

**$B \rightarrow Kl^+l^-$  decay form factors from three-flavor lattice QCD**

Jon A. Bailey,<sup>1</sup> A. Bazavov,<sup>2</sup> C. Bernard,<sup>3</sup> C. M. Bouchard,<sup>4,\*</sup> C. DeTar,<sup>5</sup> Daping Du,<sup>6</sup> A. X. El-Khadra,<sup>7</sup> J. Foley,<sup>5</sup> E. D. Freeland,<sup>8</sup> E. Gámiz,<sup>9</sup> Steven Gottlieb,<sup>10</sup> U. M. Heller,<sup>11</sup> R. D. Jain,<sup>7</sup> J. Komijani,<sup>3</sup> A. S. Kronfeld,<sup>12,13</sup> J. Laiho,<sup>6</sup> L. Levkova,<sup>5</sup> Yuzhi Liu,<sup>14</sup> P. B. Mackenzie,<sup>12</sup> Y. Meurice,<sup>15</sup> E. T. Neil,<sup>14,16</sup> Si-Wei Qiu,<sup>5,†</sup> J. N. Simone,<sup>12</sup> R. Sugar,<sup>17</sup> D. Toussaint,<sup>18</sup> R. S. Van de Water,<sup>12</sup> and Ran Zhou<sup>12,‡</sup>

(Fermilab Lattice and MILC Collaborations)

<sup>1</sup>*Department of Physics and Astronomy, Seoul National University, Seoul, South Korea*

<sup>2</sup>*Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA*

<sup>3</sup>*Department of Physics, Washington University, St. Louis, Missouri 63130, USA*

<sup>4</sup>*Department of Physics, The Ohio State University, Columbus, Ohio 43210, USA*

<sup>5</sup>*Department of Physics and Astronomy, University of Utah, Salt Lake City, Utah 84112, USA*

<sup>6</sup>*Department of Physics, Syracuse University, Syracuse, New York 13244, USA*

<sup>7</sup>*Department of Physics, University of Illinois, Urbana, Illinois 61801, USA*

<sup>8</sup>*School of the Art Institute of Chicago, Chicago, Illinois 60603, USA*

<sup>9</sup>*CAFPE and Departamento de Física Teórica y del Cosmos, Universidad de Granada, Granada, Spain*

<sup>10</sup>*Department of Physics, Indiana University, Bloomington, Indiana 47405, USA*

<sup>11</sup>*American Physical Society, Ridge, New York 11961, USA*

<sup>12</sup>*Fermi National Accelerator Laboratory, Batavia, Illinois 60510, USA*

<sup>13</sup>*Institute for Advanced Study, Technische Universität München, Garching, Germany*

<sup>14</sup>*Department of Physics, University of Colorado, Boulder, Colorado 80309, USA*

<sup>15</sup>*Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa 52242, USA*

<sup>16</sup>*RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA*

<sup>17</sup>*Department of Physics, University of California, Santa Barbara, California 93106, USA*

<sup>18</sup>*Department of Physics, University of Arizona, Tucson, Arizona 85721, USA*

(Received 6 September 2015; published 27 January 2016)

We compute the form factors for the  $B \rightarrow Kl^+l^-$  semileptonic decay process in lattice QCD using gauge-field ensembles with  $2 + 1$  flavors of sea quark, generated by the MILC Collaboration. The ensembles span lattice spacings from 0.12 to 0.045 fm and have multiple sea-quark masses to help control the chiral extrapolation. The asqtad improved staggered action is used for the light valence and sea quarks, and the clover action with the Fermilab interpretation is used for the heavy  $b$  quark. We present results for the form factors  $f_+(q^2)$ ,  $f_0(q^2)$ , and  $f_T(q^2)$ , where  $q^2$  is the momentum transfer, together with a comprehensive examination of systematic errors. Lattice QCD determines the form factors for a limited range of  $q^2$ , and we use the model-independent  $z$  expansion to cover the whole kinematically allowed range. We present our final form-factor results as coefficients of the  $z$  expansion and the correlations between them, where the errors on the coefficients include statistical and all systematic uncertainties. We use this complete description of the form factors to test QCD predictions of the form factors at high and low  $q^2$ .

DOI: 10.1103/PhysRevD.93.025026

## I. INTRODUCTION

Flavor-changing neutral-current interactions (FCNC) place important constraints on physics beyond the standard model. In the standard model, tree-level FCNC contributions vanish by the Glashow-Iliopoulos-Maiani (GIM) mechanism. Even at the one-loop level, the GIM mechanism suppresses these amplitudes, as do factors of

the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix. Thus, new-physics effects may be substantially larger than the small standard-model contribution and, hence, observable. In this paper, we present an unquenched lattice-QCD calculation of the amplitudes for the FCNC process  $B \rightarrow Kl^+l^-$ . Within the standard model and beyond, three form factors can arise, and we present results for all three. This work is part of a larger program by the Fermilab Lattice and MILC collaborations to calculate form factors for exclusive semileptonic  $B$  decays needed to test the standard model and search for new physics, all of which use the same lattice actions and parameters. It builds upon our previous work on charged-current semileptonic  $B$

\*Present address: Physics Department, College of William and Mary, Williamsburg, Virginia, USA.

†Present address: Laboratory of Biological Modeling, NIDDK, NIH, Bethesda, Maryland, USA.

‡zhouran@fnal.gov

decays,  $B \rightarrow \pi l \nu$  [1,2] and  $B \rightarrow D^{(*)} l \nu$  [3–5], which are used to determine the CKM matrix elements  $|V_{ub}|$  and  $|V_{cb}|$  [6]. It is also part of a suite of form factors needed to search for new physics in rare semileptonic  $B$ -decay processes such as  $B \rightarrow \pi l^+ l^-$  [7],  $B \rightarrow D \tau \nu$  [8] and  $B_s \rightarrow \mu^+ \mu^-$  [9].

Experimental research on rare  $B$ -meson decays is active [10,11]. The *BABAR*, Belle, and CDF collaborations have measured the differential branching ratio, the forward-backward asymmetry and other observables for both  $B \rightarrow Kl^+ l^-$  and  $B \rightarrow K^* l^+ l^-$  decays [12–16]. The LHCb Collaboration has reported more precise results for the  $B^0 \rightarrow K^0 l^+ l^-$  and  $B^\pm \rightarrow K^\pm l^+ l^-$  decays [17–19]. The high-intensity  $B$  factories will also have results in the future [20]. Thus, it is timely to improve the precision of the theoretical calculation of these processes. Recently, the HPQCD Collaboration published the first three-flavor lattice calculation of  $B \rightarrow Kl^+ l^-$  [21], also analyzing the phenomenological implications [22]. Three-flavor results for the modes with vector mesons in the final state,  $B \rightarrow K^* l^+ l^-$  and  $B_s \rightarrow \phi l^+ l^-$ , have also been presented [23].

The theoretical description of the  $B \rightarrow K^{(*)} l^+ l^-$  process is based on the operator-product expansion, which leads to a low-energy effective Hamiltonian [24–27]. Amplitudes are expressed in terms of Wilson coefficients that encode the high-scale physics and hadronic matrix elements that capture the low-energy nonperturbative QCD contributions. Hadronic matrix elements of local operators can be parametrized in terms of form factors. The uncertainties in the form factors are an important source of error in the theoretical predictions of the observables mentioned above. In order to calculate the form factors, one requires knowledge of nonperturbative QCD dynamics, and lattice QCD is the tool of choice. We focus on  $B \rightarrow Kl^+ l^-$ , rather than  $B \rightarrow K^* l^+ l^-$ , because the former is a “gold-plated” decay with a stable hadron (under strong interactions) in the final state. In the vector-meson case, the  $K^*$  is unstable, and the resonance would have to be distinguished from  $K\pi$  states.

The goal of this work is to improve our knowledge of the  $B \rightarrow Kl^+ l^-$  form factors. We use the three-flavor gauge-field ensembles generated by the MILC Collaboration with dynamical up, down, and strange quarks. We extrapolate our lattice simulation data to the physical light-quark masses and continuum using SU(2) chiral perturbation theory formulated for the process  $B \rightarrow Kl^+ l^-$ . Because the strange-quark mass is integrated out of the SU(2) theory, the use of SU(2)  $\chi$ PT, rather than SU(3), improves the convergence of the chiral expansion, thereby reducing the systematic uncertainty due to the chiral-continuum extrapolation. On currently available lattices, we directly obtain the form factors at large momentum transfer (low recoil),  $q^2 \gtrsim 17 \text{ GeV}^2$ . Following Ref. [1], we use the  $z$  expansion to extend the lattice-QCD calculation to the full range of  $q^2$ . Compared with the work of the HPQCD Collaboration

[21,22], we use twice as many ensembles, covering a larger range of lattice spacings and using lighter sea-quark masses. In particular, the smallest lattice spacing and the smallest light-quark mass in our ensemble set are smaller by a factor of two compared to the set used by HPQCD. In addition, we use the Sheikholeslami-Wohlert (SW) action [28] with the Fermilab interpretation [29] for the  $b$  quark, while the HPQCD Collaboration uses a nonrelativistic QCD  $b$  quark [30]. As discussed below, details of the chiral-continuum extrapolation and the  $z$  expansion also differ.

This paper is organized as follows. In Sec. II, we define the form factors for the  $B \rightarrow Kl^+ l^-$  decay. We then describe the lattice ensembles used in our simulations. We also discuss the formalism for the light and heavy quarks. In Sec. III, we present the numerical analysis. This section has four parts. We first present results for the simulated  $K$  and  $B$ -meson masses. Next, we determine the lattice form factors from two-point and three-point correlation functions. We then correct our form-factor data to account for the slight difference between the simulated  $b$ -quark mass and the physical  $b$ -quark mass. Last, we extrapolate the lattice simulation results to the chiral and continuum limits with SU(2) heavy meson rooted staggered chiral perturbation theory (HM $\chi$ PT). In Sec. IV, we analyze the systematic errors in our calculation and give a complete error budget for the range of momenta  $q^2 \gtrsim 17 \text{ GeV}^2$  accessible in our numerical simulations. In Sec. V, we extrapolate our form factors from low to high recoil using the  $z$  expansion [31–34]. We present our final results for  $f_+(q^2)$ ,  $f_0(q^2)$ , and  $f_T(q^2)$ , including statistical and all systematic errors, as coefficients of the  $z$  expansion and the correlations between them; this provides a complete description of our form factors valid over the entire kinematic range. In Sec. VI, we use these results to test predictions for the form factor from heavy-quark symmetry at high  $q^2$  and from QCD factorization at low  $q^2$ . Finally, we compare our form factors with other lattice-QCD and light-cone-sum-rule results, and present an outlook for future work, in Sec. VII.

Preliminary results have been reported in Refs. [35–37]. Here we present a full analysis that includes the tensor-current form factor and complete systematic error budgets.

## II. LATTICE CALCULATION

In this section, we present the methods and ingredients used in this work. We give the definitions of the form factors for the  $B \rightarrow Kl^+ l^-$  process and their relation to physical observables in Sec. II A. We then describe the lattice actions and parameters used for gluon and fermion fields in our simulation in Sec. II B. Finally, we define the lattice currents in Sec. II C.

### A. Matrix elements and form factors

An operator-product expansion analysis of the  $B \rightarrow Kl^+l^-$  decay in the standard model shows that two currents, a vector current  $\mathcal{V}^\mu = \bar{s}\gamma^\mu b$  and a tensor current  $\mathcal{T}^{\mu\nu} = i\bar{s}\sigma^{\mu\nu}b$ , contribute to the  $b \rightarrow s$  process at lowest order [10]. In general standard-model extensions, a scalar current  $\mathcal{S} = \bar{s}b$  can also arise. The matrix elements of the vector, tensor, and scalar current are characterized by three form factors  $f_+(q^2)$ ,  $f_0(q^2)$ , and  $f_T(q^2)$ , which are defined via

$$\begin{aligned} \langle K|\bar{s}\gamma^\mu b|B\rangle &= f_+(q^2)\left(p^\mu + k^\mu - \frac{M_B^2 - M_K^2}{q^2}q^\mu\right) \\ &+ f_0(q^2)\frac{M_B^2 - M_K^2}{q^2}q^\mu, \end{aligned} \quad (2.1)$$

$$\langle K|i\bar{s}\sigma^{\mu\nu}b|B\rangle = \frac{2f_T(q^2)}{M_B + M_K}(p^\mu k^\nu - p^\nu k^\mu), \quad (2.2)$$

$$\langle K|\bar{s}b|B\rangle = \frac{M_B^2 - M_K^2}{m_b - m_s}f_0(q^2), \quad (2.3)$$

where  $p$  and  $k$  are the  $B$ -meson and kaon momenta, respectively, and  $q = p - k$  is the momentum carried off by the leptons. The Ward identity relating the matrix element of a vector current to that of the corresponding scalar current ensures that  $f_0$  is the same in Eqs. (2.1) and (2.3).

For the analysis that follows, it is convenient to write the vector-current matrix element as

$$\langle K|\bar{s}\gamma^\mu b|B\rangle = \sqrt{2M_B}[v^\mu f_\parallel(E_K) + k_\perp^\mu f_\perp(E_K)], \quad (2.4)$$

where  $v^\mu = p^\mu/M_B$  is the four-velocity of the  $B$  meson,  $k_\perp^\mu = k^\mu - (k \cdot v)v^\mu$ , and  $E_K = v \cdot k$  is the kaon energy in the  $B$ -meson rest frame. From energy-momentum conservation,  $q^2 = M_B^2 + M_K^2 - 2M_B E_K$ . We obtain  $f_\parallel(E_K)$  and  $f_\perp(E_K)$  from the temporal and spatial components of the matrix element of the vector current:

$$f_\parallel(E_K) = \frac{\langle K|\bar{s}\gamma^0 b|B\rangle}{\sqrt{2M_B}}, \quad (2.5)$$

$$f_\perp(E_K) = \frac{\langle K|\bar{s}\gamma^i b|B\rangle}{\sqrt{2M_B}k^i}. \quad (2.6)$$

Similarly, we obtain the tensor form factor  $f_T$  from

$$f_T(q^2) = \frac{M_B + M_K}{\sqrt{2M_B}} \frac{\langle K|\bar{s}\sigma^{0i}b|B\rangle}{\sqrt{2M_B}k^i}. \quad (2.7)$$

Finally, the vector and scalar form factors  $f_+$  and  $f_0$  can be obtained from

$$f_+(q^2) = \frac{1}{\sqrt{2M_B}}[f_\parallel(E_K) + (M_B - E_K)f_\perp(E_K)], \quad (2.8)$$

$$\begin{aligned} f_0(q^2) &= \frac{\sqrt{2M_B}}{M_B^2 - M_K^2}[(M_B - E_K)f_\parallel(E_K) \\ &+ (E_K^2 - M_K^2)f_\perp(E_K)]. \end{aligned} \quad (2.9)$$

Equations (2.8)–(2.9) satisfy the kinematic constraint,  $f_+(0) = f_0(0)$ , automatically. At low recoil, the form factor  $f_\perp$  gives the dominant contribution to  $f_+$ .

Physical observables can be described in terms of the form factors, if we neglect nonfactorizable contributions. For example, the standard-model differential decay rate for  $B \rightarrow Kl^+l^-$  is [24,38,39]

$$\begin{aligned} \frac{d\Gamma}{dq^2} &= \frac{G_F^2 \alpha^2 |V_{tb}V_{ts}^*|^2}{2^7 \pi^5} |\mathbf{k}| \beta_+ \left\{ \frac{2}{3} |\mathbf{k}|^2 \beta_+^2 |C_{10}^{\text{eff}} f_+(q^2)|^2 \right. \\ &+ \frac{m_l^2 (M_B^2 - M_K^2)^2}{q^2 M_B^2} |C_{10}^{\text{eff}} f_0(q^2)|^2 \\ &+ |\mathbf{k}|^2 \left[ 1 - \frac{1}{3} \beta_+^2 \right] |C_9^{\text{eff}} f_+(q^2)|^2 \\ &\left. + 2C_7^{\text{eff}} \frac{m_b + m_s}{M_B + M_K} f_T(q^2) \right\}^2, \end{aligned} \quad (2.10)$$

where  $G_F$ ,  $\alpha$ , and  $V_{iq}$  are the Fermi constant, the (QED) fine structure constant, and CKM matrix elements, respectively,  $|\mathbf{k}| = \sqrt{E_K^2 - M_K^2}$  is the kaon momentum in the  $B$ -meson rest frame, and  $\beta_+^2 = 1 - 4m_l^2/q^2$ , with  $m_l$  being the lepton mass. The  $C_i^{\text{eff}}$  are effective Wilson coefficients [25]; we follow the notation of Ref. [27] in Eq. (2.10). When  $q^2$  corresponds to a charmonium resonance, further contributions must be added to Eq. (2.10). Beyond the standard model, the expression can become more complicated, but  $f_+(q^2)$ ,  $f_T(q^2)$ , and  $f_0(q^2)$  still suffice.

### B. Actions and parameters

Our calculations employ the  $N_f = 2 + 1$  flavor gauge configurations generated by the MILC Collaboration [40,41], which include the effects of dynamical  $u$ ,  $d$ , and  $s$  quarks. The one-loop improved Lüscher-Weisz action is used for the gluon fields, which leads to lattice artifacts of  $\mathcal{O}(\alpha_s a^2)$  [42]. (The gluon-loop correction is included [43], but not that of the quark loop [44].)

For light quarks ( $u$ ,  $d$  and  $s$ ), these configurations employ the  $a^2$  tadpole-improved staggered action (asqtad) [45–51], leading to discretization errors of  $\mathcal{O}(\alpha_s a^2)$  and  $\mathcal{O}(a^4)$  [52]. The sea quarks are simulated with the fourth root of the staggered fermion determinant. Several theoretical and numerical analyses support the idea that this procedure yields continuum QCD as the lattice spacing  $a \rightarrow 0$  [52–65].

TABLE I. Parameters of the QCD gauge-field ensembles and light valence-quark masses used in this work, lattice spacing  $a$ , lattice size  $N_s^3 \times N_t$ , sea-quark masses  $am'_l$  and  $am'_h$ , light-valence mass  $am_l$ , daughter mass  $am_h$ , the number of configurations and sources denoted as  $N_{\text{conf}} \times N_{\text{src}}$ , and the box size times the pion mass. On all ensembles but one, we use the same light valence- and sea-quark mass. (The only exception is on the  $a \approx 0.09$  fm ensemble with  $m'_l = 0.0465$ , where the light valence-quark mass is 0.0047 instead of 0.00465.) On the  $a \approx 0.12$  fm and  $a \approx 0.09$  fm ensembles we also use the same valence and sea strange-quark mass. On the  $a \approx 0.06$  fm and  $a \approx 0.045$  fm ensembles, we use slightly different valence strange-quark masses than in the sea; the valence masses are tuned to be closer to the physical value. The values of  $M_\pi L$  are taken from Refs. [4,52]. The gauge-field configurations can be downloaded using the DOI links provided in Refs. [66–81].

$\approx a$ (fm)	$N_s^3$	$N_t$	$am'_l$	$am'_h$	$am_l$	$am_h$	$N_{\text{conf}}$	$N_{\text{src}}$	$M_\pi L$
0.12 [66,67]	$20^3$	64	0.01	0.05	0.01	0.05	2259	4	4.5
0.12 [68]	$20^3$	64	0.007	0.05	0.007	0.05	2110	4	3.8
0.12 [69]	$24^3$	64	0.005	0.05	0.005	0.05	2099	4	3.8
0.09 [70–72]	$28^3$	96	0.0062	0.031	0.0062	0.031	1931	4	4.1
0.09 [73]	$32^3$	96	0.00465	0.031	0.0047	0.031	984	4	4.1
0.09 [74,75]	$40^3$	96	0.0031	0.031	0.0031	0.031	1015	4	4.2
0.09 [76]	$64^3$	94	0.00155	0.031	0.00155	0.031	791	4	4.8
0.06 [77,78]	$48^3$	144	0.0036	0.0180	0.0036	0.0188	673	4	4.5
0.06 [79,80]	$64^3$	144	0.0018	0.0180	0.0018	0.0188	827	4	4.3
0.045 [81]	$64^3$	192	0.0028	0.0140	0.0028	0.0130	801	4	4.6

Table I summarizes the properties of the ensembles used in this work. We use the asqtad ensembles at four lattice spacings:  $a \approx 0.12$  fm,  $a \approx 0.09$  fm,  $a \approx 0.06$  fm, and  $a \approx 0.045$  fm. The volumes of the lattices are large enough ( $M_\pi L \gtrsim 4$ ) to suppress finite-volume effects. The strange sea-quark mass is tuned to be close to its physical value. The light-to-strange sea-quark mass ratios range from  $am'_l/am'_h = 0.2$  down to 0.05, to facilitate reliable chiral extrapolations. On the  $a \approx 0.12$  fm and  $a \approx 0.09$  fm ensembles, we use unitary data, with the light and strange valence-quark masses set equal to the corresponding sea-quark masses, with one exception. On the  $a \approx 0.06$  fm and  $a \approx 0.045$  fm ensembles, however, we use valence strange-quark masses that are closer to the physical value and, thus, differ slightly from the strange-quark mass in the sea.

On each configuration, we compute the correlation functions starting at four different source locations to increase the available statistics. We first translate the gauge field by a different random four-vector on each configuration and then fix the spatial source locations at  $\mathbf{x} = \mathbf{0}$  and the temporal source locations at  $t = 0, N_t/4, N_t/2$ , and  $3N_t/4$ . The correlation between the results from different source locations is weak. The random translation of the gauge field reduces autocorrelations between successive configurations.

For the heavy  $b$  quark, we use the SW action [28] with the Fermilab interpretation [29]. The lattice action and currents are matched to the continuum QCD action via HQET [82]. The heavy-quark action can be systematically improved to arbitrarily high orders in  $1/m_b$ , or, equivalently,  $a$ , by including higher-dimensional operators in the lattice action [29,82,83] and currents [84–86]. In this work, we remove the leading discretization errors in the

action by tuning the hopping parameter  $\kappa$  and clover coefficient  $c_{\text{SW}}$ . We fix the bare  $b$ -quark mass by tuning the value of  $\kappa_b$  to reproduce the spin-averaged  $B_s$  meson kinetic mass as in Ref. [4]. We use the tadpole-improved tree-level value for  $c_{\text{SW}} = u_0^{-3}$ , where  $u_0$  is obtained from the fourth root of the plaquette. We also remove the leading discretization error in the vector and tensor currents; see Sec. II C. The values of the parameters for  $b$  quarks used in our simulations are listed in Table II.

To extrapolate the form factors calculated on the lattice to the continuum limit, we need a unified scale to compare the results from different spacings and convert to physical units. We do so with the scale  $r_1$  which is defined such that  $r_1^2 F(r_1) = 1.0$  [87,88]. Here  $F(r)$  is the force between static quarks at distance  $r$ . We first determine the relative scale  $r_1/a$  on each ensemble, and then interpolate  $r_1/a$  with a smooth function of the gauge coupling  $\beta$ ; the smoothed  $r_1$  values are independent of the light sea-quark mass. (The explicit form of the smoothing function is given in Ref. [4].) In this paper, we choose a mass-independent scheme for  $r_1/a$ , so that it is the same for all sea masses with the same approximate lattice spacing. We use the values of  $r_1/a$  to convert all lattice quantities to  $r_1$  units. We can then combine results from different ensembles and perform a chiral-continuum extrapolation. The physical value  $r_1 = 0.3117(22)$  fm [52,89] is determined by requiring that the continuum limit of the pion decay constant at the physical quark masses takes the PDG value [6]. The RBC-UKQCD Collaboration also reported the physical value  $r_1 = 0.323(8)(4)$  fm in Ref. [90]. This result is consistent with the one we use, but less precise. The values of  $r_1/a$  used in this work are provided in Table III.

TABLE II. Parameters used in the simulation of the heavy  $b$  quark [4]. We list the clover coefficient  $c_{\text{SW}}$ , input  $b$ -quark hopping parameter  $\kappa'_b$ , and rotation coefficient  $d_1$ .

