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Extended disformal approach in the scenario of rainbow gravity

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We investigate all feasible mathematical representations of disformal transformations on a space-time metric according to the action of a linear operator upon the manifold's tangent and cotangent bundles. The geometric, algebraic, and group structures of this operator and their interfaces are analyzed in detail. Then, we scrutinize a possible physical application, providing a new covariant formalism for a phenomenological approach to quantum gravity known as rainbow gravity.

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I. INTRODUCTION

The study of new kinds of symmetries associated with equations of motion is crucial in modern physics, since it can elucidate hidden features and help us find new nontrivial solutions to the equations involved [1]. We quote Noether's theorem, gauge choices, and the theory of the angular momentum operators as some important examples from a lengthy list describing physical properties of a given system that can circumvent the cumbersomeness of solving equations of motion.

The same reasoning can be applied to disformal transformations. The increasing literature on this issue has revealed new physics beyond Bekenstein's initial proposal [2,3]. Motivated by the results we have previously obtained in Ref. [4], we show here that not only can disformal transformations be defined as purely geometric transformations, but that they also allow for distinct representations (algebraic, geometric, and group) in general. In this way, we gather each mathematical aspect of disformal transformations of metric tensors in a unique object. By constructing an abstract operator acting upon the tangent spaces of a manifold, it is possible to see that it takes the form of a space-time geometry, a genuine algebraic tensor, or a group element. This unified description of the disformal maps shall be illustrated by means of the diagrams in Sec. III. The aforesaid operator singles out a preferred vector field (or a set of them) to deform a previously defined tetrad frame. Therefore, depending on the choice of such a vector, the physical notions of time, energy, and momentum can be altered. As we shall see later, new trends in quantum gravity phenomenology—for instance, rainbow gravity [5] and doubly special relativity [6]—point in this direction.

Naturally, one can also interpret the new formalism developed here for disformal transformations as purely mathematical; nevertheless, this is already sufficient to attract attention on its own. Notwithstanding, from the physical point of view, there has been a growing interest in disformal transformations due to their applications in several gravitational theories, such as Bekenstein's TeVeS formalism [7] (which furnishes a covariant formulation for modified Newtonian dynamics [8] in the weak-field limit), bimetric theories of gravity [9], scalar [10] or scalar-tensor theories [11–14] (including mimetic [15–18] and Horndeski ones [19–21]), cosmology [22–25], k-essence [26], and analogue models of gravity [27,28]. There are also situations where disformal transformations are related to the emergence of a Lorentz signature in classical field theory [29], nonmetricity [30], and particle physics [31,32], providing alternative explanations for chiral symmetry breaking [33], the anomalous magnetic moment [34], as well as the disformal invariance of matter field dynamics [4,35,36].

This paper is organized as follows. In Sec. II we review the standard geometric definitions for disformal transformations and; in Sec. II A, we introduce the aforesaid disformal operator acting on vector fields on M. We then follow the same approach to recover the inverse of a disformal metric by introducing a operator acting on covectors. In other words, we establish a new procedure such that instead of mapping a previously defined *metric* onto a disformal metric, we map a previously defined *vector field* onto a disformal vector field. Indeed, a metric tensor is an additional structure we can endow a manifold with and the existence of vector fields only relies on the differential structure of the manifold; hence, it is more fundamental. In Sec. III we show how these operators become algebraic tensors in terms of an arbitrary coordinate system and their relationship to the components of the disformal (co)metric. With the help of the disformal group structure, already satisfied by disformal metrics, we then show that all the aforesaid operators can be reduced to a single one. In Sec. IV we derive an algebraic criterium, in terms of the disformal parameters, to promptly find the causal relation between the background light cone and the disformal one. Also, we

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discuss how the introduction of a disformal operator generating a disformal metric gives rise to two consistent interpretations of causality. Finally, in Sec. V we scrutinize the case of rainbow gravity [5]—a formalism for quantum gravity phenomenology that takes into account a possible energy dependence of the space-time metric probed by a particle with such energy—and show how such a phenomenological approach has a natural explanation within the paradigm of disformal transformations.

II. DISFORMAL TRANSFORMATIONS: THEN AND NOW

Using the definition presented in Ref. [4], a single-vector disformal transformation of a given metric (which we shall simply call a disformal transformation) is an application that takes scalar functions α and β, a metric tensor g, and a globally defined time-like[1] vector field V in a space-time and associates them to another (well-defined) metric tensor $\hat{g}$ according to

$$\hat{g}(*,\cdot) = \alpha g(*,\cdot) + \frac{\beta}{g(V,V)} g(V,*) \otimes g(V,\cdot), \quad (1)$$

where $*$ and $\cdot$ represent the placements of arbitrary vector fields on which the metric tensors g and $\hat{g}$ act. The existence of a nonvanishing time-like vector field on M is guaranteed when one considers time-orientable space-times, which is reasonable from the physical point of view. If M is not time-orientable, there still exists a time-orientable twofold covering of M, where this formalism may lie (cf. the brief discussion in Ref. [37]). Taking that into account, besides mathematical convenience and physical reasonableness, we shall henceforth consider only time-orientable space-times.

