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Anyonic glueballs from an effective string model

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Relying on an effective-string approach in which glueballs—bound states of pure Yang-Mills theory—are modeled by closed strings, we give arguments suggesting that anyonic glueballs, i.e. glueballs with arbitrary spin, may exist in $(2+1)$-dimensional Yang-Mills theory. We then focus on the large-N_c limit of $SU(N_c)$ Yang-Mills theory and show that our model leads to a mass spectrum in good agreement with lattice data in the scalar sector, while it predicts the masses and spins of anyonic glueball states.

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I. INTRODUCTION

The appearance of quantum states with arbitrary spin, called anyons, is a fascinating feature of quantum mechanics in $2+1$ dimensions [1] that has been explored in a considerable amount of works: The interested reader may find useful references in Refs. [2–4]. Actually, in $2+1$ dimensions, the spin s of a given state may be arbitrary because the Lorentz group $SO(2,1)$, as a group manifold, contains a noncontractible circle S^1 whose covering $\mathbb{R}$ covers it infinitely many times. In the case of an Euclidean spacetime, the "Lorentz" group $SO(3)$ is a compact, connected (albeit nonsimply) manifold that admits at most two-valued unitary representations.

It is known in field theory that coupling a matter field to a three-dimensional vector gauge field with a Chern-Simons term leads to the appearance of states with fractional statistics [5]. The equivalent result is obtained within an $O(3)$ σ model with a Hopf term [6]. Note that a Chern-Simons term is not a necessary condition to produce anyons in field theory, as illustrated by the following examples:

- (i) Composite quantum states with arbitrary spin or arbitrary exchange statistics can be built from the genuine Abelian Higgs model without a Chern-Simons term [7,8].
- (ii) Within an Abelian gauge theory with a matter field denoted by Ψ and g^2, a constant with the dimension of mass, one defines the shifted connection $A^\theta_\mu = A_\mu + \frac{\theta}{g^2} F_\mu$, where $F_\mu = \epsilon_{\mu\nu\rho} F^{\nu\rho}/2$. The operator $\Psi(x) P\{\exp(i\int_x^y dz^\mu A^\theta_\mu)\}\bar{\Psi}(y)$ then propagates an anyon with nontrivial statistics related to the arbitrary real number θ [9].
- (iii) The spectrum of closed Nambu-Goto strings in $2+1$ dimensions necessarily contains fractional spin fields after light-cone quantization [10].

More generally, it has to be stressed that the existence of fractional-spin fields in $(2+1)$-dimensional Minkowski spacetime arises from pure group theoretical arguments that are actually independent of the particular form of the action under consideration [11,12]. These arguments will be summarized in Sec. II, while the case of closed Nambu-Goto strings, particularly important for our present work, will be discussed in Sec. III.

The purpose of the present paper is to investigate whether anyonic states exist or not in pure $(2+1)$-dimensional Yang-Mills theory. Such a problem has, to our knowledge, never been studied so far. If anyonic glueballs can be built, the next question is "What are their masses and spins?" This problem can be addressed by resorting to a closed-string effective model of glueballs. The idea that Yang-Mills theory should be equivalent to some closed-string theory at large N_c actually originates from 't Hooft and Veneziano's work on the large-N_c limit of QCD [13,14]. It has indeed been known since then that any amplitude in large-N_c Yang-Mills theory can be expressed as a sum over terms containing planar diagrams forming Riemann surfaces with various genus numbers, just as it is the case in closed-string theory.

From an effective model point of view, it is therefore tempting to assume that glueball dynamics has some stringy nonperturbative origin. The celebrated Isgur and Paton's flux tube model [15] is a first example of how, starting from a lattice-QCD-inspired approach, one is led to the conclusion that glueballs—or at least some of them—may be described by closed strings. Closed effective strings are often referred to as closed flux tubes, since they are seen as particular configurations of the chromoelectric field

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whose dynamics is expected to be that of a closed string. Interestingly, lattice computations regarding Yang-Mills theory have given some support to this picture. The interested reader may find in Refs. [16,17] a discussion of the agreement between a string model of glueballs and the lattice data of Ref. [18].

The effective model we use, inspired in particular by that of Ref. [17], is presented in Sec. IV, and numerical results are obtained in Sec. V. Concluding comments are given in Sec. VII.

II. RELATIVISTIC ANYONS

Since the seminal works of Wigner and Bargmann [19,20], it has been known that the elementary particles in Minkowski spacetime of dimension D are associated with the unitary irreducible representations (UIRs) of the spacetime isometry group $ISO(D-1,1)$, the latter group being the semidirect product of the Lorentz group $SO(D-1,1)$ with the translation group T_D. In $2+1$ dimensions, the Poincaré group therefore is $ISO(2,1) \cong SO(2,1) \ltimes T_3$. In this section we first show how to build the UIRs of the $(2+1)$-dimensional Lorentz group $SO(2,1)$; then we extend the discussion to the Poincaré group $ISO(2,1)$.

A. The case of $SO(2,1)$

Let $L_{ab} = -L_{ba}$ be the generators of $SO(2,1)$. Then $so(2,1)$, the Lorentz algebra in $2+1$ dimensions, is presented by

$$[J^a, J^b] = -i\varepsilon^{abc} J_c, \tag{1}$$

where $J^a = \frac{1}{2}\varepsilon^{abc} L_{bc}$ and $\varepsilon^{012} = 1$. Using the Minkowski metric in Cartesian coordinates $\eta = \mathrm{diag}(-++)$ and bold fonts for 3-vectors, the scalar product of $\boldsymbol{U}$ and $\boldsymbol{V}$ reads $\boldsymbol{U}\cdot\boldsymbol{V} \equiv U^a \eta_{ab} V^b$, and the Casimir operator of $SO(2,1)$ is taken to be

$$C_2[so(2,1)] \coloneqq -\boldsymbol{J}^2 = \frac{1}{2} L^{ab} L_{ab}. \tag{2}$$

We use the notation $\vec{u}, \vec{v}\ldots$ for 2-vectors in the planes at fixed values of the Minkowskian coordinate x^0.

