PHYSICAL REVIEW D **93,** 043005 (2016)

Field theory for zero sound and ion acoustic wave in astrophysical matter

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(Received 6 November 2015; published 5 February 2016)

We set up a field theory model to describe the longitudinal low-energy modes in high density matter present in white dwarf stars. At the relevant scales, ions—the nuclei of oxygen, carbon, and helium—are treated as heavy pointlike spin-0 charged particles in an effective field theory approach, while the electron dynamics is described by the Dirac Lagrangian at the one-loop level. We show that there always exists a longitudinal gapless mode in the system irrespective of whether the ions are in a plasma, crystal, or quantum liquid state. For certain values of the parameters, the gapless mode can be interpreted as a zero sound mode and, for other values, as an ion acoustic wave; we show that the zero sound and ion acoustic wave are complementary to each other. We discuss possible physical consequences of these modes for properties of white dwarfs.

DOI: 10.1103/PhysRevD.93.043005

I. INTRODUCTION AND SUMMARY

Consider a neutral system of particles made of quantum degenerate electrons, and either oxygen ($Z = 8$) or carbon ($Z = 6$) or helium ($Z = 2$) nuclei, at mass densities $\rho \sim (10^6\text{–}10^7)\ \mathrm{g/cm^3}$ and temperatures $T < (\text{a few}) \times 10^8$ K. For such high densities, average interparticle separations in the system are much smaller than the atomic scale—hence no atoms would form, even if the system were cold. On the other hand, the separations are much larger than the nuclear scale, and hence one can regard the O, C, and He nuclei as pointlike, positively charged, spin-0 particles; we will refer to them as *ions* in the present work, to emphasize that one does not need to care about their detailed nuclear structure.

For the O, C, and He ions, the baryon number, A, equals twice their charge Z, and therefore the ion mass densities, $\rho \sim (10^6\text{–}10^7)\ \mathrm{g/cm^3}$, translate into the electron *number density* that can be estimated as $J_0 \simeq (\rho/2m_p) \sim (0.1\text{–}0.3\ \mathrm{MeV})^3$, where m_p is the protons mass. The corresponding Fermi momenta are $p_F \sim (0.3\text{–}0.9)$ MeV, and hence the electrons are (nearly) relativistic. Their Fermi energy significantly exceeds their thermal energy, as well as their two-body Coulomb interaction energies; therefore, the electron system can be regarded as a quantum degenerate gas.

As for the ions, they are much heavier than the electrons, and hence their thermal de Broglie wavelengths are much shorter, so that at $T \sim (\text{a few}) \times 10^8$ K they generically form a classical gas (see Ref. [1] and references therein). However, at lower temperatures two effects need to be taken into account: (a) Their two body Coulomb interaction energies start to dominate over their thermal energy; (b) their thermal de Broglie wavelengths become comparable to their average separation, and they enter a quantum regime. As a result, below $T \sim (\text{a few}) \times 10^6$ K, the ions may form either a classical or quantum crystal [2], or a quantum liquid [3], depending on concrete values of ρ and Z.

The above described matter is believed to exist in the interiors of white dwarf stars (WDs). These are stars that finished their thermonuclear burning process and are sitting in the sky to radiate away the heat stored in them. They can be regarded as retired stars, slowly evolving from being part of luminous matter to become baryonic dark matter. There are abundant numbers of such WDs observed in our Galaxy alone, and the interiors of the majority of them consist of carbon or oxygen or a mixture of the two, with mass density $\rho \sim (10^6\text{–}10^7)\ \mathrm{g/cm^3}$, although higher densities are also present in many of them (see, e.g., Ref. [1] and references therein). A typical WD starts off at temperature $\sim (\text{a few}) \times 10^8$ K and takes from a few to 10 Gyrs to cool down to about 10^5 K or below, becoming then directly unobservable.

The WD cooling rate is strongly influenced by thermodynamic properties of the state of matter in the bulk of these stars, and in particular by its specific heat. The latter can be calculated if one knows dispersion relations of low-energy excitations of the substance in WDs [1]. As mentioned above, depending on concrete values of the temperature T, density ρ, and ion charge Z, the ions could end up being in a classical gas state or may create a classical or quantum crystal or may condense into a quantum liquid state. These different substances will have different low-energy excitations and hence different specific heats and cooling rates. Knowing accurate values of these rates is important, e.g., for precise determination of the age of the Universe.

Irrespective of the microscopic structure of the resulting state of the ions in WDs, the ion system may be treated as a uniform substance described by an appropriate equation of state, at length scales much greater than the average inter-ion separation. We would like to study long-wavelength collective fluctuations in this neutral system. Furthermore, we will be solely interested in the longitudinal low-energy excitations for reasons that will become clear below. In particular, we would like to understand the interplay between the zero sound mode, that typically exists in degenerate fermionic systems, and the ion acoustic wave that is usually present in a neutral plasma. What we will show is that these two modes are complementary to each other: When one of these modes is present, the other one is absent. However, one of them is always present. We will also show that the cores of O and C WDs with $\rho \sim (10^6–10^7)\ \mathrm{g/cm^3}$ support only the ion acoustic wave; the zero sound mode could be present in those WDs in a relatively narrow spherical shell, away from the cores if density in those domains is $\rho < 10^5\ \mathrm{g/cm^3}$.

In Sec. II, we will set up a prototype effective field theory model to calculate dispersion relations for the low-energy excitations. Our calculations will focus on temperatures below $(\text{a few}) \times 10^7$ K. The two-body Coulomb energy of the ions is of the order of $(10^4–10^5)\ \mathrm{eV} = (10^8–10^9)$ K. While the Coulomb interactions are screened by the electrons, the screening length is greater than the average inter-ion separation, thus leading only to a small reduction, at the level of 10% or so, of the unscreened two-body Coulomb interaction energy [4]. Therefore, for $T < (\text{a few}) \times 10^7$ K, the screened Coulomb energy dominates over the ion thermal energies, by at least an order of magnitude. Hence, we neglect the thermal effects. Likewise, finite temperature effects are negligible for the calculation of dispersion relations in the fermionic part of the system, since $T/J_0^{1/3} \ll 1$. Thus, in Sec. II, we formulate an effective field theory at zero temperature, to calculate the dispersion relations for low-energy modes. One can then use the standard formalism of finite temperature statistical mechanics to evaluate the effects of the dispersion relations on the values of the specific heat. This is a self-consistent procedure, as long as $T/J_0^{1/3} \ll 1$.

In Sec. III, we recover all the results of Sec. II in a Coulomb gauge. The advantage of the latter is that the properties of the longitudinal collective modes are captured by the phase of a scalar field describing the system of spin-0 charged ions. It is in this section that we show that the O and C WDs will dominantly support the ion acoustic wave in their cores, even before the ions turn into a crystal state.

Section IV is dedicated to the He WDs. There are WDs that have a helium core due to the removal of matter and energy from them by their binary companions (see discussions and references in Ref. [5]). Among these, furthermore, there is a very small subclass of the dwarf stars for which the temperature T_c, at which the de Broglie wavelengths of the nuclei begin to overlap, is higher than the would-be crystallization temperature. Then, right below T_c, the quantum-mechanical uncertainty in the position of the charged nuclei is greater that the average internuclear separation. This is exactly opposite to the crystallized state where the nuclei would have well-localized positions with slight quantum-mechanical fuzziness due to their zero-point oscillations.