The map which defines the disformal metric is well defined if $\alpha > 0$ and $\alpha + \beta > 0$ throughout the manifold. This well-definiteness means that the constructed $\hat{g}$ is a pseudo-Riemannian metric with the same signature as that of g. The scalars α and β are not necessarily functions depending only on points of the manifold, but rather arbitrary functions that could also have a functional dependence on V and its derivatives. We can write down explicitly the components of the disformal metric in a given coordinate system in its covariant and contravariant versions as

$$\hat{g}_{\mu\nu} = \alpha g_{\mu\nu} + \frac{\beta}{V^2} V_\mu V_\nu, \quad (2)$$

$$\hat{g}^{\mu\nu} = \frac{1}{\alpha} g^{\mu\nu} - \frac{\beta}{\alpha(\alpha+\beta)} \frac{V^\mu V^\nu}{V^2}, \quad (3)$$

where $V^2 \equiv g_{\mu\nu} V^\mu V^\nu$ and it is straightforward to verify that $\hat{g}_{\mu\nu}\hat{g}^{\nu\sigma} = \delta^\sigma_\mu$.

The main goal of the ensuing mathematical machinery is to introduce a new formalism to describe disformal metrics. This shall be done in terms of disformal operators acting on vectors and covectors in such a way that the disformal metric inherits their properties.

A. New facet of a disformal transformation

Hereafter, let us fix a space-time (M, g) and a nonvanishing time-like vector field $V \in \Gamma(TM)$.[2] Consider also any two scalars α and β satisfying the conditions $\alpha > 0$ and $\alpha + \beta > 0$. Then, the map (1) can be equivalently described as an action on the tangent space in each point of the manifold by the following definition:

$$\hat{g}(Y,Z) = g(\vec{D}(Y), \vec{D}(Z)), \quad (4)$$

where, for any $X \in \Gamma(TM)$, we have

$$\vec{D}(X) \doteq \sqrt{\alpha+\beta} X_\parallel + \sqrt{\alpha} X_\perp, \quad (5)$$

with

$$X_\parallel \doteq \frac{g(X,V)}{g(V,V)} V, \quad \text{and} \quad X_\perp \doteq X - \frac{g(X,V)}{g(V,V)} V, \quad (6)$$

where $X_\parallel$ is the projection of X onto V and $X_\perp$ is the projection onto the orthogonal complement of V, such that $X = X_\parallel + X_\perp$. So, instead of working with a disformal transformation on the metric, one can define the map

$$\vec{D}\colon \Gamma(TM) \to \Gamma(TM), \qquad X \mapsto \hat{X} = \vec{D}(X) \quad (7)$$

to deform vectors and recover the disformal metric $\hat{g}$. We shall denote by $\hat{X}$ the disformal vector related to X through the action of $\vec{D}$. Clearly, such a map is linear, i.e., $\vec{D}(\gamma X + Y) = \gamma \vec{D}(X) + \vec{D}(Y)$ for any scalar function γ and vectors X and Y, and from this, $\vec{D}$ defines a mixed rank-2 tensor field on M given by

$$\vec{D}\colon \Gamma(T^*M) \otimes \Gamma(TM) \to \mathcal{F}(M),$$
$$(\theta, X) \mapsto \vec{D}(\theta, X) \doteq \theta(\vec{D}(X)), \quad (8)$$

where θ is an arbitrary covector field and $\mathcal{F}(M)$ corresponds to the set of all smooth real-valued functions on M.[3]

[1]It could be extended for light-like vectors or even tensorial fields (as discussed in Refs. [4,36]), but these cases are out of the scope of this paper, for practical reasons.

[2]$\Gamma(TM)$ denotes the set of smooth sections of the tangent bundle, i.e., vector fields over M. This set is also denoted in the literature by $\mathfrak{X}(M)$ or $C^\infty(TM)$.

[3]For economy of notation, we use the same symbol for the operator defined in Eq. (7) and the corresponding mixed tensor (8).

Furthermore, $\vec{D}$ can be seen as a linear transformation on the tangent space T_pM for each $p \in M$ and, provided $\alpha > 0$ and $\alpha + \beta > 0$, a linear isomorphism.

It is simple to see that the operator $\vec{D}$ satisfying Eq. (4) is not unique. In fact, all possible operators of the form $\vec{D}(X) = f_1 X_\parallel + f_2 X_\perp$ satisfying Eq. (4) are fourfold degenerate:

$$\vec{D}_{\{\pm,\pm\}}(X) = \pm\sqrt{\alpha+\beta}X_\parallel \pm \sqrt{\alpha}X_\perp. \tag{9}$$

This degeneracy in the choice of $\vec{D}$ could lead to ambiguities in the definition of a disformal metric by a disformal operator according to Eq. (4). For reasons that shall subsequently become clear, we shall define the disformal operator as $\vec{D}(X) = \vec{D}_{\{+,+\}}(X)$, for every vector field X, and prove its uniqueness afterwards.

