frac{1}{f_p}\frac{d\hat{X}_{\ell mn}^{\mp}}{dr}\right)F_{\ell m}\right]e^{i\omega t}dt, \tag{3.22}$$

The terms $G_{\ell m}$ and $F_{\ell m}$ arise from taking spherical harmonic projections of the point particle's stress energy tensor, and as such, each carries a factor of $e^{-im\varphi_p(t)}$. Noting Eq. (3.9), it is useful to remove the $e^{-im\Omega_\varphi t}$ contribution from $G_{\ell m}$ and $F_{\ell m}$ by defining [49]

$$\bar{G}_{\ell m}(t) \equiv G_{\ell m}(t)e^{im\Omega_\varphi t}, \quad \bar{F}_{\ell m}(t) \equiv F_{\ell m}(t)e^{im\Omega_\varphi t}, \tag{3.23}$$

which are T_r-periodic. The $e^{-im\Omega_\varphi t}$ term cancels with a compensating term in the $e^{i\omega t}$ of Eq. (3.22). Thus, changing the integration variable to χ, we are left with

$$C_{\ell mn}^{\pm} = \frac{1}{W_{\ell mn}T_r}\int_0^{2\pi}\left[\frac{1}{f_p}\hat{X}_{\ell mn}^{\mp}\bar{G}_{\ell m} + \left(\frac{2M}{r_p^2 f_p^2}\hat{X}_{\ell mn}^{\mp}\right.\right.$$
$$\left.\left. - \frac{1}{f_p}\frac{d\hat{X}_{\ell mn}^{\mp}}{dr}\right)\bar{F}_{\ell m}\right]e^{in\Omega_r t}\frac{dt_p}{d\chi}d\chi, \tag{3.24}$$

and all terms in the integrand are now 2π-periodic (when considered as functions of χ).

Generally, we compute the integral (3.24) numerically. In the PN/small-eccentricity regime, however, we are able to perform the integral analytically ℓ by ℓ. In previous subsections, we have already computed PN expansions of $\hat{X}_{\ell mn}^{\mp}$, r_p, $dt_p/d\chi$, and $e^{in\Omega_r t}$. The f_p terms follow naturally from the r_p expansion. What remains is the expansion of $\bar{G}_{\ell m}$ and $\bar{F}_{\ell m}$.

In the odd-parity sector we have

$$G^o_{\ell m} = \frac{32\pi\mu\mathcal{L}f_p}{(\ell-1)\ell(\ell+1)(\ell+2)\mathcal{E}^2 r_p^5}\{\mathcal{L}\mathcal{E}r_p^2 \dot{r}_p X^{\ell m,*}_{\varphi\varphi} - f_p[5Mr_p^2 + 7M\mathcal{L}^2 + (2\mathcal{E}^2-1)r_p^3 - 2\mathcal{L}^2 r_p]X^{\ell m,*}_\varphi\},$$

$$F^o_{\ell m} = \frac{32\pi\mu\mathcal{L}f_p^3(r_p^2 + \mathcal{L}^2)}{(\ell-1)\ell(\ell+1)(\ell+2)\mathcal{E}^2 r_p^3}X^{\ell m,*}_\varphi. \tag{3.25}$$

The terms $X^{\ell m,*}_\varphi$ and $X^{\ell m,*}_{\varphi\varphi}$ are complex conjugates of odd-parity vector and tensor spherical harmonics, evaluated on the worldline. See Ref. [41] for details. We now insert the various expansions we have already computed, as usual only keeping two terms in each y and e. The resulting 2π-periodic expressions are

$$\bar{G}^o_{\ell m} = \frac{16\pi\mu}{M}\frac{\partial_\theta Y_{\ell m}(\pi/2,0)}{(\ell-1)\ell(\ell+1)(\ell+2)}\{2[-1-2(\cos\chi - im\sin\chi)e + \mathcal{O}(e^2)]y^{3/2}$$
$$+ [1 + 4(2\cos\chi + im\sin\chi)e + \mathcal{O}(e^2)]y^{5/2} + \mathcal{O}(y^{7/2})\},$$

$$\bar{F}^o_{\ell m} = 16\pi\mu\frac{\partial_\theta Y_{\ell m}(\pi/2,0)}{(\ell-1)\ell(\ell+1)(\ell+2)}\{2[1 + (\cos\chi - 2im\sin\chi)e + \mathcal{O}(e^2)]y^{1/2}$$
$$- [5 + (13\cos\chi - 2im\sin\chi)e + \mathcal{O}(e^2)]y^{3/2} + \mathcal{O}(y^{5/2})\}. \tag{3.26}$$

Having expanded all the relevant quantities, we are now in a position to perform the integral (3.24) analytically order by order. The integral itself is straightforward; we encounter nothing more than complex exponentials. The resulting normalization coefficient is

$$C^{o,+}_{2mn} = \mu\partial_\theta Y_{2m}(\pi/2,0)\sin(n\pi)\left\{\frac{16}{15}\left[\frac{1}{n} + \frac{(n-2m)}{n^2-1}e + \mathcal{O}(e^2)\right]y^{-3/2}\right.$$
$$+ \frac{4}{105}\left[-\frac{(3m^2 + 6mn + 3n^2 + 14)}{n} + \frac{(6m^3 + 3m^2n - m(12n^2 + 77) - 9n^3 + 49n)}{n^2-1}e + \mathcal{O}(e^2)\right]y^{-1/2}$$
$$\left. + \mathcal{O}(y^{1/2})\right\}. \tag{3.27}$$

Note that there are several terms in this expression which appear at first glance to diverge for integer values of n, but all values are finite in the limit. With the normalization coefficients in hand, the frequency domain extended homogeneous solutions are just the product of $\hat{X}^\pm_{\ell mn}$ and $C^\pm_{\ell mn}$ as shown in Eq. (2.17),

$$X^{o,+}_{2mn} = \mu\partial_\theta Y_{2m}(\pi/2,0)\sin(\pi n)\left\{\frac{16}{15}\left[\frac{1}{n} + \left(\frac{n-2m}{n^2-1} + \frac{2}{n}\cos\chi\right)e + \mathcal{O}(e^2)\right]y^{1/2}\right.$$
$$+ \frac{4}{105}\left[\frac{5m^2 + 10mn + 5n^2 + 98}{3n} - \left(\frac{2(3m^2 + 6mn + 3n^2 - 56)}{n}\cos\chi\right.\right.$$
$$\left.\left.\left. + \frac{10m^3 + 33m^2n + 36mn^2 + 511m + 13n^3 - 287n}{3(n^2-1)}\right)e + \mathcal{O}(e^2)\right]y^{3/2} + \mathcal{O}(y^{5/2})\right\}. \tag{3.28}$$

D. Radially periodic, time domain extended homogeneous solutions

We are now in a position to return to the time domain by summing over harmonics n, as in Eq. (2.18). Since we have performed the e-expansion to a finite order, the sum (2.18) actually truncates. We find that if we compute the frequency domain extended homogeneous solutions with eccentricity contributions up to e^N, then the only nonzero terms in Eq. (2.18) are in the range $-N \leq n \leq N$.

Another wrinkle comes into play at this stage. The full time domain extended homogeneous solutions are formed from a sum involving $e^{-i\omega t} = e^{-i(m\Omega_\varphi + n\Omega_r)t}$. As discussed in Sec. III A, however, the factor $e^{-im\Omega_\varphi t}$ is not purely oscillatory when expanded in y and e. It is therefore more useful to form the quantity

$$\bar{\Psi}^\pm_{\ell m}(t,r) \equiv \Psi^\pm_{\ell m}(t,r)e^{im\Omega_\varphi t} = \sum_n X^\pm_{\ell mn}(r)e^{-in\Omega_r t}. \tag{3.29}$$

Contrary to $\Psi_{\ell m}^{\pm}$, $\bar{\Psi}_{\ell m}^{\pm}$ is purely periodic in χ, with no linear terms. This leads one to ask: do we not need that $e^{-im\Omega_\varphi t}$ term? Surprisingly, the answer is no, at least when computing local quantities. The reason is that whenever we want to compute anything physically relevant, like the GSF, we have to multiply by spherical harmonics and sum over the ℓm modes. The spherical harmonics carry an exactly compensating term $e^{im\Omega_\varphi t}$ which cancels this piece out.