It was argued in Refs. [3,6] that such a system, instead of forming a crystalline lattice, would condense owing to the quantum-mechanical probabilistic attraction of Bose particles to occupy one and the same zero-momentum state, and leading to a quantum liquid in which the charged spin-0 nuclei would form a macroscopic quantum state with a large occupation number—the charged condensate.

The dispersion relations for the quasimodes of the charged condensate were derived in Refs. [3,6]. These results were obtained by using the unitary gauge and the Thomas-Fermi (TF) approximation for the electrons. Using these results the cooling of the He core WD stars was studied in Refs. [5,7], with the conclusion that they would cool faster due to condensation and that this prediction could be tested if a large enough sample of He core WDs existed.

However, it was subsequently shown in Ref. [8] that the TF approximation combined with the unitary gauge misses one gapless mode, which could affect the cooling calculation. In particular, instead of using the TF approximation, Ref. [8] took into account the one-loop fermion effects and unveiled the gapless mode. The same results were later confirmed in a gauge independent way in Ref. [9]. Furthermore, it was also shown in Ref. [8] that the gapless mode makes a negligible contribution to the specific heat at relevant temperatures and densities for He WDs, and therefore the predictions for the fast cooling rates obtained in Refs. [5,7] remain unchanged. The contribution of the gapless mode to cooling via neutrinos was discussed in Ref. [10]; however, this is also not a significant effect at the relevant temperatures.

In the present work, we will confirm the results of Ref. [8] on the existence of the gapless mode in a charged condensate, by doing calculations in both unitary and Coulomb gauges. The latter calculation turns out to be simpler and does not require going beyond the TF approximation to establish the existence of the gapless mode (the one-loop result is needed though in the Coulomb gauge to calculate the imaginary part of this mode). Furthermore, the Coulomb gauge calculations lead us to the arguments that the gapless mode found in Ref. [8] is an ion acoustic wave, that is present in a charged condensate even when the finite temperature effects are ignored.

A brief outlook is presented in the last section of the paper.

II. PROTOTYPE MODEL

Consider a high density medium composed of nuclei and degenerate electrons, as discussed in the previous section. Depending on the temperature and/or composition, the nuclei could be in a state of plasma, crystal, or charged condensate. In all three cases, Lorentz invariance is broken by the medium; for a crystal the rotation and translation symmetries are also broken to their discrete subgroups. We consider this medium at length scales larger than the average interparticle separations and would like to study long-wavelength longitudinal collective modes. We use an effective field theory technique to write down the Lagrangian that respects symmetries of the system at hand. In this section we discuss a prototype Lagrangian.

The ions are much heavier than the electrons, and at low momenta the effects they produce are presumed to be modeled by introducing in the Maxwell Lagrangian the "electric" and "magnetic" masses, denoted by m_0 and m_γ, respectively (electrons and their effects will be accounted for a bit later),

$$\mathcal{L}_A = \frac{1}{2}(E_j^2 - H_j^2 + m_0^2 A_0^2 - m_\gamma^2 A_j^2), \tag{1}$$

where E_j and H_j, $j = 1, 2, 3$, are the components of the electric and magnetic fields, and unless stated otherwise we use the units $c = \hbar = 1$, for which the vacuum values of the dielectric constant and magnetic permeability are set to unity.

The presence of the electric and magnetic mass terms in (1) captures the phase of the spin-0 ionic matter, as will be more clear below. This phase is what is capturing the relevant low-energy physics of longitudinal modes. In reality, the parameters m_0 and m_γ would be scale dependent quantities and should account for renormalization of the vacuum values of the dielectric constant and magnetic permeability, due to the quantum effects of the ions. However, we are interested in the low momentum and frequency limit, where these parameters can be approximated by constant m_0 and m_γ. The scale dependent renormalization of these quantities due to the electrons will be explicitly included below. We will show that the outlined approximations give a reasonable description for longitudinal modes in the plasma and crystal (Secs. II and III); however, they break down for the charged condensate (Sec. IV), where the explicit scale dependence of the "electric mass" cannot be ignored.

In particular, the prototype model allows for a simple and clear description of the conditions for the existence of a zero sound and ion acoustic wave in the degenerate plasma and crystal; with a certain modification, derived in Sec. IV, this model can also describe the collective fluctuations of the charged condensate.

The Lagrangian (1) is not gauge invariant. The only physical meaning of the latter statement is that (1) describes more degrees of freedom than the Maxwell theory. Local gauge invariance can always be restored at the expense of introducing new fields, and therefore it is a redundancy of the description (although a convenient one). In particular, Eq. (1) can always be regarded as a gauge-fixed version of a gauge-invariant Lagrangian obtained from (1) by the substitution, $A_\mu \to B_\mu = A_\mu - \partial_\mu \alpha$, with the invariance transformation, $\delta A_\mu = \partial_\mu \gamma, \delta\alpha = \gamma$, where γ is a gauge transformation parameter and $\mu = 0, 1, 2, 3$. The phase field α makes explicit the presence of the degree(s) of freedom beyond the two transverse states that can be attributed to the gauge field A_μ, when a nonzero α is retained.

Conversely, Eq. (1) can be regarded as the Lagrangian in the so-called unitary gauge, $\alpha = 0$. We use this gauge in the present section, while in Sec. III we restore back a nonzero α and use instead the Coulomb gauge for the gauge field, $\partial^j A_j = 0$. The results in the two gauges will naturally be the same, but the two derivations are different, each having its own advantages for understanding of the final results.

Irrespective of the gauge choice, the system described by (1) contains an extra longitudinal degree of freedom, in addition to the two transverse modes of a photon. The longitudinal mode can be thought of as a collective low-energy excitation of the charged ion background. Since we have not introduced yet the neutralizing electrons in (1), the spectrum of excitations is gapped by the parameter m_γ. As long as the latter is smaller than the inverse interparticle distance, all three gapped modes can still be meaningfully described by the low-energy Lagrangian.

We now introduce the electrons. Instead of writing an effective Lagrangian for them, we use the fundamental description in terms of the Dirac theory. This is justified; the ions are much heavier than the electrons, and at energy scales below the ion mass, their collective longitudinal mode is captured into an effective Lagrangian (1), while the electrons can be kept fundamental as long as they are weakly interacting and as long as their loop effects will in the end be taken into account (see below). Hence, the total low-energy prototype Lagrangian for the electron-ion system reads as follows:

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_F, \qquad \mathcal{L}_F = \bar{\psi}(i\tilde{\not{D}} - m_e)\psi. \tag{2}$$

Here we have introduced the chemical potential for the electrons μ_e via the usual prescription on the covariant derivative $\tilde{D}_0 = D_0 - i\mu_e$. We will also package the electric and magnetic fields in a Lorentz-invariant Maxwell form, with the Lorentz-breaking effects of the nuclei summarized by m_0 and m_γ; this is pending additional Lorentz-violating effects to arise due to the electrons. To account for the latter, we look at the contribution of the electrons to the Lagrangian at the one-loop level, namely via the photon self-energy $\bar{\Pi}^{\mu\nu}$. Then, the Lagrangian density for the fluctuations in the quadratic approximation is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_0^2 A_0^2 - \frac{1}{2}m_\gamma^2 A_j^2 + \frac{1}{2}A_\mu \bar{\Pi}^{\mu\nu} A_\nu. \quad (3)$$

Note that the inclusion of a one-loop expression for $\bar{\Pi}^{\mu\nu}$ in the effective Lagrangian (3) to determine the dispersion relations yields the same results for the dispersion relations, as if they were deduced from the poles of Green's functions in which the one-loop bubble diagrams have been resummed (such a resummation is often referred as the random phase approximation).