An oscillator-based method for the classification of the UIRs of $SO(2,1)$ was given in Ref. [11], which we closely follow, since it has the advantage of building up the UIRs of $SO(3)$ in complete analogy, thereby unifying the treatments of the various real forms of $so(3,\mathbb{C})$. It is of importance for us in view of drawing the reader's attention to the differences between both groups, the latter being actually at the basis of lattice QCD because of the Wick rotation leading to a Euclidean rather than hyperbolic spacetime.

Defining, as usual, the ladder operators

$$J^{\pm} = \frac{1}{\sqrt{2}}(-iJ^2 \pm J^1) \tag{3}$$

yields

$$[J^+, J^-] = \begin{cases} L_{12} & \text{for } so(2,1) \\ -L_{12} & \text{for } so(3) \end{cases} \tag{4}$$

and

$$[L_{12}, J^{\pm}] = \pm J^{\pm} \tag{5}$$

for both $so(2,1)$ and $so(3)$. The authors of Ref. [11] considered the complex algebra $so(3,\mathbb{C})$, thereby taking one and the same set of commutation relations for *both* $so(2,1)$ and $so(3)$ (with $[J^+, J^-] = L_{12}$) and distinguished the groups $SO(2,1)$ and $SO(3)$ by different reality conditions on the corresponding parameters of infinitesimal transformations. In turn, these conditions and the requirement of unitarity of irreducible representations for $SO(2,1)$ or $SO(3)$ give different reality conditions on the generators of the two groups. Effectively, this amounts to allowing real linear combinations of the noncompact generators $\{L_{01}, L_{02}\}$ for $SO(2,1)$ and only purely imaginary linear combinations of them in the case of $SO(3)$, thereby Euclideanizing $SO(2,1)$ to $SO(3)$, or stated equivalently, making $\{L_{01}, L_{02}\}$ compact. One must have $(J^+)^\dagger = J^-$ for the rotation group $SO(3)$ and $(J^+)^\dagger = -J^-$ for the three-dimensional Lorentz group $SO(2,1)$. To summarize, in a unitary representation,

$$\begin{aligned} L_{12}^\dagger &= L_{12}, \qquad (J^+)^\dagger = -J^- \quad \text{for } SO(2,1), \\ L_{12}^\dagger &= L_{12}, \qquad (J^+)^\dagger = J^- \quad \text{for } SO(3). \end{aligned} \tag{6}$$

Let $\xi = (\xi_\alpha)_{\alpha=1,2}$ be a commuting real spinor of $SO(2,1)$ and consider the linear vector space spanned by normalized vectors of the form

$$|\Phi, m\rangle = \mathcal{N}_m \xi_1^a \xi_2^b = \mathcal{N}_m (\xi_1\xi_2)^{\Phi} (\xi_1/\xi_2)^{E_0+m},$$

$$\text{where } \Phi = \frac{1}{2}(a+b), \qquad E_0 + m = \frac{1}{2}(a-b),$$
$$(a,b) \in \mathbb{C}^2, \qquad m \in \mathbb{Z}. \tag{7}$$

The integer m unambiguously labels the vectors once Φ and E_0 are specified. The inner product is defined by $\langle \Phi, m | \Phi, m' \rangle = \delta_{m,m'}$. In this representation the generators $J^{\pm}$ and J^0 are realized by the operators

$$\begin{aligned} J^0 &= \frac{1}{2}\left(\xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2}\right), \\ J^+ &= \frac{1}{\sqrt{2}} \xi_1 \frac{\partial}{\partial \xi_2}, \\ J^- &= \frac{1}{\sqrt{2}} \xi_2 \frac{\partial}{\partial \xi_1}. \end{aligned} \tag{8}$$

These act on the basis vectors as

$$\begin{cases} J^0|\Phi,m\rangle = (E_0+m)|\Phi,m\rangle, \\ J^+|\Phi,m\rangle = \frac{1}{\sqrt{2}}(\Phi-E_0-m)(\mathcal{N}_m/\mathcal{N}_{m+1})|\Phi,m+1\rangle, \\ J^-|\Phi,m\rangle = \frac{1}{\sqrt{2}}(\Phi+E_0+m)(\mathcal{N}_m/\mathcal{N}_{m-1})|\Phi,m-1\rangle. \end{cases} \tag{9}$$

One sees that (E_0+m) is the eigenvalue of the generator of spatial rotations J^0, while a quick calculation shows that $\Phi(\Phi+1)$ is the eigenvalue of the quadratic Casimir

$$C_2 = 2J^-J^+ + J^0(J^0+1) = 2J^+J^- + J^0(J^0-1). \tag{10}$$

Imposing the unitarity condition (6) for the $SO(3)$ group leads to

$$\mathcal{I}(E_0) = 0, \tag{11}$$

$$\left|\frac{\mathcal{N}_{m+1}}{\mathcal{N}_m}\right|^2 = \frac{\Phi^* - E_0 - m}{m + E_0 + \Phi + 1}. \tag{12}$$

These recursion relations can be solved only if the values of m are bounded both from above and from below. This corresponds to the well-known result that the UIRs of $SO(3)$ are finite dimensional. The spectrum of the operator J^0 is then

$$\frac{1}{2}(a-b) = -\Phi, -\Phi+1, \ldots, \Phi, \tag{13}$$

showing that $\Phi = s$ is the spin of the corresponding $SO(3)$ irreducible representation, and $\mathcal{R}E_0 = 0$.