$\approx a$ (fm)	$am'_l$	$c_{\text{SW}}$	$\kappa'_b$	$d_1$
0.12	0.01	1.531	0.0901	0.093340
0.12	0.007	1.530	0.0901	0.093320
0.12	0.005	1.530	0.0901	0.093320
0.09	0.0062	1.476	0.0979	0.096765
0.09	0.00465	1.477	0.0977	0.096708
0.09	0.0031	1.478	0.0976	0.096690
0.09	0.00155	1.4784	0.0976	0.096700
0.06	0.0036	1.4287	0.1052	0.096300
0.06	0.0018	1.4298	0.1052	0.096300
0.045	0.0028	1.3943	0.1143	0.08864

### C. Definition of currents

We define the current operators on the lattice as in Refs. [1,91]:

$$V_\xi^\mu(x) = \bar{\Psi}_\alpha(x) \gamma_{\alpha\beta}^\mu \Omega_{\beta\xi}(x) \chi(x), \quad (2.11)$$

$$T_\xi^{\mu\nu}(x) = \bar{\Psi}_\alpha(x) \sigma_{\alpha\beta}^{\mu\nu} \Omega_{\beta\xi}(x) \chi(x), \quad (2.12)$$

where the matrix  $\Omega = \gamma_4^{x_4/a} \gamma_1^{x_1/a} \gamma_2^{x_2/a} \gamma_3^{x_3/a}$  and  $\chi(x)$  is the one-component staggered fermion field. The clover  $b$ -quark field is rotated to remove discretization errors of order  $a$  from the lattice current [29]:

$$\bar{\Psi} = (1 + ad_1 \boldsymbol{\gamma} \cdot \mathbf{D}_{\text{lat}}) \bar{\psi}, \quad (2.13)$$

where  $\psi$  is the field in the Fermilab action (for the  $b$  quark),  $\mathbf{D}_{\text{lat}}$  is the symmetric, nearest-neighbor, covariant difference operator, and  $d_1$  is adjusted to remove discretization errors. In practice, we set the rotation coefficient  $d_1$  to its tadpole-improved tree-level value:

TABLE III. Relative scales  $r_1/a$  used in this work, for corresponding values of  $\beta$  [4,52]. The statistical and systematic errors on  $r_1/a$  are both 0.1%–0.3% [4]. We also list the Goldstone pion mass ( $M_\pi$ ) and root-mean-square (RMS) pion mass ( $M_\pi^{\text{RMS}}$ ) here.

$\approx a$ (fm)	$am'_l/am'_h$	$\beta$	$r_1/a$	$M_\pi$ (MeV)	$M_\pi^{\text{RMS}}$ (MeV)
0.12	0.01/0.05	6.760	2.739	389	532
0.12	0.007/0.05	6.760	2.739	327	488
0.12	0.005/0.05	6.760	2.739	277	456
0.09	0.0062/0.031	7.090	3.789	354	413
0.09	0.00465/0.031	7.085	3.772	307	374
0.09	0.0031/0.031	7.080	3.755	249	329
0.09	0.00155/0.031	7.075	3.738	177	277
0.06	0.0036/0.018	7.470	5.353	316	340
0.06	0.0018/0.018	7.460	5.307	224	255
0.045	0.0028/0.014	7.810	7.208	324	331

$$d_1 = \frac{1}{u_0} \left( \frac{1}{2 + m_0 a} - \frac{1}{2(1 + m_0 a)} \right), \quad (2.14)$$

where  $m_0 a$  is the bare lattice  $b$ -quark mass. The index  $\xi$  in Eqs. (2.11)–(2.12) corresponds to taste, and it is contracted with another taste index in the heavy-light operators coupling the  $B$  meson to the vacuum [91].

To calculate the form factors on the lattice, we have to define currents with the correct continuum limit. As in earlier work [1,82], we define

$$\mathcal{V}^\mu \doteq Z_{V^\mu} V^\mu, \quad (2.15)$$

$$\mathcal{T}^{\mu\nu} \doteq Z_{T^{\mu\nu}} T^{\mu\nu}, \quad (2.16)$$

where  $\{\mathcal{V}, \mathcal{T}\}$  and  $\{V, T\}$  are the continuum and lattice current operators, respectively. We use a mostly nonperturbative renormalization procedure to obtain the  $Z$  factors [92],

$$Z_J = \rho_J \sqrt{Z_{V_{bb}^4} Z_{V_{ss}^4}}, \quad (2.17)$$

where  $Z_{V_{bb}^4}$  and  $Z_{V_{ss}^4}$  are computed nonperturbatively, and the remaining factor  $\rho_J$  is calculated at one-loop order in mean-field improved lattice perturbation theory [84].

The light-light renormalization factor  $Z_{V_{ss}^4}$  is calculated nonperturbatively from the charge normalization condition of a  $\bar{c}s$  meson:

$$Z_{V_{ss}^4}^{-1} = \int d^3x \langle D_s | V_{ss}^4(x) | D_s \rangle \quad (2.18)$$

as in Ref. [89], but with random color wall sources and higher statistics, leading to the values listed in Table IV. The result for  $Z_{V_{ss}^4}$  is insensitive to the mass of the spectator quark in the correlation function, so we use a heavy charm quark to improve the statistical errors. The heavy-heavy renormalization factor  $Z_{V_{bb}^4}$  is computed analogously from the charge normalization condition of the  $B$  meson using

TABLE IV. The flavor-conserving renormalization factors  $Z_{V_{ss}^4}$  and  $Z_{V_{bb}^4}$  used in this work. Errors shown are statistical.

$\approx a$ (fm)	$am'_l$	$am'_h$	$\kappa'_b$	$Z_{V_{ss}^4}$	$Z_{V_{bb}^4}$
0.12	0.01	0.05	0.0901	1.741(3)	0.5065(57)
0.12	0.007	0.05	0.0901	1.741(3)	0.5119(75)
0.12	0.005	0.05	0.0901	1.741(3)	0.5026(71)
0.09	0.0062	0.031	0.0979	1.777(5)	0.4482(57)
0.09	0.00465	0.031	0.0977	1.776(5)	0.4694(100)
0.09	0.0031	0.031	0.0976	1.776(5)	0.4608(94)
0.09	0.00155	0.031	0.0976	1.776(5)	0.4491(116)
0.06	0.0036	0.018	0.1052	1.808(6)	0.4196(101)
0.06	0.0018	0.018	0.1052	1.807(7)	0.4100(103)
0.045	0.0028	0.014	0.1143	1.841(6)	0.3564(65)

TABLE V. Matching factors  $\rho_{V^4}$ ,  $\rho_{V^1}$ , and  $\rho_T$  calculated at one loop in tadpole-improved lattice perturbation theory. Here,  $\rho_T$  brings  $f_T$  to the  $\overline{\text{MS}}$  scheme at  $\mu = m_2$ , and  $m_2$  should be interpreted as the pole mass.

$\approx a$ (fm)	$am'_l$	$am'_h$	$am_h$	$\kappa'_b$	$\rho_{V^4}$	$\rho_{V^1}$	$\rho_T(\mu = m_2)$
0.12	0.010	0.050	0.050	0.0901	1.0071	0.9737	1.0334
0.12	0.007	0.050	0.050	0.0901	1.0071	0.9737	1.0333
0.12	0.005	0.050	0.050	0.0901	1.0072	0.9738	1.0333
0.09	0.0062	0.031	0.031	0.0979	0.9997	0.9759	1.0366
0.09	0.00465	0.031	0.031	0.0977	0.9998	0.9759	1.0364
0.09	0.0031	0.031	0.031	0.0976	0.9999	0.9758	1.0364
0.09	0.00155	0.031	0.031	0.0976	0.9999	0.9757	1.0364
0.06	0.0036	0.018	0.0188	0.1052	0.9956	0.9792	1.0432
0.06	0.0018	0.018	0.0188	0.1052	0.9956	0.9792	1.0433
0.045	0.0028	0.014	0.013	0.1143	0.9943	0.9843	1.0588

data generated for our  $B \rightarrow D\ell\nu$  analysis [93]. We compute  $Z_{V^4_{bb}}$  on the same jackknife samples as the form factors and propagate the statistical error directly throughout the remainder of the analysis. The values of  $Z_{V^4_{bb}}$  are shown in Table IV.

The remaining factor  $\rho_J$  (here,  $J = V^\mu, T^{\mu\nu}$ ) is close to unity [84,94], because most of the radiative corrections, particularly those from tadpole diagrams, cancel among the  $Z$  factors in Eq. (2.17). We expand the factor  $\rho_J$  perturbatively as

$$\rho_J = 1 + \alpha_s(q^*)\rho_J^{[1]} + \mathcal{O}(\alpha_s^2), \quad (2.19)$$

where  $\alpha_s$  is the QCD coupling [95,96]. Details of the one-loop perturbative calculation will be given in a separate publication [97]; the values used here are listed in Table V. In practice, we evaluate the coupling in the  $V$ -scheme [95,98] at the scale  $q^* = 2/a$  in mean-field improved lattice perturbation theory. For  $\rho_{V^4}$  we find that the one-loop corrections are less than 1%, while for  $\rho_{V^1}$  they range from 1.5% to 2.6%. The tensor current is scale dependent, and we renormalize it at the scale  $\mu = m_b$  (where according to the Fermilab prescription  $m_2 = m_b$ ). We find that for  $\rho_T$  the corrections range from 3% to 6%.

Because  $\rho_J$  is computed separately from the correlation functions, we used it to introduce a blinding procedure (as in many  $B$  physics experiments) to reduce subjective bias. Those of us carrying out the perturbative calculation [97] multiplied  $\rho_J$  by a constant prefactor. Only after we finalized the choices made in our analysis, including tests and estimates of systematic uncertainties, was the prefactor revealed to the rest of the collaboration and removed from the results reported here.

### III. ANALYSIS

In this section, we present our form-factor analysis and results. In Sec. III A, we obtain the  $B$ -meson and kaon masses and energies by fitting two-point correlation functions. In Sec. III B, we extract the lattice form factors from ratios of

three-point over two-point correlation functions. In Sec. III C, we slightly shift the full set of lattice form-factor data from the simulated  $\kappa'_b$  to the physical value. In Sec. III D, we carry out the chiral-continuum extrapolation by fitting the form factors to the expression derived in HMrS $\chi$ PT.

#### A. $B$ and $K$ meson masses

We extract meson masses and energies from two-point correlation functions defined at Euclidean time  $t$ :

$$C_2(t; \mathbf{k}) = \sum_{\mathbf{x}} \langle \mathcal{O}_P(\mathbf{x}, t) \mathcal{O}_P^\dagger(\mathbf{0}, 0) \rangle e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad (3.1)$$

where the subscript  $P$  denotes the  $K$  or  $B$  pseudoscalar meson in the interpolating operator. For the kaon we use a local interpolating operator. For the  $B$  meson we use the wave function for bottomonium given by the Richardson potential model [99] as explained in Refs. [100–102]. We generate correlators with kaon three-momenta  $\mathbf{k} = 2\pi(0, 0, 0)/L$ ,  $2\pi(1, 0, 0)/L$ ,  $2\pi(1, 1, 0)/L$ , and  $2\pi(1, 1, 1)/L$ .

The meson masses and energies are extracted from the large- $t$  behavior of the two-point correlation functions. By inserting a complete set of states, two-point correlation functions can be decomposed into a sum of energy levels as

$$C_2(t; \mathbf{k}) = \sum_m (-1)^{m(t+1)} \frac{|\langle 0 | \mathcal{O}_P | P(m) \rangle|^2}{2E_P^{(m)}} e^{-E_P^{(m)} t}. \quad (3.2)$$

The amplitudes of terms with odd  $m$  oscillate in time as  $(-1)^{m(t+1)}$  and are due to opposite-parity-state contributions to staggered correlators. Figure 1 shows sample kaon and  $B$ -meson scaled correlators  $[C_2(t) - C_2^{(0)}(t)]/C_2^{(0)}(t)$  on the  $a \approx 0.12$  fm ensemble with  $m'_l = 0.1m'_h$  and momentum  $\mathbf{k} = \mathbf{0}$ , where

$$C_2^{(0)}(t) = \frac{|\langle 0 | \mathcal{O}_P | P(0) \rangle|^2}{2E_P^{(0)}} e^{-E_P^{(0)} t} \quad (3.3)$$

is the ground-state contribution determined by our fit. The opposite-parity-state contribution is insignificant for the

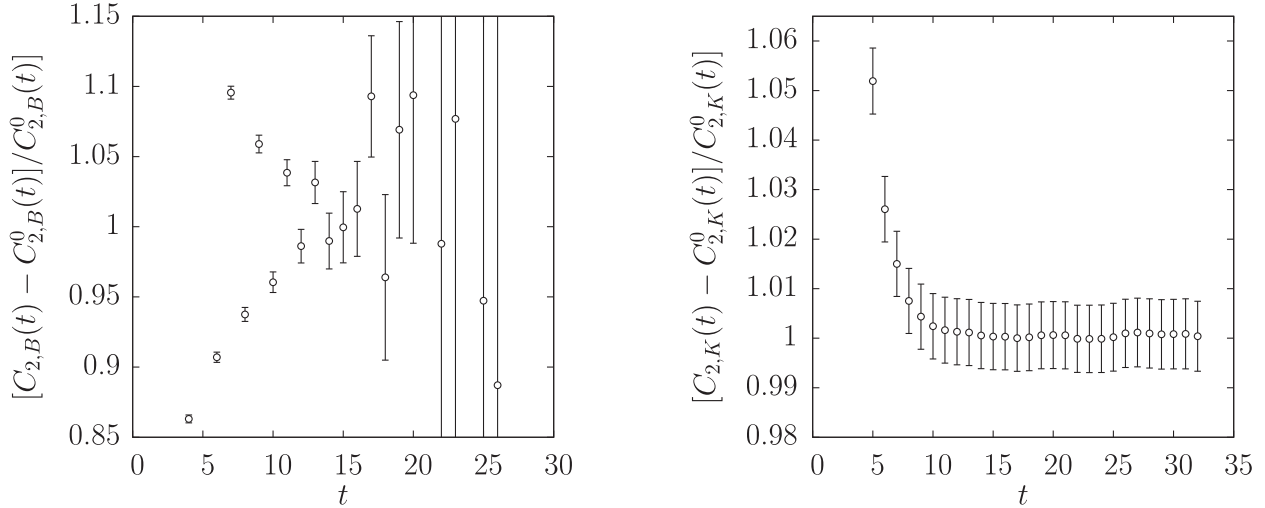


FIG. 1. Scaled correlator  $[C_2(t) - C_2^{(0)}(t)]/C_2^{(0)}(t)$  as a function of time  $t$  on the  $am_l/am_h = 0.005/0.05$   $a \approx 0.12$  fm ensemble at the unitary point. The oscillating opposite-parity-state contribution is clearly visible in the  $B$ -meson correlator (left), but it is small in the zero-momentum kaon correlator (right).

zero-momentum kaon but is visible for the  $B$  meson. We employ a simple strategy to fit the two-point correlators because the statistical errors in the kaon and  $B$ -meson energies contribute little to the errors in form factors, which stem primarily from the three-point correlators. For the kaon correlators, we perform two-state fits that include the ground state and a same-parity excited state. For the  $B$ -meson correlators, we perform three-state fits including the ground state, its excited state, and the lowest-lying opposite-parity state.

We use a single-elimination jackknife analysis to estimate the statistical errors in this work. We first average the correlation functions generated from the four sources at  $0$ ,  $N_t/4$ ,  $N_t/2$ , and  $3N_t/4$ . We fit  $C_2(t)$  in an interval  $t \in [t_{\min}, t_{\max}]$ , taking correlation from time slice to time slice into account. In general, we choose  $t_{\max}$  so that the fractional error in the correlator remains below 4%. We choose  $t_{\min}$  such that we obtain a good correlated  $p$  value. We use the same interval  $[t_{\min}, t_{\max}]$  for all kaon or  $B$ -meson fits at a given lattice spacing, and use similar physical distances for  $[t_{\min}, t_{\max}]$  on the four lattice spacings. These fit ranges are given in Table VI. We use a  $2 + 1$ -state fit for the  $B$  meson in this paper and find consistent results with the  $1 + 1$ -state, larger  $t_{\min}$  fit of Ref. [2].

Figure 2 shows sample  $B$ -meson and kaon correlator fits versus  $t_{\min}$  for fixed  $t_{\max}$  on the same  $a \approx 0.12$  fm ensemble

TABLE VI. Fit ranges  $[t_{\min}, t_{\max}]$  used in the kaon and  $B$ -meson mass and energy fits.

$\approx a$ (fm)	Kaon	$B$ meson
0.12	[7,30]	[3,15]
0.09	[10,35]	[5,20]
0.06	[17,60]	[7,30]
0.045	[20,90]	[8,40]

as in Fig. 1. The fit results and errors are stable versus  $t_{\min}$ , and show no evidence of residual excited-state contamination.

For kaons with nonzero momentum, we can either extract the energy from two-point correlation functions with nonzero momentum or we can use the kaon mass from the zero-momentum correlator and the continuum dispersion relation,  $E^2 = M^2 + k^2$ . Figure 3 shows a comparison of the kaon energy calculated from the continuum dispersion relation and from directly fitting the nonzero momentum two-point correlation functions on the ensemble discussed above. We do not observe any statistically significant deviations from the continuum dispersion relation. Further, while the statistical errors grow with increasing momentum, the kaon energies are consistent with a continuum dispersion relation within a 2% statistical accuracy even at our largest simulated lattice kaon momentum. Therefore, we use the continuum dispersion relation to obtain the kaon energies at nonzero lattice momenta because this yields smaller statistical errors than the direct fit.

The meson propagators from consecutive gauge-field configurations are, in principle, correlated, so we look for possible autocorrelations by studying the effect of the block size on our fit results. We perform this test on every ensemble. As illustrated in Fig. 4 for two of the ensembles, the central values and errors are stable with increasing block size, so we do not block the data or inflate the statistical errors in our analysis.

### B. Extracting form factors

We extract the lattice form factors  $f_{\parallel}$ ,  $f_{\perp}$ , and  $f_T$  from the ratio of three-point to two-point correlation functions. The three-point functions are defined as

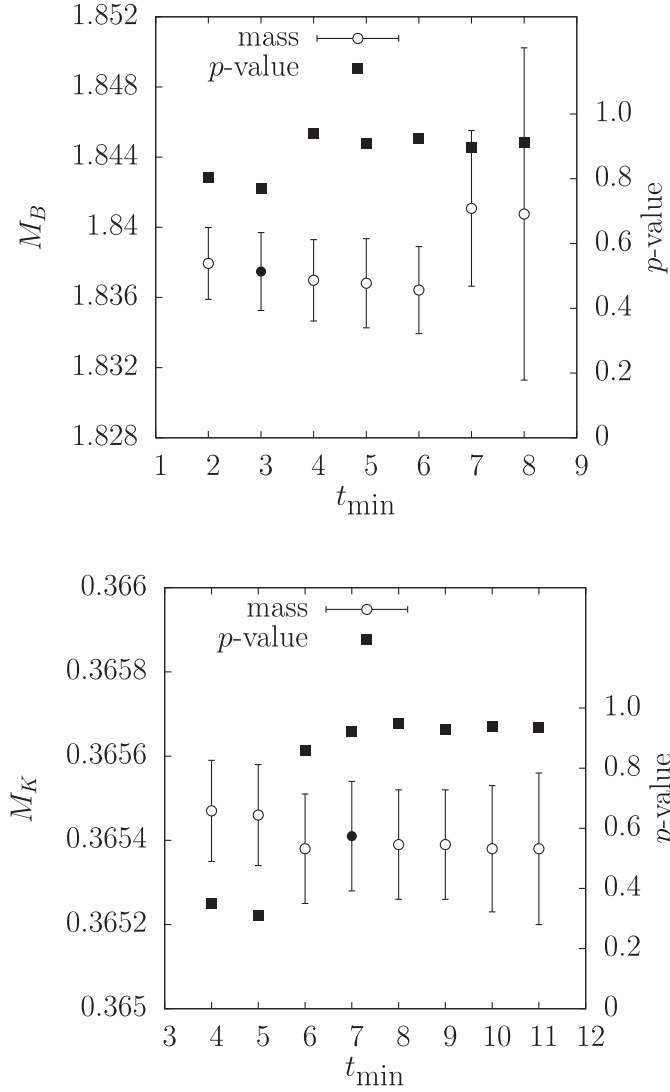


FIG. 2.  $B$ -meson (upper) and kaon (lower) mass vs  $t_{\min}$  on the  $a \approx 0.12$  fm,  $m_l' = 0.1m_h'$  ensemble for fixed  $t_{\max} = 15$  and  $30$ , respectively. The left and right vertical axes show the fitted mass and the  $p$  value (confidence level) of the fit, respectively. The filled circles show the values of  $t_{\min}$  selected for the analysis.

$$C_3^\mu(t, T; \mathbf{k}) = \sum_{\mathbf{x}, \mathbf{y}} e^{i\mathbf{k} \cdot \mathbf{y}} \langle \mathcal{O}_K(0, \mathbf{0}) V^\mu(t, \mathbf{y}) \mathcal{O}_B^\dagger(T, \mathbf{x}) \rangle, \quad (3.4)$$

$$C_3^{\mu\nu}(t, T; \mathbf{k}) = \sum_{\mathbf{x}, \mathbf{y}} e^{i\mathbf{k} \cdot \mathbf{y}} \langle \mathcal{O}_K(0, \mathbf{0}) T^{\mu\nu}(t, \mathbf{y}) \mathcal{O}_B^\dagger(T, \mathbf{x}) \rangle, \quad (3.5)$$

where the kaon source is at time slice  $0$  and the  $B$ -meson sink is at time slice  $T$ . The source-sink separations  $T$  are given in Table VII. Because we calculate the form factors in the  $B$ -meson rest frame, only the kaon has nonzero momentum  $\mathbf{k}$ .

By inserting two complete sets of states, the three-point correlation function  $C_3^\mu$  can be decomposed into sums over energy levels as

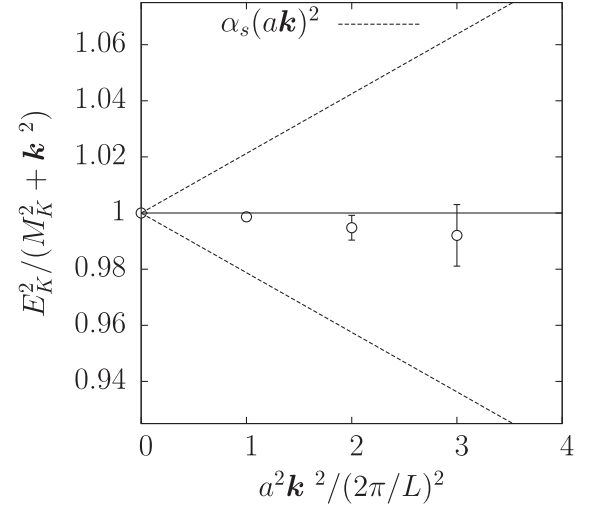


FIG. 3.  $E_K^2 / (M_K^2 + \mathbf{k}^2)$  vs kaon momentum in units of  $2\pi/L$  on the  $a \approx 0.12$  fm,  $m_l' = 0.1m_h'$  ensemble. The continuum dispersion relation is well respected through momentum  $2\pi(1, 1, 1)/L$ . The dashed lines show a power-counting estimate for the size of the momentum-dependent discretization error for comparison.

$$C_3^\mu(t, T; \mathbf{k}) = \sum_{m, n} (-1)^{m(t+1)} (-1)^{n(T-t-1)} A_{mn}^\mu e^{-E_K^{(m)} t} e^{-M_B^{(n)}(T-t)}, \quad (3.6)$$

where

$$A_{mn}^\mu = \frac{\langle 0 | \mathcal{O}_K | K^{(m)} \rangle \langle K^{(m)} | V^\mu | B^{(n)} \rangle \langle B^{(n)} | \mathcal{O}_B | 0 \rangle}{2E_K^{(m)} 2M_B^{(n)}}. \quad (3.7)$$

The contributions from the first few terms dominate  $C_3^\mu$  at times sufficiently far from both the source and sink. A similar decomposition applies to  $C_3^{\mu\nu}$ .