It will be convenient to work in the momentum space and use the Fourier transform of $\bar{\Pi}^{\mu\nu}$ that we denote by $\Pi^{\mu\nu}$. Due to the conservation of the fermion number and rotation symmetry, $\Pi^{\mu\nu}$ can be expressed in terms of two functions, $\Pi(\omega, k)$ and $\Pi^\perp(\omega, k)$, where $k = |\vec{k}|$, and takes the form

$$\Pi^{\mu\nu} = \begin{pmatrix} \Pi & \frac{\omega k_j}{k^2}\Pi \\ \frac{\omega k_i}{k^2}\Pi & \frac{\omega^2 k_i k_j}{k^4}\Pi - \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\Pi^\perp \end{pmatrix}. \quad (4)$$

If we decompose the photon into transverse, longitudinal, and timelike components, $A_j^\perp$, $A^{\rm L}$, A_0, the Lagrangian for the transverse modes decouples from that of the longitudinal and timelike components. In momentum space, we have

$$\mathcal{L}^\perp = \frac{1}{2}A_j^\perp(\omega^2 - k^2 - \Pi^\perp - m_\gamma^2)A_j^\perp, \quad (5)$$

$$\mathcal{L}_2^{\rm L} = \frac{1}{2}(A_0 A^{\rm L}) \cdot \mathbb{M} \cdot \begin{pmatrix} A_0 \\ A^{\rm L} \end{pmatrix}, \quad (6)$$

where

$$\mathbb{M} = \begin{pmatrix} k^2\left(1 + \frac{\Pi}{k^2}\right) + m_0^2 & \omega k\left(1 + \frac{\Pi}{k^2}\right) \\ \omega k\left(1 + \frac{\Pi}{k^2}\right) & \omega^2\left(1 + \frac{\Pi}{k^2}\right) - m_\gamma^2 \end{pmatrix}. \quad (7)$$

The dispersion relations of the two transverse modes are clearly given by

$$\omega^2 - k^2 - \Pi^\perp(\omega, k) - m_\gamma^2 = 0. \quad (8)$$

The dispersion relations for the remaining modes are given by the zeros of the determinant of the matrix $\mathbb{M}$,

$$\det \mathbb{M} = (m_0^2\omega^2 - m_\gamma^2 k^2)\left(1 + \frac{\Pi(\omega, k)}{k^2}\right) - m_0^2 m_\gamma^2 = 0. \quad (9)$$

If we were to neglect the contribution of the electrons, i.e., to set $\Pi(\omega, k) = \Pi^\perp(\omega, k) = 0$, we would find from (9) one massive longitudinal mode

$$\omega^2(k \to 0) = m_\gamma^2, \quad (10)$$

alongside with two massive transverse modes described by (8). Let us see now how the effects of the electrons, encoded in $\Pi(\omega, k)$ and $\Pi^\perp(\omega, k)$, modify this spectrum.

The expression for the nonvacuum contribution to the one-loop self-energy of the photon due to electrons at finite chemical potential is standard and can be found, for example, in Refs. [11,12]. In the approximation of small ω and k as compared to the Fermi energy and momenta, and for $\omega \neq v_F k$, this expression is given by

$$\Pi(\omega, k) \simeq \frac{e^2\mu_e^2}{\pi^2}\left(1 - \frac{1}{2}\frac{\omega}{k v_{\rm F}}\ln\left[\frac{\omega + k v_{\rm F}}{\omega - k v_{\rm F}}\right]\right), \quad (11)$$

where the Fermi velocity $v_{\rm F}$ is defined as $v_{\rm F} = k_{\rm F}/m_e$. The usual Debye screening mass is denoted by m_s,

$$m_s^2 \equiv \Pi(\omega = 0, k \to 0) = \frac{e^2\mu_e^2}{\pi^2}. \quad (12)$$

The above expression is for relativistic electrons, while in the nonrelativistic case, one has $m_s^2 = e^2 m_e k_F/\pi^2$.

Using Eq. (11), we can now check for both massive and massless longitudinal modes. For massive modes, we take $k \to 0$ while keeping ω finite. We find

$$\omega^2(k \to 0) = m_\gamma^2 + \frac{1}{3}m_s^2 v_{\rm F}^2. \quad (13)$$

Thus, the Debye screening mass contributes to the longitudinal mode of the massive photon.

To check for a massless pole, we set $\omega = x v_{\rm F} k$ and also introduce the following notations:

$$a^2 \equiv \frac{m_\gamma^2}{v_{\rm F}^2 m_0^2} \equiv \frac{v_0^2}{v_{\rm F}^2}, \qquad b^2 \equiv \frac{m_\gamma^2}{v_{\rm F}^2 m_s^2} \equiv \frac{v_s^2}{v_{\rm F}^2}. \quad (14)$$

We then take the $k \to 0$ limit while keeping x fixed. Then, $\det \mathbb{M} = 0$ corresponds to

$$(x^2 - a^2)\left(1 - \frac{x}{2}\ln\left[\frac{x+1}{x-1}\right]\right) = b^2. \quad (15)$$

The right-hand side of the above expression is clearly both real and positive. Thus, in order for a solution to this expression to exist, there must be some value of x, either real or complex, for which the left-hand side is both real and positive. We will investigate these solutions below.

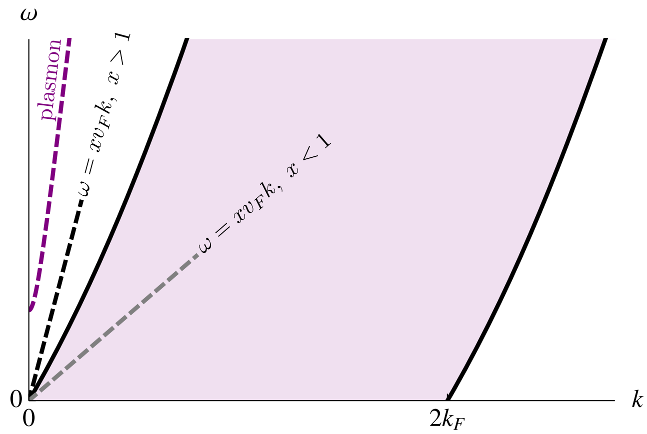

FIG. 1. Spectrum of longitudinal collective modes and particle-hole excitations.

A. Zero sound

Consider real x that is also large, $x \gg 1$. Then, Eq. (15) takes the form

$$(x^2 - a^2)\left(-\frac{1}{3x^2} - \frac{1}{5x^4} + \cdots\right) = b^2,$$

and the solution is

$$x^2 \simeq \frac{5a^2 - 3}{15b^2 + 5}. \tag{16}$$

Our assumption of $x \gg 1$ is valid as long as $a \gg \sqrt{1+3b^2}$. In that case, $x^2 \simeq a^2/(1+3b^2)$, and the dispersion relation for the massless mode is

$$\omega \simeq \frac{v_0 k}{\sqrt{1 + 3v_s^2/v_{\rm F}^2}}. \tag{17}$$

Thus, the velocity of the massless mode (both phase and group) is given by $v_0/\sqrt{1+3b^2}$. The mode exists when $v_0 \gg \sqrt{v_{\rm F}^2 + 3m_\gamma^2/m_s^2}$, because only in that case $x \gg 1$. Furthermore, in our system of electrons and ions, $v_F \gg v_s$, and (17) can be approximated by the dispersion relation $\omega \simeq v_0 k$.