For the noncompact group $SO(2,1)$, one derives from the unitarity condition (6) that

$$\mathcal{I}(E_0) = 0, \tag{14}$$

$$\left|\frac{\mathcal{N}_{m+1}}{\mathcal{N}_m}\right|^2 = \frac{m + E_0 + \frac{1}{2} - (\Phi^* + \frac{1}{2})}{m + E_0 + \frac{1}{2} + (\Phi + \frac{1}{2})}. \tag{15}$$

Keeping the notation of Ref. [11], this gives the following UIRs:

$$\begin{aligned} \mathcal{D}(C_2,E_0)&: \ C_2 < |E_0|(|E_0|-1) \quad \text{and} \quad m \in \mathbb{Z}, \\ \mathcal{D}^+(\Phi)&: \ \Phi < 0 \quad \text{and} \quad m \in \mathbb{N}_0, \\ \mathcal{D}^-(\Phi)&: \ \Phi < 0 \quad \text{and} \quad -m \in \mathbb{N}_0, \\ \mathcal{D}(\Phi)&: \ \Phi = 0 \quad \text{and} \quad m = 0. \end{aligned} \tag{16}$$

The UIRs $\mathcal{D}(C_2,E_0)$, whose spectra for J^0 are neither bounded from above nor from below, contain the principal and complementary (or supplementary) UIRs of $SO(2,1)$. As explained in the next section, we shall focus on the other possible UIRs. The representations $\mathcal{D}^\pm(\Phi)$ are called the *discrete series*, while the representation $\mathcal{D}(0)$ is the trivial, one-dimensional representation. For the discrete series $\mathcal{D}^+(\Phi)$ [$\mathcal{D}^-(\Phi)$], the spectrum of J^0 is countably infinite, bounded from below (above). Cases of particular interest for our purpose will be denoted by

$$\begin{aligned} &\mathcal{D}_s^+ \coloneqq \mathcal{D}^+(-s): \ J^0 = s+m, \\ &\mathcal{D}_s^- \coloneqq \mathcal{D}^-(-s): \ J^0 = -s-m, \\ &m \in \mathbb{N}, \qquad s > 0. \end{aligned} \tag{17}$$

The spin of the discrete series $\mathcal{D}_s^+$ representation is s (with $s>0$), that can be an integer or even an arbitrary (albeit positive), real number.

Representations of $SO(2,1)$ bounded from above and below like for $SO(3)$ exist but are nonunitary [11]. The only UIR that $SO(3)$ and $SO(2,1)$ share is the trivial one $\mathcal{D}(\Phi=0)$, which corresponds to scalar fields. Finally, it is worth mentioning that the statistical phase $\exp(2i\pi s)$ can still be associated with a state of arbitrary spin s by virtue of the spin-statistics theorem [4,21].

B. The case of $ISO(2,1)$

Let P^a be the translation generators of T_3. Then $iso(2,1)$, the Poincaré algebra in $2+1$ dimensions, is presented by

$$\begin{aligned} &[J^a, J^b] = -i\varepsilon^{abc}J_c, \qquad [J^a, P^b] = -i\varepsilon^{abc}P_c, \\ &[P^a, P^b] = 0, \end{aligned} \tag{18}$$

and the two Casimir operators of $ISO(2,1)$ read, for massive representations,

$$M^2 = -\boldsymbol{P}^2, \qquad s = -\frac{\boldsymbol{P}\cdot\boldsymbol{J}}{M}. \tag{19}$$

They respectively give, on irreducible representations, the squared mass and the spin of a state. It has been shown in Ref. [22] that states $|\Psi\rangle$ belonging to the complementary series $\mathcal{D}(C_2,E_0)$ are such that $\boldsymbol{P}^2|\Psi\rangle = 0$, $\boldsymbol{P}\cdot\boldsymbol{J}|\Psi\rangle = 0$, $\boldsymbol{J}^2|\Psi\rangle = 0$. Such states are not relevant in view of studying glueballs, since we are looking for massive representations with nonzero spin that will contain anyons. As also shown in Ref. [22], such "physical" states belong rather to the discrete series $\mathcal{D}_s^+$ or $\mathcal{D}_s^-$. Let us denote by $|M^2;\vec{p};s;J^0\rangle$ these states, the two series being distinguished by the signs of the eigenvalues of P^0 and J^0: positive (negative) for $\mathcal{D}_s^+$ ($\mathcal{D}_s^-$). Therefore, the two series $\mathcal{D}_s^\pm$ can be seen as $\mathcal{PT}$ conjugated to each other, with $\mathcal{P}$ and $\mathcal{T}$ denoting the parity and time conjugation, respectively. In the rest frame, $\vec{p}=\vec{0}$, and s reduces to J^0 ($-J^0$) for states in the $\mathcal{D}_s^+$ ($\mathcal{D}_s^-$) representation. We note

$$|M^2; s; \pm s\rangle \in \mathcal{D}_s^\pm; \tag{20}$$

such states will play a particular role in the rest of this work.

In $2+1$ dimensions, the action of parity $\mathcal{P}$ is to revert one spatial direction; we define it to act as $\boldsymbol{X} = (x^0, x^1, x^2) \to \mathcal{P}\boldsymbol{X}\mathcal{P}^{-1} = (x^0, x^1, -x^2)$. As a consequence,

$$[\mathcal{P}, \boldsymbol{P}^2] = 0, \qquad \left\{\mathcal{P}, \frac{-\boldsymbol{P}\cdot\boldsymbol{J}}{M}\right\} = 0. \tag{21}$$

Eigenstates of both (19) and the parity can be built; they represent anyons, and in the rest frame they read

$$|M^2; s; \eta_{\mathcal{P}}\rangle = \frac{1}{\sqrt{2}}(|M^2; s; s\rangle + \eta_{\mathcal{P}}|M^2; s; -s\rangle) \in \mathcal{D}_s^+ \oplus \mathcal{D}_s^-, \tag{22}$$

where $\eta_{\mathcal{P}}$ is the eigenvalue of the parity. This prescription is valid when $s \neq 0$. For states belonging to $\mathcal{D}(0)$, eigenstates of the parity can still be obtained by application of the projector $\frac{1}{2}(1+\eta_{\mathcal{P}}\mathcal{P})$, but both values of $\eta_{\mathcal{P}}$ cannot necessarily be reached, as we will see in Sec. IV by explicit computation.