We have been considering the example of odd-parity, $\ell = 2$ expanded to include two powers of y and e. We can form $\bar{\Psi}_{2m}^{o,\pm}$ by summing over $n = -1, 0, 1$, and so we find

$$\bar{\Psi}_{2m}^{o,+} = \mu\pi\partial_\theta Y_{2m}(\pi/2,0)\left\{\frac{16}{15}[1 + (\cos\chi - 2im\sin\chi)e\right.$$
$$+ \mathcal{O}(e^2)]y^{1/2} + \frac{4}{315}[98 + 5m^2 + ((15m^2 + 62)\cos\chi$$
$$\left. - im(10m^2 + 547)\sin\chi)e + \mathcal{O}(e^2)]y^{3/2} + \mathcal{O}(y^{5/2})\right\}.$$

$$(3.30)$$

Note that we need only perform the calculation once for each ℓ mode. Therefore, this expression is valid for all $-2 \leq m \leq 2$ (although in this case we are only interested in $m = \pm 1$, of course).

E. Metric perturbation reconstruction

We now wish to use the master function to reconstruct the retarded metric perturbation at the location of the particle. We first form the metric perturbation amplitudes, and then sum over spherical harmonics to form the full metric perturbation. Expressions for the metric perturbation amplitudes are given earlier in Eqs. (2.20) and (2.22). However, as with the master function, we wish to form "barred" versions of the metric perturbation amplitudes, which are periodic in T_r, e.g. $\bar{h}_{tt}^{\ell m} = h_{tt}^{\ell m}e^{im\Omega_\varphi t}$. Note that since the full master function contains an extra factor of $e^{-im\Omega_\varphi t}$, time derivatives of $\Psi_{\ell m}$ pick up counterterms when written in terms of $\bar{\Psi}_{\ell m}$. The adjusted reconstruction expressions are, for even parity,

$$\bar{K}^{\ell m,\pm}(\chi) = f_p\partial_r\bar{\Psi}_{\ell m}^{e,\pm}(\chi) + A_p\bar{\Psi}_{\ell m}^{e,\pm}(\chi),$$
$$\bar{h}_{rr}^{\ell m,\pm}(\chi) = \frac{\Lambda_p}{f_p^2}\left[\frac{\lambda+1}{r_p}\bar{\Psi}_{\ell m}^{e,\pm}(\chi) - \bar{K}^{\ell m,\pm}(\chi)\right]$$
$$+ \frac{r_p}{f_p}\partial_r\bar{K}^{\ell m,\pm}(\chi),$$
$$\bar{h}_{tr}^{\ell m,\pm}(\chi) = r_p[\partial_t\partial_r\bar{\Psi}_{\ell m}^{e,\pm}(\chi) - im\Omega_\varphi\partial_r\bar{\Psi}_{\ell m}^{e,\pm}(\chi)]$$
$$+ r_pB_p[\partial_t\bar{\Psi}_{\ell m}^{e,\pm}(\chi) - im\Omega_\varphi\bar{\Psi}_{\ell m}^{e,\pm}(\chi)],$$
$$\bar{h}_{tt}^{\ell m,\pm}(\chi) = f_p^2\bar{h}_{rr}^{\ell m,\pm}(\chi),$$

$$(3.31)$$

and for odd parity,

$$\bar{h}_t^{\ell m,\pm}(\chi) = \frac{f_p}{2}[\bar{\Psi}_{\ell m}^{o,\pm}(\chi) + r_p\partial_r\bar{\Psi}_{\ell m}^{o,\pm}(\chi)],$$
$$\bar{h}_r^{\ell m,\pm}(\chi) = \frac{r_p}{2f_p}[\partial_t\bar{\Psi}_{\ell m}^{o,\pm}(\chi) - im\Omega_\varphi\bar{\Psi}_{\ell m}^{o,\pm}(\chi)].$$

$$(3.32)$$

Note the subscript p on all r-dependent quantities indicating evaluation at $r = r_p(\chi)$. Be aware that the functional-χ notation that we use is shorthand for, e.g. $\bar{\Psi}_{\ell m}^{e,\pm}(\chi) = \bar{\Psi}_{\ell m}^{e,\pm}[t_p(\chi), r_p(\chi)]$. Therefore, it is not true that $\partial_t\bar{\Psi}_{\ell m}^{e,\pm}(\chi) = \partial_\chi\bar{\Psi}_{\ell m}^{e,\pm}(\chi)(dt_p/d\chi)^{-1}$. To the contrary, all t and r derivatives of $\bar{\Psi}_{\ell m}^{e/o,\pm}$ must be computed explicitly. The r derivatives follow from forming the frequency domain extended homogeneous solutions $\partial_r X_{\ell mn}^{e/o,\pm}$, while t derivatives are formed by taking the t derivative of Eq. (3.29).

Next, we form the metric perturbation components for each ℓm mode, as shown in Eq. (2.23). Now though, with the "barred" form of the metric perturbation amplitudes, we must multiply by the "barred" form of the various spherical harmonics, e.g. $\bar{Y}^{\ell m} = Y^{\ell m}[\pi/2, \varphi_p(t)]e^{-im\Omega_\varphi t}$. This is not to be confused with complex conjugation, which we indicate with an asterisk ($*$). As expected, this does not change quantities evaluated on the worldline. For example

$$p_{tt}^{\ell m}(\chi) = h_{tt}^{\ell m}Y^{\ell m} = \bar{h}_{tt}^{\ell m}e^{im\Omega_\varphi t}e^{-im\Omega_\varphi t}\bar{Y}^{\ell m} = \bar{h}_{tt}^{\ell m}\bar{Y}^{\ell m}.$$

$$(3.33)$$

Thus, the metric perturbation components for each ℓm mode can be formed on either side of the particle through the expressions

$$p_{ab}^{\ell m,\pm}(\chi) = \bar{h}_{ab}^{\ell m,\pm}(\chi)\bar{Y}^{\ell m,\pm}(\chi),$$
$$p_{aB}^{\ell m,\pm}(\chi) = \bar{h}_a^{\ell m,\pm}(\chi)\bar{X}_B^{\ell m,\pm}(\chi),$$
$$p_{AB}^{\ell m,\pm}(\chi) = r_p^2\Omega_{AB}\bar{K}^{\ell m,\pm}(\chi)\bar{Y}^{\ell m,\pm}(\chi).$$

$$(3.34)$$

Returning to our ongoing example, we can form the odd-parity, $(\ell, m) = (2,1)$ contribution to the t, φ and r, φ metric perturbation components. Note that the t, θ and r, θ components vanish. Using our expression from Eq. (3.30) in Eqs. (3.32) and (3.34) we find

$$p_{t\varphi}^{21,+} = \mu[-1 - e\cos\chi + \mathcal{O}(e^2)]y^{1/2}$$
$$+ \mu\left[-\frac{47}{84} + \left(\frac{5}{12}\cos\chi + \frac{5i}{28}\sin\chi\right)e + \mathcal{O}(e^2)\right]y^{3/2}$$
$$+ \mathcal{O}(y^{5/2}),$$
$$p_{r\varphi}^{21,+} = \mu[-i + (\sin\chi - 2i\cos\chi)e + \mathcal{O}(e^2)]y$$
$$+ \mu\left[-\frac{271i}{84} + \left(\frac{71}{42}\sin\chi - \frac{177i}{28}\cos\chi\right)e\right.$$
$$\left. + \mathcal{O}(e^2)\right]y^2 + \mathcal{O}(y^3).$$

$$(3.35)$$

Finally, we sum over m-modes. In practice we do this by taking twice the real part of each nonzero m-mode and

adding to the $m = 0$ mode. As mentioned in Sec. II, while RWZ gauge is known to have discontinuous metric perturbation amplitudes for each ℓm mode, we find that these discontinuities cancel out after summing over m-modes. In fact, using the expressions in Ref. [41], we find that the Dirac delta behavior of the metric perturbation amplitudes also cancels out, so that the RWZ gauge is C^0 for all $\ell \geq 2$. For the specific example of $\ell = 2$, we find the t, φ and r, φ components of the metric perturbation to be

$$p_{t\varphi}^2 = \mu\Bigg[(-2 - 2e\cos\chi + \mathcal{O}(e^2))y^{1/2}$$
$$+ \left(-\frac{47}{42} + \frac{5}{6}e\cos\chi + \mathcal{O}(e^2)\right)y^{3/2} + \mathcal{O}(y^{5/2})\Bigg],$$
$$p_{r\varphi}^2 = \mu\Bigg[(2e\sin\chi + \mathcal{O}(e^2))y + \left(\frac{71}{21}e\sin\chi + \mathcal{O}(e^2)\right)y^2$$
$$+ \mathcal{O}(y^3)\Bigg]. \tag{3.36}$$

F. Large-ℓ expressions

In the previous subsections, we used the example of $\ell = 2$ and odd parity to show how we construct retarded solutions to each metric perturbation component for that specific ℓ-mode. Our example only showed an expansion through 1PN. In reality, for such a low-order expansion, we need not specify an explicit ℓ value. In fact, through 2PN we can write down solutions for arbitrary $\ell \geq 2$. For the purposes of this work, wherein we expand to 4PN, we calculated explicit solutions for all modes $\ell \leq 3$. For all

higher modes we use generic-ℓ expressions. These expressions follow from a nearly identical calculation to the specific-ℓ case. In fact, it is only the homogeneous solutions that are different. We give an abbreviated overview of the procedure.