Note that the slope of the linear dispersion relation of the mode (17) is larger than v_F since $x \gg 1$, and thus the $\omega(k)$ line never intersects a region in which the near-the-Fermi-surface modes live (see Fig. 1). Hence, dumping of the mode into electron-hole pairs is negligible.

This excitation corresponds to the zero sound mode that exists in interacting degenerate Fermi systems, e.g., in repulsively interacting Fermi liquids and gases alike [13]. More will be said on this in the next section.

B. Ion acoustic wave

Let us now turn to the opposite regime, when x could be complex but both its real and imaginary parts are small, i.e., $x \ll 1 + i1$. In this case, we need to take into account that the log entering (15) is a multivalued function in the complex plane of its argument. Then, it is instructive to start by proving that the Eq. (15) has no solutions on the first Riemann sheet of the complex x-plane if $|x| < 1$.

To proceed with the proof, we introduce the definition

$$W(x) \equiv (x^2 - a^2)\left(1 - \frac{x}{2}\ln\left[\frac{x+1}{x-1}\right]\right). \tag{18}$$

Let us first consider the case in which $x \in \mathbb{R}$ and $-1 < x < 1$. For this range of values, we can use the following integral representation:

$$W(x) = \frac{x^2 - a^2}{2}\int_{-1}^{+1} dz \frac{z}{z - x - i\epsilon}. \tag{19}$$

The epsilon prescription is given to recover the correct imaginary part of $\Pi(x)$,

$$\begin{aligned} {\rm Im}W(x) &= \frac{x^2 - a^2}{2}\int_{-1}^{+1} dz z\pi\delta(z - x), \\ &= \frac{\pi}{2}x(x^2 - a^2). \end{aligned} \tag{20}$$

We see that for these values of x, $W(x)$ is complex, with the exception of $x = a$. However, at $x = a$, we have $W(x) = 0$, and thus these values of x do not contain a solution of (15).

For all other values of x, real or complex, we use the following representation:

$$W(x) = (x^2 - a^2)\int_0^{+1} dz \frac{z^2}{z^2 - x^2}. \tag{21}$$

For both real x and purely imaginary x, we see that $W(x)$ is real. However, only when $a > 1$ can $W(x)$ be positive and thus solve (15).

Finally, we check for complex x. Setting $x = \sigma + i\beta$, with $\sigma, \beta \in \mathbb{R}$ and $\sigma, \beta \neq 0$, and using the representation (21), we find through some basic algebra that either ${\rm Im}W(x) \neq 0$ or ${\rm Re}W(x) < 0$. Thus, only in the case in which $a > 1$ is $W(x)$ both real and positive. Otherwise, we cannot solve (15), and no massless pole exists on the first Riemann sheet. This completes the proof.

Thus, we look for solutions on the second Riemann sheet and rewrite the Eq. (15) in the following form:

$$(x^2 - a^2)\left(1 - \frac{x}{2}\left(\ln\left[\frac{1+x}{1-x}\right] - i\pi\right)\right) = b^2. \tag{22}$$

In the approximation of small x, this has a solution:

$$x \simeq \sqrt{a^2 + b^2} - i\frac{\pi b^2}{4}. \tag{23}$$

Thus, the physical mode at hand has the following dispersion relation:

$$\omega \simeq \left(\sqrt{v_0^2 + v_s^2} - i\frac{\pi v_s^2}{4 v_{\rm F}}\right)k. \tag{24}$$

The sign of the imaginary parts above is such that the mode is on the second Riemann sheet of the complex energy plane. Thus, it is a resonance, with a momentum dependent width. In what follows, we will argue that this is nothing but the ionic acoustic wave.

Consider the $m_0 \to \infty$ limit; the above mode does exist in this limit, and its dispersion relation reads

$$\omega = \left(v_s - i\frac{\pi v_s^2}{4 v_{\rm F}}\right)k. \tag{25}$$

It is instructive to go back to the Lagrangian and calculate the exact k dependence of the mode, not only its low k limit. The $m_0 \to \infty$ limit settles us in the regime $A_0 = 0$. The Lagrangian then takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_t A_j)^2 - \frac{1}{4}F_{ij}^2 - \frac{1}{2}m_\gamma^2 A_j^2 + \frac{1}{2}A_i \bar{\Pi}^{ij} A_j. \tag{26}$$

This Lagrangian describes four degrees of freedom: two massive helicity-1, one massive helicity-0, and one massless helicity-0 states; the latter is the mode with the dispersion relation (25). To see all this, we proceed by splitting the transverse and longitudinal parts of A_j as follows:

$$A_j = A_j^\perp + \frac{\partial_j \pi}{m_\gamma}. \tag{27}$$

Upon this substitution, the Lagrangian splits into two separate parts, one for $A_j^\perp$,

$$\frac{1}{2}(\partial_t A_j^\perp)^2 + \frac{1}{4}A_j^\perp \Delta A_j^\perp - \frac{1}{2}m_\gamma^2 (A_j^\perp)^2 + \frac{1}{2}A_i^\perp \bar{\Pi}^{ij} A_j^\perp,$$

and one for π,

$$\frac{1}{2}\left(\frac{\partial_t \partial_j \pi}{m_\gamma}\right)^2 - \frac{1}{2}(\partial_j \pi)^2 - \frac{(\partial_t \pi)\bar{\Pi}(\partial_t \pi)}{m_\gamma^2}. \tag{28}$$

It is straightforward to see that the dispersion relations for the two modes in $A_j^\perp$ are determined by

$$\omega_\perp^2 = k^2 + m_\gamma^2 + \Pi^\perp(\omega_\perp, k), \tag{29}$$

while the dispersion relations that follow from the Lagrangian (28) are determined by the equation

$$\omega^2 = \frac{m_\gamma^2 k^2}{k^2 + \Pi(k, \omega)}. \tag{30}$$

In the absence of Π (i.e., if the electrons are assumed to be frozen), we get from (30)

$$\omega^2 = m_\gamma^2 = \frac{Z e^2 J_0}{m_H}. \tag{31}$$

The latter is a plasmon dispersion of a charged ion gas. In this limit of "frozen" electrons ($\Pi \to 0$), there is nothing to compensate for a displaced change of ions upon their perturbation, and therefore the mode is gapped. Note that in the $\Pi = \Pi^\perp = 0$ limit the longitudinal and transverse modes have the same gap, $\omega(k = 0) = \omega_\perp(k = 0) = m_\gamma$, consistent with the rotation symmetry.

Once the dynamical electrons are returned back, then the Lagrangian (28) and the dispersion relation (30) describe two modes, one massive and one massless. Let us start with the massive mode; taking $k \to 0$ in (30) with fixed and nonzero ω, we find $\Pi \simeq -m_s^2 v_{\rm F}^2 k^2/(3\omega^2)$ and get a solution $\omega^2 = m_\gamma^2 + m_s^2 v_{\rm F}^2/3$, which agrees with (13). The latter sets the gap for the longitudinal photon.

