

Constant matter neutrino oscillations in a parametrization-free formulation

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Neutrino oscillations are now a well-established and deeply studied phenomena. Their mixing parameters, except for the CP phase, are measured with good accuracy. The three-neutrino oscillation picture in matter is currently of great interest due to the different long-baseline neutrino experiments that are already running or under construction. In this work, we reanalyze the exact expression for the neutrino probabilities (in a constant density medium) and introduce an approximate formula. Our results are shown in a formulation that is independent of the parametrization and could be useful for unitary tests of the leptonic mixing matrix. We illustrate how the approximation, besides being simple, can reproduce the neutrino probabilities with good accuracy.

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I. INTRODUCTION

Neutrino oscillations are a well-established phenomenon, with parameters that have been measured with great accuracy [1], except for the CP -violating phase which is expected to be precisely determined in the new generation of long-baseline neutrino experiments. Despite this success, all the analysis has been done in a particular parametrization [2]. In the case of neutrino physics, there are other parametrizations that could be interesting [3–5], taking into account the possibility of accounting for more neutrino families as well as the richness of the possible neutrino Majorana nature. Moreover, for a unitarity test [6], it may be more interesting to analyze the experimental data in a parameter-independent way, studying the values of the matrix entrances. It is already known that, for three families of Dirac fermions, the mixing matrix will have four independent elements [7–11]. If neutrinos have a Majorana nature, or if there are more neutrino families, the situation will be more complex [12], but a test of the “standard” picture through unitarity can be considered as a first step into the search for new physics. On the other hand, an important part of neutrino data comes from matter effects [13], making the analysis more complicated due to the need for computing numerical solutions to the neutrino evolution equation, for the case of varying density profiles, or for using approximate formulas for neutrino probabilities in constant density environments [14–19].

In this work, we first discuss the exact formula for neutrino oscillations in a constant density environment. We show our results in terms of the entrances of the leptonic mixing matrix and, therefore, they are independent of the parametrization. Afterwards, we introduce a new approximation that, besides being simple, can be formulated in terms of a series expansions and, therefore, can be

computed with a level of accuracy according to the phenomenological needs of the given problem. The expressions found here could be useful in analyzing the neutrino data either in the standard parametrization or in other contexts, such as unitary test in the neutrino sector.

II. THE EXACT CASE

The leptonic mixing matrix relates the mass and flavor states through the relation $\nu_\alpha = \sum_i U_{\alpha i} \nu_i$, where the matrix $U_{\alpha i}$ could be parametrized, for instance, in the usual convention adopted by the Particle Data Group (PDG) [2]. The evolution equation in vacuum will be given by

$$i \frac{d}{dt} \nu_j = \frac{m_j^2}{2E} \nu_j \quad (1)$$

That leads, through a well-known procedure, to the usual expression for neutrino probabilities in vacuum:

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{\ell>j}^n \text{Re}[U_{\alpha\ell}^* U_{\beta\ell} U_{\alpha j} U_{\beta j}^*] \sin^2\left(\frac{\Delta m_{\ell j}^2 L}{4E}\right) + 2 \sum_{\ell>j}^n \text{Im}[U_{\alpha\ell}^* U_{\beta\ell} U_{\alpha j} U_{\beta j}^*] \sin\left(\frac{\Delta m_{\ell j}^2 L}{2E}\right). \quad (2)$$

Notice, however, that in the case when the extra neutrino states are heavy, they do not participate in the oscillation and the sum is cut up to three:

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \sum_{\ell,j}^3 U_{\alpha\ell}^* U_{\beta\ell} U_{\alpha j} U_{\beta j}^* - 4 \sum_{\ell>j}^3 \text{Re}[U_{\alpha\ell}^* U_{\beta\ell} U_{\alpha j} U_{\beta j}^*] \sin^2\left(\frac{\Delta m_{\ell j}^2 L}{4E}\right) + 2 \sum_{\ell>j}^3 \text{Im}[U_{\alpha\ell}^* U_{\beta\ell} U_{\alpha j} U_{\beta j}^*] \sin\left(\frac{\Delta m_{\ell j}^2 L}{2E}\right), \quad (3)$$

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and the $\delta_{\alpha\beta}$ appearing in Eq. (2) is substituted by the well-known zero distance effect.