B. The disformal cometric

In Sec. V, concerning the physical applications of our analyses, it will be important to use the disformal cometric instead of the disformal metric; therefore, we briefly elaborate upon how one can analogously define a disformal operator acting on covectors to recover the information contained in Eq. (3).

For each vector X in a manifold M endowed with a metric tensor g, there exists its unique metric dual $\tilde{X} \doteq g(X,*)$. Hence the dual is a linear map $\tilde{X}$: $\Gamma(TM) \to \mathcal{F}(M)$, i.e., $\tilde{X} \in \Gamma(T^*M)$. In this way, the cometric h is defined by a linear, symmetric, and nondegenerate map:

$$h\colon \Gamma(T^*M) \otimes \Gamma(T^*M) \to \mathcal{F}(M),$$
$$(\tilde{X},\tilde{Y}) \mapsto h(\tilde{X},\tilde{Y}) = g(X,Y). \tag{10}$$

In a given coordinate system $\{x^\mu\}$, the cometric has components $g^{\mu\nu} = h(dx^\mu, dx^\nu)$, i.e., it is the contravariant components of the metric. From Eq. (3), we can define the disformal cometric intrinsically as

$$\hat{h}(*,\cdot) = \frac{1}{\alpha}h(*,\cdot) - \frac{\beta}{\alpha(\alpha+\beta)}\frac{h(\tilde{V},*)\otimes h(\tilde{V},\cdot)}{h(\tilde{V},\tilde{V})}, \tag{11}$$

where $*$ and $\cdot$ represent here the placements of arbitrary covector fields on which the cometric tensors h and $\hat{h}$ act. Analogously to what we did before, we can define the disformal covector $\hat{\omega}$ associated with ω by the application of a linear map $\tilde{D}$: $\Gamma(T^*M) \to \Gamma(T^*M)$, from which we write the disformal cometric as

$$\hat{h}(\omega,\eta) = h(\tilde{D}(\omega),\tilde{D}(\eta)), \tag{12}$$

with $\tilde{D}$ given by

$$\tilde{D}(\omega) \doteq \frac{1}{\sqrt{\alpha+\beta}}\omega_\parallel + \frac{1}{\sqrt{\alpha}}\omega_\perp \tag{13}$$

and again we have the decomposition

$$\omega_\parallel \doteq \frac{h(\omega,\tilde{V})}{V^2}\tilde{V} \quad \text{and} \quad \omega_\perp \doteq \omega - \frac{h(\omega,\tilde{V})}{V^2}\tilde{V}. \tag{14}$$

Similarly to the covariant disformal operator, we have four possible contravariant disformal operators. We shall set $\tilde{D}(\omega) = \tilde{D}_{\{+,+\}}(\omega)$ to be the disformal operator for elements in $\Gamma(T^*M)$ and again prove that it is unique.

III. MACHINERY AND UNIQUENESS OF THE DISFORMAL OPERATORS

In Ref. [4] one can find algebraic properties concerning disformal metrics as the eigenvalue problem for $\hat{g}^\mu{}_\nu = \hat{g}^{\mu\sigma}g_{\sigma\nu}$ and a group structure satisfied by them. Indeed, we now show that these metrics can be completely characterized by (disformal) operators, since they share similar properties. For practical purposes, it is useful to provide a coordinate representation for both $\vec{D}$ and $\tilde{D}$. We then start with these coordinate expressions and use them to prove some propositions about the disformal operator, and explore. their algebraic and geometric features.

A. Coordinate expressions

To derive a coordinate expression for the disformal operators $\vec{D}$ and $\tilde{D}$, let $\{x^\mu\}$ be a coordinate system, $\{\partial_\mu\}$ the tangent vectors associated with the coordinate lines, and $\{dx^\mu\}$ their duals. Thus,

$$\vec{D}(\partial_\nu) = \sqrt{\alpha}\partial_\nu + \frac{\sqrt{\alpha+\beta}-\sqrt{\alpha}}{V^2}V_\nu V.$$

Applying dx^μ to this, we get the desired expression

$$\mathfrak{D}^\mu{}_\nu \doteq dx^\mu(\vec{D}(\partial_\nu)) = \sqrt{\alpha}\delta^\mu_\nu + \frac{\sqrt{\alpha+\beta}-\sqrt{\alpha}}{V^2}V^\mu V_\nu. \tag{15}$$

In coordinates, we thus have $\hat{X}^\mu = \mathfrak{D}^\mu{}_\nu X^\nu$. Analogously, we get

$$\mathcal{D}_\nu{}^\mu \doteq \partial_\nu(\tilde{D}(dx^\mu)) = \frac{1}{\sqrt{\alpha}}\delta^\mu_\nu + \left(\frac{1}{\sqrt{\alpha+\beta}} - \frac{1}{\sqrt{\alpha}}\right)\frac{V^\mu V_\nu}{V^2} \tag{16}$$

when defining $\hat{\omega}_\mu = \mathcal{D}_\mu{}^\nu\omega_\nu$. It should be remarked that with our definitions, vectors (covectors) transform upon the action of $\vec{D}$ ($\tilde{D}$). Although it is possible to transform covectors (vectors) by means of $\mathfrak{D}^\mu{}_\nu$ ($\mathcal{D}_\nu{}^\mu$) this shall not be of our general interest here.