In the odd parity, through 1PN the homogeneous solutions are of the form [32,35]

$$\hat{X}_\ell^{o,+} = (\eta^2 \mathcal{X}_1)^\ell\Bigg[1 + \left(\frac{\mathcal{X}_2}{4\ell - 2} + \frac{\mathcal{X}_1(\ell - 1)(\ell + 3)}{\ell + 1}\right)\eta^2$$
$$+ \mathcal{O}(\eta^4)\Bigg],$$
$$\hat{X}_\ell^{o,-} = (\eta^2 \mathcal{X}_1)^{-\ell-1}\Bigg[1 + \left(\frac{4 - \ell^2}{\ell}\mathcal{X}_1 - \frac{\mathcal{X}_2}{4\ell + 6}\right)\eta^2$$
$$+ \mathcal{O}(\eta^4)\Bigg]. \tag{3.37}$$

Note that when $\ell = 2$, these reduce to the expressions in Eq. (3.15). We take these expressions, evaluate them along the particle's worldline, then normalize them by performing the integral (3.24). For the even parity, the procedure is equivalent, after using Eq. (3.16) to form the homogeneous solutions. Critically, during the entire calculation we keep the leading term [either $(\eta^2 \mathcal{X}_1)^\ell$ or $(\eta^2 \mathcal{X}_1)^{-\ell-1}$] factored out of the PN expansion. This term eventually cancels out once the frequency domain extended homogeneous solutions are formed. Following the procedure through, we find that the infinity-side odd-parity master function is

$$\bar{\Psi}_{\ell m}^{o,+} = \frac{16\pi\mu\partial_\theta Y_{2m}(\pi/2, 0)}{(\ell - 1)\ell(\ell + 1)(2\ell + 1)}\Bigg\{2[1 + (\cos\chi - 2im\sin\chi)e + \mathcal{O}(e^2)]y^{1/2}$$
$$+ \frac{1}{\ell(\ell + 1)(\ell + 2)(2\ell - 1)(2\ell + 3)}[5(2m^2 + 5)\ell^2 + 2(5m^2 + 13)\ell + 12\ell^5 + 32\ell^4 + 19\ell^3 - 24$$
$$+ (3((10m^2 + 19)\ell^2 + 2(5m^2 + 13)\ell + 4\ell^5 + 4\ell^4 - 7\ell^3 - 16)\cos\chi$$
$$- 2im(2(5m^2 + 18)\ell^2 + (10m^2 + 17)\ell + 28\ell^5 + 96\ell^4 + 87\ell^3 - 24)\sin\chi)e + \mathcal{O}(e^2)]y^{3/2} + \mathcal{O}(y^{5/2})\Bigg\}. \tag{3.38}$$

Again, this reduces to the specific expression given in Eq. (3.30) when $\ell = 2$. We use the master function expressions to form the metric perturbation contributions for each ℓm mode. In order to perform the m sum for generic ℓ, we make use of the procedure of Nakano *et al.* [25] in Appendix F. Our final expressions for $p_{\mu\nu}^\ell$ are again a double expansion in y and e. The generic-ℓ equivalents of Eq. (3.36) are

$$p_{t\varphi}^\ell = \mu\Bigg\{[-2 - 2e\cos\chi + \mathcal{O}(e^2)]y^{1/2}$$
$$+ \left[-\frac{3(2\ell^4 + 4\ell^3 + 7\ell^2 + 5\ell - 8)}{2\ell(\ell + 1)(2\ell - 1)(2\ell + 3)} + \frac{14\ell^4 + 28\ell^3 - 43\ell^2 - 57\ell + 48}{2\ell(\ell + 1)(2\ell - 1)(2\ell + 3)}e\cos\chi + \mathcal{O}(e^2)\right]y^{3/2} + \mathcal{O}(y^{5/2})\Bigg\},$$
$$p_{r\varphi}^\ell = \mu\Bigg\{[2e\sin\chi + \mathcal{O}(e^2)]y + \left[\frac{11\ell^4 + 22\ell^3 + 20\ell^2 + 9\ell - 24}{\ell(\ell + 1)(2\ell - 1)(2\ell + 3)}e\sin\chi + \mathcal{O}(e^2)\right]y^2 + \mathcal{O}(y^3)\Bigg\}. \tag{3.39}$$

As before, we find the metric perturbation to be single valued for each ℓ. Taking into account three different cases: low-order modes (Appendix A), specific-ℓ values (in our case $\ell = 2$, 3), and generic-ℓ for all the rest, the full retarded metric perturbation is formed from a simple sum over ℓ,

$$p_{\mu\nu} = \sum_{\ell} p_{\mu\nu}^{\ell}. \tag{3.40}$$

G. Resummation of the small eccentricity expansions

For our work here, we used Mathematica to expand all quantities in the small-e limit, keeping powers up to e^{10}. As eccentricity order increases, the task of simplifying large expressions requires substantial computational resources. Indeed, finding the generic-ℓ even-parity normalization coefficient (our most taxing calculation) took some 10 days and 20 GB of memory. Nonetheless, the virtue of this approach is that we need only perform that calculation once at each ℓ. Still, with such computational overhead, the question remains: can this approach be useful when considering high-eccentricity orbits? Inspired by recent work of Forseth *et al.* [50] (see also Ref. [51]), we have sought to "resum" our final results at each PN order so as to capture the $e \to 1$ behavior. (We note also that it is likely possible, and perhaps simpler, to achieve the same result by expanding in p^{-1} instead of y [52,53].) As an example, consider the 0PN expression for $p_{t\varphi}^2$, shown here with all 10 powers of eccentricity,

$$p_{t\varphi}^2 = -\mu \left[2 + 2e\cos\chi + e^2 + e^3\cos\chi + \frac{3}{4}e^4 \right.$$
$$+ \frac{3}{4}e^5\cos\chi + \frac{5}{8}e^6 + \frac{5}{8}e^7\cos\chi + \frac{35}{64}e^8$$
$$\left. + \frac{35}{64}e^9\cos\chi + \frac{63}{128}e^{10} + \mathcal{O}(e^{11}) \right] y^{1/2} + \mathcal{O}(y^{3/2}).$$
$$\tag{3.41}$$

It is a relatively simple task to guess that this series is probably the small-e expansion of

$$p_{t\varphi}^2 = -\frac{2\mu(1 + e\cos\chi)}{(1 - e^2)^{1/2}} y^{1/2} + \mathcal{O}(y^{3/2}). \tag{3.42}$$

A similar analysis of our generic-ℓ solutions provided closed-form expressions for all the 0PN and 1PN retarded metric perturbation components.

Beyond 1PN, finding closed-form solutions becomes harder. Examining the PN literature on eccentric orbits (see, e.g., Eq. (356) of Blanchet [54]), it is clear that starting at 2PN the $e \to 1$ singular behavior becomes more subtle than some simple inverse factor of $1 - e^2$. And yet, as a first approximation, factoring out the appropriate leading power

of $1 - e^2$ at each PN order [each order comes with an additional factor of $(1 - e^2)^{-1}$], yields dramatically improved convergence for high eccentricities.