III. ANYONS FROM CLOSED STRINGS

As shown in Ref. [10], fractional spin does appear in the spectrum of closed $(2+1)$-dimensional Nambu-Goto strings in the light-cone gauge. More precisely, the authors of Ref. [10] have performed the light-cone quantization of the following Hamiltonian version of the Nambu-Goto action:

$$S[\boldsymbol{X}, \boldsymbol{P}; l, u] = \int d\tau \int \frac{d\phi}{2\pi}\left\{\dot{\boldsymbol{X}}\cdot\boldsymbol{P} - \frac{l}{2}[\boldsymbol{P}^2 + (2\pi\sigma\boldsymbol{X}')^2] - u\boldsymbol{X}'\cdot\boldsymbol{P}\right\}, \tag{23}$$

where σ is the string tension and where the string coordinates $\boldsymbol{X}$ are a function of τ and $\phi \in [0, 2\pi]$. This last action is equivalent to the standard Nambu-Goto action provided l, the Lagrange multiplier accounting for the S^1 diffeomorphism invariance, is nowhere vanishing. The other Lagrange multiplier, u, stands for the τ reparametrization invariance. The reader can find in Ref. [23] a detailed and rigorous presentation of the Hamiltonian quantization of the Polyakov action for the (super)string, where the Hamiltonian action (23) appears upon fixing the constraint related to Weyl invariance of the classical Polyakov string.

A first observation made in Ref. [10] is that the mass spectrum of the theory reads

$$M^2 = 4\pi\sigma(N + \bar{N} - a), \tag{24}$$

with the usual number operators N and $\bar{N}$. The constraint

$$N = \bar{N}, \tag{25}$$

equating the number of left- and right-movers, as a consequence of the S^1 diffeomorphism invariance, must be added to Eq. (24). The constant a is actually not constrained by the theory. Indeed, it is well known that a light-cone quantization in a D-dimensional spacetime would have led to the critical value $a = (D-2)/12$ necessary to restore the Lorentz invariance at quantum level. However, the authors of Ref. [10] have fixed $D = 3$ *a priori*, which has a strong impact: The problematic commutators are *de facto* absent, and Poincaré invariance is satisfied at the quantum level without having to fix a unless the theory is supersymmetric, a case that we are not dealing with here.

The spectrum can be built by requiring the string states to be simultaneously eigenstates of M^2 and s, given by (19). This last operator is cubic in the a's and couples the different states with the same N. The eigenvalues of the operator s finally give the spins of the closed string states with a given mass. Inspection of these eigenvalues shows that there necessarily are fractional spin fields in the spectrum of the first-quantized closed string in 3D. This is the key result of Ref. [10]. More precisely, the first levels of the closed-string spectrum contain states with the following spins:

(i) Only $s = 0$ for $N = 0$ and $N = 1$.
(ii) Two $s = 0$ states and two $s = \frac{3}{\sqrt{4-a}}$ states for $N = 2$.
(iii) Three $s = 0$ states, four $s = \sqrt{\frac{179}{12\sqrt{6-a}}}$ states and two $s = \sqrt{\frac{179}{3\sqrt{6-a}}}$ states for $N = 3$.

States with $s \neq 0$ actually appear in doublets of opposite helicities, standing for the two discrete series $\mathcal{D}_s^{\pm}$. We recall that both discrete series are characterized by the same eigenvalue of the operator on the right-hand side of the second equation of (19), but differ by the sign of J^0. The interested reader will find the explicit expression of all the above states in terms of the string oscillators in Ref. [10].

There is actually an infinite but countable set of closed-string states, some of which have fractional spin, since there is no value of a leading to only integer or half-integer spins. In view of what we recalled in Sec. II, this result is natural: Imposing Poincaré invariance to the first-quantized closed string in 3D should logically lead to states belonging to anyonic representations. Note, however, that the noncritical nature of the bosonic string in $2+1$ dimensions comes in the light-cone quantization prescription. BRST quantization, on the other hand, forbids low-dimensional, critical Polyakov strings; see Ref. [23].

IV. THE MODEL

A. Glueballs and closed strings

Beyond the pioneering work [14], the relevance of relating Yang-Mills theory at large N_c to a closed-string theory has been studied also in Ref. [24], where the following picture is developed. On the one hand, at large

N_c, Yang-Mills dynamics can be reformulated in terms of a reduced model, typically a quenched Eguchi-Kawai model [25]. On the other hand, an appropriate limit $N_c \to \infty$ of $SU(N_c)$ [24,26] is isomorphic to the algebra of area-preserving diffeomorphisms. Both results allow us to reformulate the quenched Eguchi-Kawai action as a Nambu-Goto action. However, $SU(\infty)$ Yang-Mills is not fully equivalent to a Nambu-Goto string, since the integration measure of its partition function is not that of a Nambu-Goto string [24]. Other approaches clearly show that a closed Nambu-Goto string can only be a leading-order approximation of Yang-Mills theory even at large N_c, see e.g. Ref. [27] and references therein.

Another point of view is that of Ref. [28], in which Yang-Mills theory in $2+1$ dimensions is reduced to a $(1+1)$-dimensional Yang-Mills theory with scalar adjoint matter. The spectrum of the latter theory is shown to contain bound states (glueballs) that can be interpreted as closed strings. Nevertheless, as observed in Ref. [24], the Nambu-Goto string alone cannot provide an effective description of Yang-Mills theory. A better-known reason is the standard result that Poincaré invariance is fulfilled at the quantum level for $D=26$ only. This issue was solved in Ref. [29], where it was shown that adding a term

$$\delta\mathcal{L}_{PS} \propto \frac{(\partial_\alpha\partial_\beta X^\mu \partial^\beta X_\mu)^2}{(\partial_\gamma X_\mu)^2} \tag{26}$$

to the Polyakov Lagrangian restores Poincaré invariance for any spacetime dimension D. The Polchinski-Strominger term [29] has been computed in conformal gauge and recovered in static gauge [30]. Note that such an extra term is not needed in the case we focus on, since, within the light-cone gauge quantization scheme used in Ref. [10], Poincaré invariance is already satisfied at the quantum level for the 3D Nambu-Goto action.