We use the averages introduced in Ref. [1] to suppress the contribution from oscillating states in correlation functions. We average the value of the two-point correlator on successive time slices:

$$\begin{aligned} \bar{C}_2(t) &\equiv \frac{e^{-M_p^{(0)} t}}{4} \left[ \frac{C_2(t)}{e^{-M_p^{(0)} t}} + \frac{2C_2(t+1)}{e^{-M_p^{(0)}(t+1)}} + \frac{C_2(t+2)}{e^{-M_p^{(0)}(t+2)}} \right] \\ &= \frac{Z_p^2}{2M_p^{(0)}} e^{-M_p^{(0)} t} + \mathcal{O}(\Delta M_p^2), \end{aligned} \quad (3.8)$$

where  $Z_p = \langle 0 | \mathcal{O}_p | P \rangle$  is the ground-state amplitude of the kaon or  $B$  meson, and  $\Delta M_p$  is the energy difference between the ground and first oscillating state. For three-point functions, we also average the value of the correlator for two neighboring sink locations  $T$  and  $T+1$ :



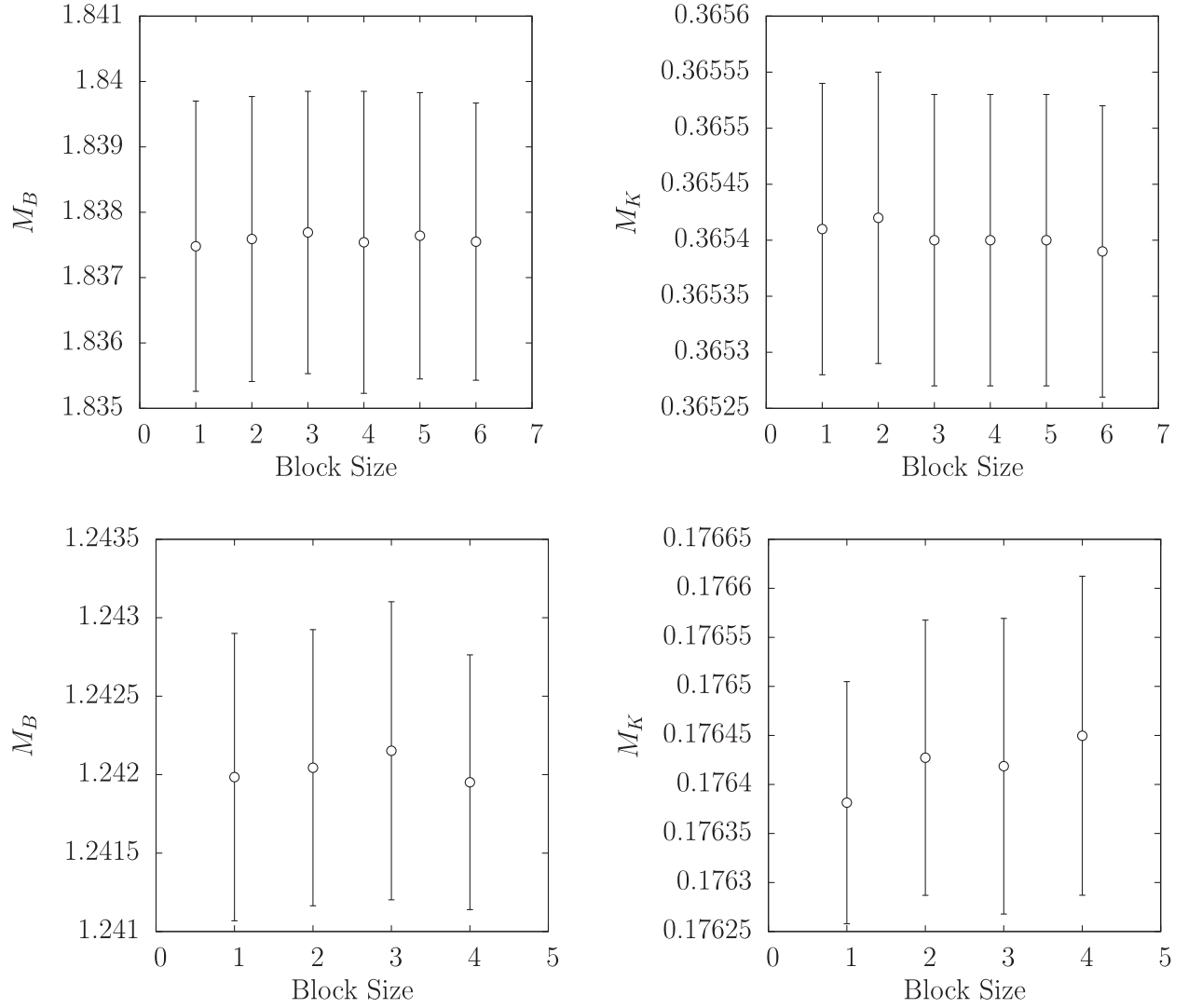


FIG. 4.  $M_B$  (left column) and  $M_K$  (right column) vs block size on the  $a \approx 0.12$  fm (top row) and  $a \approx 0.06$  fm (bottom row),  $m_l' = 0.1m_h'$  ensemble. The fit results are stable as the block size increases.

$$\begin{aligned} \bar{C}_3^{\mu(\nu)}(t, T; \mathbf{k}) \equiv & \frac{1}{8} [e^{-E_K^{(0)}t} e^{-M_B^{(0)}(T-t)}] \times \left[ \frac{C_3^{\mu(\nu)}(t, T; \mathbf{k})}{e^{-E_K^{(0)}t} e^{-M_B^{(0)}(T-t)}} + \frac{C_3^{\mu(\nu)}(t, T+1; \mathbf{k})}{e^{-E_K^{(0)}t} e^{-M_B^{(0)}(T+1-t)}} + \frac{2C_3^{\mu(\nu)}(t+1, T; \mathbf{k})}{e^{-E_K^{(0)}(t+1)} e^{-M_B^{(0)}(T-t-1)}} \right. \\ & \left. + \frac{2C_3^{\mu(\nu)}(t+1, T+1; \mathbf{k})}{e^{-E_K^{(0)}(t+1)} e^{-M_B^{(0)}(T-t)}} + \frac{C_3^{\mu(\nu)}(t+2, T; \mathbf{k})}{e^{-E_K^{(0)}(t+2)} e^{-M_B^{(0)}(T-t-2)}} + \frac{C_3^{\mu(\nu)}(t+2, T+1; \mathbf{k})}{e^{-E_K^{(0)}(t+2)} e^{-M_B^{(0)}(T-t-1)}} \right] \end{aligned} \quad (3.9)$$

$$= A_{00} e^{-E_K^{(0)}t} e^{-M_B^{(0)}(T-t)} + (-1)^{T+1} A_{11} e^{-E_K^{(1)}t} e^{-M_B^{(1)}(T-t)} \left( \frac{\Delta M_B}{2} \right) + \mathcal{O}(\Delta E_K^2, \Delta E_K \Delta M_B, \Delta M_B^2). \quad (3.10)$$

We then form the ratios

$$\bar{R}^{\mu(\nu)}(t, T; \mathbf{k}) \equiv \frac{\bar{C}_3^{\mu(\nu)}(t, T; \mathbf{k})}{\sqrt{\bar{C}_2^K(t; \mathbf{k}) \bar{C}_2^B(T-t)}} \sqrt{\frac{2E_K^{(0)}}{e^{-E_K^{(0)}t} e^{-M_B^{(0)}(T-t)}}}, \quad (3.11)$$

TABLE VII. Pairs of source-sink separations  $T$ ,  $T + 1$  and fit ranges used in the  $\bar{R}^{\mu(\nu)}$  fits.

$\approx a$ (fm)	$T$ , $T + 1$	$[t_{\min}, t_{\max}]$
0.12	18, 19	[8, 12]
0.09	25, 26	[10, 16]
0.06	36, 37	[16, 24]
0.045	48, 49	[20, 32]

where  $E_K^{(0)}$  and  $M_B^{(0)}$  are obtained from fits to Eq. (3.2) with  $E_K^{(0)} = \sqrt{M_K^{(0)} + \mathbf{k}^2}$ . From Eqs. (3.8) and (3.10), the ratio  $\bar{R}^{\mu(\nu)}$  contains a  $t$ -independent term proportional to the desired matrix element, and other higher-order terms from the excited states.

We show an example of the ratio  $\bar{R}^{\mu(\nu)}$  on the  $a \approx 0.12$  fm,  $m_l' = 0.1m_s'$  ensemble in Fig. 5. There is a short plateau region in the middle between 0 and  $T$ , with kaon excited-state contributions visible on the left and  $B$ -meson excited-state contributions visible on the right. The  $B$ -meson excited-state contributions, however, are smaller as indicated by the less dramatic falloff of the correlator on the right-hand side. We therefore choose to fit the correlator closer to the  $B$ -meson side including the contribution from a single  $B$ -meson excited state, but sufficiently far from the kaon that we can neglect kaon excited states. The fit function is given by

$$\bar{R}^{\mu(\nu)}(t, T; \mathbf{k}) = D_0^{\mu(\nu)} [1 - D_1 e^{-\Delta M_B (T-t)}] \quad (3.12)$$

where  $D_0^{\mu(\nu)}$ ,  $D_1$ , and  $\Delta M_B$  are fit parameters. Although the second term in Eq. (3.12) models all excited states, we expect  $\Delta M_B$  to be close to the mass difference of the first excited state.

We employ a correlated, constrained fit [103,104] to Eq. (3.12), with priors determined as follows. For the prior on  $D_0^{\mu(\nu)}$ , we select a point from the middle of the plateau region and use its central value with the error inflated by a factor of two. For  $D_1$ , we use a prior of central value zero and width one. For  $\Delta M_B$ , we use the central value and width of  $M_B^{(1)} - M_B^{(0)}$  obtained from the corresponding two-point correlator fit. We minimize the augmented  $\chi_{\text{aug}}^2$  [104],

$$\chi_{\text{aug}}^2 = \chi^2 + \sum_i \frac{(P^{(i)} - \tilde{P}^{(i)})^2}{\sigma_i^2}, \quad (3.13)$$

where  $P^{(i)}$  is the  $i$ th fit parameter, and  $\tilde{P}^{(i)}$  and  $\sigma_i$  are the prior central value and width. We measure the goodness of fit using the  $\chi_{\text{aug}}^2/\text{dof}$  or  $p$  value, obtaining  $p$  from  $\chi_{\text{aug}}^2$  and the number of degrees of freedom equal to the sum of the number of data points and prior constraints minus the number of fit parameters. We choose the fit interval

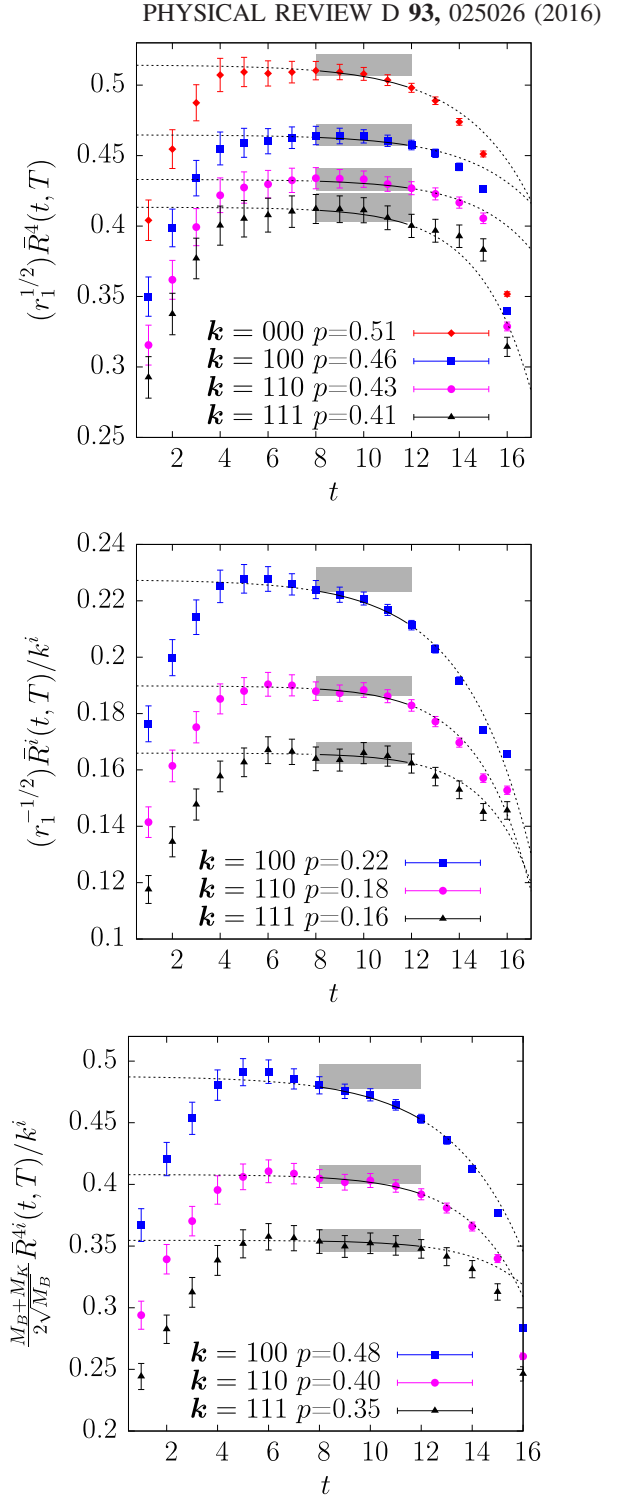


FIG. 5. Form-factor ratio  $\bar{R}^{\mu(\nu)}$  fits on the  $a \approx 0.12$  fm,  $m_l' = 0.1m_h'$  ensemble. From top to bottom, the three plots show the ratios for the temporal vector, spatial vector, and tensor currents. In the top plot, the data sets correspond to lattice kaon momenta  $\mathbf{k} = 2\pi(0, 0, 0)/L$ ,  $2\pi(1, 0, 0)/L$ ,  $2\pi(1, 1, 0)/L$  and  $2\pi(1, 1, 1)/L$ ; nonzero momentum is required to extract the form factors in the bottom two plots, so there are only three sets of data in each of them. The gray horizontal bands show the fit results with statistical errors for  $C_0^{\mu(\nu)}$  in Eq. (3.12). The black solid and dashed curves show the fit result within and extended beyond the fit range, respectively.

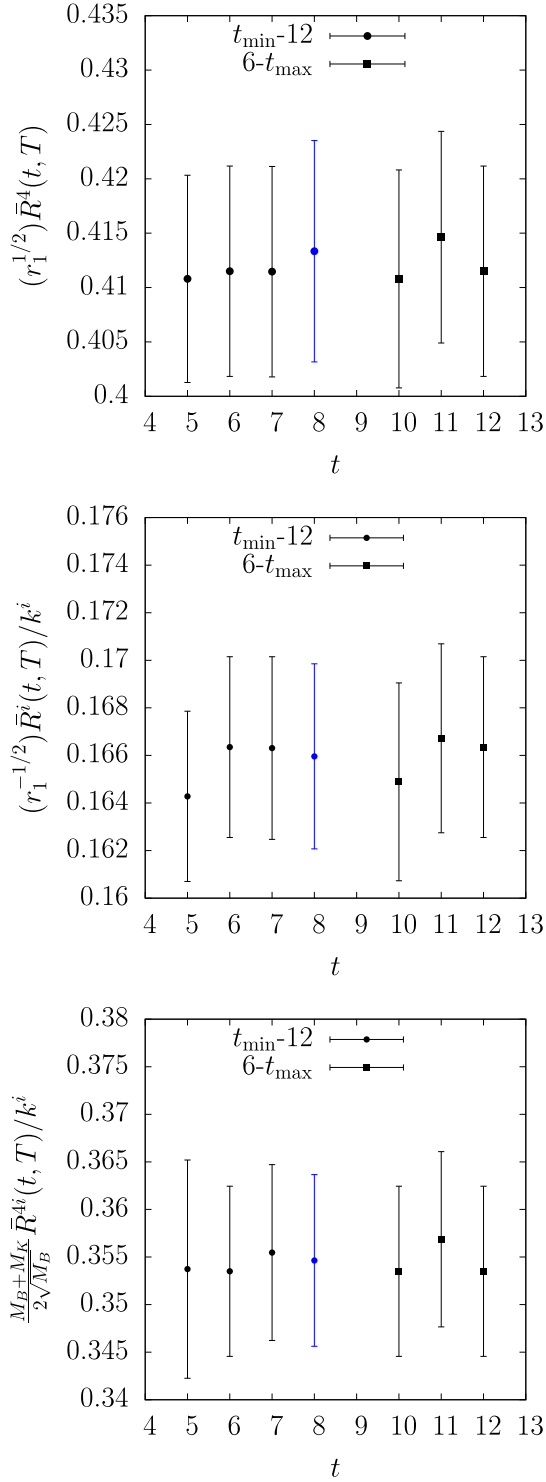


FIG. 6. Fit results of  $\bar{R}^{\mu(\nu)}$  from different fit ranges on the  $a \approx 0.12$  fm,  $m'_l = 0.1m'_h$  ensemble with lattice kaon momentum  $p = \frac{2\pi}{L}(1, 1, 1)$ . From top to bottom, the three plots show the ratios for the temporal vector, spatial vector, and tensor currents. We vary the fit range by changing  $t_{\min}$  and  $t_{\max}$ . The blue data point denotes the result from the fit range used in this paper.

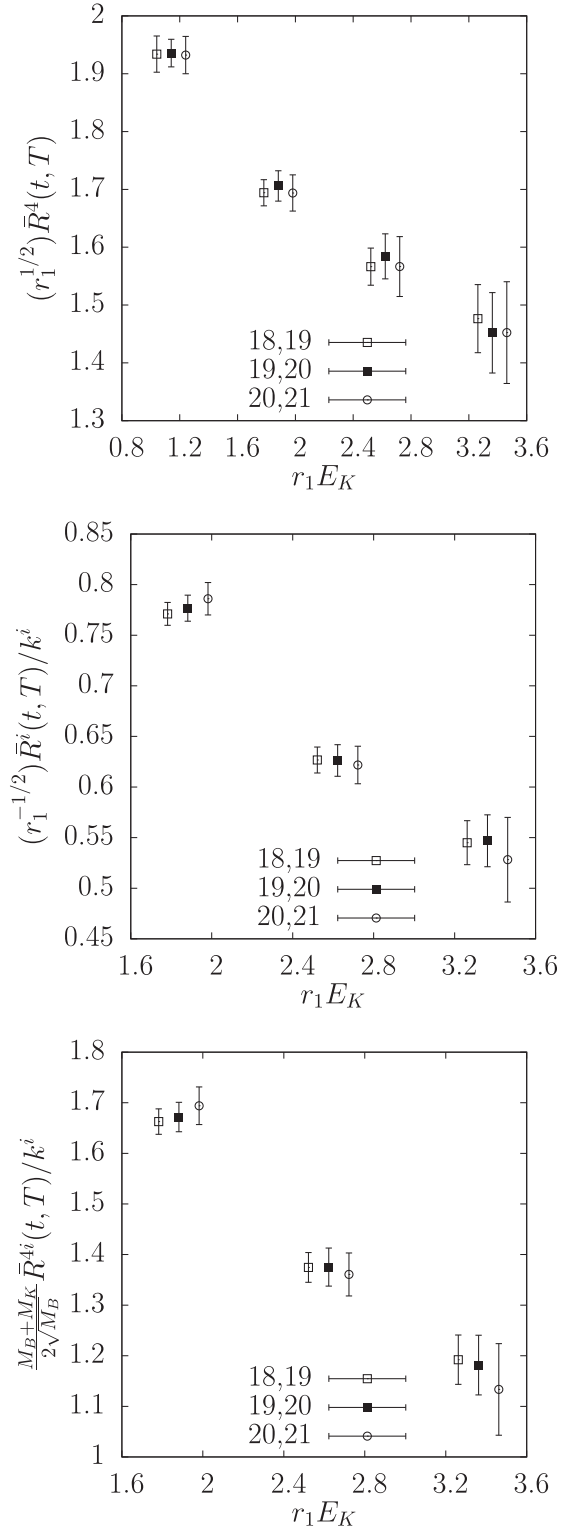


FIG. 7. Form-factor ratio  $\bar{R}^{\mu(\nu)}$  fits on the  $a \approx 0.12$  fm,  $m'_l/m'_h = 0.007/0.05$  ensemble. From top to bottom, the three plots show the ratios for the temporal vector  $f_{\parallel}$ , spatial vector  $f_{\perp}$ , and tensor  $f_T$  currents. The fit results for different pairs of source-sink separations  $T, T + 1$  are shown as a function of  $E_K$ . The results from larger sink combinations are slightly displaced to the right for clarity.

TABLE VIII. The simulation  $\kappa'_b$  and physical  $\kappa_b$  [4]. We also include  $\kappa_{\text{crit}}$  and  $u_0$  from the plaquette in this table for convenience, because they are used in the calculation of the  $b$ -quark kinetic mass.

$\approx a$ (fm)	$am'_l/am'_h$	$\kappa'_b$	$\kappa_b$	$\kappa_{\text{crit}}$	$u_0$
0.12	0.01/0.05	0.0901	0.0868(9)(3)	0.14091	0.8677
0.12	0.007/0.05	0.0901	0.0868(9)(3)	0.14095	0.8678
0.12	0.005/0.05	0.0901	0.0868(9)(3)	0.14096	0.8678
0.09	0.0062/0.031	0.0979	0.0967(7)(3)	0.139119	0.8782
0.09	0.00465/0.031	0.0977	0.0966(7)(3)	0.139134	0.8781
0.09	0.0031/0.031	0.0976	0.0965(7)(3)	0.139173	0.8779
0.09	0.00155/0.031	0.0976	0.0964(7)(3)	0.139190	0.877805
0.06	0.0036/0.018	0.1052	0.1052(5)(2)	0.137632	0.88788
0.06	0.0018/0.018	0.1052	0.1050(5)(2)	0.137678	0.88764
0.045	0.0028/0.014	0.1143	0.1116(3)(2)	0.136640	0.89511

$[t_{\text{min}}, t_{\text{max}}]$  such that we obtain a good  $p$  value, using the same fit range for all momenta on the same ensemble. We select approximately the same physical fit ranges on the ensembles with different lattice spacings. Figure 5 shows sample fits of the three form-factor ratios on the  $a \approx 0.12$  fm,  $m'_l = 0.1m'_s$  ensemble. Figure 6 shows an example of the stability of the fit result against the variations of the fit range. We choose the preferred fit range to be  $[t_{\text{min}}, t_{\text{max}}] = [8, 12]$ , where we find a good  $p$  value. The fit ranges and source-sink separations used on other ensembles are given in Table VII.

To study the effects of residual excited-state contamination, we generated three-point correlators on the  $a \approx 0.12$  fm,  $m'_l = 0.14m'_s$  ensemble with several source-sink separations  $T = 18, 19, 20, 21$ . We repeat the correlator fits with three sink-location combinations  $(T, T+1) = (18, 19), (19, 20),$  and  $(20, 21)$ , and the results are shown in Fig. 7 for four different momenta. We find no statistically significant differences for all operators and momenta except for  $f_{\perp}$  and  $f_T$  at  $\mathbf{p} = 2\pi(1, 0, 0)/L$ . These differences, however, are still sufficiently small that increasing the error on all  $\mathbf{p} = 2\pi(1, 0, 0)/L$  points in the chiral-continuum fit does not change the physical form-factor results.

The fit parameters  $C_0^{\mu(\nu)}$  are proportional to the matrix elements  $\langle K^0 | J | B^0 \rangle$ . The lattice form factors are obtained as

$$f_{\parallel}^{\text{lat}}(E_K) = D_0^4(\mathbf{k}), \quad (3.14)$$

$$f_{\perp}^{\text{lat}}(E_K) = \frac{D_0^i(\mathbf{k})}{k^i}, \quad (3.15)$$

$$f_T^{\text{lat}}(E_K) = \frac{M_B + M_K}{\sqrt{2M_B}} \frac{D_0^{4i}(\mathbf{k})}{k^i}. \quad (3.16)$$

The factor  $(M_B + M_K)/\sqrt{2M_B}$  in  $f_T$  in Eq. (3.16), which stems from Eq. (2.7), is evaluated with the physical meson masses to avoid introducing  $m_q$  dependence not captured in the  $\chi$ PT formula.