We note, of course, that our inability to find closed-form expressions beyond 1PN does not imply their nonexistence. Indeed, from standard PN calculations, the metric of two bodies in eccentric motion is known through 3PN with arbitrary mass ratios. See Ref. [22] and references therein where the metric is given in standard harmonic coordinates. Here we simply present the results we were able to deduce by resumming our small-eccentricity expansion, and without making use of known results from other research.

IV. THE GENERALIZED REDSHIFT INVARIANT

Before continuing to our results, we introduce the specific gauge invariant that we will compute. We first briefly cover some GSF background and then give the exact expression we use in our calculations.

A. Abbreviated background on gravitational self-force invariants

When working at zeroth order in the mass ratio q, the particle moves on a geodesic of the background spacetime $g_{\mu\nu}$, as we have assumed up to this point. Once we allow the particle to have a small, but finite mass μ, the motion is no longer geodesic in $g_{\mu\nu}$, but it is geodesic in the effective spacetime $\mathrm{g}_{\mu\nu} = g_{\mu\nu} + p_{\mu\nu}^R$. Here $p_{\mu\nu}^R$ is the *regular* part of the metric perturbation, due to Detweiler and Whiting [55]. The regular metric perturbation comes from removing the particle's own singular field $p_{\mu\nu}^S$ from the retarded metric perturbation $p_{\mu\nu}$. An appropriate gradient of $p_{\mu\nu}^R$ provides the GSF; see e.g. [43,56].

When computing local gauge invariants, it is convenient to "turn off" the dissipation due to the GSF and look at effects due solely to the conservative GSF. This is done by forming $p_{\mu\nu}^{R,\mathrm{cons}} = (p_{\mu\nu}^{R,\mathrm{ret}} + p_{\mu\nu}^{R,\mathrm{adv}})/2$. Here $p_{\mu\nu}^{R,\mathrm{ret}}$ is the regular metric perturbation computed with retarded boundary conditions, while $p_{\mu\nu}^{R,\mathrm{adv}}$ is computed with advanced boundary conditions. All gauge-invariant GSF quantities mentioned in the introduction are defined with respect to the conservative effective spacetime $\mathrm{g}_{\mu\nu}^{\mathrm{cons}} = g_{\mu\nu} + p_{\mu\nu}^{R,\mathrm{cons}}$. However, when computing an orbit average, the dissipative contribution to $\mathrm{g}_{\mu\nu}^{\mathrm{cons}}$ averages to zero, and so we can compute such quantities directly from $\mathrm{g}_{\mu\nu}$.

B. Practical details of the $\langle U \rangle$ calculation

We are now ready to calculate the so-called "generalized redshift invariant." This quantity is an eccentric orbit extension of Detweiler's [4] original invariant u^t. The generalization $\langle U \rangle$ is due to Barack and Sago [21], and is the orbit average of u^t with respect to proper time τ.

Critically, this average is taken on the GSF-*perturbed* orbit. In the language introduced above, the average is performed along the geodesic motion in the effective spacetime $\mathsf{g}_{\mu\nu}$. When mapping between the background metric $g_{\mu\nu}$ and the effective metric $\mathsf{g}_{\mu\nu}$, we take Ω_i to be fixed. This implies that T_r has the same value in the two spacetimes, but the τ radial period $\mathcal{T}_r$ does not. Thus, we write the orbit average of u^t with respect to proper time as

$$\langle U \rangle \equiv \langle u^t \rangle = \frac{T_r}{\mathcal{T}_r + \delta \mathcal{T}_r}, \qquad (4.1)$$

where $\delta \mathcal{T}_r$ is an $\mathcal{O}(q)$ shift in $\mathcal{T}_r$ due to the conservative GSF. (Note that here we use T_r and $\mathcal{T}_r$ to indicate background quantities, a notation which differs from [22].)

Before continuing we wish to emphasize a point stressed in Refs. [21,22]. It is not enough to simply compute a gauge-invariant quantity like $\langle U \rangle$. Rather, one must find a functional way to relate that quantity to the particular orbit being considered. A natural gauge-invariant characterization of perturbed orbits is the two fundamental frequencies, Ω_r and Ω_φ (recall that $\delta \Omega_r = \delta \Omega_\varphi = 0$ due to frequency fixing). If one can parametrize a GSF-perturbed orbit with observable frequencies, then the nontrivial gauge invariant $\langle U \rangle$ can be taken as a function of those two parameters. In this way, it is possible for those working in different gauges (such as those used in PN literature) to compare results in a meaningful way.

We now expand $\langle U \rangle$ into a background part, and a part due to the GSF,

$$\langle U \rangle(\Omega_i) = \langle U \rangle_0(\Omega_i) + q\langle U \rangle_{\text{gsf}}(\Omega_i), \qquad \langle U \rangle_0 = \frac{T_r}{\mathcal{T}_r}, \qquad (4.2)$$

where $\Omega_i \equiv \{\Omega_r, \Omega_\varphi\}$. When Ω_i are taken to be fixed, Akcay *et al.* [22] show that the $\mathcal{O}(q)$ shift in $\langle U \rangle$ can be reduced to

$$q\langle U \rangle_{\text{gsf}} = -\frac{\langle U \rangle_0}{\mathcal{T}_r} \delta \mathcal{T}_r = \frac{T_r}{\mathcal{T}_r^2} \langle H^R \rangle, \qquad (4.3)$$

a simplification over the original Barack and Sago expression. Here we follow the notation of Refs. [21,57] and introduce

$$H^R \equiv \frac{1}{2} p_{\mu\nu}^R u^\mu u^\nu = \frac{1}{2} p_{\mu\nu} u^\mu u^\nu - H^S, \qquad (4.4)$$

where H^S is due to the singular field $p_{\mu\nu}^S$.

The calculation of the singular field is a subtle task. Its removal is most often performed with mode-sum regularization [58], wherein singular terms are subtracted ℓ by ℓ (though other techniques exist; see Ref. [59] for an overview). Barack and Sago first computed the leading-order

regularization parameter for H^S when they introduced $\langle U \rangle$. Since then, higher-order regularization parameters (improving convergence rates for cases when the ℓ sum must be truncated) have been computed by Heffernan *et al.* [57]. But, for our purposes, where we know all ℓ, we only need the leading-order term,

$$H^S = \sum_\ell H_{[0]} \equiv \sum_\ell \frac{2\mu}{\pi \sqrt{\mathcal{L}^2 + r_p^2}} \mathcal{K}\left(\frac{\mathcal{L}^2}{\mathcal{L}^2 + r_p^2}\right). \qquad (4.5)$$

The $H_{[0]}$ regularization parameter notation follows Ref. [57]; the subscript [0] indicates that $H_{[0]}$ multiplies zero powers of ℓ. Combining Eq. (4.5) with (4.4) gives

$$\langle H^R \rangle = \sum_\ell \left(\frac{1}{2} \langle p_{\mu\nu}^\ell u^\mu u^\nu \rangle - \langle H_{[0]} \rangle\right). \qquad (4.6)$$

For our present calculation, it is a simple task to use our expansions for r_p and $\mathcal{L}$ and obtain a PN expansion for $H_{[0]}$. Both $p_{\mu\nu}^\ell u^\mu u^\nu$ and $H_{[0]}$ are then averaged over one τ-period with the use of the PN expansion for $d\tau_p/d\chi$ from Eq. (2.6), thus forming $\langle p_{\mu\nu}^\ell u^\mu u^\nu \rangle$ and $\langle H_{[0]} \rangle$, and forming all that we need for a practical calculation of $\langle U \rangle_{\text{gsf}}$.

Lastly, we note that Akcay *et al.* [22] adjust for the nonasymptotic flatness of the Lorenz gauge monopole by adding a correction term to Eq. (4.3). In Appendix A we show that the original RWZ monopole [37] is also not asymptotically flat, but we are able to correct that with a slight gauge transformation, and so we use Eq. (4.3) exactly as is. However, we note that the radiative modes of RWZ gauge are *not* asymptotically flat [60–62]. This is curious, because Barack and Sago established the gauge invariance of $\langle U \rangle_{\text{gsf}}$ for a certain class of gauges which respect the periodicity of the orbit and are well behaved at spatial infinity. Still, we have empirical evidence from several calculations (including this one) that RWZ gauge falls into the class of gauges for which $\langle U \rangle_{\text{gsf}}$ is invariant. The question remains, why must we correct the nonasymptotic flatness of the monopole, but not other modes? At this point the answer is not clear.