For small ω and k, on the other hand, we get a massless mode as follows: In this regime, the real part of Π, $Re\Pi \simeq m_s^2$, and therefore the real part of the corresponding dispersion relation reads

$$\omega = \sqrt{\frac{m_\gamma^2 k^2}{k^2 + m_s^2}} \simeq v_s k. \tag{32}$$

This dispersion relation coincides with the $m_0 \to \infty$ limit of the one in (24), up to the imaginary part that can also be straightforwardly recovered by keeping it in Π. Therefore, this mode is an ion acoustic wave; sometimes it is also referred as an ionic sound [14]. It corresponds to a collective excitation of the ions and neutralizing fast electrons, producing a gapless longitudinal mode.

A few comments before we turn to the next section. We reiterate that the propagator in the considered regime has no low-energy poles on the first Riemann sheet, neither real nor complex ones. The obtained complex pole describing the ionic sound is on the second sheet. The function that the propagator is proportional to

$$\frac{1}{(x^2 - a^2)\Pi - m_\gamma^2}$$

has a branch cut and exhibits a "bumplike" behavior for real values of energy and momentum. Why does this mode have a resonancelike nature? The reason is simple: The phase velocity of the mode equals $x v_F$, that is less than v_F. The range of allowed energies of the electron-hole pairs near the Fermi surface (depicted in Fig. 1) has the upper boundary with the slope near the origin that equals v_F. Thus, a wave with the dispersion $\omega \simeq x v_F k$ would be damped due to near-the-Fermi-surface excitations, as long as $x < 1$.

This also explains why this mode does not exist for $x \gg 1$. If it existed, it would have been faster than the near-the-Fermi-surface excitations, which then would not be able to catch up with the mode to give rise to a neutral acoustic wave; instead, in the $x \gg 1$ regime, only the zero sound mode exists. Thus, the zero sound and ion acoustic wave are complementary to each other.

III. PHASE FIELD AND LONGITUDINAL MODES

The derivations in the previous section were performed in the unitary gauge; i.e., the phase of the charged scalar field describing collective motion of the ions was gauge fixed to zero. This was reflected in the gauge-noninvariant terms in the Lagrangian (1). The latter was regarded as a gauge-fixed version on a gauge-invariant Lagrangian, obtained from (1) by a substitution, $A_\mu \to B_\mu \equiv A_\mu - \partial_\mu \alpha$, with the unitary gauge corresponding to $\alpha = 0$. In this section, we would like to keep nonzero α, but we use instead the Coulomb gauge $\partial^j A_j = 0$. This will enable us to obtain and understand the results of the previous section in a more clear way.

A. Warmup example

We start with a field theory containing a complex scalar field, degenerate fermions, and an Abelian gauge field that couples to both the scalar and fermions in a conventional way. The charged scalar field is thought to model properties of charged nuclei, while fermions model the electrons, and the Abelian gauge boson models a photon. For illustrative purposes, we consider the case when the $U(1)$ gauge symmetry is spontaneously broken, and the gauge field acquires the mass term as a result of this breaking. Furthermore, we consider the parameter space for which the scalar field radial mode is a heavy state that can be decoupled from the rest of the fields. In this approximation, and in the Coulomb gauge, the relevant part of the Lagrangian reads as follows:

$$\frac{1}{2} m^2 B_\mu^2 + \frac{1}{2} A_0 (-\Delta + \bar{\Pi}) A_0. \tag{33}$$

The first term is the gauge boson mass term, while the second term contains a part of the Maxwell Lagrangian, as well as the term generated due to one-loop renormalization of the polarization operator via the fermion-antifermion pair. These are all the terms in the Lagrangian that contain A_0 and α.

Integrating out A_0, we get for $m^2 \neq 0$

$$A_0 = \frac{m^2}{(-\Delta + m^2 + \bar{\Pi})} \partial_0 \alpha \tag{34}$$

[we neglect an irrelevant zero mode of the inverse of the operator multiplying $\partial_0 \alpha$ on the right-hand side of (34)]. Substituting (34) back into the Lagrangian (33) and calculating the dispersion relation for the remaining field α, we obtain the following expression:

$$\omega^2 \frac{k^2 + \Pi(\omega, k)}{k^2 + \Pi(\omega, k) + m^2} = k^2. \tag{35}$$

Let us now study this relation in two different regimes, $x \gg 1$, and $x \ll 1$, where as before we introduce $\omega = x v_F k$. For the $x \gg 1$ case, one gets $\Pi \simeq -m_s^2/(3x^2)$. Hence, the solution to (35) reads

$$x^2 = \frac{1}{v_F^2 + 3m^2/m_s^2}. \tag{36}$$

When $m_s \gg m$ and $v_F \ll 1$, we get the condition, $x \gg 1$, required by the approximation made. As per the arguments of the previous section, this is the dispersion relation of the zero sound mode,

$$\omega = \frac{v_F}{\sqrt{v_F^2 + 3m^2/m_s^2}} k. \tag{37}$$

We see that in the Coulomb gauge the zero sound mode is described by the phase α. Since $x \gg 1$, the phase/group velocity of this mode is greater than v_F, and hence, this mode experiences no damping in the approximation we use.

In the opposite regime, $x \ll 1$, the gapless mode does not exist, as shown by *reductio ad absurdum*: Assuming that $x \ll 1$, one gets $\Pi \simeq m_s^2 (1 + \mathcal{O}(x))$, and the solution $x = v_F^{-1} \sqrt{(1 + m^2/m_s^2)} > 1$, which contradicts the initial assumption that x were small.

The lack of the ion sound wave in this model has a reason: We used the approximation when the Lorentz-invariant vacuum condensate of the charge scalar is the only source for the mass term [the first term in (33)]; this implies that the number of dynamical scalars that can fluctuate is zero, in the approximation used (the Lorentz-invariant Higgs vacuum has zero scalar number). Hence, one should not expect to have the ion acoustic wave, in the approximation when the number of ions that can fluctuate is ignored (the charge that neutralizes the electrons in this approximation is not dynamical).

B. More realistic model

We now consider the case when the electric and magnetic masses of the photon are different, and the phase is explicitly kept, while the Coulomb gauge is assumed for

A_j, as in the previous subsection. The relevant part of the Lagrangian in this gauge then reads

$$\frac{1}{2}(m_0^2 B_0^2 - m_\gamma^2 B_j^2) + \frac{1}{2} A_0(-\Delta + \bar{\Pi}) A_0. \tag{38}$$

As before, we integrate out A_0 to get

$$A_0 = \frac{m_0^2}{(-\Delta + m_0^2 + \bar{\Pi})} \partial_0 \alpha. \tag{39}$$

Substituting this back into (38) and deducing the dispersion relation for α, we obtain

$$\omega^2 \frac{k^2 + \Pi(\omega, k)}{k^2 + \Pi(\omega, k) + m_0^2} = v_0^2 k^2, \qquad v_0^2 \equiv \frac{m_\gamma^2}{m_0^2}. \tag{40}$$

In a realistic system such as a plasma and solid, $m_0 \gg m_\gamma$, and the ion sound speed $v_0 \ll 1$.

For $x \gg 1$, we know that $\Pi \simeq -m_s^2/(3x^2)$, and in this approximation, the solution to (40) reads

$$x^2 = \frac{1}{v_F^2/v_0^2 + 3m_0^2/m_s^2}. \tag{41}$$

Hence, when $m_s > m_0$ and $v_0 \gg v_F$, we get $x \gg 1$, and the zero sound dispersion relation

$$\omega \simeq \frac{v_F}{\sqrt{v_F^2/v_0^2 + 3m_0^2/m_s^2}} k. \tag{42}$$

This coincides with the dispersion relations for the zero sound mode found in (17).