With the coordinate expressions for $\vec{D}$ and $\tilde{D}$ at our disposal, we state the following.

Proposition 1. $\mathfrak{D}^{\mu}{}_{\nu}$ and $\mathcal{D}_{\nu}{}^{\mu}$ satisfy

$$\mathfrak{D}^{\mu}{}_{\sigma}\mathcal{D}_{\nu}{}^{\sigma} = \delta^{\mu}_{\nu}, \tag{17}$$

and hence act as mutual inverses.

Proof.—This is straightforward from Eqs. (15) and (16).

Using Eqs. (4) and (12) and the coordinate expressions for $\mathfrak{D}^{\mu}_{\nu}$ and $\mathcal{D}^{\mu}_{\nu}$, it is easy to show the following.

Proposition 2. The diagrams

$$\begin{array}{ccc} \Gamma(TM) & \xrightarrow{\vec{D}} & \Gamma(TM) \\ & \searrow^{\hat{g}} & \downarrow g \\ & & \Gamma(T^*M) \end{array} \qquad \begin{array}{ccc} \Gamma(T^*M) & \xrightarrow{\tilde{D}} & \Gamma(T^*M) \\ \hat{h}\downarrow & \swarrow_{h} & \\ \Gamma(TM) & & \end{array}$$

are not commutative, i.e., $g(\vec{D}(V),\cdot) \neq \hat{g}(V,\cdot)$ and $h(\tilde{D}(\omega),*) \neq \hat{h}(\omega,*)$.

From the theory of differentiable manifolds it is known that for each point $p \in M$, the tangent T_pM and cotangent T^*_pM spaces of a differentiable manifold M are naturally isomorphic linear spaces, although this isomorphism is basis dependent. In the presence of a metric tensor on M, there is a canonical isomorphism between T_pM and T^*_pM, namely, for a given vector $X \in T_pM$, there is a unique $\omega \in T^*_pM$ satisfying $g(X,\cdot) = \omega$. Since we are now dealing with a manifold endowed with two metric tensors, the above proposition indicates an ambiguity when taking duals since the g dual of a disformal vector is not the $\hat{g}$ dual of the vector. This ambiguity is always present when the manifold under consideration has more than one metric tensor. Therefore, it is important to make clear which metric tensor is being used when raising and lowering indices.

However, there is a commutative manner to deform vectors and covectors and take their duals. It is not difficult to show that the diagram

$$\begin{array}{ccc} \Gamma(TM) & \xrightarrow{\vec{D}} & \Gamma(TM) \\ \hat{g}\downarrow & & \downarrow g \\ \Gamma(T^*M) & \xrightarrow[\tilde{D}]{} & \Gamma(T^*M) \end{array} \tag{18}$$

is *commutative*. In fact, we could have started with Eq. (4) and the definition of $\vec{D}$ given by Eq. (5) and then defined $\tilde{D}$ to be the only operator able to make Eq. (18) a commutative diagram. In doing so, one can recover the coordinate expression for $\tilde{D}$ and the other properties associated with it. For completeness, we stress that if the direction of the arrows labeled by g and $\hat{g}$ is reversed, and g and $\hat{g}$ are replaced by their inverses, the diagram is also commutative. Thus, one can start with the cometric and the definition of $\tilde{D}$ to define $\vec{D}$ and everything else in terms of it.

B. Disformal group structure revisited

In this section we review one of the mathematical structures underlying disformal transformations (for more details the reader is referred to Refs. [4,36]). Let $\vec{D}_i$ be disformal operators with disformal parameters α_i and β_i, for $i = 1, 2$, given by

$$\vec{D}_1(\cdot) = \sqrt{\alpha_1+\beta_1}(\cdot)_{\parallel} + \sqrt{\alpha_1}(\cdot)_{\perp}, \tag{19}$$

$$\vec{D}_2(\cdot) = \sqrt{\alpha_2+\beta_2}(\cdot)_{\parallel} + \sqrt{\alpha_2}(\cdot)_{\perp}. \tag{20}$$

Since the action of $\vec{D}_i$ on a vector field is also a vector field, it is easy to verify that for any vector field X the following holds:

$$\begin{aligned}\vec{D}_1(\vec{D}_2(X)) &= \vec{D}_2(\vec{D}_1(X)) \\ &= \sqrt{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}X_{\parallel} + \sqrt{\alpha_1\alpha_2}X_{\perp}. \end{aligned}\tag{21}$$

We can use this equation to define the composition of two disformal operators as

$$\begin{aligned}(\vec{D}_1 \bullet \vec{D}_2)(X) &= (\vec{D}_2 \bullet \vec{D}_1)(X) \\ &= \sqrt{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}X_{\parallel} + \sqrt{\alpha_1\alpha_2}X_{\perp}. \end{aligned}\tag{22}$$

One can also verify that the set of all disformal operators with the composition law given above is closed. It means that the composition of two disformal operators is itself another operator. The commutativity and associativity are easily checked, characterizing an Abelian group structure for that set, where the identity operator has parameters $\alpha = 1$ and $\beta = 0$ and the inverse of an operator with parameters α and β has parameters $\alpha' = \alpha^{-1}$ and $\beta' = -\beta[\alpha(\alpha+\beta)]^{-1}$. A similar result is surely obtained for $\tilde{D}$, since they share the same structure.