### C. $b$ -quark mass correction

The  $b$ -quark hopping parameter used in our simulations  $\kappa'_b$  differs slightly from the physical value  $\kappa_b$  because our production runs started before a more precise tuning of the  $b$ -quark hopping parameter  $\kappa_b$  was completed. For our desired accuracy, we need to apply a correction. To this end, we have carried out runs with multiple values of  $\kappa'_b$  on the  $a \approx 0.12$  fm ensemble with  $m'_l/m'_h = 0.2$ . In addition to the production value of  $\kappa'_b = 0.0901$ , we repeated the run with  $\kappa'_b = 0.0820$  and  $0.0860$ , allowing us to bracket the physical value  $\kappa_b = 0.0868$ . The form factors depend on the  $b$ -quark kinetic mass  $m'_2$ . At the tree level

$$\frac{1}{m'_2 a} = \frac{2}{m'_0 a (2 + m'_0 a)} + \frac{1}{1 + m'_0 a}, \quad (3.17)$$

where

$$m'_0 a = \frac{1}{2u_0} \left( \frac{1}{\kappa'} - \frac{1}{\kappa_{\text{crit}}} \right). \quad (3.18)$$

The values of  $u_0$  and  $\kappa_{\text{crit}}$  are given in Table VIII. Following Ref. [2], we expand the form factor in  $m_2^{-1}$  about  $m'_2$ :

$$f(m'_2, E_K) = f(m_2, E_K) \left[ 1 - \frac{\partial \ln f}{\partial \ln m_2} \left( \frac{m_2}{m'_2} - 1 \right) \right], \quad (3.19)$$

where  $m_2$  denotes the physical  $b$ -quark kinetic mass. We determine the slope,  $\frac{\partial \ln f}{\partial \ln m_2}$ , in our companion work on the semileptonic decay  $B \rightarrow \pi l \nu$  [2]. Because the slope depends mildly on the daughter-quark mass, and the daughter-quark mass is tuned close to its physical value in our calculation, we neglect the daughter-quark dependence of the slope in this work. Finally, we quote  $\frac{\partial \ln f}{\partial \ln m_2}$  of  $f_{\parallel}$ ,  $f_{\perp}$ , and  $f_T$  at the simulated daughter-quark mass as 0.115 (9), 0.139(13), and 0.126(13) [2]. We find relative shifts due to  $b$ -quark mass tuning of about 0.5%–1.5% on the different ensembles.

### D. Chiral-continuum extrapolations

The lattice form factors are computed numerically on ensembles with degenerate up- and down-quark masses that are heavier than the value in nature, as well as at nonzero lattice spacing. To obtain physical results, we first compute the form factors on several lattice spacings with varying up-/down-quark masses and close-to-physical strange-quark masses, and then extrapolate to the physical light-quark mass and continuum (and interpolate to the physical strange-quark mass) using HMrS $\chi$ PT [105,106].

For the chiral-continuum extrapolation we use an HMrS $\chi$ PT formula valid to leading order (LO) in  $1/m_b$  and next-to-leading order (NLO) in the light-quark masses, kaon energy, and lattice spacing, supplemented by next-to-next-to-leading order (NNLO) analytical terms. We have tested both SU(3) HMrS $\chi$ PT [106], which includes the effects of dynamical pions, kaons, and  $\eta$  mesons, and SU(2) HMrS $\chi$ PT, in which the mesons with strange quarks are integrated out. In addition, we also consider hard-kaon HMrS $\chi$ PT, which applies to semileptonic decays with energetic kaons. We find that NLO SU(3) HMrS $\chi$ PT, even supplemented with NNLO analytical terms, does not provide a good description of the data for  $f_{\parallel}$  [35–37], and the  $p$  value of the fit is  $10^{-9}$ . On the other hand, SU(2)

HMrS $\chi$ PT describes the data well even at NLO. We therefore choose SU(2) HMrS $\chi$ PT to perform the chiral-continuum extrapolations.

The kaon energies in our numerical simulations are much larger than the rest mass of the physical kaon. Therefore standard HMrS $\chi$ PT, which is derived for the situation in which the kaon momenta are soft, may not provide a good description of our data throughout the available kinematic range. We therefore also consider hard-kaon HMrS $\chi$ PT, which applies for semileptonic decays with energetic kaons. Recently, Bijmans and Jemos derived the continuum NLO hard-kaon (pion) HMrS $\chi$ PT formulas for both  $B \rightarrow K$  and  $B \rightarrow \pi$  processes [107,108]. We derive the corresponding NLO staggered SU(2) and SU(3) hard-kaon (pion) HMrS $\chi$ PT formulas in the Appendix. It turns out that the chiral logarithms in NLO hard-kaon SU(2) HMrS $\chi$ PT are identical to those in standard soft-kaon SU(2) HMrS $\chi$ PT for  $B \rightarrow K$  decays. This is likely the reason that the standard NLO SU(2) expressions describe our data even at such large kaon energies. Reference [109] found that the hard-pion theory can break down at three-loop level, but we only work at one-loop level here.

The NLO SU(2) HMrS $\chi$ PT formulas for  $B \rightarrow K$  decays take the form

$$r_1^{1/2} f_{\parallel} = \frac{g_{\pi}[C_{\parallel}^{(0)}(1 + \text{logs}) + C_{\parallel}^{(1)}\chi_l + C_{\parallel}^{(2)}\chi_h + C_{\parallel}^{(3)}\chi_E + C_{\parallel}^{(4)}\chi_{a^2} + C_{\parallel}^{(5)}\chi_E^2]}{f_{\pi}r_1(E_K + \Delta_{B_{s0}^*})r_1}, \quad (3.20)$$

$$r_1^{-1/2} f_{\perp} = \frac{g_{\pi}[C_{\perp}^{(0)}(1 + \text{logs}) + C_{\perp}^{(1)}\chi_l + C_{\perp}^{(2)}\chi_h + C_{\perp}^{(3)}\chi_E + C_{\perp}^{(4)}\chi_{a^2} + C_{\perp}^{(5)}\chi_E^2]}{f_{\pi}r_1(E_K + \Delta_{B_s^*})r_1}, \quad (3.21)$$

where “logs” denotes nonanalytic functions of the light-quark mass and lattice spacing; the explicit expressions are given in Eqs. (A28) and (A33)–(A34). The dimensionless expansion parameters  $\chi_i$  in Eqs. (3.20)–(3.21) are

$$\chi_l = \frac{2\mu m_l}{8\pi^2 f_{\pi}^2}, \quad (3.22)$$

$$\chi_h = \frac{2\mu m_h}{8\pi^2 f_{\pi}^2}, \quad (3.23)$$

$$\chi_{a^2} = \frac{a^2 \bar{\Delta}}{8\pi^2 f_{\pi}^2}, \quad (3.24)$$

$$\chi_E = \frac{\sqrt{2}E_K}{4\pi f_{\pi}}, \quad (3.25)$$

where  $a^2 \bar{\Delta}$  is the averaged taste-symmetry breaking parameter,  $a^2 \bar{\Delta} \equiv \frac{1}{16} \sum_{\xi} a^2 \Delta_{\xi}$  and  $\mu$  denotes the leading-order

QCD LEC; see Eqs. (A3)–(A14) for the definition. If HMrS $\chi$ PT gives a good description of the data, we expect the  $C^{(i)}$ ,  $i > 0$ , to be of order unity. The SU(2) $\chi$ PT formulas do not contain  $m_h$  explicitly; however, the low-energy constants (LECs) depend on  $m_h$ . Because the strange-quark masses on different ensembles are slightly different from each other, we include a term proportional to  $\chi_h$  in the set of analytic terms to account for the leading strange-quark mass dependence of the LECs and enable an interpolation to the physical strange-quark mass.

Equations (3.20)–(3.21) each contain a pole in  $E_K$ . The poles appear at negative energy  $-\Delta_{B_{s(0)}^*}$  with

$$\Delta_{B_{s(0)}^*} \equiv \frac{M_{B_{s(0)}^*}^2 - M_B^2 - M_K^2}{2M_B} \approx M_{B_{s(0)}^*} - M_B. \quad (3.26)$$

The pole arises from low-lying states with flavor content  $\bar{b}s$  and quantum numbers that depend upon the form factor:

TABLE IX. Fixed parameters used in the chiral fit [4].  $\mu$  is the leading-order low-energy constant in QCD.  $r_1^2 a^2 \Delta_\Xi$  and  $r_1^2 a^2 \delta_{V/A}$  are the taste splittings and hairpin parameters for asqtad staggered fermions.

	$a \approx 0.12$ fm	$a \approx 0.09$ fm	$a \approx 0.06$ fm	$a \approx 0.045$ fm	Continuum
$r_1 \mu$	6.831904	6.638563	6.486649	6.417427	6.015349
$r_1^2 a^2 \Delta_P (10^{-2})$	0	0	0	0	0
$r_1^2 a^2 \Delta_A (10^{-2})$	22.70460	7.469220	2.634800	1.040930	0
$r_1^2 a^2 \Delta_T (10^{-2})$	36.61620	12.37760	4.297780	1.697920	0
$r_1^2 a^2 \Delta_V (10^{-2})$	48.02591	15.93220	5.743780	2.269190	0
$r_1^2 a^2 \Delta_S (10^{-2})$	60.08212	22.06520	7.038790	2.780810	0
$r_1^2 a^2 \delta_V$	0.00	0.00	0.00	0.00	0
$r_1^2 a^2 \delta_A$	-0.28	-0.09	-0.03	-0.01	0

for  $f_\perp$  and  $f_T$ , the relevant  $B_s^*$  meson has  $J^P = 1^-$ , while for  $f_\parallel$ , the  $B_{s0}^*$  state has  $J^P = 0^+$ . In the chiral-continuum fits, we fix  $M_B$  to its experimentally measured value 5.27958 GeV [6] (recall that we tuned the lattice  $b$ -quark mass using the experimental  $B_s$ -meson mass). We also use the experimentally measured value of the lowest-lying vector meson  $M_{B_s^*} = 5.4154$  GeV [6], which is stable apart from  $B_s^* \rightarrow B_s \gamma$ , for the pole position in the fits of  $f_\perp$  and  $f_T$  to Eq. (3.21). Although a scalar  $B_{s0}^*$  state has not been observed in experiments, theoretical predictions estimate its mass to be just below the  $B$ - $K$  production threshold [110,111]. Therefore, in the fit of  $f_\parallel$ , we use the prediction  $M_{B_{s0}^*} = 5.711(23)$  GeV from a recent three-flavor lattice-QCD calculation [112] for the pole position in Eq. (3.20).

Following the approach of Refs. [103,104], we constrain the parameters of the chiral-continuum fit with Bayesian priors and minimize the augmented  $\chi_{\text{aug}}^2$  defined in Eq. (3.13). The chiral logarithms in Eqs. (3.20)–(3.21) depend upon the universal  $B$ - $B^*$ - $\pi$  coupling  $g_\pi$ , which we constrain with a Gaussian prior of central value 0.45 and width 0.08. This prior is consistent with a direct lattice calculation [113–115], yet conservative enough to accommodate other lattice results [116,117]. The chiral logarithms also depend on the mass splittings between mesons of different tastes and on the leading-order LEC  $\mu$ . These parameters depend only on the light-quark action, and we fix them to the values determined in the MILC light-pseudoscalar analysis [52]; see Table IX. In the  $f_\parallel$  chiral-continuum extrapolation, we account for the uncertainty on the scalar  $B_{s0}^*$  mass by taking a generous prior width of three times the theoretical error reported in Ref. [112], or  $\pm 69$  MeV.

We constrain the coefficients of the LO and NLO analytic terms  $C^{(0)} - C^{(5)}$  using priors with central values zero and widths two. To allow for higher-order contributions in the chiral expansion, we also include the complete set of NNLO analytic terms. These are proportional to  $\chi_1^2$ ,  $\chi_1 \chi_{a^2}$ ,  $\chi_1 \chi_E$ ,  $\chi_1 \chi_E^2$ ,  $\chi_{a^2} \chi_E$ ,  $\chi_{a^2} \chi_E^2$ ,  $\chi_E^3$ ,  $\chi_E^4$ , and  $\chi_{a^2}^2$ . We use prior central values of 0 with widths 1 for the coefficients

of the NNLO analytic terms. The systematic error from truncating the chiral expansion will be discussed in Sec. IV.

Staggered  $\chi$ PT incorporates taste-breaking discretization effects from the light valence and sea quarks, but the lattice data also contain generic light-quark and gluon discretization effects as well as discretization effects from the heavy quark. We account for generic light-quark and gluon discretization errors by adding the term  $z\alpha_s(a\Lambda_{\text{QCD}})^2$  in the HMrS $\chi$ PT formulas with coefficient prior central value zero and width one. Similarly, to account for heavy-quark discretization effects in both the action and heavy-light currents, we add terms of order  $a^2$  and  $\alpha_s a$  with coefficients constrained by heavy-quark power counting [83]. At this order there are five functions ( $f_B, f_Y, f_3, f_E, f_X$ ) that depend upon the bare heavy-quark mass; their explicit forms are given in Appendix A of Ref. [1]. Dimensional analysis can be used to estimate the heavy-quark error

$$\text{error}_i \propto f_i(m_0 a) (a\Lambda)^{\dim \mathcal{O}_i - 4}, \quad (3.27)$$

where  $f_i$  is related to the mismatch between coefficients of the continuum operators in the action and currents and their lattice counterparts, and  $\Lambda$  is a typical QCD scale for heavy-light mesons which we take to be  $\Lambda = 500$  MeV. As in Ref. [89], we add terms  $z_i \times \text{error}_i$  to the HMrS $\chi$ PT formulas for  $f_{\parallel, \perp, T}$ . The priors on the  $z_i$  have central values zero and widths equal to the square root of the number of times each function appears. (See Appendix A of Ref. [1].) Because the discretization errors are included via the constrained fit in the chiral-continuum extrapolations, our results for the extrapolated form factors include the systematic uncertainties from light and heavy discretization effects.

In summary, we use expressions derived in SU(2) HMrS $\chi$ PT for the central chiral-continuum extrapolations of the form factors  $f_\parallel$ ,  $f_\perp$ , and  $f_T$ ; these are shown in Fig. 8. The SU(2) theory describes our data well: the  $p$  values of the fits are 0.91, 0.94, and 0.98 for  $f_\parallel$ ,  $f_\perp$ , and  $f_T$ , respectively. Our fit results for  $g_\pi$  are 0.47(5), 0.46(4),

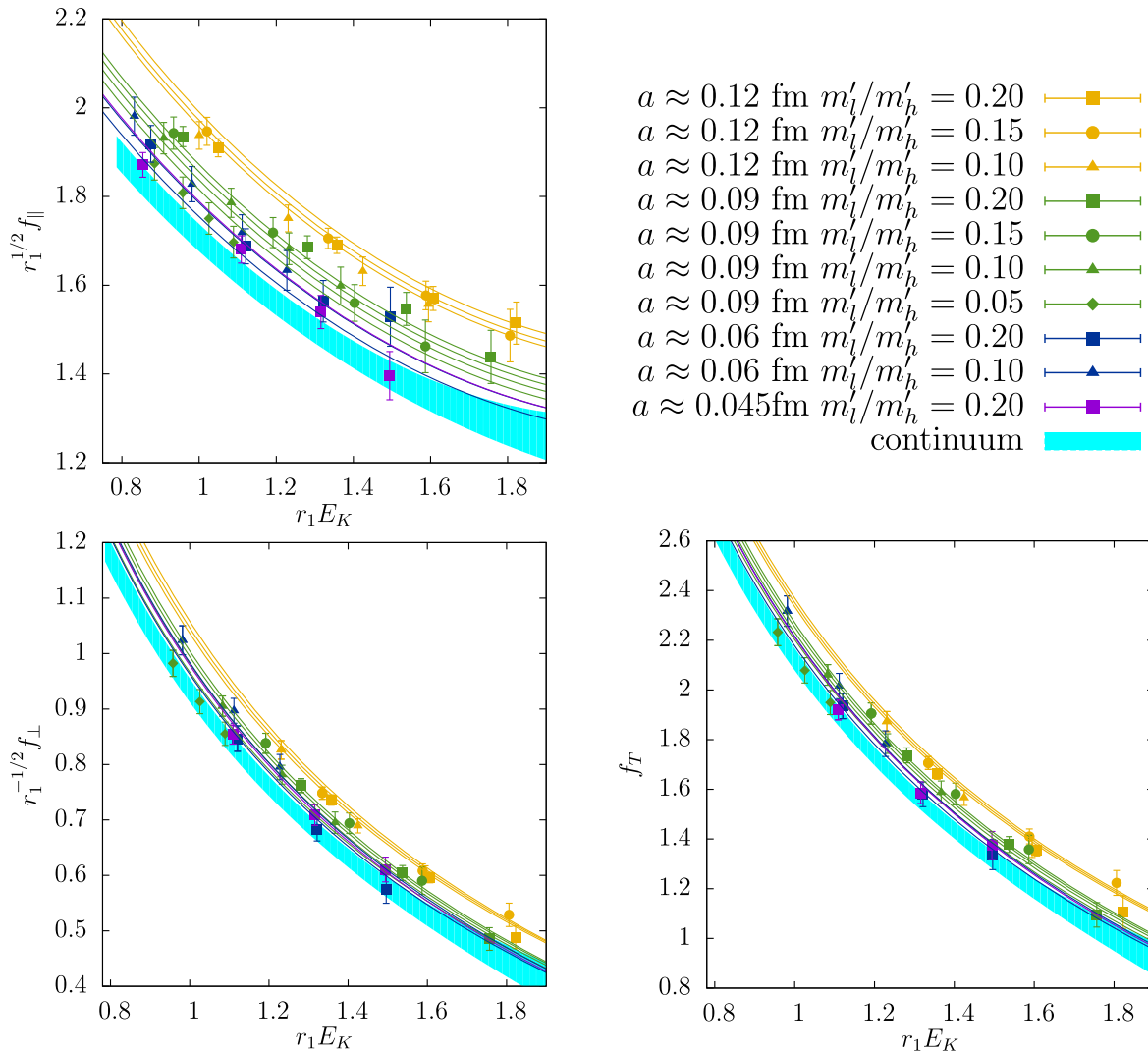


FIG. 8. Chiral-continuum extrapolations of  $f_{\parallel}$  (upper left),  $f_{\perp}$  (lower left), and  $f_T$  (lower right) using NLO SU(2) HMR $\chi$ PT plus NNLO analytical terms. The squares, circles, triangles, and diamonds denote the  $m_l'/m_h' = 0.2, 0.14, 0.1, \text{ and } 0.05$  data, respectively. The colored fit lines correspond to the different lattice spacings as indicated in the legend. The cyan band shows the continuum extrapolated curve with statistical error, which includes the systematic uncertainties due to  $g_{\pi}$ , and the heavy-quark, light-quark, and gluon discretization errors. Fit lines should pass through the data points of the corresponding color.

and 0.47(3), respectively. At this stage, only the statistical,  $g_{\pi}$ , chiral truncation, and discretization errors have been included. In the next section, we estimate the size of the remaining uncertainties before employing the  $z$  expansion in Sec. V to extend our results over the full kinematic range.

#### IV. FORM-FACTOR ERROR BUDGET

In this section, we estimate the systematic errors in the form factors, discussing each source of uncertainty in a separate subsection. We first discuss the error from the chiral-continuum extrapolation, which also includes heavy-quark, light-quark, and gluon discretization errors. We then discuss the remaining systematic uncertainties from the heavy-light current renormalization, lattice-scale

determination, light- and strange-quark mass determinations, finite-volume effects, and  $b$ -quark mass determination, discussing each in a separate subsection. As discussed previously, the systematic errors from  $g_{\pi}$  and heavy- and light-quark discretization effects are included in the statistical errors of the chiral-continuum extrapolation result through the constrained fit. Finally, we visually summarize the error budgets for the three form factors as a function of  $q^2$  in Fig. 11.

##### A. Chiral-continuum extrapolation

We use NLO SU(2) HMR $\chi$ PT supplemented by all possible NNLO analytic terms, as well as heavy-quark, light-quark, and gluon discretization terms, in our preferred chiral extrapolations of  $f_{\perp}$ ,  $f_{\parallel}$ , and  $f_T$ .

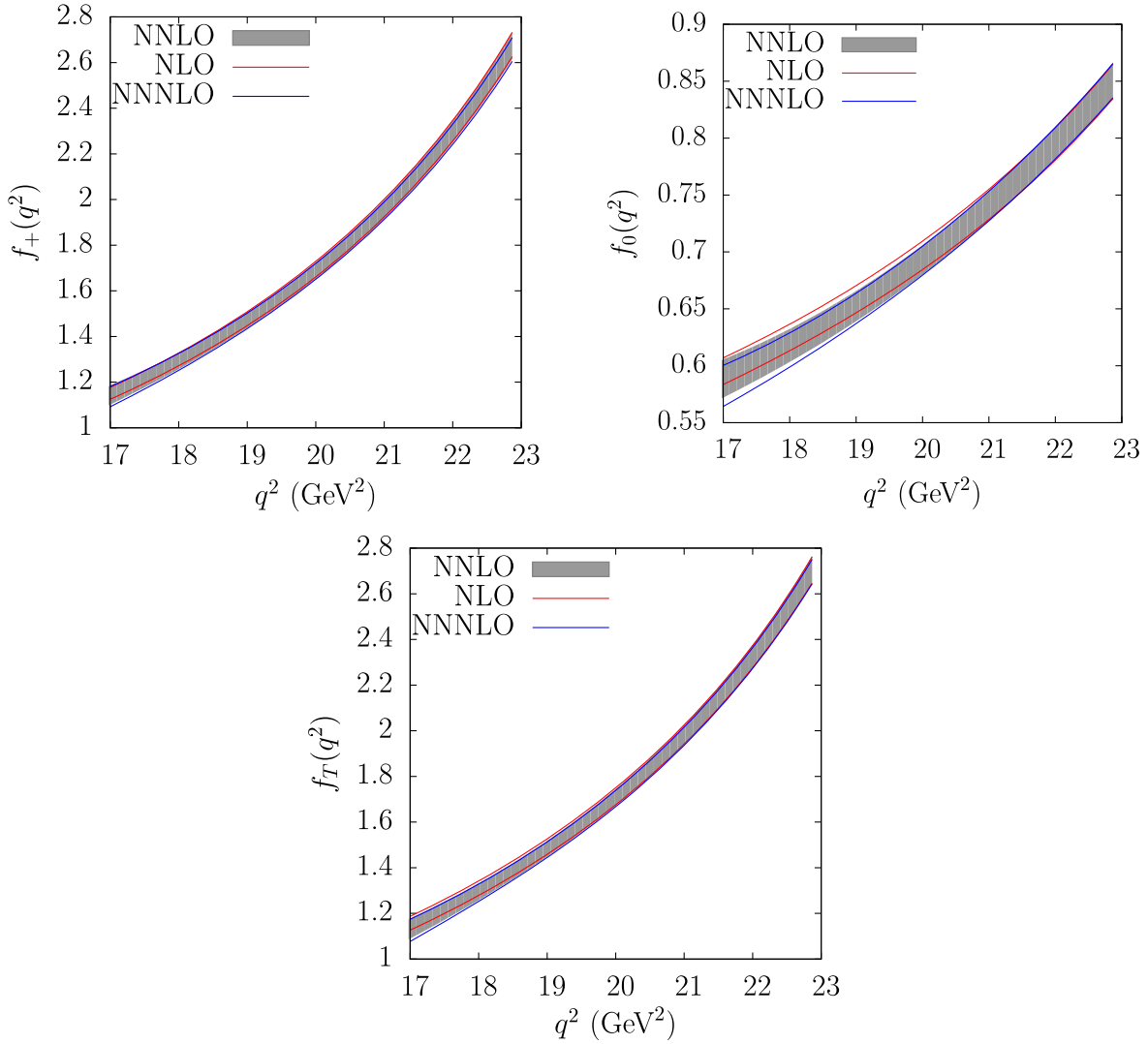


FIG. 9. Chiral-continuum extrapolations with NLO, NNLO, or NNNLO analytic terms for  $f_+$  (upper left),  $f_0$  (upper right), and  $f_T$  (lower panel). In each plot, the grey band shows the statistical error from the preferred NNLO SU(2)  $\chi$ PT. The red and blue lines show the error from the fits with NLO and NNNLO analytic terms, respectively.