Unlike in the previous subsection, however, a solution also exists in the opposite limit, $x \ll 1$: Indeed, assuming that $x \ll 1$, one gets $\Pi \simeq m_s^2(1 + \mathcal{O}(x))$ and hence the solution $x = \frac{v_0}{v_F}\sqrt{(1 + m_0^2/m_s^2)}$, which, if $v_0\sqrt{(1 + m_0^2/m_s^2)} \ll v_F$, gives rise to the following physical mode:

$$\omega \simeq v_0 \sqrt{1 + \frac{m_0^2}{m_s^2}} k. \tag{43}$$

This is nothing but the dispersion relation for the longitudinal ion sound wave discussed in Sec. II.

One can give heuristic arguments for the complementarity of the two modes that we have derived above. For this, we consider the regime in which v_s can be neglected as compared with v_0 and v_F. This is a meaningful approximation for a plasma and crystal since m_γ is suppressed by the heavy ion mass scale, while m_s is determined by the electron mass and chemical potential. Then, the phase velocity of both the zero sound mode and the ion acoustic wave is approximately v_0. When $v_0 \ll v_F$, we get $x \ll 1$, and the ionic sound is present; this makes sense—the electrons are faster than the wave, and thus they can readily follow a perturbation of the ions to screen its charge and create a gapless neutral mode, the ionic sound. In the opposite limit, $v_0 \gg v_F$, the electrons are slower than the would-be ionic sound wave, and therefore they cannot be effectively screening ion perturbations, and it makes sense that the ion acoustic wave does not exist. Instead, the electron fluctuations themselves—which are now effectively screened by the ambient mobile ion charge distribution—form a collective mode, the zero sound.

Having derived the dispersions relations, let us apply them to the system of the oxygen and carbon ions at densities $\rho \sim (10^6–10^7)$ g/cm^3. As we have already discussed, the corresponding Fermi momentum for the electrons is $p_F \sim (0.3–0.9)$ MeV. Therefore, the electrons are (nearly) relativistic, with $v_F \sim 1$, to a good accuracy. This implies that the value of x given in (41) can never be greater than the unity, since $v_0 < 1$. Therefore, we conclude that the zero sound cannot be supported in the cores of the O and C WDs. Instead, the dominant longitudinal mode in this case is the ion acoustic wave (43), with the sounds speed approximated by $v_0 \ll 1$. The exact value of the sound speed cannot be calculated in our effective Lagrangian approach; however, our finding confirms the suggestion made in Ref. [2] that a longitudinal wave, the one that is similar to the longitudinal acoustic wave of a crystal, can be used to study the cooling of the O and C WDs even when the interior is in a strongly interacting plasma state and the crystal is not yet formed [i.e., from temperatures (a few) $\times 10^7$ K, down to (a few) $\times 10^6$ K, after which the crystal forms].

As to the zero sound mode, its existence requires lower densities. While WDs have fairly uniform density profiles in their bulks (excluding their "atmospheres" that are dominantly made of H and He, with some small fractions of "metals"), nevertheless the interiors are not exactly uniform. Typically, one can have variation of density amounting to a factor of 5 in the ratio of the maximal density to average density. Therefore, it might not be foolish to think of relatively narrow spherical shells, away from the cores of low density WDs, in which densities might be $\rho < 10^5$ g/cm^3. In such shells, the zero sound mode could be supported instead of the ion acoustic wave. It could also be interesting to look for the existence of the zero sound mode in the low mass brown dwarfs [15], where densities are $\rho \sim 10^3$ g/cm^3.

IV. CHARGED CONDENSATE

We now consider in detail the spectrum of the charged condensate. We incorporate the effect of the background density of electrons through their one-loop contribution to the photon self-energy. An analogous approach was used in Refs. [16,17] to determine the electrostatic potential of this system at finite temperature. More recently, this approach was applied in Ref. [8] to argue that the dynamical electrons give rise to a previously unnoticed massless mode. Here we confirm this finding and argue that the obtained massless mode is the ion acoustic wave discussed in the previous section.

A classical nonzero vacuum expectation value of a field Φ can serve as an order parameter for the condensation of the helium-4 nuclei, describing a state with a large occupation number. Fluctuations of the order parameter describe the collective modes of the condensate. In the nonrelativistic approximation, the effective Lagrangian for the nuclei Φ and electrons ψ can be written as follows [6]:

$$\mathcal{L} = \mathcal{P}(X) - \frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(iD - m_e)\psi, \tag{44}$$

where

$$X = \frac{i}{2}(\Phi^* D_0\Phi - (D_0\Phi)^*\Phi) - \frac{|D_j\Phi|^2}{2m_H}. \tag{45}$$

The covariant derivative of the scalar field is given by $D_\mu = \partial_\mu - 2ieA_\mu$. $\mathcal{P}(X)$ stands for a general polynomial function of its argument. The coefficients of this polynomial are dimensionful numbers that are inversely proportional to powers of a short-distance cutoff of the effective field theory. We normalize the first coefficient to 1 so that

$$\mathcal{P}(X) = X + c_2X^2 + \cdots. \tag{46}$$

The Lagrangian is invariant under global $U_s(1)$ transformations, responsible for the conservation of the number of scalars. Another global $U_e(1)$ guarantees the electron number conservation. Accordingly, we can introduce chemical potentials for both the scalars and electrons, μ_s and μ_e, respectively, via the usual prescription on the covariant derivative $D_0 \to D_0 - i\mu$.

We could also have included a quartic interaction for the scalars $\lambda(\Phi^*\Phi)^2$. However, as long as the quartic coupling is small, $\lambda n \ll m_H^3$, our results will not be affected by this term.

It is useful to represent the scalar as $\Phi = \Sigma e^{i\Gamma}$ and to work first in the unitary gauge where the phase of the scalar is set to zero, $\Gamma = 0$. At a later stage, we will move instead to the Coulomb gauge where things will become easier.

When the scalar chemical potential is zero, $\mu_s = 0$, there is a solution to the equations of motion of (44) with a nonzero expectation value for the scalar field

$$\langle\Sigma\rangle = \sqrt{\frac{J_0}{2}}, \tag{47}$$

where J_0 is the background electron density.

Let us consider the quadratic action around this background solution. We introduce perturbations of the scalar field as follows:

$$\Sigma(x) = \sqrt{\frac{J_0}{2}} + \sqrt{m_H}\tau(x). \tag{48}$$

The factor of $\sqrt{m_H}$ has been introduced for convenience. Let us also integrate out the electrons as was done in Ref. [8]. Thus, the Lagrangian density for the fluctuations in the quadratic approximation reads

$$\mathcal{L}_2 = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_0^2A_0^2 - \frac{1}{2}m_\gamma^2A_j^2 + \frac{1}{2}A_\mu\Pi^{\mu\nu}A_\nu - \frac{1}{2}(\partial_j\tau)^2 + 2m_Hm_\gamma A_0\tau. \tag{49}$$

Here

$$m_\gamma^2 \equiv (2e)^2\frac{J_0}{2m_H}, \qquad m_0^2 = c_2J_0m_Hm_\gamma^2. \tag{50}$$

When the coefficient c_2 is such that $m_0 = m_\gamma$, the dispersion relations coincide with those of a relativistic theory; however, in general $m_0 \neq m_\gamma$.