As mentioned before, there is a group structure associated with disformal metrics. Comparing the composition law (22) with the approach developed in Ref. [4], it is neater and more elegant if we deal with operators instead of the group action. Finally, it should be noticed that particularly interesting examples of disformal subgroups take place when all conformal coefficients are equal to 1—which renders disformal metrics similar to those from the spin-2 field theory formulation, but with finite inverse metric—besides the cases in which the disformal coefficients are zero ($\beta's = 0$), coinciding with the usual conformal group.

C. Uniqueness of the disformal operator

It has been shown that disformal metrics are related to an Abelian group structure [4]. More precisely, there is an Abelian group acting on the space of metrics on M. Besides, in the previous section, we have seen that $\vec{D}$ and $\tilde{D}$ also satisfy an Abelian group structure, and in Eq. (4) we proposed that a disformal metric arises when we deform vectors and use the background metric. So, if we want to characterize the disformal metric in terms of a disformal operator, $\vec{D}$ must be well defined and $\hat{g}$ must inherit some of its properties. Recalling that there are four possible disformal operators satisfying Eq. (4) and that we have claimed $\vec{D}_{\{+,+\}}$ is unique in a certain sense, we then state and prove the following.

Theorem 1. $\vec{D}_{\{+,+\}}$ is the only disformal operator which satisfies Eq. (4) and the disformal group structure. The same holds for the disformal cometric and the operator $\tilde{D}_{\{+,+\}}$.

Proof.—Consider the set G of all admissible disformal operators given by Eq. (9) with a composition law (21). By admissible we mean those operators whose disformal parameters (α's and β's) might satisfy $\alpha > 0$ and $\alpha + \beta > 0$. For instance,

$$
\begin{aligned}
&(\vec{D}^1_{\{-,+\}} \bullet \vec{D}^2_{\{+,-\}})(X) \\
&\quad = (\vec{D}^2_{\{+,-\}} \bullet \vec{D}^1_{\{-,+\}})(X) \\
&\quad = -\sqrt{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}X_\parallel - \sqrt{\alpha_1\alpha_2}X_\perp = \vec{D}_{\{-,-\}}(X),
\end{aligned}
$$

for any $X \in \Gamma(TM)$, where the disformal parameters are $\alpha' = \alpha_1\alpha_2$ and $\beta' = \alpha_1\beta_2 + \beta_1\alpha_2 + \beta_1\beta_2$. It is easy to ascertain that $(\mathrm{G}, \bullet)$ is an Abelian group isomorphic to the Klein four-group (also isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$). As a result of this isomorphism, each element but the identity has order two and the only one whose square is itself is the identity; in our case these requirements are only fulfilled by $\vec{D}_{\{+,+\}}$. Therefore, the only closed composition of disformal operators holds when they are both of the form $\vec{D}_{\{+,+\}}$. The proof that it satisfies a group structure was given in the previous section. This is entirely analogous to the $\tilde{D}$ operator.

A geometrical argument to rule out the other candidates for the disformal operator (the ones with at least one negative sign) is that, because of the negative sign, they must include a reflection in the direction perpendicular to V and/or a change in direction along with V. Therefore, whenever they are applied an even number of times, a positive sign must appear, implying that the operation is not closed. Thus, this ensures that there is a unique and well-defined disformal operator characterizing the disformal metric.

D. The disformal operator as the square root of the disformal metric

We have shown that given a local coordinate system $\{x^\mu\}$ the coordinate expression for the disformal operator $\vec{D}$ takes the form (15). Lowering the μ index with the background metric g, we thus get

$$
\mathfrak{D}_{\mu\nu} \doteq g_{\mu\sigma}\mathfrak{D}^\sigma{}_\nu = \alpha' g_{\mu\nu} + \frac{\beta'}{V^2}V_\mu V_\nu,
$$

where $\alpha' = \sqrt{\alpha} > 0$ and $\alpha' + \beta' = \sqrt{\alpha+\beta} > 0$ and, therefore, $\mathfrak{D}_{\mu\nu}$ can be seen as a disformal metric tensor on M.