Another reason to go beyond the Nambu-Goto string may then be to reach a more accurate description of the dynamics of the effective QCD string. For example, as seen from a semiclassical expansion around a closed folded string, the Polchinski-Strominger term produces corrections to the well-known mass formula $M^2 \propto J$, J being the string angular momentum. The corrections appear as powers of J smaller than 1 and have been computed in Ref. [31]. More generally, the analysis performed in Ref. [32] of the terms allowed by classical Lorentz invariance reveals that the first nontrivial correction to the Nambu-Goto Lagrangian in $2+1$ dimensions is a term involving the induced world-sheet metric h and the scalar curvature R constructed from it:

$$\delta\mathcal{L} \propto \sqrt{-h}R^2. \tag{27}$$

However, in the present exploratory work, we are mainly interested in a qualitative description of the glueball spectrum, so it is worth asking whether adding such a term brings relevant information or not. It appears from Ref. [33] that, when expanding the energy of an effective closed string in terms of its classical length L, the energy formula is universal up to $1/L^5$ terms in $2+1$ dimensions, and deviations from universality only appear at order $1/L^7$. According to lattice computations [34], the mass of the lowest-lying glueball at large N_c is given by $M/\sqrt{\sigma} \sim 4$, which provides the estimate $\sqrt{\sigma}L \sim 4$, a length range such that $1/(\sqrt{\sigma}L)^7$ corrections to the standard Nambu-Goto energy formula are negligible [35].

We aim at building an effective model in which the nonperturbative dynamics of $(2+1)$-dimensional YM theory is that of a closed bosonic string. From what we have just been arguing, it is thus sufficient to adopt, in a first approach, the quantization scheme of Ref. [10] that will allow us to reach this goal.

B. Glueball states

In order to match string states and glueball states according to standard terminology, one has to associate $s^{\mathcal{PC}}$ quantum numbers to a given string state. On top of the reversal of any spatial momentum, the parity operator $\mathcal{P}$ for closed strings is defined by

$$\mathcal{P} = (-1)^{N+\bar{N}}. \tag{28}$$

It anticommutes with the helicity operator [10]. As a consequence, for any given eigenvalue of N (equivalently M^2), states with nonzero spin form parity doublets (22). The $s=0$ cases must be treated separately; see below.

Charge conjugation $\mathcal{C}$ has to be introduced by hand by recalling that, in $2+1$ dimensions, a closed flux tube is actually a loop of fundamental color flux that closes on itself. Hence, it has an intrinsic orientation which is that of the chromoelectric field [17]. So a given state in the closed-string spectrum can either correspond to a flux tube with clockwise ($\circlearrowright$) orientation or to one with anticlockwise orientation ($\circlearrowleft$). The action of the charge conjugation is to revert this orientation, basically by turning fundamental color charges into conjugated ones [17],

$$\mathcal{C}|\circlearrowright; M^2; s; s\rangle = |\circlearrowleft; M^2; s; s\rangle, \tag{29}$$

while parity also flips J^0:

$$\mathcal{P}|\circlearrowright; M^2; s; s\rangle = |\circlearrowleft; M^2; s; -s\rangle. \tag{30}$$

Note that, in our framework, time reversal would just flip J^0.

In summary, starting from a closed-string state $|\circlearrowright; M^2; s; s\rangle$ found in Ref. [10], one can build a $s^{\eta_{\mathcal{P}}\eta_{\mathcal{C}}}$ glueball with mass M^2 provided that the linear combination

TABLE I. Glueball quantum numbers predicted by our flux tube model, with $a = -2.071$ and $b = 0.746$, compared to the pure gauge lattice studies [18,34] in the large-N_c limit. Masses are given in units of the string tension.

		$M/\sqrt{\sigma}$			$M/\sqrt{\sigma}$	
N	$s^{\eta_\mathcal{P}\eta_\mathcal{C}}$	Model	Lattice	$s^{\eta_\mathcal{P}\eta_\mathcal{C}}$	Model	Lattice
0	0^{++}	4.081	4.108(20) [34]	0^{--}	5.950	5.953(71) [34]
1	0^{++*}	6.464	6.211(46) [34]	0^{--*}	7.780	7.77(14) [34]
2	0^{++**}	8.180	8.35(20) [34]	0^{--**}	9.256	8.96(65) [18]
	0^{-+}	8.180	9.02(30) [34]	0^{+-}	9.256	9.47(116) [18]
	$1.22^{\pm+}$	8.180		$1.22^{\pm-}$	9.256	

$$|M^2; s^{\eta_\mathcal{P}\eta_\mathcal{C}}\rangle = \frac{1}{2}(1+\eta_\mathcal{C}\mathcal{C})(1+\eta_\mathcal{P}\mathcal{P})|\circlearrowright; M^2; s; s\rangle \quad (31)$$

is nonzero. At this stage, charge conjugation just adds an additional $\mathbb{Z}_2$ degree of freedom to the spectrum.

The explicit form of the eigenstates of M^2 and s is given in Ref. [10] and will not be recalled here for the sake of brevity. We have checked that, from these $|\circlearrowright; M^2; s; s\rangle$ states, one can form the following multiplets:

(i) $\{0^{++}, 0^{--}\}$ for $N = 0$.
(ii) $\{0^{++*}, 0^{--*}\}$ for $N = 1$.
(iii) $\{0^{++**}, 0^{-+}, 0^{--**}, 0^{+-}, \frac{3}{\sqrt{4-a}}^{\pm\pm}\}$ for $N = 2$.
(iv) …

The * is used to distinguish excited states of a given $s^{\eta_\mathcal{P}\eta_\mathcal{C}}$. It is readily seen that, if glueball dynamics is that of a closed string, the low-lying spectrum should be filled by (pseudo) scalar states, while the first states with nonzero spin are expected to arise at higher masses, corresponding to level 2 in our formalism. At this stage, the state with $s = 3/\sqrt{4-a}$ can still be a boson with spin $n \in \mathbb{N}_0$, provided that $a = 4 - 9/n^2$. However, $n > 1$ leads to $a > 0$, implying unphysical glueball states with $M^2 < 0$ at level 0. Even if the $N = 2$ glueball with $J \neq 0$ is not an anyon but a spin-1 boson, then anyons necessarily appear at level 3, so they cannot be avoided in the glueball spectrum.