A. Spectrum in the unitary gauge

We now consider the spectrum of these fluctuations. We decompose the photon into transverse, longitudinal, and timelike components. In addition, let us also integrate out the scalar mode τ so that we can easily compare our results to those of the previous section. Again, the transverse modes of the photon decouple entirely. Their dispersion relations are given by

$$\omega^2 - k^2 - \Pi^\perp(\omega, k) - m_\gamma^2 = 0. \tag{51}$$

For the remaining modes, we have

$$\mathcal{L}_2^{\mathrm{L}} = \frac{1}{2}(A_0A^{\mathrm{L}})\cdot\mathbb{M}\cdot\begin{pmatrix}A_0\\A^{\mathrm{L}}\end{pmatrix}, \tag{52}$$

where now

$$\mathbb{M} \equiv \begin{pmatrix} k^2\left(1+\frac{\Pi}{k^2}\right) + \frac{4M^4}{k^2} + m_0^2 & \omega k\left(1+\frac{\Pi}{k^2}\right) \\ \omega k\left(1+\frac{\Pi}{k^2}\right) & \omega^2\left(1+\frac{\Pi}{k^2}\right) - m_\gamma^2 \end{pmatrix}, \tag{53}$$

with $M \equiv \sqrt{m_\gamma m_H}$. The matrix $\mathbb{M}$ differs from the analogous matrix in our prototype model only via the appearance of the $4M^4/k^2$ term.

The dispersion relations of these modes are again given by the zeros of the determinant of the matrix $\mathbb{M}$. To check for the presence of massive modes, we use expression (11) in the matrix (53) and take the $k \to 0$ limit, keeping ω finite. Setting $\det\mathbb{M} = 0$, we find a massive mode,

$$\omega^2(k \to 0) = m_\gamma^2 + \frac{1}{3} m_s^2 v_{\rm F}^2. \quad (54)$$

This can be thought of as the mass of the longitudinal component of the photon that now carries 3 degrees of freedom. The mass receives two contributions, one from the charged nuclei, m_γ^2, and another one from charged electrons, $m_s^2 v_{\rm F}^2/3$. Hence, we find that the one-loop contribution shifts up the mass of the longitudinal mode. This mode was argued to be heavy to contribute to specific heat of the charged condensate at relevant temperatures [6]; we see that the one-loop contribution makes it even heavier, justifying further that it can be neglected.

To determine the possibility of a massless pole, we set $\omega = x v_F k$ and then take the $k \to 0$ limit. Then, $\det \mathbb{M} = 0$ corresponds to

$$x^2 \left(1 - \frac{x}{2} \ln \left[\frac{x+1}{x-1} \right] \right) = \frac{m_\gamma^2}{m_s^2}. \quad (55)$$

Thus, we see that the charged condensate corresponds to our prototype model of Sec. II in the particular case with $a = 0$. As argued above, there is a massless resonance solution, with the dispersion relation $\omega \simeq (v_s - i \frac{\pi v_s^2}{4 v_{\rm F}}) k$, in agreement with Ref. [8]. This is also in agreement with the dispersion relation obtained in Sec. II, describing the ion acoustic wave in a plasma. In our view, this mode has the same physical origin and interpretation as the ion sound wave of ordinary degenerate plasma [14]. We further strengthen this latter point in Sec. IV C by unveiling the hydrodynamics origin of the ion acoustic wave in a charged condensate.

B. Spectrum in the Coulomb gauge

Here, we derive the results of the previous subsection in the Coulomb gauge. This makes the connection to the calculations in the prototype model presented in Sec. III clearer and helps to explain the origin of the ion sound wave in the charged condensate.

As before, we use $B_\mu = A_\mu - \partial_\mu \alpha$. The relevant part of the Lagrangian in the Coulomb gauge reads as follows,

$$\frac{1}{2} (\bar{m}_0 B_0^2 - m_\gamma^2 B_j^2) + \frac{1}{2} A_0 (-\Delta + \bar{\Pi}) A_0, \quad (56)$$

where the key difference from the model of Secs. II and III is that electric mass in the charged condensate has essential dependence on the momentum:

$$\bar{m}_0 = \left(m_0^2 + \frac{4M^4}{-\Delta} \right). \quad (57)$$

This suggests that for small momenta, we always end up in the regime of large electric mass for the photon; as discussed in Sec. II, this implies that the zero sound mode will be absent, but the ionic acoustic wave should be present, in agreement with the results of Sec. IV A. Let us see this explicitly in the Coulomb gauge.

We first integrate out A_0:

$$A_0 = \frac{m_0^2}{(-\Delta + m_0^2 + \frac{4M^4}{-\Delta} + \bar{\Pi})} \partial_0 \alpha. \quad (58)$$

Substituting this back into the Lagrangian (56) and deducing the dispersion relation for α, we get

$$\omega^2 \frac{k^2 + \Pi(\omega, k)}{k^2 + \Pi(\omega, k) + m_0^2 + \frac{4M^4}{k^2}} = v_0^2 \frac{m_0^2}{m_0^2 + \frac{4M^4}{k^2}} k^2. \quad (59)$$

As before, we use the notation $\omega = x v_F k$. For both k and ω approaching zero, we only get one dispersion relation, and that is with $x \ll 1$. This dispersion relation reads

$$\omega \simeq \frac{m_\gamma}{m_s} k. \quad (60)$$

This is the ionic acoustic wave, in agreement with previously obtained results.

A final comment before we move to the hydrodynamics considerations. In the approach adopted above, the fermions were treated in a one-loop approximation, while the scalars were treated in terms of a low-energy effective field theory. The quantum effects of the condensed scalars were captured by the order parameter Lagrangian. It is instructive to check that the same results are obtained if the scalars are also treated via the one-loop calculations, as was done in Refs. [16,17]. We briefly use this method in the Coulomb gauge. Then, the poles of the full propagator are determined by

$$k^2 + \Pi^B(\omega, k) + \Pi(\omega, k) = 0, \quad (61)$$

where the polarization operator for the bosons, $\Pi^B(\omega, k)$, can be calculated straightforwardly via the corresponding one-loop diagrams [16,17]. The part of the loops that is due to the existence of the condensate reads as follows:

$$\Pi^B(\omega, k) = \frac{m_\gamma^2}{2} \left[\frac{(2m_H - \omega)^2}{(\omega - m_H)^2 - k^2 - m_H^2} + \frac{(2m_H + \omega)^2}{(\omega + m_H)^2 - k^2 - m_H^2} - 2 \right]. \quad (62)$$

Substituting this into (61) and taking the small momentum limit, we get the pole at

$$\omega \simeq \frac{m_\gamma}{m_s} k. \tag{63}$$

The latter coincides with the result already obtained above (60).

C. Hydrodynamic considerations

The purpose of this subsection is to demonstrate that the ionic sound found in the prototype model, as well as its counterpart emerging in the charged condensate, can be understood in terms of standard hydrodynamics, with the only difference that the charged condensate hydrodynamics equations need to retain the pressure gradient term even when finite temperature effects are ignored, as will be shown below.

We are looking at a degenerate plasma of electrons and positively charged nuclei (or ions); as already mentioned, depending on temperature T, the ion mass m_H, and charge Z, the system of nuclei could be in a classical gas state or may create a Wigner crystal or may be in a condensed quantum liquid state. In any case, at length scales much greater than the inter-ion separation, the ion system may be treated as a uniform substance described by an appropriate equation of state. We would like to understand the spectrum of long-wavelength longitudinal collective fluctuations in this system.