V. RESULTS

Following the procedure described above, we have used Mathematica to compute the metric perturbation along with its first t and r derivatives through 4PN while keeping powers in eccentricity up to e^{10}. Our PN order required us to compute solutions to the modes $\ell = 2$ and $\ell = 3$ explicitly, while all higher ℓ modes are described by a general-ℓ expression.
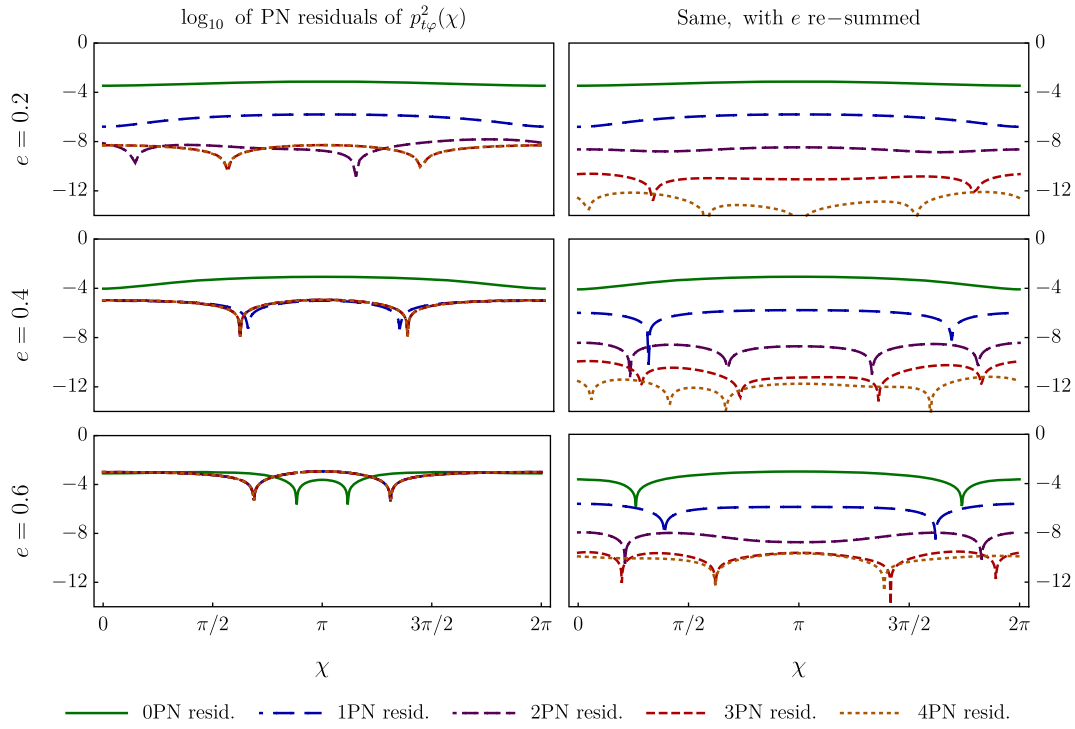
FIG. 1.   The effect of resumming the small-e expansion at each PN order as seen by comparing to numerical data, all computed at $p = 1000$. We see that especially as eccentricity increases, our resummation greatly improves convergence. Also, note the consistency of our convergence throughout the orbit. Our results are no less effective at periapsis than apoapsis. The dips in the residuals are from zero crossings, and not meaningful. See the discussion in the text for more details.

For each specific-ℓ, as well as the generic-ℓ case, we performed the calculation for both even and odd parities, on both sides of the particle. After summing over m-modes we noticed the surprising result that RWZ gauge is in fact C^0 for each ℓ, despite being discontinuous with delta functions at the ℓm level. We subsequently confirmed that this was true to all PN orders using the expressions in Ref. [41].

We show the convergence of one of the metric perturbation components with PN order in Fig. 1. We used a numerical code developed for recent work [50] to compute the $\ell = 2$ contribution to the metric perturbation component t, φ at three different eccentricities. We then subtracted successive PN terms derived analytically for this work and computed the relative error. In the left column we see that even for a moderately low eccentricity of $e = 0.2$ the PN convergence stalls at 2PN due to the small-e expansion. At $e = 0.6$ the convergence stalls after the subtraction of only the 0PN term. In the right column we see that factoring out the $e \rightarrow 1$ singular behavior greatly improves the convergence. Note that at $e = 0.6$ the convergence still appears to stall around 3PN. In order to probe such eccentricities at the 4PN level, we will evidently need more than 10 powers of e, or perhaps a more precise capturing of the $e \rightarrow 1$ singular behavior for 2PN and beyond.

We now provide our results for the invariant $\langle U \rangle_{\mathrm{gsf}}$, computed using the procedure described in Sec. IV. The specific expression is given below in Eq. (5.1) (γ is the Euler-Mascheroni constant). We performed the regularization in two ways so as to check our removal of the singular field. First, we fit out the constant-with-ℓ term by taking the large-ℓ limit of our generic-ℓ expressions. That fit-out regularization parameter exactly agreed with the proper time average of $H_{[0]}$ when expanded in y and e. Using Mathematica we are able to take the ℓ sum all the way to infinity, and thus have no error due to truncation.

We have compared our expression to the published 3PN values of Akcay et al. [22], which were computed by starting in standard harmonic coordinates. That reference also provides numerical data for $\langle U \rangle_{\mathrm{gsf}}$, computed in Lorenz gauge, which we compare to in Fig. 2. The recent RWZ gauge work by Bini et al. [39] provides an analytic 4PN, $\mathcal{O}(e^2)$ value of $\langle U \rangle_{\mathrm{gsf}}$, which we agree with as well. Lastly, van de Meent and Shah [23], who work in radiation gauge, provide numerical predictions for 4PN terms through e^6 and we agree with all of their values within the provided error bars.
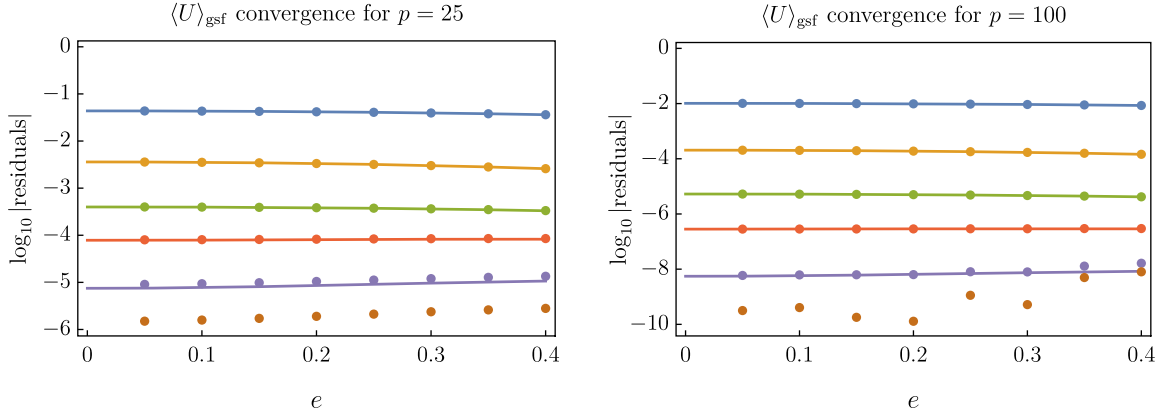
FIG. 2.   Comparison of our PN expressions with numerical data from Table II of Akcay *et al.* [22]. Dots are numerical values and lines are our analytic calculations. The top row is $\log_{10}$ of the absolute value of the full $\langle U \rangle_{\mathrm{gsf}}$. Successive rows show residuals after subtracting each term in the PN series. The consistency of our agreement out to $e = 0.4$ is only possible because of our resummation of the small-e expansion. Unevenness of the final residuals for $p = 100$ is likely a numerical artifact.