V. GLUEBALL SPECTRUM

A. Numerical results

Glueball states obtained in the previous section follow the simple mass formula (24). Hence, the glueball spectrum is completely known from our model once the value of a is fixed. As usually done in the field, this can be achieved by comparing our results to the $(2+1)$-dimensional glueball spectrum computed in pure gauge lattice QCD in Refs. [18,34] and further analyzed in Refs. [16,36].

A clear feature of the lattice spectrum is the appearance of Regge trajectories, i.e. a linear dependence between the squared mass M^2 and the spin s of a glueball, with a slope compatible with the value $8\pi\sigma$ of a classical closed string [16,36]. However, the spin "measured" on the lattice is necessarily integer due to the Euclidean spacetime induced by Wick's rotation. That is why, as discussed in Sec. II, comparisons between our model and lattice results should be restricted to $s = 0$ states: $SO(2,1)$ and $SO(3)$ only share the $\mathcal{D}(0)$ UIR. These states are listed in Table I. It is readily seen that, as predicted by the closed-string picture, the lightest states with $\mathcal{C} = +$ ($\mathcal{C} = -$) are 0^{++} (0^{--}) ones, while the first 0^{-+} (0^{+-}) glueball is much heavier.

As pointed out in Ref. [17], the lattice spectrum shows a large splitting between $\mathcal{C} = +$ and $\mathcal{C} = -$ states, which are degenerate according to the mass formula (24). As argued in Ref. [17], this is the stage at which it has to be remembered that flux tubes may be more complex objects than Nambu-Goto strings because of their intrinsic orientation. Processes that induce a mixing between $\circlearrowright$ and $\circlearrowleft$ states can be figured out: One can think of a $\circlearrowright$ flux tube shrinking to a "ball-like" configuration where information about the orientation is lost, then expanding into a $\circlearrowleft$ flux tube. The simplest way of implementing such a mixing is to add a constant coupling of the form

$$\begin{pmatrix} M^2 & 4\pi\sigma b \\ 4\pi\sigma b & M^2 \end{pmatrix}, \quad (32)$$

the above mass (squared) operator being expressed in the $\{|\circlearrowright\rangle, |\circlearrowleft\rangle\}$ basis. The eigenstates are $\mathcal{C} = +$ states, with mass $M^2_{\mathcal{C}=+} = 4\pi\sigma(N + \bar{N} - a - b)$, and the $\mathcal{C} = -$ states, with mass $M^2_{\mathcal{C}=+} = 4\pi\sigma(N + \bar{N} - a + b)$. The effect of the mixing introduced is thus simply to shift the intercept of $\mathcal{C} = -$ states with respect to that of $\mathcal{C} = +$ states.

The model built here is obviously very simple and should be regarded as valid only in a first approximation. Spin-dependent corrections, in particular, should be present in a more refined model. It is nevertheless interesting to notice the good agreement between our mass formula and existing lattice data once a and b are fitted; see Table I. A prediction of the present model is that there should exist two degenerate $1.22^{\pm-}$ glueballs with a mass around 8.18 in units of the string tension, as well as $1.22^{\pm-}$ glueballs with a mass around 9.26.

For completeness, we mention that an attempt to compute the large-N_c glueball spectrum in $2+1$ dimensions by resorting to a formulation of lattice gauge theory in the light-cone gauge has been made previously [37].

Among other results, the ratios $M_{0^{--}}/M_{0^{++}} = 1.35(5)$ and $M_{0^{--*}}/M_{0^{++}} = 1.82(6)$ are found, while our approach leads to the similar values 1.46 and 1.90, respectively, keeping the same values of a and b. Anyonic states were not built in Ref. [37]; to our knowledge it is an open question to know whether anyonic states can be built in light-cone gauge lattice theory or not.

B. Comments on the mass spectrum

Although the present flux tube model is close to the one proposed in Ref. [17], a fundamental difference occurs at the level of the quantization of the closed string. Indeed, in Ref. [17], a spectrum was found in agreement with lattice data by using the Isgur-Paton closed flux-tube model [15]. This is not surprising, since the authors of Ref. [17] perform a nonrelativistic, Schrödinger-like quantization of the fluctuations of a closed circular string, and in such a scheme the spin of a state is identified with $s = |N - \bar{N}|$, so it is necessarily an integer and the constraint $N = \bar{N}$ is not present. Only the constraint $N + \bar{N} \neq 1$ is imposed by the model [17]. Hence, the angular momentum appearing in the resulting Hamiltonian is integer and matches existing lattice data.

When N_c is finite, our main assumption—i.e. identifying glueballs with closed flux tubes—may appear less sound. It has to be noticed, however, that the quantum numbers and mass hierarchy of the glueball states are identical whatever N_c is [18,34]. The case $N_c = 2$ is special, since the fundamental representation is real. Then, no orientation can be given to a flux tube, and only the $\mathcal{C} = +$ sector is present. The universal structure of the glueball spectrum for $N_c > 2$ may suggest that the stringy picture developed here is still relevant at finite N_c, and thus that anyonic glueballs are a generic feature of $SU(N_c)$ Yang-Mills theory in $2+1$ dimensions. Even the $SU(2)$ lattice scalar mass spectrum can be recovered by using $b = 0$ (no $\mathcal{C} = -$ sector) and $a = -1.9$ in our model. Note that the spectrum obtained in the present section is expected to be the same in the large-N_c limit of $SU(N_c)$, $SO(N_c)$ and $Sp(N_c)$ Yang-Mills theories, that have been proven to be equivalent in the strong coupling limit [38].