Using the composition law between disformal metrics (given in Ref. [4])

$$
(\lfloor\alpha_1,\beta_1\rfloor \odot \lfloor\alpha_2,\beta_2\rfloor)g = (\alpha_1\alpha_2)g_{\mu\nu} + \frac{\beta_1\alpha_2 + \beta_1\beta_2 + \alpha_1\beta_2}{V^2}V_\mu V_\nu,
$$

we obtain that the square of $\mathfrak{D}$ is precisely $\hat{g}$:

$$
(\lfloor\alpha',\beta'\rfloor \odot \lfloor\alpha',\beta'\rfloor)g = \hat{g}_{\mu\nu}. \tag{23}
$$

Another way to see this is by looking at the eigenvalues of $\hat{g}^\mu{}_\nu = \hat{g}_{\sigma\nu}g^{\sigma\mu}$ and $\mathfrak{D}^\mu{}_\nu$: from the definition (5), the eigenvalue problem associated with the operator $\vec{D}$ is trivially solvable. We see that V is an eigenvector related to the eigenvalue $\lambda_V = \sqrt{\alpha+\beta}$, while the other eigenvalues are degenerate and equal to $\sqrt{\alpha}$, with linearly independent eigenvectors lying on the orthogonal complement of V. The same analysis can be carried out for $\tilde{D}$. It means that the eigenvalues of $\hat{g}^\mu{}_\nu$ (see Ref. [4]) are exactly the eigenvalues of $\mathfrak{D}^\mu{}_\nu$ squared, and hence $\mathfrak{D}^2 = \hat{g}$ as operators.

IV. REMARKS ON THE CAUSAL STRUCTURE

We now study the relationship between light cones of the background geometry and the disformal one, and how Eq. (4) gives rise to two interesting (and equivalent) interpretations of causality in the context of disformally related metrics and operators. For the first task, consider a disformal metric as in Eq. (1) with signature $(+,-,-,-)$. Let us fix a point $p \in M$ and at that point consider an orthonormal basis $\{e_A\}$ with respect to the background metric g, where $e_0 = V/\sqrt{g(V,V)}$. Thus, for any $X \in T_pM$, we can write $X = X^A e_A$ and obtain

$$
\hat{g}(X,X) = (\alpha+\beta)(X^0)^2 - \alpha\delta_{IJ}X^I X^J,
$$

where δ_{IJ} is the Kronecker delta, for $I, J = 1, 2, 3$.

Since we want to compare light cones of both metrics, let us assume that X is a null-like vector with respect to $\hat{g}$, that is, $(X^0)^2 - \delta_{IJ}X^I X^J = -\beta(X^0)^2/\alpha$. Therefore, at $p \in M$, we have the following conditions:

(1) If $\beta = 0$, then X is also a null-like vector with respect to g and the disformal light cone is the same as the background one.
(2) If $\beta < 0$, then X is a time-like vector with respect to g and the disformal light cone lies inside the background one.
(3) If $\beta > 0$, then X is a space-like vector with respect to g and the background light cone lies inside the disformal one.

For the second task, note that the existence of the tensor field $\vec{D}$, such that a disformal metric in Eq. (1) can be written as Eq. (4) for any fields X, Y, allows us to interpret the left-hand side of Eq. (4) as a new metric tensor for M. Thus, the light cones of the metric $\hat{g}$ are, in general, different from those of g and hence have a different causal structure on M.

For practical reasons, we provide now a simple and merely illustrative example, without any physical meaning *a priori*. Let us fix the background metric to be the Minkowski one $[\eta = \mathrm{diag}(1, -1, -1, -1)]$ and, at a fixed point p, we have $V(p) = (2, 1, 0, 0)$, $X(p) = 5(-1, 1, 1, 0)$, $\alpha(p) = 2$, and $\beta(p) = 3$. It is easily verified that X is space-like with respect to g and time-like with respect to $\hat{g}$ at p, which means that a vector field could be space-like in the background metric and time-like in the disformal one. This situation is depicted in the top panel of Fig. 1. Conversely, the right-hand side of Eq. (4) indicates that we can consider just the background metric g, but applied to deformed vectors. Therefore, although light cones are preserved, causal relations change because the vectors have done so. In the bottom panel of Fig. 1, the vector is originally space-like and its disformal counterpart is time-like. Surely, the important feature is that the causal relation must agree whether you apply the disformal metric to vectors or the background metric to the deformed vectors.

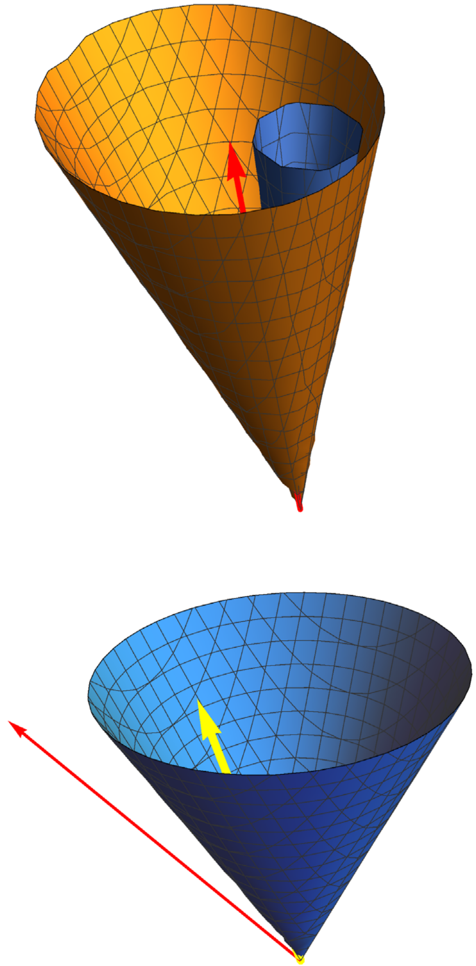

FIG. 1. On top, the vector is kept and the light cone is changed. In this case, X is space-like in g and time-like in $\hat{g}$. On the bottom, the light cone is kept and the vector is deformed. The original vector (red) is space-like and its deformed counterpart (yellow) is time-like.