First, to estimate truncation effects, we compare fit results using NLO HMrS $\chi$ PT, our preferred fit function with NNLO analytic terms, and the same fit function with the addition of the complete set of next-to-next-to-next-to-leading order (NNNLO) analytic terms in Fig. 9. We see that the errors in the preferred fit with NNLO analytic terms are already saturated, since they are the same as the errors in the fit with NNNLO analytic terms. Hence, truncation effects are included in the statistical fit errors from our preferred fit.

In addition, we also consider two alternative fit *Ansätze* for the chiral-continuum extrapolation. First, we consider NLO SU(3) hard-pion HMrS $\chi$ PT, which provides a good description of our data, although the standard NLO SU(3) expressions do not. We use the result from the SU(2)

HMrS $\chi$ PT fit as our preferred fit, because the SU(2) theory converges faster than the SU(3) theory as studied in Ref. [118]. We compare the fit results from NLO hard-kaon SU(3) HMrS $\chi$ PT plus NNLO analytical terms and our preferred fit, and find differences between the central values of about 1%–2% for all form factors and  $q^2$ . Second, we consider the effect of the  $E_K$  range of the lattice-QCD data to the extrapolated continuum result by omitting the  $\mathbf{k} = 2\pi(1, 1, 1)/L$  data from our fit. We find the differences are below 1%–2%. Figure 10 summarizes the differences between the form factors obtained from the alternative chiral-continuum fits and the central results. Overall, the shifts of the continuum form-factor central values are within the quoted statistical errors of the preferred chiral fit that includes truncation effects.



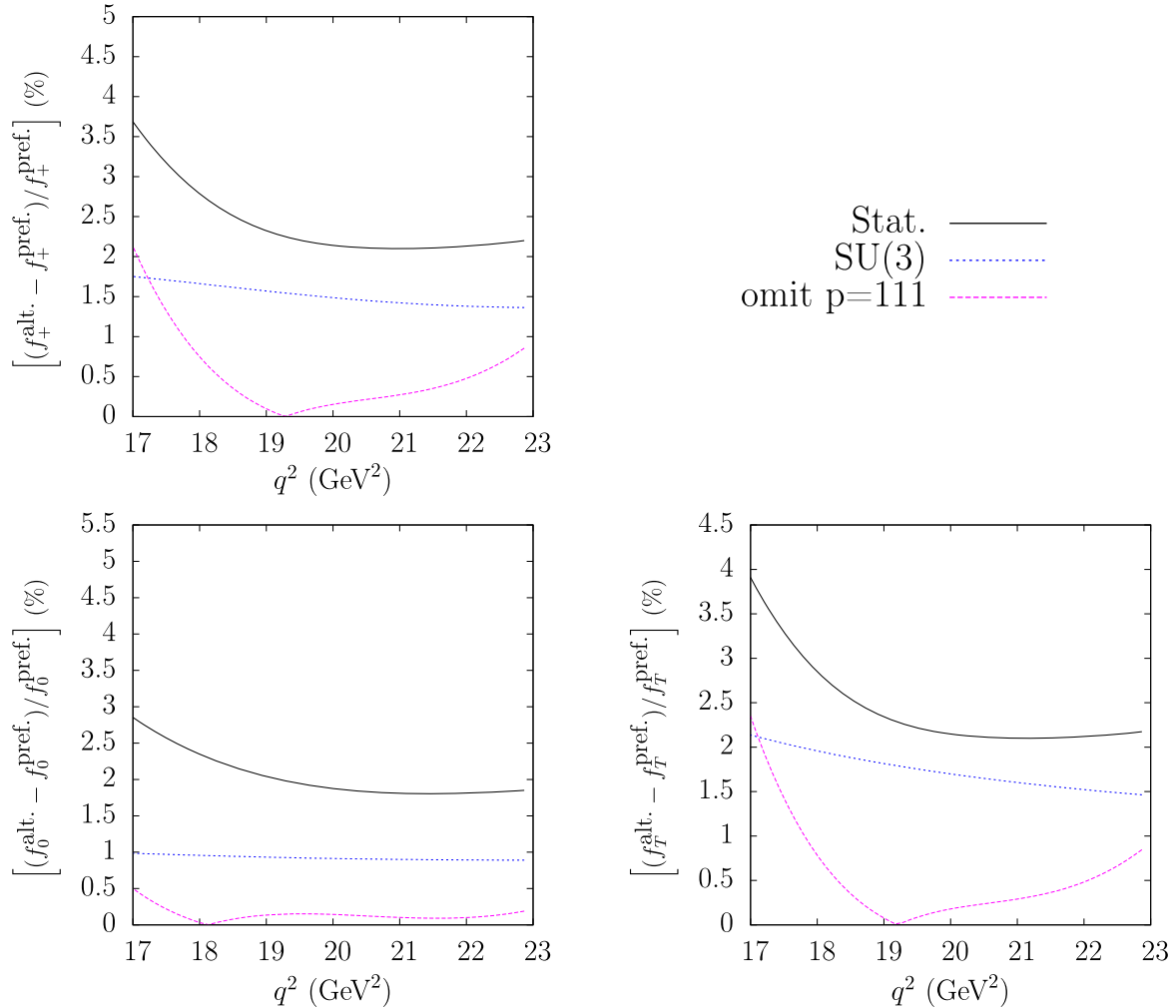


FIG. 10. Deviations of alternate chiral-continuum extrapolations from the central results for  $f_+$  (upper left),  $f_0$  (lower left), and  $f_T$  (lower right). In each plot, the black curve shows the statistical error from the preferred NNLO SU(2)  $\chi$ PT. The blue and pink lines show the % difference from the central fit obtained by using SU(3) hard-kaon  $\chi$ PT and omitting  $\mathbf{k} = 2\pi(1, 1, 1)/L$  data, respectively.

### B. Heavy-light current renormalization

To obtain the continuum form factors, we multiply the lattice form factors by the renormalization constant given in Eq. (2.17), using the values of  $\rho_J$ ,  $Z_{V^4}$ , and  $Z_{V_{ss}^4}$  listed in Tables IV–V. The statistical error on  $Z_{V_{ss}^4}^{1/2}$  is about 0.2%. By using the jackknife blocks of  $Z_{V_{bb}^4}$  calculated on the same ensembles, we incorporate the statistical error from  $Z_{V_{bb}^4}^{1/2}$  automatically in our fit results.

The  $\rho_J$  are calculated at one-loop order in perturbation theory. They are close to unity by design, since they are defined as ratios of renormalization factors. Indeed their one-loop corrections are small, as shown in Table V. We estimate the error due to truncating the perturbative expansion as  $2\rho_{J,\max}^{[1]}\alpha_s^2$  in order to avoid sensitivity due to accidental cancellations. We obtain  $\rho_{J,\max}^{[1]}$  as follows. For the scale-independent vector currents ( $V^i$  and  $V^4$ ), we

simply look for the largest value of the one-loop coefficients for both currents on all of the ensembles. We find that the spatial vector current has a larger one-loop coefficient with  $\rho_{V,\max}^{[1]} = 0.1$ . We evaluate  $\alpha_s$  at the  $a \approx 0.06$  fm lattice spacing (the next to finest), which yields an error of 1% for both components of the vector current. For the scale-dependent tensor current, the perturbative corrections include logarithmic contributions due to their anomalous dimension, which are responsible for the growth of  $\rho_T$  towards smaller lattice spacings seen in Table V. In order to estimate the truncation error, we remove the effect of the anomalous dimension by setting  $\mu = 2/a$ . We find that  $\rho_{T,\max}^{[1]} = 0.2$ , which corresponds to a truncation error of 2% on  $\rho_T$ . In summary, we assign a perturbative truncation error of 1% on  $f_+$ , 0 and an error of 2% on  $f_T$ .

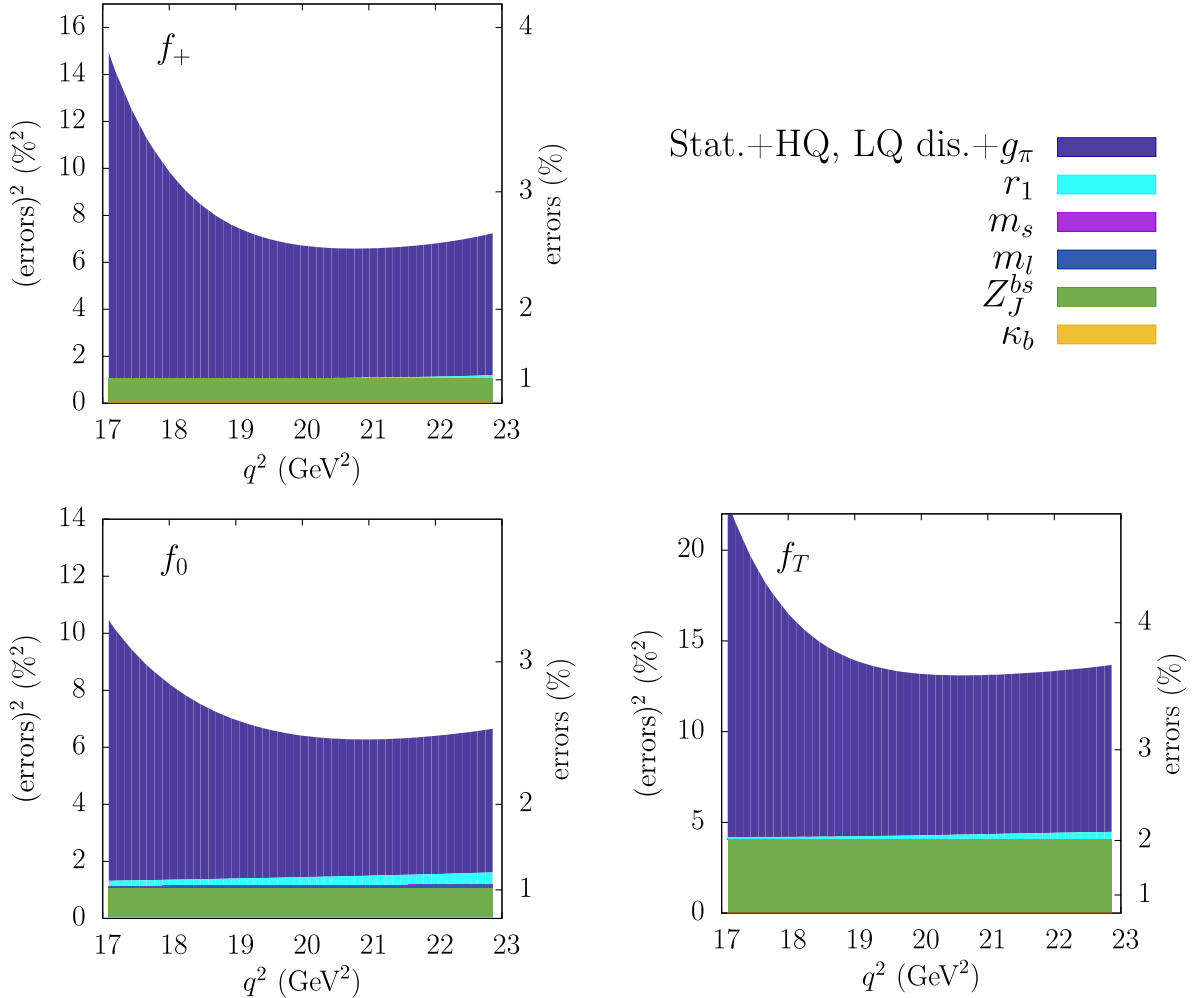


FIG. 11. Statistical and systematic error contributions to  $f_+$  (upper left),  $f_0$  (lower left), and  $f_T$  (lower right). The left vertical axis label shows the squares of the errors added in quadrature, while the right vertical axis label shows the errors themselves. The filled, stacked curves from bottom to top show the total error when we add each individual source of error in quadrature one by one.

### C. Scale uncertainty

We use  $r_1 = 0.3117(22)$  fm in the continuum from Ref. [89] to convert lattice quantities to physical units, where the quoted error includes both statistics and systematics. We repeat our analysis varying  $r_1$  by plus and minus one standard deviation from its central value and use the larger change of each form factor as an estimate of the systematic error due to the scale uncertainty. We find differences of less than 1% for  $f_{\parallel}$ ,  $f_{\perp}$ , and  $f_T$  throughout the simulated  $q^2$  region.

### D. Light- and strange-quark mass uncertainties

After the chiral-continuum fit, we evaluate the form factors at the physical quark masses  $r_1 \hat{m} = 0.000965(33)$  and  $r_1 m_s = 0.0265(8)$  determined from the analysis of the light pseudoscalar meson spectrum [4,52]. We vary the quark masses by plus and minus one standard deviation and find the differences in all three form factors due to

changing  $m_l$  and  $m_s$  to be below 0.6% in the simulated  $q^2$  region.

### E. Finite-volume effects

The lattices used in this work have finite spatial volumes with  $M_{\pi} L \gtrsim 4$ . We estimate the size of finite-volume effects using HMrS $\chi$ PT. In chiral perturbation theory, finite-volume contributions change loop-momentum integrals to sums which have been calculated in Refs. [106,119]. We employ continuum integrals in the preferred chiral-continuum extrapolations. To estimate the size of finite-volume effects, we evaluate the form factors with the LECs we obtain from the preferred chiral fits, and compare the results from the infinite-volume formulas and the finite-volume formulas on all ensembles used in this work. We try both SU(2) HMrS $\chi$ PT and SU(3) hard-kaon HMrS $\chi$ PT. We find that in all cases finite-volume effects are below 0.001%. Therefore, we neglect finite-volume effects in the total error budget.

### F. $b$ -quark mass correction

We correct the form factors from the simulated  $\kappa'_b$  to the physical  $\kappa_b$  before we perform the chiral-continuum extrapolation. Including these corrections accounts for the dominant effect from  $b$ -quark mistuning, but small errors in the form factors remain due to the uncertainties in the  $\kappa_b$ -correction factors. The statistical errors in the slopes  $\frac{\partial \ln f}{\partial \ln m_2}$  are at most about 10% for  $f_{\perp,T}$  at  $2\pi(1, 1, 1)/L$ , while the sizes of the  $\kappa_b$  shifts applied to the data points are about 1%–2%. We therefore take the systematic error from the  $\kappa_b$  correction to be  $2\% \times 10\% = 0.2\%$ , which is conservative enough to accommodate the largest possible error in the shift.

### G. Summary of the systematic error budget

Figure 11 visually summarizes the results for the statistical and systematic errors. For all three form factors, the combined chiral-continuum extrapolation error is the largest source of systematic uncertainty. The total errors in the form factors  $f_+$ ,  $f_0$ , and  $f_T$  are below 5% for all  $q^2 > 17 \text{ GeV}^2$ , and are  $\sim 3\%$  near  $q_{\text{max}}^2$ . We quote numerical results for the form factors including all systematic errors over the entire  $q^2$  range in the following section, after the  $q^2$ -extrapolation to the full kinematic range using the  $z$  expansion.

### V. $z$ EXPANSION OF FORM FACTORS

The form factors obtained from the chiral-continuum fit are reliable for high momentum transfer,  $q^2 \gtrsim 17 \text{ GeV}^2$ . We only simulated kaons with momenta up to  $2\pi(1, 1, 1)/L$ , because, at higher momenta, the two- and three-point correlators become noisier and are subject to larger discretization errors. Further, the HMrS $\chi$ PT formalism used to take the continuum limit does not apply when  $E_K$  is too large. In particular, for  $E_K \gtrsim 1.2 \text{ GeV}$  the expansion parameter  $\chi_E \gtrsim 1$ , so the terms analytic in  $\chi_E^n$  increase with higher powers of  $n$ . Because of these limitations, a way to extend the form factors to high kaon energy, or, equivalently,  $q^2 = 0$ , is needed. In this paper, we follow Ref. [31] and map  $q^2$  to a new variable  $z$  such that  $|z| \leq 1$ . Constraints from unitarity, analyticity, and heavy-quark physics ensure that the expansion of the form factors in terms of  $z$  converges. Thus we can use the  $z$  expansion to obtain a model-independent parametrization of our form factors valid over the entire kinematic range. This technique is now standard for analyzing  $B \rightarrow \pi l\nu$  decays [6,120,121].

We first define the new variable  $z$  via the conformal mapping [31]

$$z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}, \quad (5.1)$$

where  $t_{\pm} = (M_B \pm M_K)^2$  and  $t_0$  is a free parameter that can be chosen to minimize  $|z|$  for the semileptonic-decay region. In this work, we use  $t_0 = (M_B + M_K)(\sqrt{M_B} - \sqrt{M_K})^2$

[33], which maps the physical semileptonic-decay region  $0 \leq q^2 \leq 22.8 \text{ GeV}^2$  to  $|z| < 0.15$ . The small range of  $|z|$  helps control the truncation error in the  $z$  expansion.

Using the new variable  $z$ , we expand the form factors as [33]

$$f_+(q^2) = \frac{1}{P_+(q^2)} \sum_{m=0}^{K-1} b_m^+ \left[ z^m - (-1)^{m-K} \frac{m}{K} z^K \right], \quad (5.2)$$

$$f_0(q^2) = \frac{1}{P_0(q^2)} \sum_{m=0}^{K-1} b_m^0 z^m, \quad (5.3)$$

$$f_T(q^2) = \frac{1}{P_T(q^2)} \sum_{m=0}^{K-1} b_m^T \left[ z^m - (-1)^{m-K} \frac{m}{K} z^K \right]. \quad (5.4)$$

The function  $P_{+,0,T}(q^2) = 1 - q^2/M^2$  accounts for poles below and near the  $B$ - $K$  production threshold. For the  $z$  fits of  $f_+$  and  $f_T$ , we fix the location of the vector  $B_s^*$  pole to the measured value  $M_{B_s^*} = 5.4154 \text{ GeV}$  [6]. For the  $f_0$  fit, we fix the location of the scalar  $B_{s,0}^*$  pole to the lattice-QCD prediction  $M_{B_{s,0}^*} = 5.711 \text{ GeV}$  from Ref. [112]. We find that varying its location by three times the quoted theoretical error ( $\pm 69 \text{ MeV}$ ) does not change the extrapolated form factor.

The expression for  $f_+$  in Eq. (5.2) was derived by Bourrely, Caprini and Lellouch in Ref. [33], and is commonly called the BCL parametrization. In the BCL expression for  $f_+$  in Eq. (5.2), the coefficient of the term proportional to  $z^K$  is related to that of the lower-order terms. This constraint is due to the conservation of momentum and the analyticity of the form factors [33]. There is no analogous constraint for  $f_0$ . We use the same expression for  $f_T$  as for  $f_+$  because they are proportional to each other at leading order in the heavy-quark expansion. These expressions were also used to analyze the lattice form factors for  $B \rightarrow Kl^+l^-$  in Refs. [21,22].

Unitarity constrains the coefficients of the  $z$  expansion such that

TABLE X. Lowest-order coefficients  $B_{mn}$  for  $B \rightarrow Kl^+l^-$  decay using  $M_B = 5.27958 \text{ GeV}$ ,  $M_K = 0.497614 \text{ GeV}$ , and  $t_0 = (M_B + M_K)(\sqrt{M_B} - \sqrt{M_K})^2$ . The outer function used in the calculation is from Ref. [32] with  $\chi_{f_+} = 5.025 \times 10^{-4}$  and  $\chi_{f_0} = 1.4575 \times 10^{-2}$ . Although these  $\chi_i$ s are derived for the  $B \rightarrow \pi l\nu$  process, the calculation in Ref. [34] shows the difference between  $\chi_i$ s of the  $B \rightarrow Kl^+l^-$  and  $B \rightarrow \pi l\nu$  process is less than 10%. Therefore, we quote the inputs from Ref. [32] to obtain these  $B_{mn}$ . All  $B_{mn}$  not listed here can be obtained from the relations  $B_{m(m+n)} = B_{0n}$  and  $B_{mn} = B_{nm}$ .

	$B_{00}$	$B_{01}$	$B_{02}$	$B_{03}$	$B_{04}$	$B_{05}$
$f_{+,T}$	0.0161	-0.0003	-0.0104	0.0002	0.0022	0.0002
$f_0$	0.0921	0.0132	-0.0483	-0.0168	-0.0001	0.0024

$$\sum_{m,n=0}^{\infty} B_{mn} b_m b_n \lesssim 1, \quad (5.5)$$

where the values of  $B_{mn}$  are calculated using the Taylor expansion of the function discussed in Ref. [33] and given in Table X. We employ the same coefficients  $B_{mn}$  for  $f_T$  and  $f_+$ . The outer function  $\phi$  defined in Ref. [122] is used in the derivation of the  $B_{mn}$ . Although the  $\phi$  of  $f_0$  in Ref. [122] was derived without a scalar pole, its form is not altered by the presence of the pole, because  $|z|$  always equals 1 on the unit circle. In Ref. [122], Becher and Hill showed that, in the limit of large  $b$ -quark mass, the sizes of the  $z$  coefficients for  $f_+$  are even smaller than the expectation from (5.5). Heavy-quark effective theory provides an estimate of the sum [122]:

$$\sum B_{mn} b_m b_n = \frac{1}{\pi} \int_{t_+}^{\infty} \frac{dt}{t - t_0} \times \text{Im} \left( \sqrt{\frac{t_+ - t_0}{t_+ - t}} \right) |\phi_i(t) f_i(t)|^2, \quad (5.6)$$

where  $i = +, 0$ , or  $T$ , and the  $\phi$  is an outer function. To calculate the integral in Eq. (5.6), we need to know the form factors in the range  $[t_+, \infty]$ . For  $f_+$ , we assume that  $f_{\perp}$  gives the dominant contribution and has only the single  $B_s^*$  pole. Taking the limit  $M_B \rightarrow \infty$  gives the following simple form for  $f_+(q^2)$ :

$$f_+(q^2) \approx \frac{M_B}{\sqrt{2M_B}} f_{\perp}(E_K) \approx \frac{M_B}{\sqrt{2M_B}} \frac{C_{\perp}^{(0)} g_{\pi}}{f_{\pi}(E_K + \Delta_{B_s^*})}. \quad (5.7)$$

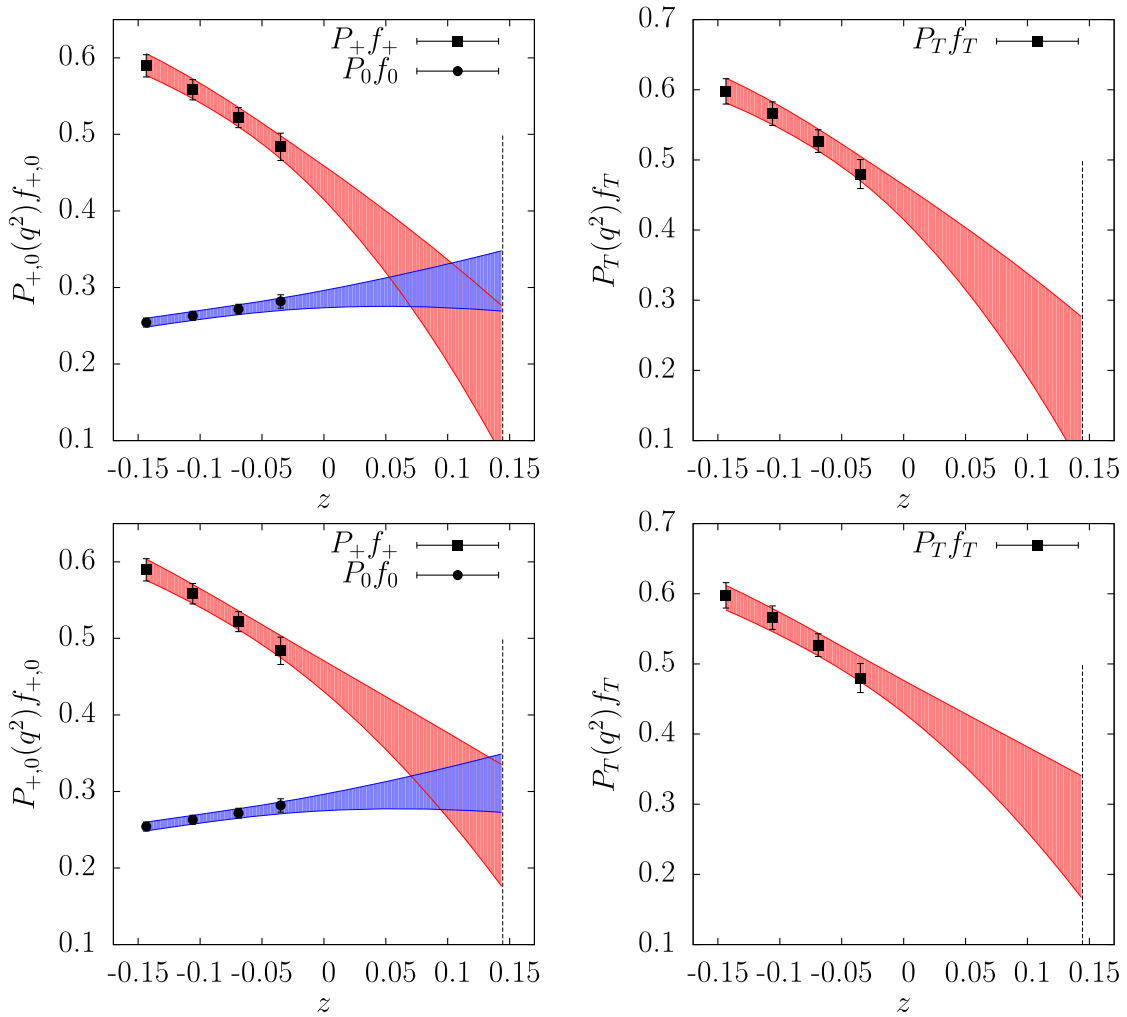


FIG. 12. Separate  $z$ -expansion fits of  $f_+$ ,  $f_0$  (left) and  $f_T$  (right) without (upper) and with (lower) HQ constraints on the sum of coefficients for  $f_+$  and  $f_T$ . The synthetic data points are generated at large  $q^2$  (small  $z$ ) in the region of simulated lattice momenta. The kinematic condition  $f_+(q^2 = 0) = f_0(q^2 = 0)$  is satisfied better when the HQ constraint is applied to  $f_+$ . (Recall that the factor  $P_{+,0} = 1$  at  $q^2 = 0$ .)