VI. RELATION WITH ’T HOOFT AND WILSON LOOPS

It is now worth wondering how much the existence of anyonic states in YM theory relies on our effective closed string description. There exist other ways to build anyons. One of the simplest ways, at the nonrelativistic level, is to minimally couple a particle to a vortex-like vector potential: The resulting vortex-plus-particle system constitutes an anyon [2]. This coupling can be achieved in Yang-Mills theory too. Let us start from the 3D ’t Hooft operator $\phi(\vec{x})$ defined through the nonstandard commutation relation [39]

$$W(C_t)\phi(\vec{x}) = \mathrm{e}^{\frac{2\pi i n(\vec{x};C_t)}{N_c}}\phi(\vec{x})W(C_t), \tag{33}$$

where $W(C_t) = \mathrm{Tr}\mathrm{P}\exp ig\oint_{C_t} A$ is a standard Wilson loop with C_t a closed spacelike curve. By “spacelike,” it is meant that all the points of C_t have the same temporal coordinate $x^0 = t$. Moreover, in the equation above, $n(\vec{x};C_t)$ is the number of times that the closed curve C_t winds around $\vec{x}$ in a clockwise fashion minus the number of times it winds around $\vec{x}$ anticlockwise. Note also that $[\phi(\vec{x}), \phi(\vec{y})] = 0$, which reflects the locality of the operator ϕ [39]. Explicit representations of $\phi(\vec{z})$ can be found in Refs. [9,40].

We now define the operator

$$G_{C_t}(\vec{z}) = \phi(\vec{z})W(C_t), \tag{34}$$

where $\vec{z}$ may or may not be enclosed by C_t, a closed spacelike curve fixed once for all. Since spacelike Wilson loops commute at equal time [39], it is readily shown that $G_{C_t}(\vec{z})$ may have a nontrivial statistical phase: It is indeed such that, for two separated points $\vec{z}_1$ and $\vec{z}_2$,

$$G_{C_t}(\vec{z}_1)G_{C_t}(\vec{z}_2) = \mathrm{e}^{\frac{2\pi i}{N_c}[n(\vec{z}_2;C_t)-n(\vec{z}_1;C_t)]}G_{C_t}(\vec{z}_2)G_{C_t}(\vec{z}_1). \tag{35}$$

The statistical phase will be nontrivial as soon as $n(\vec{z}_2;C_t) \neq n(\vec{z}_1;C_t)$. From the generalized spin statistics theorem [21], it can be concluded that the operator $G_{C_t}(\vec{z})$ creates a color-singlet state with spin $s = (k/N_c) + n$ with $k, n \in \mathbb{N}$—that is, a value that can be nonzero and neither integer nor half-integer.

Just as the correlator of spacelike Wilson loops contains scalar glueballs [18], it can be expected that the correlator $\langle 0|G^\dagger_{C_t}(\vec{z})G_{C_0}(\vec{z})|0\rangle$ will propagate anyonic glueballs with spin k/N_c. If that turned out to be true, this would show that our main result is not fully dependent on the model used. In the context of the Abelian Higgs model with a Chern-Simons term, the propagation of anyonic states is described in Ref. [4], where in particular it is shown that the physical Hilbert space of one-anyon states is decomposed into orthogonal sectors labeled by the vorticity q:

$$\mathcal{H}^{(\mu)} = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}^{(\mu)}_q, \tag{36}$$

where $\mu/4\pi$ is the coefficient multiplying the Chern-Simons term $\int A \wedge dA$ in the action and where the vorticity eigenvalue q labels the homotopy classes for the map $\mathbf{S}^1 \to \mathbf{S}^1$, expressing the asymptotic behavior of the complex scalar field at spatial infinity. Note that the spin of a state is then given by $\mu q^2/2 \bmod \mathbb{Z}$. The previous considerations on anyon propagation can be made even more rigorous on the lattice in 3D Euclidean space; see Sec. VII of Ref. [4].

VII. SUMMARY AND OUTLOOK

In this note, we have developed a closed-string model of glueballs in $2+1$ dimensions based on the light-cone quantization of the Nambu-Goto string performed in

Ref. [10]. Since closed strings are actually used to model the dynamics of Yang-Mills fields the orientation has been added as an extra quantum number in order to account for the fact that we are dealing with effective rather fundamental strings. This addition has two consequences: the possibility of defining the charge-conjugation of a state, and the addition of a mixing mechanism eventually splitting the masses of states with different eigenvalues under charge conjugation. Our model has two free parameters that, once fitted, allow us to satisfactorily reproduce the masses of the eight zero-spin glueballs currently observed in large-N_c lattice calculations. As a consequence of our model, anyonic glueballs must be present with a mass and spin that both depend on the intercept $\frac{M^2}{4\pi\sigma}|_{N=\bar{N}=0}$.

We believe that the existence of such states is not an artifact of the closed-string picture proposed, but rather, that it is a generic property of Yang-Mills theory in $2+1$ dimensions. Hence, the existence of anyonic glueballs could be confirmed (or not) in the future by resorting to lattice calculations, either in light-cone gauge or in the more standard temporal gauge, provided that appropriate correlators are built. As a starting point for future calculations, an inspiring explicit form for the t' Hooft operator can be found in Ref. [40], while similar results have been proposed in the framework of the Abelian Higgs model in Ref. [41].