Let us first ignore the temperature effects and consider the case when the ions are in the plasma or crystal state. All the hydrodynamic equations presented below for this case are well known but are given to emphasize the difference of these states from the charged condensate, to which we will turn by the end of the section.

The continuity equations for the electron and ion number densities—denoted, respectively, by n_e and n_H—read as

$$\partial_t n_{e,H} + \partial_j (n_{e,H} v_j) = 0, \tag{64}$$

while the momentum equation for the ions is

$$\partial_t v_j + (v_k \partial_k) v_j = -\frac{Ze}{m_H} E_j. \tag{65}$$

Consider small localized perturbations, small overdensities $\delta n_{e,H}$ of the electrons and ions over their background values set by J_0 and J_0/Z, respectively,

$$n_e = J_0 + \delta n_e, \qquad n_H = \frac{1}{Z} J_0 + \delta n_H. \tag{66}$$

For both the plasma and crystal states at long wavelength, the Poisson equation for the electrostatic potential created by these perturbations reads as follows:

$$\Delta A_0 = -Ze\delta n_H + e\delta n_e. \tag{67}$$

For simplicity, we consider relativistic electrons here. The electron overdensity can be related to the local potential via the Thomas-Fermi approximation,

$$E_F \equiv \mu = -eA_0 + p_F, \tag{68}$$

and using that $n_e = p_F^3/3\pi^2$, we find

$$\delta n_e \simeq 3\mu^2 e A_0. \tag{69}$$

Substituting this expression into (67), one gets

$$(\Delta - m_s^2) A_0 = -Ze\delta n_H, \tag{70}$$

where $m_s^2 \equiv e^2\mu^2/\pi^2$ is the Debye screening mass squared, due to the electrons. We now look at the system of three equations (64), (65), and (70) and consider their linearization above the background (bg) with

$$n_e^{bg} = J_0, \qquad n_H^{bg} = \frac{J_0}{Z}, \qquad A_0^{bg} = 0, \qquad v_j^{bg} = 0. \tag{71}$$

In the linearized equations, we transform to the Fourier models for all perturbations, as for instance,

$$\delta n_e(x,t) = \int d^3k d\omega \tilde{\delta n}_e(k,\omega) e^{i(\omega t - k_j x_j)}. \tag{72}$$

As a result, we get the following dispersion relation from the linearized system of equations (64), (65), and (70):

$$\omega^2 = \frac{m_\gamma^2 k^2}{k^2 + m_s^2}, \quad \text{where } m_\gamma^2 \equiv \frac{(Ze)^2 (J_0/Z)}{m_H}. \tag{73}$$

This describes a gapless collective mode; at small momentum, the dispersion relations reads

$$\omega \simeq \frac{m_\gamma}{m_s} k, \tag{74}$$

that is the dispersion relations of the ion acoustic wave (ionic sound) for plasma, while for a crystal it describes a longitudinal acoustic phonon.

Things are a bit different, however, for the case of the charged condensate. While the continuity and Poisson equations remain unchanged, the momentum equation gets modified due to the pressure term on the right-hand side; this can be shown from the Lagrangian formulation of the charged condensate given in Sec. IV (the pressure term would certainly exist in ordinary plasma as a finite temperature effect, but in charge condensate, it is nonzero even when the finite temperature effects are ignored). The corresponding momentum equation reads

$$\partial_t v_j + (v_k \partial_k) v_j = -\partial_j b_0 - \frac{Ze}{m_H} E_j. \tag{75}$$

The difference is due to the fact that a gradient of the pressure is not negligible for perturbations in the charged condensate; the respective term is kept as the first term on the right-hand side in (75), and it is determined by the gradient of the gauge-invariant potential, $b_0 = (m_\gamma^2 - 4M^2/\Delta)\delta n_H$.

We now combine this new momentum equation (75) with the Poisson (70) and continuity (64) equations and easily derive the modified dispersion relation for the longitudinal mode in the charged condensate:

$$\omega^2 \simeq \frac{k^4}{4m_H^2} + \frac{m_\gamma^2 k^2}{k^2 + m_s^2}. \tag{76}$$

The result coincides with the real part of the dispersion relation for the massless mode found in Ref. [8]. It also coincides, in the low momentum approximation, with the dispersion relations for the ion acoustic waves found in Secs. II, III, and IV of the present work.

An important approximation made above is the Thomas-Fermi method. This method will in general miss fermion dynamics near the Fermi surface. In particular, as we see, it does not capture the width of the ion acoustic wave due to its dumping by the fermionic quasiparticles. However, this effect was already taken into account by the one-loop consideration in Ref. [8] and above in the present work.

V. OUTLOOK

White dwarf stars constitute interesting physical objects and are also important for inferring key astrophysical and cosmological data. The theory of cooling of O and C WDs is well known [2] and agrees well with observations [18,19]. While at temperatures above $\sim 10^7$ K the ions can be regarded as being in a classical gas state, and below $\sim 10^6$ being in a bcc crystal state, between these two temperatures—from $\sim 10^7$ K, down to $\sim 10^6$ K—one is dealing with a strongly interacting plasma of ions that is not easy to study using fundamental electromagnetic interactions. Instead, we used an effective Lagrangian approach that is in general well suited to study long-wavelength excitations, even for strongly interacting systems. Our finding of the absence of the zero sound mode, and the presence of the longitudinal ion acoustic wave in the interacting plasma regime, confirms a suggestion made in Ref. [2] that the longitudinal acoustic wave can be used to describe physics of WDs in this interval of temperatures where neither gas nor crystal descriptions are valid.

As to the zero sound mode, its existence requires somewhat lower densities, $\rho < 10^5$ g/cm^3. While WDs have fairly uniform density profiles in their interiors, they are not exactly uniform and are described by an adiabatic equation of state. Thus, the density can vary from the core to the outskirts of the bulk by a factor of 5 or more. Therefore, there might exist a subclass of low density WDs, with relatively narrow spherical shells away from the cores, in which densities might be $\rho < 10^5$ g/cm^3; if so, then the zero sound mode could be supported in those domains instead of the ion acoustic wave. It could also be interesting to see if the zero sound mode could be supported in the low mass brown dwarfs stars studied in Ref. [15], where densities are $\rho \sim 10^3$ g/cm^3.

Finally, the interaction between the electrons due to the exchange of the zero sound mode (or the ion acoustic wave) would be strong if the momentum transfer in the two-by-two electron scattering amplitude is near the pole of the zero sound (or the ion acoustic wave), i.e., is at $\omega \simeq x v_F k$, with $x \gg 1$ (or with $x \ll 1$ for the ion acoustic wave). This interaction, if attractive for a certain domain of momenta, might lead to the formation of bound states, or loosely bound states such as Cooper pairs. While naive kinematical arguments suggest that the momentum transfer is not close to the zero sound pole as long as $x \gg 1$ (or with $x \ll 1$ for the ion acoustic wave), the issue needs careful study, presumably via the Schwinger-Dyson equation, to see if such interactions can be used to trigger Cooper instability and produce a gap of a physically meaningful magnitude.

ACKNOWLEDGMENTS

The authors would like to thank Paul Chaikin, Sergei Dubovsky, Andy Millis, and Boris Spivak for helpful remarks. Work of G. G. is supported by NASA Grant No. NNX12AF86G S06 and NSF Grant No. PHY-1316452. R. A. R. is supported by DOE Grant No. DE-SC0011941. G. G. thanks the Physics Department of East China Norma University for warm hospitality.

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