$$\langle U \rangle_{\mathrm{gsf}} = -y - \frac{2(1-2e^2)}{1-e^2}y^2 + \frac{1}{(1-e^2)^2}\left[-5 + 16e^2 - \frac{85e^4}{8} - \frac{9e^6}{16} - \frac{17e^8}{128} - \frac{7e^{10}}{256} + \mathcal{O}(e^{12}) \right]y^3$$

$$+ \frac{1}{(1-e^2)^3}\left[-\frac{121}{3} + \frac{41\pi^2}{32} + 36e^2 + e^4\left(-\frac{453}{8} - \frac{123\pi^2}{256} \right) + e^6\left(\frac{251}{8} - \frac{41\pi^2}{256} \right) + e^8\left(\frac{909}{128} - \frac{369\pi^2}{4096} \right)\right.$$

$$\left. + e^{10}\left(\frac{217}{64} - \frac{123\pi^2}{2048} \right) + \mathcal{O}(e^{12}) \right]y^4 + \frac{1}{(1-e^2)^4}\left[-\frac{64}{5} - \frac{488e^2}{15} + \frac{242e^4}{15} + \frac{122e^6}{15} + \frac{71e^8}{20} + \frac{523e^{10}}{240} + \mathcal{O}(e^{12}) \right]y^5\log y$$

$$+ \frac{1}{(1-e^2)^4}\left[-\frac{1157}{15} - \frac{128\gamma}{5} + \frac{677\pi^2}{512} - \frac{256}{5}\log 2 + e^2\left(\frac{13102}{45} - \frac{976\gamma}{15} + \frac{2239\pi^2}{3072} + \frac{496}{3}\log 2 - \frac{1458}{5}\log 3 \right)\right.$$

$$+ e^4\left(-\frac{7679}{90} + \frac{484\gamma}{15} - \frac{21941\pi^2}{6144} - \frac{20724}{5}\log 2 + \frac{5103}{2}\log 3 \right) + e^6\left(\frac{21859}{120} + \frac{244\gamma}{15} + \frac{13505\pi^2}{12288} + \frac{157564}{5}\log 2 \right.$$

$$\left. - \frac{1586061}{160}\log 3 - \frac{1953125}{288}\log 5 \right) + e^8\left(-\frac{1937767}{23040} + \frac{71\gamma}{10} + \frac{13129\pi^2}{65536} - \frac{13131809}{90}\log 2 - \frac{1355697}{1280}\log 3 \right.$$

$$\left. + \frac{48828125}{768}\log 5 \right) + e^{10}\left(-\frac{3734707}{115200} + \frac{523\gamma}{120} + \frac{31055\pi^2}{393216} + \frac{14107079851}{27000}\log 2 + \frac{202776271479}{1024000}\log 3 \right.$$

$$\left.\left. - \frac{59986328125}{221184}\log 5 - \frac{678223072849}{9216000}\log 7 \right) + \mathcal{O}(e^{12}) \right]y^5 + \mathcal{O}(y^6). \tag{5.1}$$

We note that (with an eye to the discussion in Sec. IV), our parametrization (5.1) is somewhat lacking. Since y is derived from Ω_φ, it is gauge invariant, but e, however, is not. When considering eccentric orbits, it is customary to give results in terms of y and $\lambda \equiv 3y/k$, where $k \equiv \Omega_\varphi/\Omega_r - 1$ is a measure of periapsis advance. It is a straightforward, if tedious, task to transform our results to λ, but in the process we lose the physical intuition that e provides. In practice it is more convenient to convert from λ to e for the purposes of comparison. In Appendix B, we do provide $\langle U \rangle_{\mathrm{gsf}}$ as a function of p and e, though for easy comparison with GSF codes.

In addition to comparing with other calculations of $\langle U \rangle_{\mathrm{gsf}}$, our result is useful for EOB comparison and calibration. Notably, recent 4PN EOB results have introduced terms through e^6 in the nongeodesic "$\hat{Q}$ potential." See Eq. (8.1c) of Damour *et al.* [63]. The full $\hat{Q}$ can be separated into terms linear-in-ν (with ν being the symmetric mass ratio), which are accessible through our GSF calculation, and terms which are $\mathcal{O}(\nu^2)$ and beyond, which would require at least the second-order GSF to compute. Thus, we write $\hat{Q}(u,p_r') = \nu q(u,p_r') + \mathcal{O}(\nu^2)$. (The details of the EOB notation used here and throughout the rest of this section can be found in Ref. [63].) Our expression for $q(u,p_r')$, through e^{10} [equivalently $(\mathbf{n}' \cdot \mathbf{p}')^{10}$] is

$$q(u, p'_r) = 8u^2(\mathbf{n}' \cdot \mathbf{p}')^4 + \left(-\frac{5308}{15} + \frac{496256}{45}\log 2 - \frac{33048}{5}\log 3\right)u^3(\mathbf{n}' \cdot \mathbf{p}')^4 + \left(-\frac{827}{3} - \frac{2358912}{25}\log 2\right.$$

$$+ \frac{1399437}{50}\log 3 + \frac{390625}{18}\log 5\bigg)u^2(\mathbf{n}' \cdot \mathbf{p}')^6 + \left(-\frac{35772}{175} + \frac{21668992}{45}\log 2 + \frac{6591861}{350}\log 3\right.$$

$$- \frac{27734375}{126}\log 5\bigg)u(\mathbf{n}' \cdot \mathbf{p}')^8 + \left(-\frac{231782}{1575} - \frac{408889317632}{212625}\log 2 - \frac{22187736351}{28000}\log 3 + \frac{7835546875}{7776}\log 5\right.$$

$$+ \frac{96889010407}{324000}\log 7\bigg)(\mathbf{n}' \cdot \mathbf{p}')^{10} + \mathcal{O}[u^{-1}(\mathbf{n}' \cdot \mathbf{p}')^{12}]. \tag{5.2}$$

We derived Eq. (5.2) using the procedure described in Le Tiec's recent work [64]. Using Eq. (5.27) from that reference, we were able to transcribe our 4PN, e^4 contribution to $\langle U \rangle_{\mathrm{gsf}}$ to the $u^3(\mathbf{n}' \cdot \mathbf{p}')^4$ contribution to q, confirming the same term in Eq. (8.1c) of Ref. [63]. We then followed the prescription of Le Tiec to derive transcription equations equivalent to his (5.27) for e^6, e^8, and e^{10}. With these, we confirmed the $u^2(\mathbf{n}' \cdot \mathbf{p}')^6$ coefficient (the e^6 term) of q, given in Ref. [63]. The last two terms in Eq. (5.2) are previously unknown coefficients, corresponding to e^8 and e^{10} at 4PN.

VI. CONCLUSIONS AND OUTLOOK

We have presented a method for solving the first-order field equations in a PN/small-eccentricity expansion when the source is a point particle in bound motion on a Schwarzschild background. In this work we have kept terms through 4PN and e^{10}, but our method will extend naturally to higher orders. Important to the effectiveness of our results was the resumming of the e-series at each PN order. Our method lends itself to many further calculations of eccentric orbit invariants. Since we already have computed derivatives of the metric perturbation (though they were not used in computing $\langle U \rangle$ here), a natural next step is to compute an eccentric orbit generalization of the spin-invariant ψ [15].

Moving beyond Schwarzschild to Kerr is a more challenging task. The analytic merger of PN theory with black hole perturbation theory on Kerr has a long history (e.g., [28,31]), though the focus has typically been on fluxes and nonlocal dissipative GSF. The reason is largely due to the challenge of reconstructing the radiation-gauge metric perturbation from the Teukolsky variable. Recent work by Pound et al. [65] has helped to clarify the subtleties of the process, and it may now be possible to extend our method to compute the Kerr metric perturbation, although the task is formidable.

APPENDIX A: LOW-ORDER MODES

In this appendix we give Zerilli's [37] analytic solutions to the $\ell = 0$, 1 equations. We find that we must shift the monopole, as is done in Lorenz gauge [66], in order to find an asymptotically flat solution. It is straightforward to take the expressions given here and expand them at the particle's location in a PN series using the expressions in Sec. III A.