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[1] F. Wilczek, Quantum Mechanics of Fractional Spin Particles, Phys. Rev. Lett. **49**, 957 (1982).
[2] S. Forte, Quantum mechanics and field theory with fractional spin and statistics, Rev. Mod. Phys. **64**, 193 (1992).
[3] A. Khare, *Fractional Statistics and Quantum Theory*, 2nd ed. (World Scientific, Singapore, 2005).
[4] J. Frohlich and P. A. Marchetti, Quantum field theory of anyons, Lett. Math. Phys. **16**, 347 (1988); Quantum field theories of vortices and anyons, Commun. Math. Phys. **121**, 177 (1989).
[5] G. W. Semenoff, Canonical Quantum Field Theory with Exotic Statistics, Phys. Rev. Lett. **61**, 517 (1988).
[6] Y.-S. Wu and A. Zee, Comments on the Hopf Lagrangian and fractional statistics of solitons, Phys. Lett. **147B**, 325 (1984).
[7] A. Liguori, M. Mintchev, and M. Rossi, Anyon quantum fields without a Chern-Simons term, Phys. Lett. B **305**, 52 (1993).
[8] M. N. Chernodub, M. I. Polikarpov, A. I. Veselov, and M. A. Zubkov, Strings and Aharonov-Bohm effect in Abelian Higgs model, Phys. Lett. B **432**, 182 (1998).
[9] N. Itzhaki, Anyons, 't Hooft loops and a generalized connection in three-dimensions, Phys. Rev. D **67**, 065008 (2003).
[10] L. Mezincescu and P. K. Townsend, Anyons from Strings, Phys. Rev. Lett. **105**, 191601 (2010); Quantum 3D superstrings, Phys. Rev. D **84**, 106006 (2011).
[11] A. O. Barut and C. Fronsdal, On non-compact groups: II. Representations of the $2+1$ Lorentz group, Proc. R. Soc. A **287**, 532 (1965).
[12] B. Binegar, Relativistic field theories in three-dimensions, J. Math. Phys. **23**, 1511 (1982).
[13] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. **B72**, 461 (1974).
[14] G. Veneziano, An introduction to dual models of strong interactions and their physical motivations, Phys. Rep. **9**, 199 (1974); Large N expansion in dual models, Phys. Lett. **52B**, 220 (1974); Some aspects of a unified approach to gauge, dual and Gribov theories, Nucl. Phys. **B117**, 519 (1976).
[15] N. Isgur and J. E. Paton, A flux tube model for hadrons in QCD, Phys. Rev. D **31**, 2910 (1985).
[16] H. B. Meyer and M. J. Teper, Glueball Regge trajectories and the Pomeron: A lattice study, Phys. Lett. B **605**, 344 (2005).
[17] R. W. Johnson and M. J. Teper, String models of glueballs and the spectrum of SU(N) gauge theories in $(2+1)$-dimensions, Phys. Rev. D **66**, 036006 (2002).
[18] M. J. Teper, SU(N) gauge theories in $(2+1)$-dimensions, Phys. Rev. D **59**, 014512 (1998).
[19] E. P. Wigner, On unitary representations of the inhomogeneous Lorentz group, Ann. Math. **40**, 149 (1939); On unitary representations of the inhomogeneous lorentz group, Nucl. Phys. B, Proc. Suppl. **6**, 9 (1989).
[20] V. Bargmann and E. P. Wigner, Group theoretical discussion of relativistic wave equations, Proc. Natl. Acad. Sci. U.S.A. **34**, 211 (1948).
[21] J. Mund, The spin-statistics theorem for anyons and plektons in $d = 2+1$, Commun. Math. Phys. **286**, 1159 (2009).
[22] J. L. Cortes and M. S. Plyushchay, Linear differential equations for a fractional spin field, J. Math. Phys. **35**, 6049 (1994); P. A. Horvathy, M. S. Plyushchay, and M. Valenzuela, Bosons, fermions and anyons in the plane, and supersymmetry, Ann. Phys. (Amsterdam) **325**, 1931 (2010).
[23] L. Brink and M. Henneaux, *Principles Of String Theory* (Plenum, New York, 1988).
[24] I. Bars, Strings from reduced large N gauge theory via area preserving diffeomorphisms, Phys. Lett. B **245**, 35 (1990);

D. B. Fairlie and C. K. Zachos, Infinite dimensional algebras, sine brackets and SU(∞), Phys. Lett. B **224**, 101 (1989).

[25] G. Bhanot, U. M. Heller, and H. Neuberger, The quenched Eguchi-Kawai model, Phys. Lett. **113B**, 47 (1982).

[26] D. B. Fairlie and C. K. Zachos, Infinite dimensional algebras, sine brackets and $SU(\infty)$, Phys. Lett. B **224**, 101 (1989).

[27] Y. Makeenko, *Methods of Contemporary Gauge Theory* (Cambridge University Press, Cambridge, England, 2002).

[28] S. Dalley and I. R. Klebanov, String spectrum of $(1+1)$-dimensional large N QCD with adjoint matter, Phys. Rev. D **47**, 2517 (1993).

[29] J. Polchinski and A. Strominger, Effective String Theory, Phys. Rev. Lett. **67**, 1681 (1991).

[30] S. Dubovsky, R. Flauger, and V. Gorbenko, Effective string theory revisited, J. High Energy Phys. 09 (2012) 044.

[31] S. Hellerman and I. Swanson, String Theory of the Regge Intercept, Phys. Rev. Lett. **114**, 111601 (2015); J. Sonnenschein and D. Weissman, Glueballs as rotating folded closed strings, J. High Energy Phys. 12 (2015) 011.

[32] O. Aharony and M. Dodelson, Effective string theory and nonlinear Lorentz invariance, J. High Energy Phys. 02 (2012) 008.

[33] O. Aharony and Z. Komargodski, The effective theory of long strings, J. High Energy Phys. 05 (2013) 118.

[34] B. Lucini and M. Teper, SU(N) gauge theories in $(2+1)$-dimensions: Further results, Phys. Rev. D **66**, 097502 (2002).

[35] A. Athenodorou, B. Bringoltz, and M. Teper, Closed flux tubes and their string description in $D = 2+1$ SU(N) gauge theories, J. High Energy Phys. 05 (2011) 042.

[36] H. B. Meyer and M. J. Teper, Glueball Regge trajectories in $(2+1)$-dimensional gauge theories, Nucl. Phys. **B668**, 111 (2003).

[37] S. Dalley and B. van de Sande, Precision study of large N Yang-Mills theory in $(2+1)$-dimensions, Phys. Rev. D **63**, 076004 (2001).

[38] C. Lovelace, Universality at large N, Nucl. Phys. **B201**, 333 (1982).

[39] G. 't Hooft, On the phase transition towards permanent quark confinement, Nucl. Phys. **B138**, 1 (1978).

[40] H. Reinhardt, On 't Hooft's loop operator, Phys. Lett. B **557**, 317 (2003).

[41] E. T. Akhmedov, M. N. Chernodub, M. I. Polikarpov, and M. A. Zubkov, Quantum theory of strings in Abelian Higgs model, Phys. Rev. D **53**, 2087 (1996).