We then use our determination of  $C_{\perp}^{(0)}$  from our preferred chiral-continuum fit to obtain the estimate

$$\sum B_{mn}b_m b_n \approx 0.012. \quad (5.8)$$

This result means that Eq. (5.5) is only a loose bound for  $f_+$ . In addition, it is consistent with a power-counting estimate [122], which anticipates  $\sum B_{mn}b_m b_n$  to be of order  $(\Lambda/m_b)^3$ . The analogous calculation for  $f_T$  gives a similar result. The analysis below will show that the heavy quarks (HQ) constraint on  $f_+$  (and  $f_T$ ), Eq. (5.7), together with the kinematic constraint,  $f_0(0) = f_+(0)$ , suffices to keep the  $z$  fit under control.

We assume a log-normal distribution on  $\sum B_{mn}b_m b_n$  to ensure that  $\sum B_{mn}b_m b_n$  is always positive. The contribution from this prior to the augmented  $\chi^2$  is

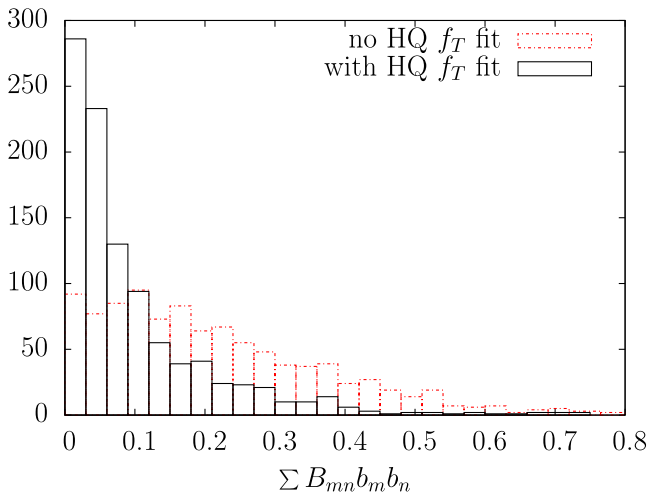
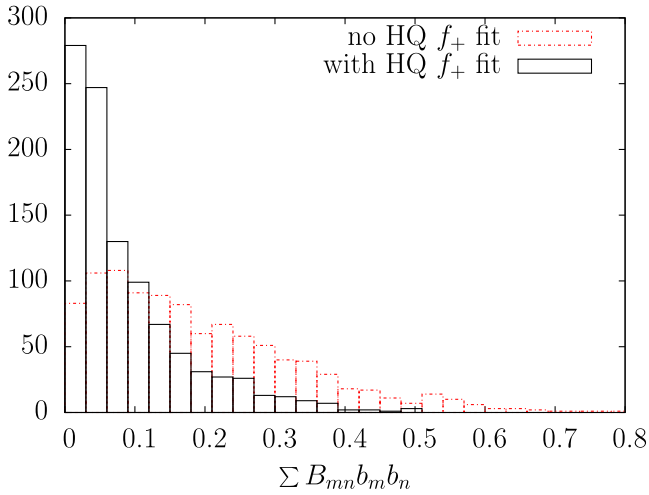


FIG. 13. Histogram of the sum of coefficients  $B_{mn}b_m b_n$  for  $f_+$  and  $f_T$  from fits with and without the HQ constraint. Use of the HQ constraint moves the distribution of  $B_{mn}b_m b_n$  to smaller values.

$$\chi_{B_{mn}b_m b_n}^2 = \frac{[\ln(\sum B_{mn}b_m b_n) - \mu]^2}{\sigma^2}, \quad (5.9)$$

where  $\mu$  is the central value and  $\sigma$  is the width of the prior. For  $f_+$  and  $f_T$ , we choose  $\mu$  and  $\sigma$  in Eq. (5.9) as  $\ln(0.02)$  and  $\ln(\frac{0.07}{0.02})$ . This choice is conservative enough to accommodate the uncertainties in the estimates.

We first generate from the continuum, physical quark-mass limit of the chiral extrapolation a few synthetic data points in the energy range of the simulated lattice data ( $q^2 \gtrsim 16.8 \text{ GeV}^2$ ). With the lattice spacing set to zero and the quark masses fixed to their physical values in Eqs. (3.20)–(3.21), the physical form factors depend upon at most six independent functions of the kaon energy  $E_K$ . These are proportional to  $1/(E_K + \Delta_{B_s^*})$ ,  $E_K^0$ ,  $E_K$ ,  $E_K^2$ ,  $E_K^3$ , and  $E_K^4$ . To the degree that the coefficients in front of these functions are correlated, the number of independent modes may be even fewer than six. If we generate too many synthetic data points, the covariance matrix will be singular. We therefore generate four synthetic data points each for  $f_+$ ,  $f_0$ , and  $f_T$  at  $q^2 = (22.86, 21.13, 19.17, 17.09) \text{ GeV}^2$ . These cover the simulated lattice-momentum range and are approximately evenly spaced in  $q^2$ . We also fit with synthetic data from a smaller and larger range and find consistent results.

The full covariance matrix of the synthetic data points includes both the statistical and systematic error:

$$C_{mn}^{\text{full}} = C_{mn}^{\text{stat}} + C_{mn}^{\text{syst}}, \quad (5.10)$$

where  $m, n$  denote the four  $q^2$  values. The systematic error contribution is calculated as

$$C_{mn}^{\text{syst}} = \sum_i \sigma_m^i \sigma_n^i \quad (5.11)$$

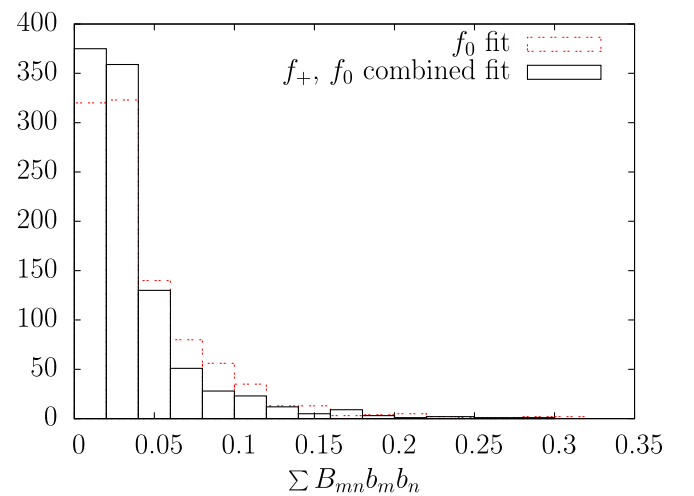


FIG. 14. Histogram of the sum of coefficients  $\sum B_{mn}b_m b_n$  for  $f_0$  from an independent fit and from a combined fit with  $f_+$  that imposes the kinematic constraint at  $q^2 = 0$ .

TABLE XI. Results of  $z$ -expansion fits of the  $B \rightarrow K$  form factors  $f_+$  (top panel),  $f_0$  (middle panel), and  $f_T$  (lower panel) using the formulas defined in Eqs. (5.2)–(5.4) with  $t_0 = (M_B + M_K)(\sqrt{M_B} - \sqrt{M_K})^2$  [33],  $M_{B_s^*} = 5.4154$  GeV in  $f_{+,T}$ ,  $M_{B_s^*} = 5.711$  GeV in  $f_0$ ,  $M_B = 5.27958$  GeV, and  $M_K = 0.497614$  GeV [6].

	Unconstrained		Constrained	
			HQ	HQ + kinematic
	$K = 3$	$K = 3$	$K = 3$	$K = 4$
$b_0^+$	0.437(22)	0.451(20)	0.466(14)	0.466(15)
$b_1^+$	-1.41(33)	-1.15(27)	-0.89(13)	-0.89(16)
$b_2^+$	-2.5(1.4)	-1.4(1.1)	-0.21(55)	-0.19(61)
$b_3^+$	...	...	...	0.3(1.1)
$\sum B_{mn} b_m b_n$	0.16	0.07	0.02	0.03
$f_+(0)$	0.18(10)	0.256(80)	0.335(36)	0.336(44)
$b_0^0$	0.285(11)	0.286(11)	0.292(10)	0.292(11)
$b_1^0$	0.19(14)	0.20(13)	0.28(12)	0.28(13)
$b_2^0$	-0.17(49)	-0.15(48)	0.15(44)	0.18(68)
$b_3^0$	...	...	...	0.2(1.7)
$\sum B_{mn} b_m b_n$	0.02	0.02	0.02	0.02
$f_0(0)$	0.309(39)	0.311(38)	0.335(36)	0.336(44)
$b_0^T$	0.440(25)	0.453(23)	0.460(19)	0.459(20)
$b_1^T$	-1.47(37)	-1.17(30)	-1.09(24)	-1.11(24)
$b_2^T$	-2.7(1.6)	-1.4(1.2)	-1.11(97)	-1.15(95)
$b_3^T$	...	...	...	-0.2(1.1)
$\sum B_{mn} b_m b_n$	0.18	0.07	0.05	0.05
$f_T(0)$	0.17(11)	0.254(87)	0.279(67)	0.276(68)
$p$ value	0.57	0.39	0.34	0.97

where the index  $i$  runs over the sources of systematic error discussed in Sec. IV. Because we assume that the systematic errors are 100% correlated between  $q^2$  values, all nontrivial correlations between points are due to statistical fluctuations of the chiral-continuum fit results.

TABLE XII. The coefficients  $b_i$  from the  $z$ -expansion fit (the first line) and their correlation matrix. The upper index +, 0, and  $T$  denote the form factors  $f_{+,0,T}$ . They are from the  $z$ -expansion fit formulas defined in Eqs. (5.2)–(5.4). We use  $t_0 = (M_B + M_K)(\sqrt{M_B} - \sqrt{M_K})^2$  [33],  $M_{B_s^*} = 5.4154$  GeV in  $f_{+,T}$ ,  $M_{B_s^*} = 5.711$  GeV in  $f_0$ ,  $M_B = 5.27958$  GeV and  $M_K = 0.497614$  GeV [6].

	$b_0^+$	$b_1^+$	$b_2^+$	$b_0^0$	$b_1^0$	$b_2^0$	$b_0^T$	$b_1^T$	$b_2^T$
Mean	0.466	-0.885	-0.213	0.292	0.281	0.150	0.460	-1.089	-1.114
Error	0.014	0.128	0.548	0.010	0.125	0.441	0.019	0.236	0.971
$b_0^+$	1	0.450	0.190	0.857	0.598	0.531	0.752	0.229	0.117
$b_1^+$		1	0.677	0.708	0.958	0.927	0.227	0.443	0.287
$b_2^+$			1	0.595	0.770	0.819	-0.023	0.070	0.196
$b_0^0$				1	0.830	0.766	0.582	0.237	0.192
$b_1^0$					1	0.973	0.324	0.372	0.272
$b_2^0$						1	0.268	0.332	0.269
$b_0^T$							1	0.590	0.515
$b_1^T$								1	0.897
$b_2^T$									1

We first fit  $f_+$ ,  $f_0$ , and  $f_T$  simultaneously in a combined fit using  $K = 3$  (three free parameters) in Eqs. (5.2)–(5.4) without any constraints on the coefficients. Table XI presents the results of these fits. We plot the fit results in Fig. 12. Although we do not impose the kinematic condition  $f_+(q^2 = 0) = f_0(q^2 = 0)$ , it is approximately satisfied with separate fits. Adding HQ constraints on the  $f_+$  and  $f_T$  fit makes the results even more consistent with the kinematic condition (see Fig. 12), and reduces the errors on  $f_+$ ,  $f_T$  at low  $q^2$ . We then fit  $f_+$ ,  $f_0$ , and  $f_T$  simultaneously with the kinematic constraint, still including the HQ constraints on  $f_+$  and  $f_T$ , which further decreases the extrapolation error in the form factors at low  $q^2$ . We implement the kinematic constraint by setting a prior of  $f_+ - f_0$  at  $q^2 = 0$  with central value zero and width of 0.00001.

We show the  $\sum B_{mn} b_m b_n$  bootstrap distribution of  $f_+$  and  $f_T$  from two fits with and without the HQ constraint in Fig. 13. Adding the HQ constraint moves the distribution of  $\sum B_{mn} b_m b_n$  to smaller values. We also compare the  $\sum B_{mn} b_m b_n$  distribution of  $f_0$  from two fits in Fig. 14. One is a fit with  $f_0$  only; the other is a combined  $f_+$  and  $f_0$  fit with the kinematic constraint. Adding the kinematic constraint decreases  $\sum B_{mn} b_m b_n$  from the separate  $f_0$  fit. Again, the result shows that the unitary constraint on the  $\sum B_{mn} b_m b_n$  of  $f_0$  is a loose bound.

We also check the truncation error by repeating the fit with  $K = 4$ . Because in  $K = 3$  fits, the coefficients  $b_2^i$  are not well determined by data, and the results are zero within error, we add a prior of 0(2) on  $b_4^i$  coefficients as in Ref. [21] to control the fluctuations of the higher-order terms. All of the coefficients  $b$  from fits with  $K = 3$  and 4 are summarized in Table XI. The results from different  $K$  are consistent with each other. The coefficients  $b_{i,3}$  are zero within error and have little impact on the central value of the final result. We therefore conclude that the  $z$  truncation error is well controlled.

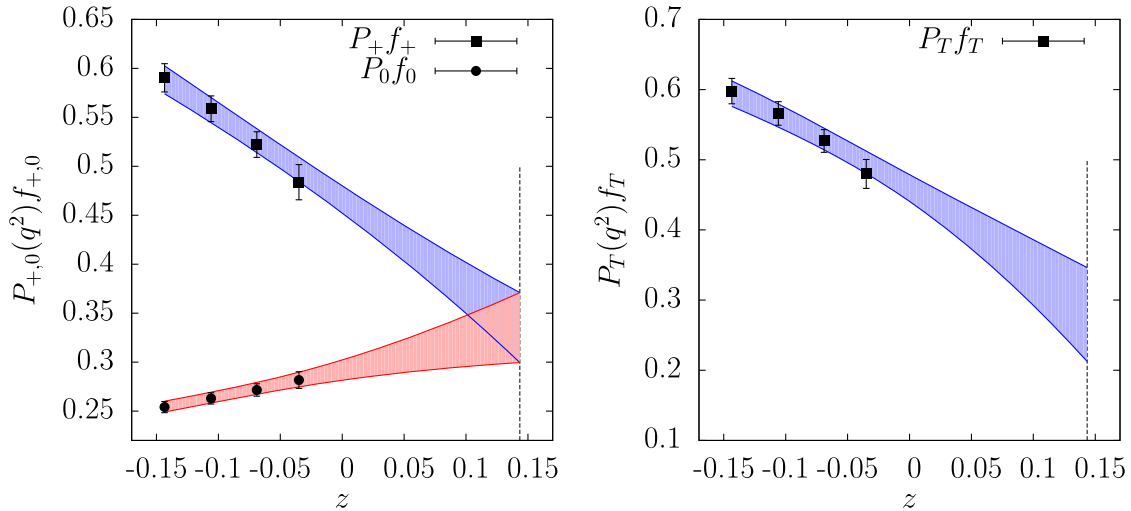


FIG. 15.  $f_+$ ,  $f_0$ , and  $f_T$   $z$ -expansion fits. The synthetic data points are generated at large  $q^2$  (small  $z$ ) from LECs of the HMrS $\chi$ PT fit result. The kinematic constraint  $f_+(q^2 = 0) = f_0(q^2 = 0)$  is applied exactly in the combined  $f_+$  and  $f_0$   $z$ -expansion fit. The vertical dashed lines correspond to  $q^2 = 0$ . We use three coefficients [ $K = 3$  in Eqs. (5.2)–(5.4)] for  $f_+$ ,  $f_0$ , and  $f_T$ .

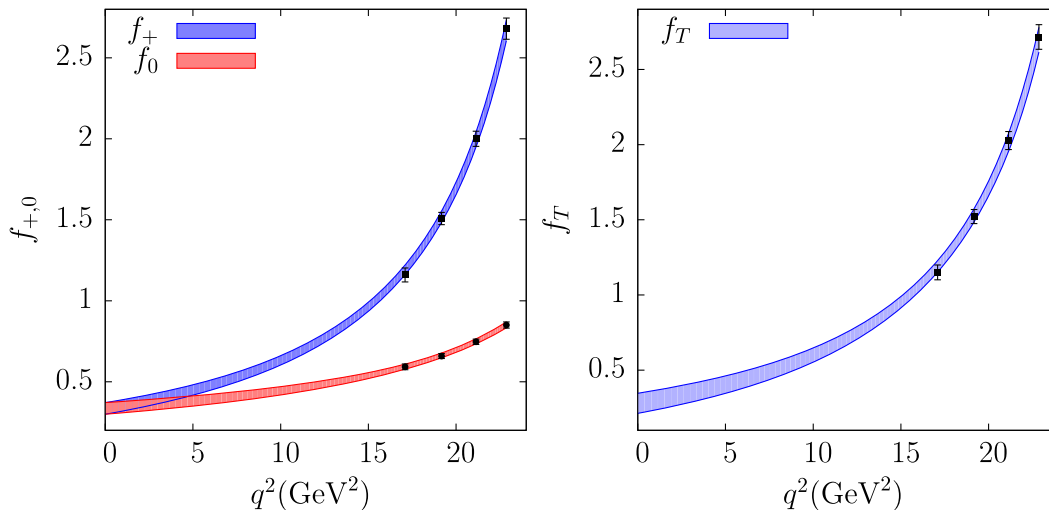


FIG. 16.  $f_+$ ,  $f_0$ , and  $f_T$  vs  $q^2$  based on the  $z$  expansion. The kinematic constraint  $f_+(q^2 = 0) = f_0(q^2 = 0)$  is applied exactly in the fit. We use three coefficients [ $K = 3$  in Eqs. (5.2)–(5.4)] for  $f_+$ ,  $f_0$ , and  $f_T$ .

We record our final, preferred results from  $K = 3$   $z$  fits including both the heavy-quark and kinematic constraints in the third column of Table XI, and we give the corresponding correlation matrix in Table XII. Together with the pole masses (also in Table XI) and Eqs. (5.2)–(5.4), this information allows the reader to reconstruct our form-factor results throughout the full kinematic range. Our final form-factor results as a function of  $z$  and  $q^2$  are plotted in Figs. 15–16.

## VI. TESTS OF QCD PREDICTIONS FOR FORM-FACTOR RATIOS

Because lattice-QCD calculations of the  $B \rightarrow K$  semi-leptonic form factors have until recently been unavailable, theoretical calculations of  $B \rightarrow Kl^+l^-$  observables sometimes use expectations from heavy-quark symmetries to

relate them to others that can be constrained from experiment or computed with QCD models (see, e.g., Ref. [123]). Heavy-quark symmetry is also commonly used in phenomenological calculations of the related decays  $B \rightarrow \pi l^+l^-$ ,  $B \rightarrow K^*l^+l^-$ , and  $B \rightarrow K^*\gamma$  [39,123–129]. Here we use our lattice-QCD form factors to directly test these heavy-quark symmetry relations in  $B \rightarrow K$  decay at both high and low  $q^2$ .

### A. Low-recoil predictions from heavy-quark symmetry

In the soft-kaon ( $E_K \ll M_B$ ) and chiral limits, the vector and scalar form factors can be related using heavy-quark effective theory and chiral perturbation theory [130,131]:

$$\lim_{q^2 \rightarrow M_B^2} \frac{f_0}{f_+} = \left( \frac{f_{B_s}}{f_{B_s^*}} \right) \frac{1 - q^2/M_{B_s^*}^2}{g_\pi} + \mathcal{O}(\Lambda^2/m_b^2), \quad (6.1)$$

where the decay-constant ratio accounts for heavy-quark corrections of  $\mathcal{O}(1/m_b)$ . Heavy-quark spin symmetry relates the vector and tensor form factors in the soft-kaon limit as [123,132]:

$$\lim_{q^2 \rightarrow M_B^2} \frac{f_T}{f_+}(q^2, \mu) = \kappa(\mu) \frac{M_B(M_B + M_K)}{q^2} + \mathcal{O}(\Lambda/m_b), \quad (6.2)$$

where the scale-dependent coefficient  $\kappa(\mu)$  incorporates corrections of  $\mathcal{O}(\alpha_s^2)$  to the leading Isgur-Wise relation [133] and is given in Eq. (2.5) of Ref. [123]. We can estimate the size of higher-order corrections in the heavy-quark expansion from power counting. Taking  $\Lambda = 500$  MeV and  $m_b = 4.2$  GeV gives  $\Lambda/m_b \sim 12\%$  and  $(\Lambda/m_b)^2 \sim 1\%$ . Equations (6.1)–(6.2) also receive corrections from the kaon recoil energy that are of  $\mathcal{O}(E_K/m_b)$ . For  $q_{\max}^2 \geq q^2 \geq 14$  GeV<sup>2</sup>, this ratio varies from  $12\% \leq E_K/m_b \leq 40\%$ , so such corrections are expected to be significant even at low kaon recoil.