If we would like to consider matter effects, we need to add the charged current potential due to electrons that, again in the mass basis:

$$i \frac{d}{dt} \nu_j = \frac{1}{2E} \left(m_j^2 \nu_j + \sum_k A U_{ej}^* U_{ek} \nu_k \right), \quad (4)$$

with $A = 2EV_{CC}$. It is known that, in this case, we can express the probability in a similar form,

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{\ell > j}^n \text{Re}[V_{\alpha\ell}^* V_{\beta\ell} V_{\alpha j} V_{\beta j}^*] \sin^2 \left(\frac{\Delta M_{\ell j}^2 L}{4E} \right) + 2 \sum_{\ell > j}^n \text{Im}[V_{\alpha\ell}^* V_{\beta\ell} V_{\alpha j} V_{\beta j}^*] \sin \left(\frac{\Delta M_{\ell j}^2 L}{2E} \right), \quad (5)$$

by defining

$$V = UW^T, \quad (6)$$

where W is an unitary matrix. We can also find the corresponding expression for Eq. (3) in the presence of matter:

$$P_{\nu_\alpha \rightarrow \nu_\beta} = \sum_{\ell, j}^3 U_{\alpha\ell}^* U_{\beta\ell} U_{\alpha j} U_{\beta j}^* - 4 \sum_{\ell > j}^3 \text{Re}[V_{\alpha\ell}^* V_{\beta\ell} V_{\alpha j} V_{\beta j}^*] \sin^2 \left(\frac{\Delta M_{\ell j}^2 L}{4E} \right) + 2 \sum_{\ell > j}^3 \text{Im}[V_{\alpha\ell}^* V_{\beta\ell} V_{\alpha j} V_{\beta j}^*] \sin \left(\frac{\Delta M_{\ell j}^2 L}{2E} \right), \quad (7)$$

where, as expected, the zero distance term remains unchanged, thanks to the unitarity of the W matrix.

To find the expressions for the matrix V , we follow the procedure described in Ref. [15]. We will arrive to the same expressions, except that we maintain the matrix elements of U_{ai} in a parameter independent form. Although the following procedure is straightforward, it will allow to see the nonunitary case in a more transparent way. Even if we work in the standard parametrization, the expression will be useful, as the numerical computations will be slightly simplified by substituting the parametrization at the end.

We start by noticing that the term inside the parenthesis in the right-hand side of equation (4) defines a matrix

$$\begin{pmatrix} A|U_{e1}|^2 & AU_{e1}^* U_{e2} & AU_{e1}^* U_{e3} \\ AU_{e2}^* U_{e1} & \Delta m_{21}^2 + A|U_{e2}|^2 & AU_{e2}^* U_{e3} \\ AU_{e3}^* U_{e1} & AU_{e3}^* U_{e2} & \Delta m_{31}^2 + A|U_{e3}|^2 \end{pmatrix} \quad (8)$$

with a characteristic polynomial given by

$$\lambda'^3 - \frac{\lambda'^2}{2E} (\Delta m_{21}^2 + \Delta m_{31}^2 + A) + \frac{\lambda'}{4E^2} [\Delta m_{31}^2 \Delta m_{21}^2 + A(\Delta m_{21}^2(1 - |U_{e2}|^2) + \Delta m_{31}^2(1 - |U_{e3}|^2))] - \frac{1}{8E^3} \Delta m_{21}^2 \Delta m_{31}^2 A |U_{e1}|^2 = 0 \quad (9)$$

where we have subtracted a term m_1^2 from the main diagonal in order to simplify the equation. We redefine the eigenvalues λ' as $\lambda = 2E\lambda'$, which implies that $\lambda_i = M_i^2$.

As is already known, for a polynomial of the form

$$\lambda^3 - \alpha\lambda^2 + \beta\lambda - \gamma = 0, \quad (10)$$

the solutions for λ real, are given by

$$\lambda_n = \frac{\alpha}{3} + \frac{2}{3} \sqrt{\alpha^2 - 3\beta} \times \cos \left[\frac{1}{3} \arccos \left(\frac{2\alpha^3 - 9\alpha\beta + 27\gamma}{2\sqrt{(\alpha^2 - 3\beta)^3}} \right) + \frac{2n\pi}{3} \right], \quad n = 0, 1, 2, \quad (11)$$

that in our case imply

$$\begin{aligned} \alpha &= \Delta m_{21}^2 + \Delta m_{31}^2 + A(|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2) \\ \beta &= \Delta m_{31}^2 \Delta m_{21}^2 + A \Delta m_{21}^2 (|U_{e1}|^2 + |U_{e3}|^2) \\ &\quad + A \Delta m_{31}^2 (|U_{e1}|^2 + |U_{e2}|^2) \\ \gamma &= A \Delta m_{21}^2 \Delta m_{31}^2 |U_{e1}|^2 \\ \eta &= \cos \left[\frac{1}{3} \arccos \left(\frac{2\alpha^3 - 9\alpha\beta + 27\gamma}{2\sqrt{(\alpha^2 - 3\beta)^3}} \right) \right]. \end{aligned} \quad (12)$$