We thus conclude the mathematical aspects of disformal transformations we intended to develop here. Now, we shall see how to apply this simple and elegant formalism to the realm of quantum gravity phenomenology.

V. APPLICATION TO RAINBOW GRAVITY

As we have stated previously, disformal transformations can naturally be seen as pure mathematics that are attractive on their own and have various physical applications. In particular, we have seen here different representations for performing disformal transformations in the context of differential geometry, essentially by means of metric tensors.

Nevertheless, we are interested in the applications of the aforesaid formalism in what concerns phenomenological approaches to quantum gravity, specifically rainbow gravity [5]. We believe that disformal transformations as presented above provide a unified language for deforming a background space-time metric in this scenario and can shed light on some fundamental problems there, like covariance and causality. With this in mind, we dedicate the forthcoming sections to discussing these issues.

A. Rainbow gravity and disformal metrics

The formulation of rainbow gravity is a phenomenological modification of general relativity that incorporates some properties of the doubly special relativity (DSR) program [5]. DSR models deform the kinematics of special relativity, modifying also the energy-momentum conservation laws and the Lorentz symmetry group, by admitting an invariant energy scale associated with quantum-gravitational effects: the Planck scale. The motivation behind this stems from the path used to go from Galilean relativity to special relativity, namely, modifying the kinematic equations of the former in order to arrive at an invariant velocity scale. Following the same lines, it is possible to deform the latter by taking into account an invariant energy scale, which is generally believed to correspond to quantum-space-time effects, and thus derive the DSR without violation of the relativity principle. For pioneering works see Refs. [6,38,39], and for a broad review see Ref. [40].

Following the prescription presented in Ref. [38] to perform such modifications, one can deform the momentum space of a particle with momentum $\pi = (p_0, p_i)$ using a function U that depends on the ratio between the particle energy p_0 and the Planck energy $\kappa \equiv 1/\sqrt{G}$, where G is Newton's constant, as follows[4]:

$$U(p_0, p_i) = (f_1(p_0/\kappa)p_0, f_2(p_0/\kappa)p_i), \tag{24}$$

leading to the modified dispersion relation (MDR)

$$\|\pi\|^2 = \eta^{AB}[U(\pi)]_A[U(\pi)]_B = (f_1)^2 p_0^2 - (f_2)^2|\vec{p}|^2. \tag{25}$$

In order to guarantee the invariance of this MDR, Lorentz symmetry transformations also need to be deformed. Although this deformation was initially intended to take place in the Minkowski space, the idea of rainbow gravity is that such a MDR can be described by energy-dependent tetrad fields, which in turn produce an energy-dependent (rainbow) metric of the form

$$ds^2 = \frac{(dx^0)^2}{f_1^2} - \frac{1}{f_2^2}\delta_{ij}dx^i dx^j, \tag{26}$$

where δ_{ij} is the Kronecker delta, for i, $j = 1, 2, 3$. This means that space-time is deformed in a way that is the inverse of that in momentum space (for details, see Ref. [5]). The U transformation defined in Eq. (24) resembles the ones we have considered throughout this paper, for a suitable choice of the disformal operator $\tilde{D}$.

In fact, considering a time-like 1-form field $\tilde{V}$ as defining a preferred direction in space-time leads to the definition of energy as the projection of the four-momentum π onto the direction of the corresponding normalized 1-form vector $\nu \doteq \tilde{V}/\sqrt{h(\tilde{V}, \tilde{V})}$, that is, $p_0 \doteq h(\pi, \nu)$. Therefore, the covector responsible for the disformal transformation introduces a natural time-like direction to the reference frame. Thus, using the orthonormal basis $\{\nu, \theta^I\}$, an immediate conclusion one can get from this analysis is that the disformal momentum assumes the form

$$\hat{\pi} = \tilde{D}(\pi) = \frac{1}{\sqrt{\alpha+\beta}}p_0\nu + \frac{1}{\sqrt{\alpha}}p_I\theta^I, \tag{27}$$

where α and β are now scalar functions depending on π and ν, and are linked to the rainbow functions f_1 and f_2 through Eq. (24):

$$\alpha = (f_2)^{-2} \quad \text{and} \quad \beta = (f_1)^{-2} - (f_2)^{-2}. \tag{28}$$