Figure 17, left, compares the quantity  $(f_0/f_+) \times (1 - q^2/M_{B_s^*}^2)^{-1}$  obtained from our lattice form factors with the theoretical prediction Eq. (6.1). For the theoretical estimate, we take  $f_{B_s^*}/f_{B_s} = 0.953(23)$  from the recent four-flavor lattice-QCD determination in Ref. [134] and  $g_\pi = 0.45(8)$  as in our chiral-continuum fit. The width of the theoretical band

is from the uncertainty on  $g_\pi$ , and does not include any other errors. Figure 17, right, compares the quantity  $(f_T/f_+) \times (q^2)/(M_B(M_B + M_K))$  obtained from our lattice form factors with the theoretical prediction Eq. (6.2) using  $m_b = 4.18$  GeV and  $\alpha_s^{(4)}(m_b) = 0.2268$ , such that  $\kappa(m_b) \approx 0.88$  [123,124]. We do not show any errors on the theoretical prediction.

The observed lattice form-factor ratios  $f_0/f_+$  and  $f_T/f_+$  at  $q_{\max}^2$  are lower than the theoretical expectations by 38% and 15%, respectively; by  $q^2 = 14.5$  GeV<sup>2</sup> the differences grow to 51% and 46%, respectively. Although the observed disagreement with the theoretical expectation for the tensor form-factor ratio is large, it is within the size expected (from simple power counting) for higher-order corrections due to the kaon recoil energy. The scalar form-factor ratio, however, differs from the theoretical expectation by a much larger amount. In Fig. 25 of Ref. [2] we compare the quantity  $(f_0/f_+) \times (1 - q^2/M_{B_s^*}^2)^{-1}$  for the related decay  $B \rightarrow \pi l \nu$  with the heavy-quark prediction in the soft-pion limit. The observed agreement near  $q_{\max}^2$  is better, which suggests that the discrepancy is indeed due to the light pseudo-scalar-meson recoil energy, which is larger for  $B \rightarrow K$  than for  $B \rightarrow \pi$ . Thus our lattice form-factor results suggest that one should be cautious in using heavy-quark relations derived in the soft-pion/kaon limit for phenomenological predictions, especially for decays with  $K$  or  $K^*$  final-state mesons.

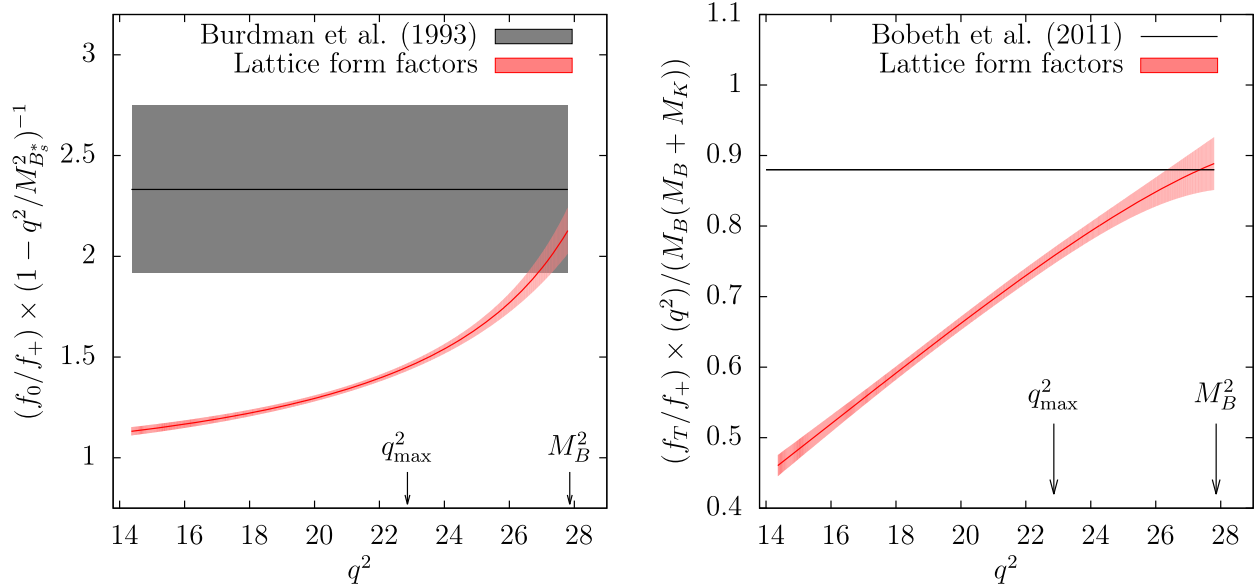


FIG. 17. Comparison of lattice form-factor ratios with theoretical predictions from heavy-quark symmetry at low recoil. Left:  $(f_0/f_+)/ (1 - q^2/M_{B_s^*}^2)^{-1}$  versus  $q^2$  from lattice QCD (red curve with error band) and heavy-quark symmetry plus  $\chi$ PT [130] (gray horizontal band). The width of the theoretical band includes the uncertainty on  $g_\pi = 0.45(8)$  but no other theory errors. Right:  $(f_T/f_+) \times (q^2)/(M_B(M_B + M_K))$  versus  $q^2$  from lattice QCD (red curve with error band) and the improved Isgur-Wise relation [123] (black horizontal line).



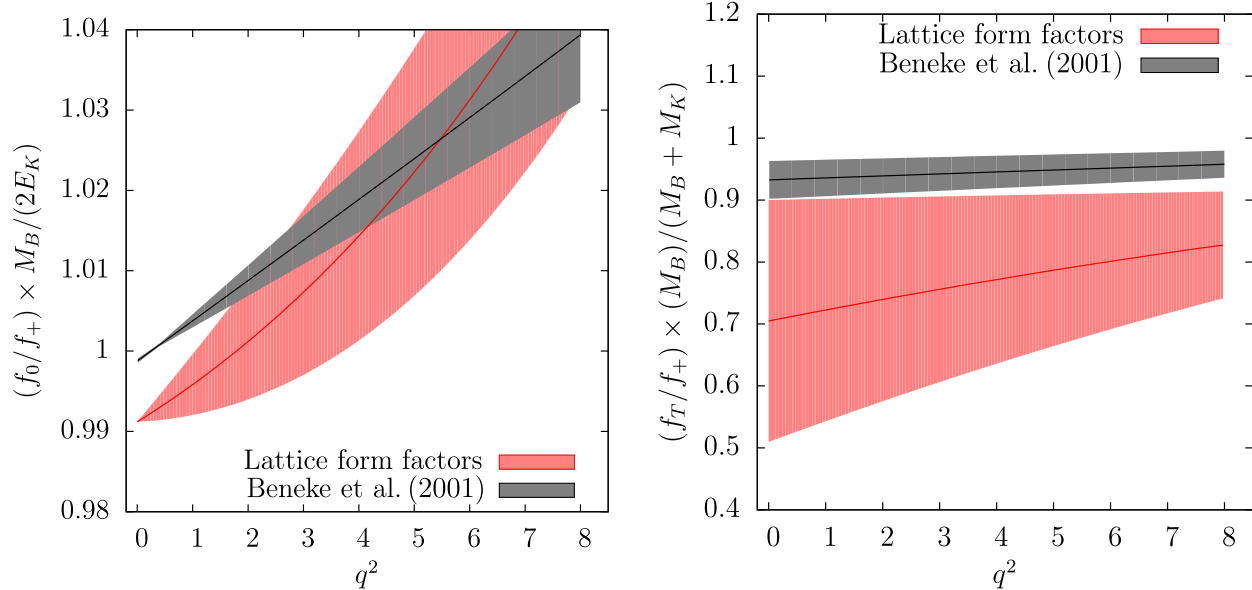


FIG. 18. Comparison of lattice form-factor ratios with theoretical predictions from heavy-quark symmetry at large recoil. Left:  $(f_0/f_+) \times M_B / (2E_K)$  versus  $q^2$  from lattice QCD (red curve with error band) and theoretical prediction with  $\mathcal{O}(\alpha_s)$  corrections [136] (gray curve with error band). Right:  $(f_T/f_+) \times (M_B) / (M_B + M_K)$  versus  $q^2$  from lattice QCD (red curve with error band) and theoretical prediction with  $\mathcal{O}(\alpha_s)$  corrections [136] (gray curve with error band).

### B. Large-recoil predictions from QCD factorization

In the large-recoil limit ( $E_K \gg M_K$ ), heavy-quark symmetry relates the vector, scalar, and tensor form factors to a single universal form factor [135]:

$$\lim_{E_K \gg M_K} \frac{f_0}{f_+} = \frac{2E_K}{M_B} + \mathcal{O}(\Lambda/m_b), \quad (6.3)$$

$$\lim_{E_K \gg M_K} \frac{f_T}{f_+} = \frac{M_B + M_K}{M_B} + \mathcal{O}(\Lambda/m_b). \quad (6.4)$$

The  $\mathcal{O}(\alpha_s)$  corrections to these leading large-recoil expressions were derived using QCD factorization (QCDF) in Ref. [136], and the resulting expressions are given in Eqs. (62)–(63) of that work. Higher-order corrections in the heavy-quark expansion are expected to be about  $\Lambda/m_b \sim 12\%$ , while  $\mathcal{O}(\alpha_s^2)$  corrections to the QCDF predictions from Ref. [136] are expected to be about 5%.

Figure 18 compares the lattice-form-factor ratios with the theoretical large-recoil predictions from Ref. [136]. For the  $\mathcal{O}(\alpha_s)$  corrections, we take the decay constants  $f_B = 190.5(4.2)$  MeV from FLAG [121] and  $f_K = 156.2(7)$  MeV from the PDG [6]. We take the first inverse moment of the  $B$ -meson distribution amplitude  $\lambda_B^{-1}(2.2 \text{ GeV}) = [0.51(12) \text{ GeV}]^{-1}$  from LCSR [137], where the quoted theory error covers the spread of other determinations from QCD/light-cone sum rules and the operator-product expansion [138–140]. We take the first and second moments of the kaon distribution amplitude

$a_1^K(2 \text{ GeV}) = 0.061(4)$  and  $a_2^K(2 \text{ GeV}) = 0.18(7)$  from a recent three-flavor lattice-QCD calculation [141]. We use our own determination of  $f_+(q^2 = 0) = 0.335(36)$ . We take  $\alpha_{s\overline{\text{MS}}}^{(4)}(m_b) = 0.2268$  as described above and  $\alpha_{s\overline{\text{MS}}}^{(4)}(2.2 \text{ GeV}) = 0.279$  [142]. The left panel of Fig. 18 shows the quantity  $(f_0/f_+) \times (M_B) / (2E_K)$ , while the right panel shows  $(f_T/f_+) \times (M_B) / (M_B + M_K)$ . The widths of the theoretical bands in Fig. 18 are from the uncertainty on  $\lambda_B^{-1}$  and  $f_+(q^2 = 0)$ , and do not include any other errors.

For  $(f_0/f_+) \times M_B / (2E_K)$ , the lattice-QCD result differs from the theoretical predictions by at most 1%, which is well within the expected size of heavy-quark corrections. For  $(f_T/f_+) \times (M_B) / (M_B + M_K)$ , the lattice-QCD result is marginally consistent with the theoretical expectation of Ref. [136]. A more recent NNLO calculation within soft-collinear effective theory updates the large-recoil predictions to include  $\mathcal{O}(\alpha_s^2)$  corrections [143]. The new  $q^2 = 0$  result for  $f_T/f_+ M_B / (M_B + M_K) = 0.817$  is in better agreement with the ratio obtained from lattice QCD. Overall, the uncertainty on the lattice-QCD tensor form-factor ratio at low  $q^2$  is too large to draw any quantitative conclusions. (The vector and tensor form factors are not strongly correlated at low  $q^2$ .) Thus, while the scalar form-factor ratio suggests that the large-recoil predictions may be reliable, some caution is nevertheless warranted in their use for phenomenology given the limited number of tests they have undergone.

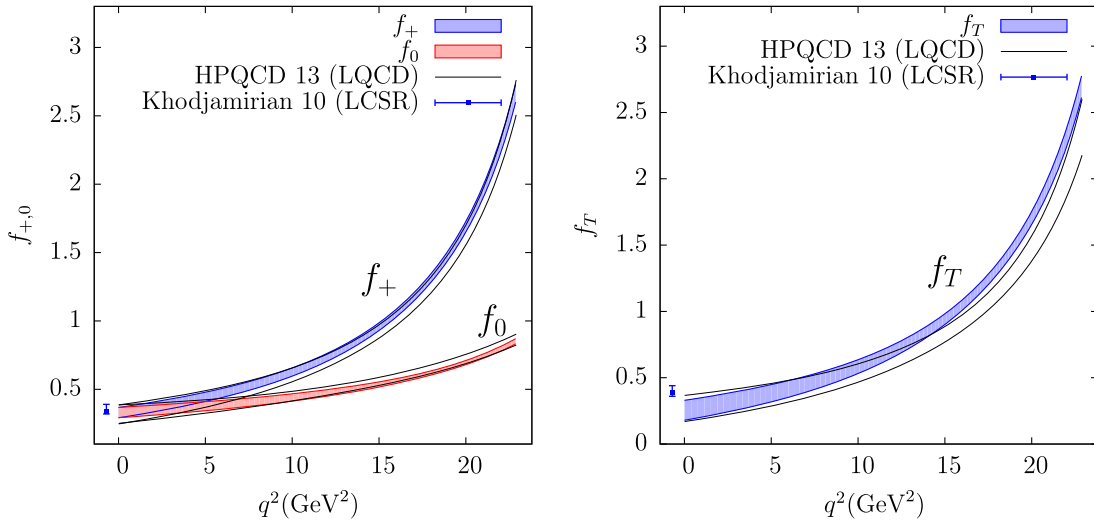


FIG. 19. Our form factors compared with light-cone-sum-rule results [144] and the other unquenched lattice-QCD calculation [21].

## VII. SUMMARY AND OUTLOOK

As discussed in Sec. V, Table XII presents our final results for the form factors  $f_+(q^2)$ ,  $f_0(q^2)$ , and  $f_T(q^2)$  for the semileptonic process  $B \rightarrow Kl^+l^-$ . These entries, which consist of the coefficients of the BCL  $z$  expansion, Eqs. (5.2)–(5.4), together with the correlations among them, can be used to reconstruct our form factors with errors for all values of  $0 \leq q^2 \leq q^2_{\max}$ . This information can also be used to compute form-factor ratios and (differential) rates with squares of linear combinations of the form factors.

Figure 19 shows a comparison of our results with others in the literature. At  $q^2 = 0$ , our result is consistent with a light-cone-sum-rule result from Khodjamirian *et al.* [144]. For all  $q^2$ , our results are consistent with the only other unquenched lattice-QCD calculation from the HPQCD Collaboration [21]. Our form factors are somewhat more precise than HPQCD's, especially at high  $q^2$ , because we used more ensembles with finer lattice spacings and lighter quark masses. The total errors, including both statistical and systematic errors, are less than 4% at high  $q^2$ , and at low  $q^2$  about 10% for  $f_+$  and 30% for  $f_T$ .

More generally, our results can be used to compute any  $B \rightarrow Kll$  observable, including asymmetries and decay rates, for all possible dilepton final states ( $l = \ell, \tau, \nu$ ), and even lepton-flavor-violating modes [145]. We present a thorough analysis of observables for  $B \rightarrow K$  semileptonic decays in a companion publication [146], where we also present ratios of observables for  $B \rightarrow Kll$  to  $B \rightarrow \pi ll$  decay processes. The three form factors  $f_+$ ,  $f_0$ , and  $f_T$  suffice to parametrize the factorizable hadronic contributions to  $B \rightarrow K$  semileptonic decays in any extension of the standard model. Other hadronic uncertainties, such as violations of quark-hadron duality due to intermediate charmonium resonances, must, of course, also be reliably estimated to obtain complete standard-model and new-physics predictions for  $B \rightarrow K$  processes. If deviations from the standard

model are observed in any  $B \rightarrow Kll$  decay channel, accurate results for the form factors will be essential to disentangling the underlying physics.

The main sources of uncertainty in our form factors are from the chiral-continuum extrapolation and extrapolation to low  $q^2$ . We plan to reduce these uncertainties with newer gauge-field ensembles that are being generated by the MILC Collaboration [147,148]. These ensembles use the highly improved staggered quark (HISQ) action for the light, strange, and charm quarks. This action is designed to have smaller discretization effects which will help reduce the size of the continuum extrapolation errors [149]. In addition, the HISQ ensembles include ensembles with physical pion masses, which will eliminate the need for the chiral extrapolation and the associated errors. Indeed, these ensembles have already been used to improve the precision for kaon [150] and charmed-meson [151] physics. In particular, we found with  $D$ - and  $D_s$ -meson decay constants [151] that an analysis with physical and unphysical quark masses provides better statistical precision with no penalty in systematic errors.

## ACKNOWLEDGMENTS

We thank Wolfgang Altmannshofer, Martin Beneke, and Enrico Lunghi for useful conversations about the form factors and phenomenology of  $b \rightarrow s$  processes. We thank Heechang Na for useful conversations about correlator fit techniques. We thank Richard J. Hill for clarifying discussions about the unitarity constraints in the  $z$  expansion. Computations for this work were carried out with resources provided by the USQCD Collaboration, the Argonne Leadership Computing Facility, the National Energy Research Scientific Computing Center, and the Los Alamos National Laboratory, which are funded by the Office of Science of the U.S. Department of Energy; and with resources provided by the National Institute for Computational Science, the Pittsburgh Supercomputer

Center, the San Diego Supercomputer Center, and the Texas Advanced Computing Center, which are funded through the National Science Foundation's Teragrid/XSEDE Program. This work was supported in part by the U.S. Department of Energy under Awards No. DE-FG02-91ER40628 (C. B. and J. K.), No. DE-FC02-12ER41879 (C. D., J. F., and L. L.), No. DE-FG02-91ER40661 (S. G. and R. Z.), No. DE-SC0010120 (S. G.), No. DE-FC02-06ER41443 (R. Z.), No. DE-FG02-91ER40677 (A. X. K., C. M. B., D. D., E. D. F., and R. D. J.), No. DE-FG02-13ER42001 (A. X. K. and D. D.), No. DE-FG02-ER41976 (D. T.), and No. DE-SC0010114 (Y. L. and Y. M.); by the National Science Foundation under Grants No. PHY-10034278, No. PHY10-67881 (C. D. and L. L.), No. PHY-1316748 (R. S.), No. PHY-1212389 (R. Z.), and No. PHY-1417805 (J. L. and D. D.); by the URA Visiting Scholars program, (Y. L., Y. M., A. X. K., D. D., and C. M. B.); by the MICINN (Spain) under Grant No. FPA2010-16696 and the Ramón y Cajal program (E. G.); by the Junta de Andalucía (Spain) under Grants No. FQM-101 and No. FQM-6552 (E. G.); by European Commission (EC) under Grant No. PCIG10-GA-2011-303781 (E. G.); by the German Excellence Initiative and the European Union Seventh Framework Programme under grant agreement No. 291763 as well as the European Union's Marie Curie COFUND program (A. S. K.); and by the Basic Science Research Program of the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grants No. 2014027937 and No. 2015024974) and the Creative Research Initiatives Program (Grants No. 2014001852 and No. 2015001776) of the NRF grant funded by the Korean government (MEST; J. A. B.). This manuscript has been coauthored by an

employee of Brookhaven Science Associates, LLC, under Contract No. DE-AC02-98CH10886 with the U.S. Department of Energy. Fermilab is operated by Fermi Research Alliance, LLC, under Contract No. DE-AC02-07CH11359 with the United States Department of Energy.

### APPENDIX: $B \rightarrow Kll$ FORM FACTORS IN SU(2) $S_\chi$ PT

We use expressions derived in HMr $S_\chi$ PT [106] as the low-energy effective theory of QCD in which the degrees of freedom are pions and kaons for our chiral-continuum extrapolations. SU(3) HMr $S_\chi$ PT [106,152,153] was applied to  $B \rightarrow \pi l\nu$  semileptonic decays [1]. More recently, SU(2) HMr $S_\chi$ PT [118,154,155] was also considered as an alternative effective theory in studies of heavy meson physics. We derive the SU(2) HMr $S_\chi$ PT formulas for form factors calculated with staggered quarks in this appendix. These formulas can be used for  $B \rightarrow \pi l\nu$ ,  $B \rightarrow Kl^+l^-$  and  $D$ -meson semileptonic decays. Our results are consistent with earlier studies of HMr $S_\chi$ PT for continuum QCD and Wilson quarks [155] after taking the continuum limit of the HMr $S_\chi$ PT expressions. The differences in the detailed expressions can be absorbed into redefinitions of the scale or LECs.

#### 1. $f_{\parallel}$ and $f_{\perp}$ in SU(3) HMr $S_\chi$ PT

The SU(3) HMr $S_\chi$ PT expression of  $f_{\parallel}$  is the same as in Eq. (3.20). We only list the expression of the log terms for unitary points ( $m_{l,h} = m'_{l,h} = m_{l,s}$ ) here.

For the  $B \rightarrow \pi$  process, the chiral logs in SU(3) HMr $S_\chi$ PT are given by [106]:

$$\begin{aligned} \log S_{\parallel, \text{SU}(3)}^{B \rightarrow \pi} = & \frac{1}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} \left[ \frac{1-3g_\pi^2}{2} [2I_1(m_{\pi, \Xi}) + I_1(m_{K, \Xi})] + 2I_2(m_{\pi, \Xi}, E) + I_2(m_{K, \Xi}, E) \right] + \frac{1+3g_\pi^2}{4} \left[ I_1(m_{\pi, l}) - \frac{1}{3} I_1(m_{\eta, l}) \right] \right. \\ & \left. + \sum_{j \in \{\pi, \eta, \eta'\}} \left[ a^2 \delta'_V R_j^{[3,1]}(\{m_{\pi, V}, m_{\eta, V}, m_{\eta', V}\}; \{m_{S, V}\}) \left( \frac{3(g_\pi^2 - 1)}{2} I_1(m_{j, V}) - 2I_2(m_{j, V}, E) \right) \right] + [V \rightarrow A] \right\}. \quad (\text{A1}) \end{aligned}$$

For the  $B \rightarrow K$  process, the chiral logs in SU(3) HMr $S_\chi$ PT are given by

$$\begin{aligned} \log S_{\parallel, \text{SU}(3)}^{B \rightarrow K} = & \frac{1}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} \left[ \frac{2-3g_\pi^2}{2} I_1(m_{K, \Xi}) - 3g_\pi^2 I_1(m_{\pi, \Xi}) + \frac{1}{2} I_1(m_{S, \Xi}) + 2I_2(m_{K, \Xi}, E) + I_2(m_{S, \Xi}, E) \right] - \frac{1}{2} I_1(m_{S, l}) \right. \\ & + \frac{3g_\pi^2}{4} I_1(m_{\pi, l}) + \frac{8-3g_\pi^2}{12} I_1(m_{\eta, l}) + I_2(m_{\eta, l}, E) - I_2(m_{S, l}, E) \\ & + a^2 \delta'_V \left[ \frac{I_1(m_{\eta', V}) - I_1(m_{\eta, V}) + I_2(m_{\eta', V}, E) - I_2(m_{\eta, V}, E)}{m_{\eta', V}^2 - m_{\eta, V}^2} - \sum_{j \in \{S, \eta, \eta'\}} R_j^{[3,1]}(\{m_{S, V}, m_{\eta, V}, m_{\eta', V}\}; \{m_{\pi, V}\}) \right. \\ & \left. \times \left( \frac{1}{2} I_1(m_{j, V}) + I_2(m_{j, V}, E) \right) + \frac{3g_\pi^2}{2} \sum_{j \in \{\pi, \eta, \eta'\}} R_j^{[3,1]}(\{m_{\pi, V}, m_{\eta, V}, m_{\eta', V}\}; \{m_{S, V}\}) I_1(m_{j, V}) \right] + [V \rightarrow A] \right\}. \quad (\text{A2}) \end{aligned}$$

The masses and integrals that appear in Eqs. (A1)–(A2) are as follows. The flavor off-diagonal meson masses are

$$m_{\pi,\Xi}^2 = \mu(m_l + m_l) + a^2\Delta_\Xi, \quad (\text{A3})$$

$$m_{K,\Xi}^2 = \mu(m_l + m_s) + a^2\Delta_\Xi, \quad (\text{A4})$$

$$m_{S,\Xi}^2 = \mu(m_s + m_s) + a^2\Delta_\Xi, \quad (\text{A5})$$

where  $m_l$  and  $m_s$  are sea-quark masses and the taste label  $\Xi$  has values  $P$ ,  $V$ ,  $T$ ,  $A$  and  $I$ . The masses of flavor-neutral mesons in the taste vector channel are [156]

$$m_{\pi^0,V}^2 = m_{U,V}^2 = m_{D,V}^2 = \mu(m_l + m_l) + a^2\Delta_V, \quad (\text{A6})$$

$$m_{S,V}^2 = \mu(m_s + m_s) + a^2\Delta_V, \quad (\text{A7})$$

$$m_{\eta,V}^2 = \frac{1}{2} \left( m_{U,V}^2 + m_{S,V}^2 + \frac{3}{4}a^2\delta'_V - Z \right), \quad (\text{A8})$$

$$m_{\eta',V}^2 = \frac{1}{2} \left( m_{U,V}^2 + m_{S,V}^2 + \frac{3}{4}a^2\delta'_V + Z \right), \quad (\text{A9})$$

$$Z \equiv \sqrt{(m_{S,V}^2 - m_{U,V}^2)^2 - \frac{a^2\delta'_V}{2}(m_{S,V}^2 - m_{U,V}^2) + \frac{9(a^2\delta'_V)^2}{16}}. \quad (\text{A10})$$

The taste-axial case just requires substituting  $A$  for  $V$ . For the taste-singlet case, we have

$$m_{\pi^0,I}^2 = m_{U,I}^2 = m_{D,I}^2 = \mu(m_l + m_l) + a^2\Delta_I, \quad (\text{A11})$$

$$m_{S,I}^2 = \mu(m_h + m_h) + a^2\Delta_I, \quad (\text{A12})$$

$$m_{\eta,I}^2 = \frac{m_{U,I}^2}{3} + \frac{2m_{S,I}^2}{3}, \quad (\text{A13})$$

$$m_{\eta',I}^2 = m_0^2. \quad (\text{A14})$$

The momentum integrals  $I_1$  and  $I_2$  that appear in the chiral log terms are defined as

$$I_1(m) = m^2 \ln \left( \frac{m^2}{\Lambda^2} \right), \quad (\text{A15})$$

$$I_2(m, \Delta) = -2\Delta^2 \ln \left( \frac{m^2}{\Lambda^2} \right) - 4\Delta^2 F \left( \frac{m}{\Delta} \right) + 2\Delta^2, \quad (\text{A16})$$

$$F(x) = \begin{cases} \sqrt{1-x^2} \tanh^{-1}(\sqrt{1-x^2}) & 0 \leq x \leq 1, \\ -\sqrt{x^2-1} \tanh^{-1}(\sqrt{x^2-1}) & x \geq 1, \end{cases} \quad (\text{A17})$$

where  $\Lambda$  is the renormalization scale.