This leads us to the three eigenvalue equations which we are going to label as

$$\begin{aligned} M_1^2 \equiv \lambda_1 &= \frac{\alpha}{3} - \frac{1}{3} \sqrt{\alpha^2 - 3\beta\eta} - \frac{\sqrt{3}}{3} \sqrt{\alpha^2 - 3\beta} \sqrt{1 - \eta^2}, \\ M_2^2 \equiv \lambda_2 &= \frac{\alpha}{3} - \frac{1}{3} \sqrt{\alpha^2 - 3\beta\eta} + \frac{\sqrt{3}}{3} \sqrt{\alpha^2 - 3\beta} \sqrt{1 - \eta^2}, \\ M_3^2 \equiv \lambda_3 &= \frac{\alpha}{3} + \frac{2}{3} \sqrt{\alpha^2 - 3\beta\eta}. \end{aligned} \quad (13)$$

In order to construct the diagonalizing matrix, we need the corresponding eigenvectors that will be given by

$$\begin{aligned}
 |\lambda_1\rangle &= \frac{1}{C_1} \begin{pmatrix} \Lambda_1 \\ AU_{e2}^* U_{e1} (M_1^2 - \Delta m_{31}^2) \\ AU_{e3}^* U_{e1} (M_1^2 - \Delta m_{21}^2) \end{pmatrix} \\
 |\lambda_2\rangle &= \frac{1}{C_2} \begin{pmatrix} \Lambda_2 \\ AU_{e1}^* U_{e2} (M_2^2 - \Delta m_{31}^2) \\ AU_{e3}^* U_{e2} M_2^2 \end{pmatrix}, \\
 |\lambda_3\rangle &= \frac{1}{C_3} \begin{pmatrix} AU_{e1}^* U_{e3} (M_3^2 - \Delta m_{21}^2) \\ AU_{e2}^* U_{e3} M_3^2 \\ \Lambda_3 \end{pmatrix}. \quad (14)
 \end{aligned}$$

Here, we define the normalization constants, C_j , as

$$C_j = \sqrt{\Lambda_j^2 + A^2 |U_{ej}|^2 \sum_{i \neq j} |U_{ei}|^2 (M_j^2 - \Delta m_{ki}^2)^2}, \quad (15)$$

for $k \neq i$

and we also define

$$\Lambda_j = M_j^4 - \sum_{i \neq j} \left[M_j^2 (\Delta m_{i1}^2 + A |U_{ei}|^2) - A \Delta m_{i1}^2 |U_{ek}|^2 - \frac{1}{2} \Delta m_{i1}^2 \Delta m_{k1}^2 \right], \quad \text{for } k \neq i. \quad (16)$$

Now we can write the explicit form of the matrix W , that in abbreviated form can be written as

$$(W^T)_{kj} = \frac{\Lambda_k}{C_k} \delta_{kj} + (1 - \delta_{kj}) A \frac{U_{ek} U_{ej}^* (M_k^2 - \sum_i [\Delta m_{i1}^2 \epsilon_{ijk}^2])}{C_k} \quad (17)$$

or, writing it explicitly,

$$W = \begin{pmatrix} \frac{\Lambda_1}{C_1} & \frac{AU_{e1}^* U_{e2} (M_2^2 - \Delta m_{31}^2)}{C_2} & \frac{AU_{e1}^* U_{e3} (M_3^2 - \Delta m_{21}^2)}{C_3} \\ \frac{AU_{e2}^* U_{e1} (M_1^2 - \Delta m_{31}^2)}{C_1} & \frac{\Lambda_2}{C_2} & \frac{AU_{e2}^* U_{e3} M_3^2}{C_3} \\ \frac{AU_{e3}^* U_{e1} (M_1^2 - \Delta m_{21}^2)}{C_1} & \frac{AU_{e3}^* U_{e2} M_2^2}{C_2} & \frac{\Lambda_3}{C_3} \end{pmatrix}. \quad (18)$$

We have arrived to the explicit form of the diagonalizing matrix W , such that

$$W^{-1} H_M W = \frac{1}{2E} \begin{pmatrix} M_1^2 & 0 & 0 \\ 0 & M_2^2 & 0 \\ 0 & 0 & M_3^2 \end{pmatrix}. \quad (19)$$