1. Monopole

In the monopole case $\ell = m = 0$ and only h_{tt}^{00}, h_{tr}^{00}, h_{rr}^{00}, and K^{00} are defined. Zerilli chooses to set $h_{tr}^{00} = K^{00} = 0$. The remaining nonzero solutions are

$$h_{tt}^{00} = 4\sqrt{\pi}\mu\left[\frac{\mathcal{E}}{r} - \frac{f}{\mathcal{E}f_p r_p}(2\mathcal{E}^2 - U_p^2)\right]\theta[r - r_p(t)],$$

$$h_{rr}^{00} = \frac{4\sqrt{\pi}\mu\mathcal{E}}{f^2 r}\theta[r - r_p(t)]. \tag{A1}$$

Recall the distinction between quantities with a subscript p, such as f_p which is a function of the particle location r_p, and those without subscripts, like f which is a function of the Schwarzschild coordinate r. We seek asymptotically flat solutions which fall off at least as $1/r$. The amplitude h_{rr}^{00} satisfies this, but as in Lorenz gauge h_{tt}^{00} does not. We therefore look for a gauge transformation to remove the

term

$$-4\sqrt{\pi}\mu f \frac{2\mathcal{E}^2 - U_p^2}{\mathcal{E}f_p r_p}\theta[r - r_p(t)]. \qquad (A2)$$

The push equations for the three nonzero amplitudes are [40]

$$\Delta h_{tt}^{00} = -2\partial_t\xi_t^{00} + f\frac{2M}{r^2}\xi_r^{00},$$

$$\Delta h_{tr}^{00} = -f\partial_r(f^{-1}\xi_t^{00}) - \partial_t\xi_r^{00},$$

$$\Delta h_{rr}^{00} = -2\partial_r\xi_r^{00} - \frac{2M}{fr^2}\xi_r^{00}. \qquad (A3)$$

From this we see that we can take $\xi_r^{00} = 0$ and demand that ξ_t^{00} satisfy

$$\Delta h_{tt}^{00} = -2\partial_t\xi_t^{00} = 4\sqrt{\pi}\mu f\frac{2\mathcal{E}^2 - U_p^2}{\mathcal{E}f_p r_p}. \qquad (A4)$$

Note that the Zerilli solution is zero on the horizon side of the particle, but this push will add a nonzero term there. As far as pushing h_{tt}^{00} goes, we can get away with simply specifying the time derivative of ξ_t^{00}. However, a gauge transformation involving ξ_t^{00} will also affect h_{tr}^{00}. Looking at the h_{tr}^{00} push, we will need $-f\partial_r(f^{-1}\xi_t^{00})$, i.e.

$$\Delta h_{tr}^{00} = -f\partial_r(f^{-1}\xi_t^{00}) = \frac{2\sqrt{\pi}\mu}{\mathcal{E}}f\partial_r\left[\int_0^t\frac{2\mathcal{E}^2 - U_p^2}{f_p r_p}dt'\right] = 0. \qquad (A5)$$

So this choice of gauge transformation actually leaves h_{tr}^{00} unchanged. Then, multiplying by Y^{00}, in our asymptotically flat gauge the nonzero monopole solutions are

$$p_{tt}^{00}(x^\mu) = 2\mu\frac{\mathcal{E}}{r}\theta[r - r_p(t)] + 2\mu f\frac{2\mathcal{E}^2 - U_p^2}{\mathcal{E}f_p r_p}\theta[r_p(t) - r],$$

$$p_{rr}^{00}(x^\mu) = \frac{2\mu\mathcal{E}}{f^2 r}\theta[r - r_p(t)]. \qquad (A6)$$

2. Odd-parity dipole

When $(\ell, m) = (1, 0)$ the odd-parity amplitude h_2^{10} is not defined and we have residual gauge freedom. Zerilli used this freedom to set $h_r^{10} = 0$. The only nonzero remaining amplitude is

$$h_t^{10} = 4\mu\mathcal{L}\sqrt{\frac{\pi}{3}}\left(\frac{1}{r}\theta[r - r_p(t)] + \frac{r^2}{r_p^3}\theta[r_p(t) - r]\right). \qquad (A7)$$

This amplitude decays as r^{-1} at large radii and as such is asymptotically flat. Multiplying by X_φ^{10} gives the only nonzero metric perturbation component,

$$p_{t\varphi}^{10}(x^\mu) = -2\mu\mathcal{L}\sin^2\theta\left(\frac{1}{r}\theta[r - r_p(t)] + \frac{r^2}{r_p^3}\theta[r_p(t) - r]\right). \qquad (A8)$$

We note that (as brought to our attention by Leor Barack), this dipole solution is in fact singular at the horizon (when viewed in horizon-regular, ingoing Eddington-Finkelstein coordinates). While this does not affect our first-order-in-q calculation, any second-order GSF analysis will require a horizon-regular solution.

3. Even-parity dipole

When $(\ell, m) = (1, 1)$ the even-parity amplitude G^{11} is not defined. Zerilli sets $K^{11} = 0$ (in addition to the usual $j_t^{11} = j_r^{11} = 0$). The remaining nonzero even-parity dipole amplitudes are

$$h_{tt}^{11} = -2\mu\sqrt{\frac{2\pi}{3}}\frac{rf_p}{f}\left[\frac{\mathcal{E}r_p}{r^3} + \frac{6\mathcal{L}^2 M + 6Mr_p^2 - 3\mathcal{L}^2 r_p + (2\mathcal{E}^2 - 3)r_p^3}{\mathcal{E}r_p^5} - \frac{6i\mathcal{L}\dot{r}_p}{r_p^3}\right]e^{-i\varphi_p(t)}\theta[r - r_p(t)],$$

$$h_{tr}^{11} = -\frac{2\sqrt{6\pi}\mu}{rf^2 r_p}(i\mathcal{L}f_p^2 - \mathcal{E}r_p\dot{r}_p)e^{-i\varphi_p(t)}\theta[r - r_p(t)],$$

$$h_{rr}^{11} = -\frac{2\sqrt{6\pi}\mu\mathcal{E}r_p f_p}{r^2 f^3}e^{-i\varphi_p(t)}\theta[r - r_p(t)]. \qquad (A9)$$

We observe that h_{tt}^{11} is not asymptotically flat, while the other two amplitudes are. However, the even-parity dipole is the so-called "pure gauge" mode. It is a straightforward calculation to form its contribution to $\langle U \rangle$, and we find that it vanishes in a pointwise sense.

APPENDIX B: THE GENERALIZED REDSHIFT INVARIANT AS AN EXPANSION IN p^{-1}

$$\langle U \rangle_{\text{gsf}} = -(1-e^2)p^{-1} + (-2+4e^2-2e^4)p^{-2} + \left[-5 + 7e^2 + \frac{e^4}{4} - \frac{5e^6}{2} + \frac{15e^8}{64} + \frac{3e^{10}}{64} + \mathcal{O}(e^{12}) \right]p^{-3}$$

$$+ \left[-\frac{121}{3} + \frac{41\pi^2}{32} + e^2\left(-\frac{5}{3} - \frac{41\pi^2}{32} \right) + e^4\left(\frac{705}{8} - \frac{123\pi^2}{256} \right) + e^6\left(-\frac{475}{12} + \frac{41\pi^2}{128} \right) \right.$$

$$+ \left. e^8\left(-\frac{1171}{384} + \frac{287\pi^2}{4096} \right) + e^{10}\left(-\frac{115}{128} + \frac{123\pi^2}{4096} \right) + \mathcal{O}(e^{12}) \right]p^{-4}$$

$$+ \left[\frac{64}{5} + \frac{296e^2}{15} - \frac{146e^4}{3} + 8e^6 + \frac{55e^8}{12} + \frac{329e^{10}}{240} + \mathcal{O}(e^{12}) \right]p^{-5}\log p$$

$$+ \left[-\frac{1157}{15} - \frac{128\gamma}{5} + \frac{677\pi^2}{512} - \frac{256}{5}\log 2 + e^2\left(-\frac{11141}{45} - \frac{592\gamma}{15} + \frac{29665\pi^2}{3072} + \frac{3248}{15}\log 2 - \frac{1458}{5}\log 3 \right) \right.$$

$$+ e^4\left(\frac{247931}{360} + \frac{292\gamma}{3} - \frac{89395\pi^2}{6144} - \frac{64652}{15}\log 2 + \frac{28431}{10}\log 3 \right)$$

$$+ e^6\left(-\frac{52877}{180} - 16\gamma + \frac{3385\pi^2}{4096} + \frac{178288}{5}\log 2 - \frac{1994301}{160}\log 3 - \frac{1953125}{288}\log 5 \right)$$

$$+ e^8\left(-\frac{24619}{384} - \frac{55\gamma}{6} + \frac{327115\pi^2}{196608} - \frac{15967961}{90}\log 2 + \frac{11332791}{1280}\log 3 + \frac{162109375}{2304}\log 5 \right)$$

$$+ e^{10}\left(-\frac{1933}{3840} - \frac{329\gamma}{120} + \frac{172697\pi^2}{393216} + \frac{18046622551}{27000}\log 2 + \frac{203860829079}{1024000}\log 3 \right.$$

$$\left. \left. - \frac{74048828125}{221184}\log 5 - \frac{678223072849}{9216000}\log 7 \right) + \mathcal{O}(e^{12}) \right]p^{-5} + \mathcal{O}(p^{-6}) \qquad (\text{B1})$$