Furthermore, from the definition of the particle mass as the norm of π, a MDR naturally appears from this map in complete analogy with Eq. (25):

$$\hat{m}_\pi^2 \doteq \hat{h}(\pi, \pi) = \left(\frac{1}{\alpha+\beta}\right)p_0^2 - \frac{1}{\alpha}\delta^{IJ}p_I p_J. \tag{29}$$

Finally, the equivalence between the formalism we developed here and rainbow gravity is fulfilled by deriving the induced space-time metric [see Eq. (26)]

$$d\hat{s}^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu = (\alpha+\beta)(dx^0)^2 - \alpha\delta_{ij}dx^i dx^j. \tag{30}$$

Thus, we could unequivocally identify the energy that appears in Eq. (24) as $p_0 = h(\pi, \nu)$ as well as the respective time-like direction that defines the deformation. We stress that although this formalism seems to impose a preferred inertial frame in space-time, which would break the local canonical Lorentz symmetry, in the light of a DSR formulation this is not at all the case, since a deformed version of the Lorentz transformation is the one that preserves the local relativity principle: this is the main conceptual achievement of DSR. Not taking the existence of deformations into account would lead to a formalism with Lorentz invariance violation and, consequently, a preferred reference frame. It should also be noticed that the formalism we developed here is intrinsically geometric and that it is fully covariant under coordinate transformations.

B. Some examples

We now make use of the literature on rainbow gravity (cf. Refs. [38,41]) to illustrate with some examples how this relation works in practice.

The case with $f_1 = f_2$: This will not alter the light cone. If f_1 is equal to f_2, then $\beta = 0$ and the disformal transformation reduces to a conformal one. A well-known example of this was first proposed in Ref. [38]:

$$f_1(E/\kappa) = f_2(E/\kappa) = \frac{1}{1 - E/\kappa}, \tag{31}$$

implying that

$$\alpha = \left[1 - \frac{h(\pi, \nu)}{\kappa}\right]^2. \tag{32}$$

This choice of deformation yields a maximum energy for a one-particle system, given by κ, and the causal structure is (of course) kept invariant.

The case with $f_2 = 1$: This second example (cf. details in Ref. [41]) has an invariant spatial contribution for the dispersion relation. Let

[4]We consider natural units: $c = \hbar = 1$.

$$f_1(E/\kappa) = \frac{e^{E/\kappa} - 1}{E/\kappa} \quad \text{and} \quad f_2(E/\kappa) \equiv 1. \tag{33}$$

In terms of the disformal functions, we get

$$\alpha = 1, \quad \text{and} \quad \beta = \left(\frac{h(\pi,\nu)/\kappa}{e^{h(\pi,\nu)/\kappa} - 1}\right)^2 - 1. \tag{34}$$

For this dispersion relation, one can calculate the speed of light as $(dE/dp)|_{m=0} \approx 1 - E/\kappa$. Therefore, ultraviolet photons propagate with a speed smaller than infrared ones within this model. Note that this is completely consistent with the causal structure analyzed in Sec. IV. Since $-1 < \beta < 0$, we have that $\alpha + \beta < \alpha$ and, therefore, the disformal light cone lies inside the undeformed one.

The case with $f_1 = 1$: This third example is the opposite of the previous one (see Ref. [42] and references therein), in the sense that the time contribution is now kept invariant. Consider

$$f_1 = 1, \quad \text{and} \quad f_2 = \left[1 + \left(\frac{|\vec{p}|}{\kappa}\right)^4\right]^{\frac{1}{2}}. \tag{35}$$

In terms of the metric coefficients, we obtain

$$\alpha = \frac{\kappa^4}{k^4 + [h^2(\pi,\nu) - h(\pi,\pi)]^2}, \tag{36a}$$

$$\beta = \frac{[h^2(\pi,\nu) - h(\pi,\pi)]^2}{\kappa^4 + [h^2(\pi,\nu) - h(\pi,\pi)]^2}. \tag{36b}$$

For this dispersion relation the deformed speed of light is $(dE/dp)|_{m=0} \approx 1 + 5(p/\kappa)^4/2$, which means that high-energy photons propagate with a speed larger than low-energy ones. Again, this is compatible with the causal structure, once $0 < \beta < 1$ and, consequently, $\alpha + \beta > \alpha$. Therefore, the disformal light cone lies outside the undeformed one.

VI. CONCLUDING REMARKS

We have shown that disformal metrics can be written in terms of a linear isomorphism acting on the tangent space and that they actually inherit the properties of what we called the disformal operator. From the reasons presented in the text, this operator can be seen as a more fundamental quantity than the disformal metric, providing a mathematical framework for disformal transformations. We then analyzed this new facet of disformal transformations in the light of the causal structure, where it gives rise to an alternative interpretation of the modified causal cones in purely algebraic terms.

Finally, as a direct application of the formalism developed previously, we verified that the most relevant models in rainbow gravity are perfectly described in terms of disformal transformations. In this vein, it was possible to obtain the missing covariant approach for such a phenomenological theory, with a well-behaved causal structure and a clear mathematical interpretation of the physical quantities involved.

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