This matrix relates the mass states in vacuum with the matter ones in the form $|\nu'_M\rangle = W|\nu_M\rangle$, where the primed

vector refers to the matter mass states. It is easy to see that the vacuum case is restored when $A = 0$. With this relation we can find the oscillation probabilities in matter as a function of the elements of the vacuum rotation matrix, without the use of any parameterization and without using the unitary relation. Therefore, they could be useful to study the unitarity of the mixing matrix, a topic that could be of interest now that we are entering into a precision era in neutrino physics. As we have already mentioned, this method is well known [15], although the treatment had been done in a specific parametrization.

III. AN APPROXIMATION

Once we have discussed the exact solution for the constant density matter case, we proceed to find an approximate formula for the probabilities. In order to preserve the parametrization-free structure, we look for an approximation for the cubic roots λ_i in Eq. (11).

We start by noticing, from Eq. (12), that if $\Delta m_{21}^2 \rightarrow 0$, then $\gamma \rightarrow 0$ and the cubic equation (10) is reduced to the quadratic case. If this is the case, Eq. (13) will reduce to the two typical solutions for a quadratic equation plus a third solution, given by $\lambda_1 = 0$. In particular, we will have the expression for η :

$$\eta = \cos \left[\frac{1}{3} \arccos \left(\frac{2\alpha^3 - 9\alpha\beta}{2\sqrt{(\alpha^2 - 3\beta)^3}} \right) \right]. \quad (20)$$

In this simple case it is easy to find that

$$\eta = \cos \theta = \frac{-\frac{1}{2}\alpha}{\sqrt{\alpha^2 - 3\beta}}. \quad (21)$$

Now we can consider that γ is not zero, but it is ‘‘small,’’ say $\gamma \ll \alpha\beta$. This seems a natural hypothesis since $\Delta m_{21}^2 \ll \Delta m_{31}^2$. We can try to find the correction ε that fulfills both

$$\cos \theta = \frac{-\frac{1}{2}\alpha + \varepsilon}{\sqrt{\alpha^2 - 3\beta}} \quad (22)$$

and

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \simeq \frac{2\alpha^3 - 9\alpha\beta + 27\gamma}{2\sqrt{(\alpha^2 - 3\beta)^3}}. \quad (23)$$

If we work only up to first-order terms, below $(\frac{\gamma}{\beta})^2$, it is easy to find that

$$\varepsilon = \frac{3\gamma}{2\beta} \quad (24)$$

fits both conditions. Therefore, the eigenvalues will be approximately given by

$$\begin{aligned}
M_1^2 &\equiv \lambda_3 \approx \frac{2}{3}\varepsilon, \\
M_2^2 &\equiv \lambda_1 \approx \frac{1}{2}\left(\alpha - \frac{2}{3}\varepsilon\right) \\
&\quad - \frac{1}{2}\sqrt{\left(\alpha + \frac{2}{3}\varepsilon\right)^2 - 4\left[\beta + \left(\frac{2}{3}\varepsilon\right)^2\right]}, \\
M_3^2 &\equiv \lambda_2 \approx \frac{1}{2}\left(\alpha - \frac{2}{3}\varepsilon\right) \\
&\quad + \frac{1}{2}\sqrt{\left(\alpha + \frac{2}{3}\varepsilon\right)^2 - 4\left[\beta + \left(\frac{2}{3}\varepsilon\right)^2\right]}, \quad (25)
\end{aligned}$$

This seems to be a reasonable approximation that leads to the equation

$$\lambda^3 - \alpha\lambda^2 + \beta\lambda - \left(\frac{2}{3}\beta\varepsilon - \frac{4}{9}\alpha\varepsilon^2 + \frac{8}{27}\varepsilon^3\right) = 0. \quad (26)$$

With the expression of ε at first order in $\frac{\gamma}{\beta}$, Eq. (24), we have

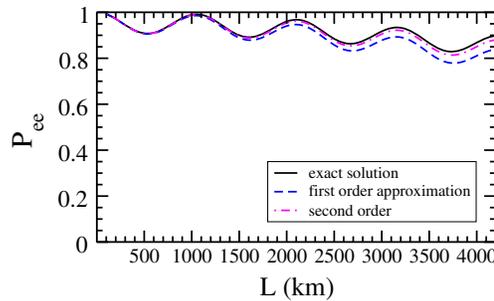
$$\lambda^3 - \alpha\lambda^2 + \beta\lambda - \left(\gamma - \frac{\alpha\gamma^2}{\beta^2} + \frac{\gamma^3}{\beta^3}\right) = 0. \quad (27)$$