[1] Ligo home page, http://www.ligo.caltech.edu/.
[2] A. Buonanno and T. Damour, Phys. Rev. D **59**, 084006 (1999).
[3] S. L. Detweiler, Classical Quantum Gravity **22**, S681 (2005).
[4] S. Detweiler, Phys. Rev. D **77**, 124026 (2008).
[5] L. Blanchet, S. Detweiler, A. Le Tiec, and B. F. Whiting, Phys. Rev. D **81**, 064004 (2010).
[6] L. Blanchet, S. Detweiler, A. Le Tiec, and B. F. Whiting, Phys. Rev. D **81**, 084033 (2010).
[7] A. G. Shah, J. L. Friedman, and B. F. Whiting, Phys. Rev. D **89**, 064042 (2014).
[8] L. Blanchet, G. Faye, and B. F. Whiting, Phys. Rev. D **89**, 064026 (2014).
[9] L. Blanchet, G. Faye, and B. F. Whiting, Phys. Rev. D **90**, 044017 (2014).
[10] S. Mano, H. Suzuki, and E. Takasugi, Prog. Theor. Phys. **96**, 549 (1996).
[11] S. Mano, H. Suzuki, and E. Takasugi, Prog. Theor. Phys. **95**, 1079 (1996).
[12] L. Barack and N. Sago, Phys. Rev. Lett. **102**, 191101 (2009).
[13] S. Akcay, L. Barack, T. Damour, and N. Sago, Phys. Rev. D **86**, 104041 (2012).
[14] A. I. Harte, Classical Quantum Gravity **29**, 055012 (2012).
[15] S. R. Dolan, N. Warburton, A. I. Harte, A. Le Tiec, B. Wardell, and L. Barack, Phys. Rev. D **89**, 064011 (2014).
[16] D. Bini and T. Damour, Phys. Rev. D **90**, 024039 (2014).
[17] S. R. Dolan, P. Nolan, A. C. Ottewill, N. Warburton, and B. Wardell, Phys. Rev. D **91**, 023009 (2015).
[18] P. Nolan, C. Kavanagh, S. R. Dolan, A. C. Ottewill, N. Warburton, and B. Wardell, Phys. Rev. D **92**, 123008 (2015).
[19] T. Damour, Phys. Rev. D **81**, 024017 (2010).
[20] L. Barack, T. Damour, and N. Sago, Phys. Rev. D **82**, 084036 (2010).
[21] L. Barack and N. Sago, Phys. Rev. D **83**, 084023 (2011).
[22] S. Akcay, A. Le Tiec, L. Barack, N. Sago, and N. Warburton, Phys. Rev. D **91**, 124014 (2015).
[23] M. van de Meent and A. G. Shah, Phys. Rev. D **92**, 064025 (2015).
[24] N. Sago, H. Nakano, and M. Sasaki, Phys. Rev. D **67**, 104017 (2003).
[25] H. Nakano, N. Sago, and M. Sasaki, Phys. Rev. D **68**, 124003 (2003).
[26] W. Hikida, S. Jhingan, H. Nakano, N. Sago, M. Sasaki, and T. Tanaka, Prog. Theor. Phys. **111**, 821 (2004).

[27] W. Hikida, S. Jhingan, H. Nakano, N. Sago, M. Sasaki, and T. Tanaka, Prog. Theor. Phys. **113**, 283 (2005).

[28] K. Ganz, W. Hikida, H. Nakano, N. Sago, and T. Tanaka, Prog. Theor. Phys. **117**, 1041 (2007).

[29] R. Fujita, Prog. Theor. Phys. **128**, 971 (2012).

[30] R. Fujita, Prog. Theor. Exp. Phys. **2015**, 33E01 (2015).

[31] N. Sago and R. Fujita, Prog. Theor. Exp. Phys. **2015**, 073E03 (2015).

[32] D. Bini and T. Damour, Phys. Rev. D **87**, 121501 (2013).

[33] D. Bini and T. Damour, Phys. Rev. D **89**, 064063 (2014).

[34] D. Bini and T. Damour, Phys. Rev. D **90**, 124037 (2014).

[35] C. Kavanagh, A. C. Ottewill, and B. Wardell, Phys. Rev. D **92**, 084025 (2015).

[36] T. Regge and J. Wheeler, Phys. Rev. **108**, 1063 (1957).

[37] F. Zerilli, Phys. Rev. D **2**, 2141 (1970).

[38] S. Hopper, *Talk presented at the 18th Capra meeting held at Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, Japan*, http://www2.yukawa.kyoto-u.ac.jp/~capra18/S_Hopper.pdf.

[39] D. Bini, T. Damour, and A. Geralico, arXiv:1511.04533.

[40] K. Martel and E. Poisson, Phys. Rev. D **71**, 104003 (2005).

[41] S. Hopper and C. R. Evans, Phys. Rev. D **82**, 084010 (2010).

[42] C. Cutler, D. Kennefick, and E. Poisson, Phys. Rev. D **50**, 3816 (1994).

[43] L. Barack and N. Sago, Phys. Rev. D **81**, 084021 (2010).

[44] C. Darwin, Proc. R. Soc. A **249**, 180 (1959).

[45] I. S. Gradshteyn, I. M. Ryzhik, A. Jeffrey, and D. Zwillinger, *Table of Integrals, Series, and Products*, 7th ed. (Elsevier Academic Press, New York, 2007), ISBN 012-373637-4.

[46] R. Fujita and W. Hikida, Classical Quantum Gravity **26**, 135002 (2009).

[47] L. Barack, A. Ori, and N. Sago, Phys. Rev. D **78**, 084021 (2008).

[48] S. Chandrasekhar, Proc. R. Soc. A **343**, 289 (1975).

[49] S. Hopper, E. Forseth, T. Osburn, and C. R. Evans, Phys. Rev. D **92**, 044048 (2015).

[50] E. Forseth, C. R. Evans, and S. Hopper, arXiv:1512.03051.

[51] C. Evans, *Talk presented at the 18th Capra meeting held at Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, Japan*, http://www2.yukawa.kyoto-u.ac.jp/~capra18/Charles_Evans.pdf.

[52] L. Barack (private communication).

[53] C. Evans (private communication).

[54] L. Blanchet, Living Rev. Relativity **17**, 2 (2014).

[55] S. L. Detweiler and B. F. Whiting, Phys. Rev. D **67**, 024025 (2003).

[56] T. Osburn, E. Forseth, C. R. Evans, and S. Hopper, Phys. Rev. D **90**, 104031 (2014).

[57] A. Heffernan, A. Ottewill, and B. Wardell, Phys. Rev. D **86**, 104023 (2012).

[58] L. Barack and A. Ori, Phys. Rev. D **61**, 061502 (2000).

[59] B. Wardell, Fund. Theor. Phy. **179**, 487 (2015).

[60] R. J. Gleiser, C. O. Nicasio, R. H. Price, and J. Pullin, Phys. Rep. **325**, 41 (2000).

[61] B. Preston and E. Poisson, Phys. Rev. D **74**, 064010 (2006).

[62] S. Hopper and C. R. Evans, Phys. Rev. D **87**, 064008 (2013).

[63] T. Damour, P. Jaranowski, and G. Schfer, Phys. Rev. D **91**, 084024 (2015).

[64] A. Le Tiec, Phys. Rev. D **92**, 084021 (2015).

[65] A. Pound, C. Merlin, and L. Barack, Phys. Rev. D **89**, 024009 (2014).

[66] N. Sago, L. Barack, and S. L. Detweiler, Phys. Rev. D **78**, 124024 (2008).