We can go one step further and find the expression for ε at second order in $\frac{\gamma}{\beta}$. In this case, we propose that

$$\varepsilon = \frac{3\gamma}{2\beta} + a_2\left(\frac{\gamma}{\beta}\right)^2 \quad (28)$$

and demand that Eq. (26) reduces to the usual cubic expression, Eq. (10), up to second-order terms. This condition is fulfilled when $a_2 = \frac{3\alpha}{2\beta}$. Therefore, at second order, we have

$$\varepsilon = \frac{3\gamma}{2\beta} + \frac{3\alpha\gamma^2}{2\beta^2}. \quad (29)$$



We can continue with this procedure and find recursively the coefficients a_k for any order of approximation that we would like to have. That is, we can write ε as an infinite polynomial that, in principle, should give an exact solution. The polynomial would have the form

$$\varepsilon = \sum_{k=1}^{\infty} a_k(\alpha, \beta) \left[\frac{\gamma}{\beta}\right]^k \quad (30)$$

with

$$a_1 = \frac{3}{2}$$

$$a_2 = \frac{3\alpha}{2\beta}$$

$$a_k(\alpha, \beta) = \frac{3}{2\beta} \left(\frac{4\alpha}{9} \sum_{\substack{ij \\ i+j=k}} a_i a_j - \frac{8}{27} \sum_{\substack{ijl \\ i+j+l=k}} a_i a_j a_l \right);$$

$$k > 2. \quad (31)$$

Once we have defined the approximation, we would like to know how well it behaves with respect to the exact formula. Although we have worked out all the computation in a formulation that is independent of the parametrization, we adopt now the standard PDG [2] parametrization in order to substitute the current values for the neutrino oscillation parameters. Therefore, in this case, we explicitly adopt the unitary condition by making the following substitutions in Eq. (12):

$$\begin{aligned}
|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2 &\equiv 1, \\
|U_{e1}|^2 + |U_{e3}|^2 &\equiv 1 \\
-|U_{e2}|^2|U_{e1}|^2 + |U_{e2}|^2 &\equiv 1 - |U_{e3}|^2. \quad (32)
\end{aligned}$$

Once we introduce the standard parametrization for the mixing matrix, U , we adopt as central values of the mixing angles the ones reported by Ref. [1] ($\sin^2 \theta_{12} = 0.320$, $\sin^2 \theta_{23} = 0.613$, $\sin^2 \theta_{13} = 0.0246$) as

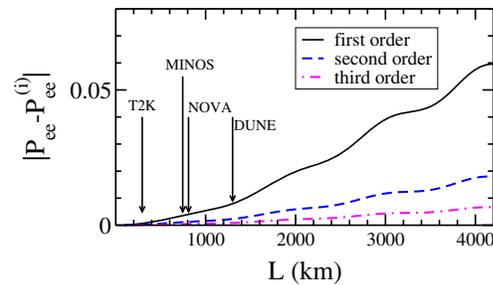


FIG. 1. Comparison of the exact electron neutrino survival probability in the three flavor case. The left panel shows the exact survival probability prediction for the central values of the mixing angles and mass squared differences and the approximated result for our approximation at order one and two. In the right panel we show the absolute difference between the exact solution and the approximated prescription at first ($P_{ee}^{(1)}$), second ($P_{ee}^{(2)}$), and third order ($P_{ee}^{(3)}$). The baseline for different experiments and for the future DUNE experimental proposal is shown as a reference. The neutrino energy has been fixed to $E_\nu = 1$ GeV and the electron density has been taken to be 5.92×10^9 eV³.

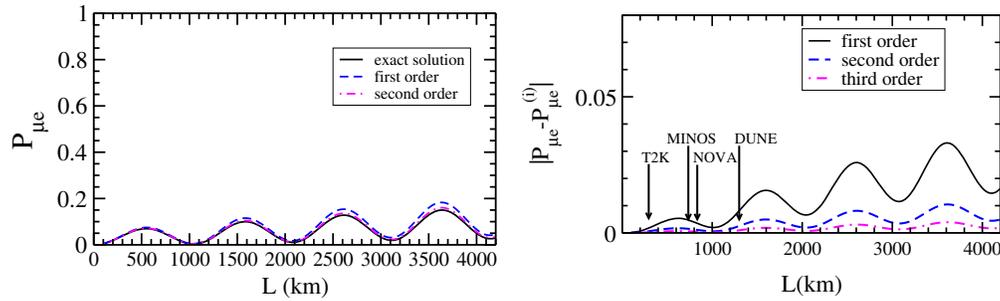


FIG. 2. Comparison of the exact muon to electron neutrino conversion probability in the three flavor case. The left panel shows the exact survival probability prediction for the central values of the mixing angles and mass squared differences and the approximated result for our approximation at order one and two. In the right panel we show the absolute difference between the exact solution and the approximated prescription at first ($P_{\mu e}^{(1)}$), second ($P_{\mu e}^{(2)}$), and third order ($P_{\mu e}^{(3)}$). The baseline for different experiments and for the future DUNE experimental proposal is shown as a reference. The neutrino energy has been fixed to $E_\nu = 1$ GeV and the electron density has been taken to be 5.92×10^9 eV³.

well as the corresponding squared mass differences ($\Delta m_{21}^2 = 7.62 \times 10^{-5}$ eV², $\Delta m_{31}^2 = 2.55 \times 10^{-3}$ eV²). For the value of the CP phase we have taken $\delta = 3\pi/2$. We have computed the survival probability P_{ee} and the conversion probability $P_{\mu e}$ and compared our approximated results with the exact formulation, for a neutrino energy of 1 GeV. The results are shown in Fig. (1) and Fig. (2) where we have plotted these probabilities as functions of the baseline. From these figures, it is possible to notice that the approximation works reasonably well at first order (especially for baselines below one thousand kilometers) and has a great improvement when we consider next-order approximations.

IV. CONCLUSIONS

In this work we have considered the case of three neutrino evolution in a constant matter potential. We have first reviewed the exact formulation and wrote the standard neutrino probabilities in a parametrization-free scheme. We have obtained an approximated formula for this scenario that can be easily extended to the desired order of approximation, based on the coefficients for the eigenvalue problem, instead of considering specific oscillation parameters, such as Δm_{21}^2 . This approximation can be used either for the parametrization-free scenario (that could be useful in unitarity tests) or in a particular parametrization such as the one adopted by the PDG. We have shown that the formalism is simple and can be worked out at any order of approximation, depending on the needs of the specific problem. The formalism could also be used for scenarios of physics beyond the Standard Model such as the case of extra neutral heavy leptons [5,20].

Finally, we can study the validity of the three orders of approximation for different energies and baselines. In order to compare with other results [21], we use the electron neutrino conversion probability into muon neutrinos and compute the absolute difference between our

approximation and the exact conversion formula. Our results are summarized in Fig. (3) where we show the regions with an absolute difference in the range 0.001–0.01. We use the oscillation parameters already quoted above. Comparing this result with the approximation discussed in Ref. [21], it is possible to notice that our formula, at first order, is not competitive in this channel; however, for second and third order, our approximation works well, especially for energies at one GeV and above.

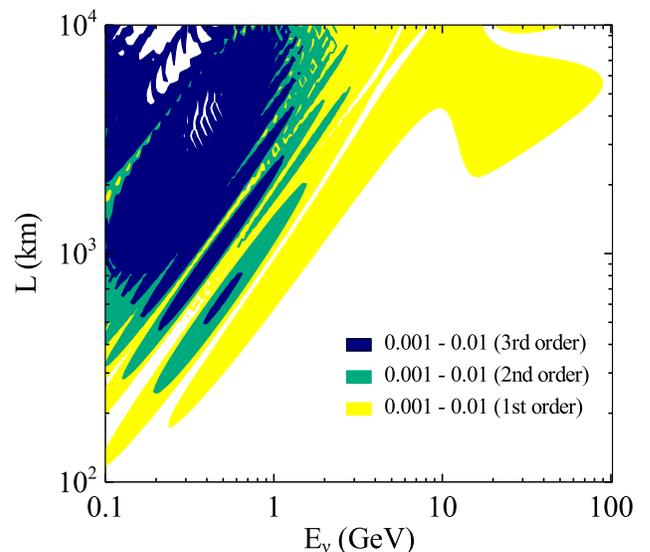


FIG. 3. Absolute difference between the exact conversion probability and the approximation discussed in this work. We have considered conversion from electron to muon neutrinos and computed the first-, second-, and third-order approximation for the oscillation parameters discussed in the text. We show the regions where the absolute difference lies in the range between 0.001 and 0.01. It is possible to see that the approximation works well for a wide range of values of distance and energy, especially at third order.

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