Similarly,  $f_\perp$  on the unitary points in NLO SU(3) HMRs $\chi$ PT is given by [106]

$$f_\perp = \frac{C^{(0)}}{f} \left[ \frac{1}{E + \Delta_{B^*} + D} \right] + \frac{C^{(0)}}{f(E + \Delta_{B^*} + D)} \\ \times (\text{logs} + C^{(1)}\chi_l + C^{(2)}\chi_s + C^{(3)}\chi_E + C^{(4)}\chi_E^2 + C^{(5)}\chi_{a^2}), \quad (\text{A18})$$

where  $\Delta_{B^*} = M_{B^*} - M_B$ . [The SU(3) expression has one extra chiral log term  $D$  comparing with the SU(2) expression we used in our analysis.] There are two chiral log related terms parametrized by  $D$  and logs in Eq. (A18). For the  $B \rightarrow \pi$  process, the SU(3) expressions are [106]

$$D_{\text{SU}(3)}^{B \rightarrow \pi} = -\frac{3g_\pi^2 E}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} [2J_1^{\text{sub}}(m_{\pi,\Xi}, E) + J_1^{\text{sub}}(m_{K,\Xi}, E)] - \frac{1}{2} J_1^{\text{sub}}(m_{\pi,I}, E) + \frac{1}{6} J_1^{\text{sub}}(m_{\eta,I}, E) \right. \\ \left. + \sum_{j \in \{\pi, \eta, \eta'\}} [(-a^2\delta'_V) R_j^{[3,1]}(\{m_{\pi,V}, m_{\eta,V}, m_{\eta',V}\}; \{m_{S,V}\}) J_1^{\text{sub}}(m_{j,V}, E)] + [V \rightarrow A] \right\}, \quad (\text{A19})$$

and

$$\text{logs}_{\perp, \text{SU}(3)}^{B \rightarrow \pi} = \frac{1}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} \left[ -\frac{1+3g_\pi^2}{2} [2I_1(m_{\pi,\Xi}) + I_1(m_{K,\Xi})] - \frac{1}{2} g_\pi^2 J_1^{\text{sub}}(m_{\pi,I}, E) + \frac{1}{6} g_\pi^2 J_1^{\text{sub}}(m_{\eta,I}, E) \right. \right. \\ \left. \left. + \frac{1+3g_\pi^2}{12} [3I_1(m_{\pi,I}) - I_1(m_{\eta,I})] \right. \right. \\ \left. \left. + \sum_{j \in \{\pi, \eta, \eta'\}} \left[ a^2\delta'_V R_j^{[3,1]}(\{m_{\pi,V}, m_{\eta,V}, m_{\eta',V}\}; \{m_{S,V}\}) \left( g_\pi^2 J_1^{\text{sub}}(m_{j,V}, E) + \frac{1+3g_\pi^2}{2} I_1(m_{j,V}) \right) \right] + [V \rightarrow A] \right\}. \quad (\text{A20})$$

For the  $B \rightarrow K$  process, the SU(3) expressions are

$$D_{\text{SU}(3)}^{B \rightarrow K} = -\frac{3g_\pi^2(E)}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} [2J_1^{\text{sub}}(m_{K,\Xi}, E) + J_1^{\text{sub}}(m_{S,\Xi}, E)] + \frac{2}{3} J_1^{\text{sub}}(m_{\eta,l}, E) - J_1^{\text{sub}}(m_{S,l}, E) \right. \\ \left. + \sum_{j \in \{S,\eta,\eta'\}} [(-a^2 \delta'_V) R_j^{[3,1]}(\{m_{S,V}, m_{\eta,V}, m_{\eta',V}\}; \{m_{\pi,V}\}) J_1^{\text{sub}}(m_{j,V}, E)] + [V \rightarrow A] \right\}, \quad (\text{A21})$$

and

$$\text{logs}_{\perp, \text{SU}(3)}^{B \rightarrow K} = \frac{1}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} \left[ -\frac{2 + 3g_\pi^2}{2} I_1(m_{K,\Xi}) - \frac{1}{2} I_1(m_{S,\Xi}) - 3g_\pi^2 I_1(m_{\pi,\Xi}) \right] \right. \\ - \frac{1}{3} g_\pi^2 J_1^{\text{sub}}(m_{\eta,l}, E) + \frac{3g_\pi^2}{4} I_1(m_{\pi,l}) - \frac{4 + 3g_\pi^2}{12} I_1(m_{\eta,l}) + \frac{1}{2} I_1(m_{S,l}) \\ + a^2 \delta'_V \left[ \frac{g_\pi^2}{m_{\eta',V}^2 - m_{\eta,V}^2} (J_1^{\text{sub}}(m_{\eta,V}, E) - J_1^{\text{sub}}(m_{\eta',V}, E)) \right. \\ \left. + \frac{3g_\pi^2}{2} \sum_{j \in \{\pi,\eta,\eta'\}} R_j^{[3,1]}(\{m_{\pi,V}, m_{\eta,V}, m_{\eta',V}\}; \{m_{S,V}\}) I_1(m_{j,V}) \right. \\ \left. + \frac{1}{2} \sum_{j \in \{S,\eta,\eta'\}} R_j^{[3,1]}(\{m_{S,V}, m_{\eta,V}, m_{\eta',V}\}; \{m_{\pi,V}\}) I_1(m_{j,V}) \right] + [V \rightarrow A] \left. \right\}. \quad (\text{A22})$$

The definition of the meson mass terms and  $I_1$  are the same as for the  $f_{\parallel}$  case. The  $f_{\perp}$  expression has an extra function  $J_1$  that is defined as

$$J_1(m, \Delta) = \left( -m^2 + \frac{2}{3} \Delta^2 \right) \ln \left( \frac{m^2}{\Lambda^2} \right) + \frac{4}{3} (\Delta^2 - m^2) F \left( \frac{m}{\Delta} \right) - \frac{10}{9} \Delta^2 + \frac{4}{3} m^2, \quad (\text{A23})$$

$$J_1^{\text{sub}}(m, \Delta) \equiv J_1(m, \Delta) - \frac{2\pi m^3}{3\Delta}. \quad (\text{A24})$$

## 2. $f_{\parallel}$ and $f_{\perp}$ in SU(2) HMrS $\chi$ PT

We derive the SU(2) formula for  $f_{\parallel}$  and  $f_{\perp}$  based on the SU(3) expression. We also use the same expression for  $f_T$  as for  $f_{\perp}$  as discussed in Sec. III D. To obtain the SU(2) limit of an SU(3) expression, we treat the strange-quark mass as infinitely heavy. The SU(2) form does not contain  $m_s$  explicitly, but all LECs depend implicitly on  $m_s$ . Because our lattice data have slightly different  $m_s$  on different ensembles, we keep the analytic term which is proportional to  $m_s$ . Next, we consider all terms in the SU(3) chiral log expression. If a term is proportional to  $m_s$  or  $\ln m_s$  in the large  $m_s$  limit, it is absorbed into the redefinition of other LECs. If a term is proportional to  $1/m_s$  or  $1/\ln m_s$  in the large  $m_s$  limit, it does not appear in the SU(2) expression. We now derive the form of the chiral log terms in the SU(2) limit.

For the chiral log terms in  $f_{\parallel}$ , because we take  $m_s$  to infinity, all  $m_s$  related terms, such as  $m_{K,\Xi}$ ,  $m_{S,\Xi}$ ,  $m_{\eta,l}$  and  $m_{\eta',V/A,l}$ , go to infinity. They are absorbed into LECs. Only  $m_{\eta,V/A}$  is finite and goes like  $\sqrt{m_U^2 + \frac{\delta'_{V/A}}{2}}$ . We now consider all contributing chiral log terms.

- (i)  $I_1(m)$  goes like  $m^2 \ln m^2$ , so only  $I_1(M_\pi)$  survives.
- (ii)  $I_2(m, E)$  diverges as  $2\pi m E$  when  $m \rightarrow \infty$ , so only  $I_2(M_\pi, E)$  survives.

- (iii) The ratio

$$\frac{I_1(m_{\eta',V}) - I_1(m_{\eta,V}) + I_2(m_{\eta',V}, E) - I_2(m_{\eta,V}, E)}{m_{\eta',V}^2 - m_{\eta,V}^2}$$

diverges as  $2 \ln m_s$  at large  $m_s$ , so it is removed.

- (iv) We find that

$$\lim_{m_s \rightarrow \infty} a^2 \delta'_{V/A} R_j^{[3,1]}(\{m_{S,V}, m_{\eta,V}, m_{\eta',V}\}; \{m_{\pi,V}\}) \\ = \begin{cases} 4, & j = S \\ -\frac{a^4 \delta_{V/A}^2}{2m_s^4} = 0, & j = \eta \\ -4, & j = \eta'. \end{cases} \quad (\text{A25})$$

When this term multiplies  $I_1$  or  $I_2$ , it is divergent as  $m_s \rightarrow \infty$  for  $j = S$  or  $j = \eta'$ . For  $j = \eta$ , the  $I_1$  and  $I_2$  are finite, but the total contribution is zero as  $m_s \rightarrow \infty$ . So these terms are removed.

- (v) We find that

$$\lim_{m_s \rightarrow \infty} a^2 \delta'_{V/A} R_j^{[3,1]}(\{m_{\pi,V}, m_{\eta,V}, m_{\eta',V}\}; \{m_{S,V}\}) \\ = \begin{cases} 2, & j = \pi \\ -2, & j = \eta \\ -\frac{a^4 \delta_{V/A}^2}{4m_s^4} = 0, & j = \eta', \end{cases} \quad (\text{A26})$$

so only  $j = \pi$  and  $j = \eta$  terms contribute in the SU(2) theory.

In summary, for the  $B \rightarrow \pi$  process, the chiral log in SU(2) HMrS $\chi$ PT is given by

$$\begin{aligned} \log s_{\parallel, \text{SU}(2)}^{B \rightarrow \pi} = & \frac{1}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} [(1 - 3g_\pi^2)I_1(m_{\pi, \Xi}) + 2I_2(m_{\pi, \Xi}, E)] + \frac{1 + 3g_\pi^2}{4} I_1(m_{\pi, I}) \right. \\ & \left. + 2 \left[ \frac{3(g_\pi^2 - 1)}{2} I_1(m_{\pi, V}) - 2I_2(m_{\pi, V}, E) \right] - 2 \left[ \frac{3(g_\pi^2 - 1)}{2} I_1(m_{\eta, V}) - 2I_2(m_{\eta, V}, E) \right] + [V \rightarrow A] \right\}. \end{aligned} \quad (\text{A27})$$

For the  $B \rightarrow K$  process, the chiral log in SU(2) HMrS $\chi$ PT is given by

$$\log s_{\parallel, \text{SU}(2)}^{B \rightarrow K} = \frac{1}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} [-3g_\pi^2 I_1(m_{\pi, \Xi})] + \frac{3g_\pi^2}{4} I_1(m_{\pi, I}) + \frac{3g_\pi^2}{2} [2I_1(m_{\pi, V}) - 2I_1(m_{\eta, V})] + [V \rightarrow A] \right\}. \quad (\text{A28})$$

We then derive the expression for the  $f_\perp$  chiral log terms in SU(2) HMrS $\chi$ PT. We use the same treatment of analytic terms as was done for  $f_\parallel$ . To calculate the SU(2) chiral log terms, we consider the large  $m_s$  limit of  $J_1$ :

$$\lim_{m \rightarrow \infty} J_1(m, E) \rightarrow -m^2 \ln m^2 \rightarrow -\infty, \quad (\text{A29})$$

$$\lim_{m_s \rightarrow \infty} \frac{J_1^{\text{sub}}(m_{\eta, V}, E) - J_1^{\text{sub}}(m_{\eta', V}, E)}{m_{\eta', V}^2 - m_{\eta, V}^2} \rightarrow 2 \ln m_s \rightarrow \infty. \quad (\text{A30})$$

So all  $J_1$  related terms are absorbed into the redefinition of LECs and disappear.

Via a procedure similar to that for  $f_\parallel$ , we obtain the SU(2) chiral log terms in  $f_\perp$  for the  $B \rightarrow \pi$  channel:

$$\begin{aligned} D_{\text{SU}(2)}^{B \rightarrow \pi} = & -\frac{3g_\pi^2 E}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} [2J_1^{\text{sub}}(m_{\pi, \Xi}, E)] - \frac{1}{2} J_1^{\text{sub}}(m_{\pi, I}, E) \right. \\ & \left. - [2J_1^{\text{sub}}(m_{\pi, V}, E) - 2J_1^{\text{sub}}(m_{\eta, V}, E)] + [V \rightarrow A], \right\} \end{aligned} \quad (\text{A31})$$

$$\begin{aligned} \log s_{\perp, \text{SU}(2)}^{B \rightarrow \pi} = & \frac{1}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} \left[ -\frac{1 + 3g_\pi^2}{2} [2I_1(m_{\pi, \Xi})] \right] - \frac{1}{2} g_\pi^2 J_1^{\text{sub}}(m_{\pi, I}, E) \right. \\ & + \frac{1 + 3g_\pi^2}{12} [3I_1(m_{\pi, I})] + \left[ 2 \left( g_\pi^2 J_1^{\text{sub}}(m_{\pi, V}, E) + \frac{1 + 3g_\pi^2}{2} I_1(m_{\pi, V}) \right) \right. \\ & \left. \left. - 2 \left( g_\pi^2 J_1^{\text{sub}}(m_{\eta, V}, E) + \frac{1 + 3g_\pi^2}{2} I_1(m_{\eta, V}) \right) \right] + [V \rightarrow A] \right\}. \end{aligned} \quad (\text{A32})$$

Similarly, the SU(2) chiral log terms in  $B \rightarrow K$  are

$$D_{\text{SU}(2)}^{B \rightarrow K} = 0, \quad (\text{A33})$$

$$\log s_{\perp, \text{SU}(2)}^{B \rightarrow K} = \frac{1}{(4\pi f)^2} \left\{ \frac{1}{16} \sum_{\Xi} [-3g_\pi^2 I_1(m_{\pi, \Xi})] + \frac{3g_\pi^2}{4} I_1(m_{\pi, I}) + \frac{3g_\pi^2}{2} [2I_1(m_{\pi, V}) - 2I_1(m_{\eta, V})] + [V \rightarrow A] \right\}. \quad (\text{A34})$$

Equations (A31)–(A32) and (A34) are written with a structure similar to their SU(3) counterparts, which makes it easier to implement a unified computer code for the various choices of  $\chi$ PT studied in this paper.

### 3. Form factors in hard-pion/kaon ChPT

The hard-kaon (pion) continuum HM $\chi$ PT for  $B \rightarrow K$  and  $B \rightarrow \pi$  semileptonic decays was derived in Refs. [107,108]. The pion or kaon with large  $E$  is integrated out from the theory and its effects are absorbed into the LECs. We derive the hard-kaon (pion) limit of the HMrS $\chi$ PT in this section. We first study the asymptotic behavior of the integrals which contain  $E_\pi$  or  $E_K$ . We find that

$$I_2(m, E) \rightarrow A_0 E^2 \ln(E^2) + A_1 E^2 + A_2 \ln E - m^2 \ln\left(\frac{m^2}{\Lambda^2}\right), \quad (\text{A35})$$

$$J_1^{\text{sub}}(m, E) \rightarrow B_0 E^2 \ln(E^2) + B_1 E^2 + B_2 \ln E + B_3, \quad (\text{A36})$$

in the large  $E$  limit, where the coefficients  $A_i$  and  $B_i$  are either constants or analytic functions of  $m$ . The divergent terms in the large  $E$  limit decouple from the expression. The analytic terms in  $m$  are absorbed into the redefinition of the LECs. So the rules to derive the hard-kaon (pion) HMrS $\chi$ PT are the following:

- (i) Replace the term  $I_2(m, E)$  by  $-I_1(m)$ .
- (ii) Remove the  $J_1^{\text{sub}}(m, E)$  term.

To compare our results with Refs. [107,108], we set all taste splitting parameters, hairpin parameters and lattice spacings to zero. We then can reproduce the continuum hard-kaon (pion) HMrS $\chi$ PT results.

$$\text{logs}_{\perp, \text{SU}(3)}^{B \rightarrow \pi} = -\left(\frac{3}{4} + \frac{9}{4}g_\pi^2\right) \frac{I_1(m_\pi)}{(4\pi f)^2} - \left(\frac{1}{2} + \frac{3}{2}g_\pi^2\right) \frac{I_1(m_K)}{(4\pi f)^2} - \left(\frac{1}{12} + \frac{1}{4}g_\pi^2\right) \frac{I_1(m_\eta)}{(4\pi f)^2}, \quad (\text{A37})$$

$$\text{logs}_{\perp, \text{SU}(3)}^{B \rightarrow K} = -\left(\frac{9}{4}g_\pi^2\right) \frac{I_1(m_\pi)}{(4\pi f)^2} - \left(1 + \frac{3}{2}g_\pi^2\right) \frac{I_1(m_K)}{(4\pi f)^2} - \left(\frac{1}{3} + \frac{1}{4}g_\pi^2\right) \frac{I_1(m_\eta)}{(4\pi f)^2}, \quad (\text{A38})$$

$$D = 0. \quad (\text{A39})$$

Our derivation shows that  $\text{logs}_{\perp, \text{SU}(3)}^{B \rightarrow \pi} = \text{logs}_{\parallel, \text{SU}(3)}^{B \rightarrow \pi}$  and  $\text{logs}_{\perp, \text{SU}(3)}^{B \rightarrow K} = \text{logs}_{\parallel, \text{SU}(3)}^{B \rightarrow K}$  in the continuum, which is also found in Refs. [107,108].

- 
- [1] J. A. Bailey *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **79**, 054507 (2009).
  - [2] J. A. Bailey *et al.* (Fermilab Lattice and MILC Collaboration), *Phys. Rev. D* **92**, 014024 (2015).
  - [3] C. Bernard *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **79**, 014506 (2009).
  - [4] J. A. Bailey *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **89**, 114504 (2014).
  - [5] J. A. Bailey *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **92**, 034506 (2015).
  - [6] K. A. Olive *et al.* (Particle Data Group), *Chin. Phys. C* **38**, 090001 (2014).
  - [7] J. A. Bailey *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. Lett.* **115**, 152002 (2015).
  - [8] J. A. Bailey *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **85**, 114502 (2012).
  - [9] J. A. Bailey *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. Lett.* **109**, 071802 (2012).
  - [10] T. Hurth and M. Nakao, *Annu. Rev. Nucl. Part. Sci.* **60**, 645 (2010).
  - [11] M. Antonelli *et al.*, *Phys. Rep.* **494**, 197 (2010).
  - [12] B. Aubert *et al.* (BABAR Collaboration), *Phys. Rev. Lett.* **102**, 091803 (2009).
  - [13] J.-T. Wei *et al.* (Belle Collaboration), *Phys. Rev. Lett.* **103**, 171801 (2009).
  - [14] T. Aaltonen *et al.* (CDF Collaboration), *Phys. Rev. Lett.* **106**, 161801 (2011).
  - [15] T. Aaltonen *et al.* (CDF Collaboration), *Phys. Rev. Lett.* **107**, 201802 (2011).
  - [16] J. P. Lees *et al.* (BABAR Collaboration), *Phys. Rev. D* **86**, 032012 (2012).
  - [17] R. Aaij *et al.* (LHCb Collaboration), *J. High Energy Phys.* **07** (2012) 133.
  - [18] R. Aaij *et al.* (LHCb Collaboration), *J. High Energy Phys.* **02** (2013) 105.
  - [19] R. Aaij *et al.* (LHCb Collaboration), *J. High Energy Phys.* **06** (2014) 133.
  - [20] J. L. Hewett, H. Weerts *et al.*, [arXiv:1205.2671](https://arxiv.org/abs/1205.2671).
  - [21] C. Bouchard, G. P. Lepage, C. Monahan, H. Na, and J. Shigemitsu (HPQCD Collaboration), *Phys. Rev. D* **88**, 054509 (2013).
  - [22] C. Bouchard, G. P. Lepage, C. Monahan, H. Na, and J. Shigemitsu (HPQCD Collaboration), *Phys. Rev. Lett.* **111**, 162002 (2013).
  - [23] R. R. Horgan, Z. Liu, S. Meinel, and M. Wingate, *Phys. Rev. D* **89**, 094501 (2014).
  - [24] B. Grinstein, M. J. Savage, and M. B. Wise, *Nucl. Phys.* **B319**, 271 (1989).
  - [25] A. J. Buras, M. Misiak, M. Münz, and S. Pokorski, *Nucl. Phys.* **B424**, 374 (1994).
  - [26] C. Bobeth, M. Misiak, and J. Urban, *Nucl. Phys.* **B574**, 291 (2000).
  - [27] W. Altmannshofer, P. Ball, A. Bharucha, A. J. Buras, D. M. Straub, and M. Wick, *J. High Energy Phys.* **01** (2009) 019.
  - [28] B. Sheikholeslami and R. Wohlert, *Nucl. Phys.* **B259**, 572 (1985).
  - [29] A. X. El-Khadra, A. S. Kronfeld, and P. B. Mackenzie, *Phys. Rev. D* **55**, 3933 (1997).

- [30] G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea, and K. Hornbostel, *Phys. Rev. D* **46**, 4052 (1992).
- [31] C. G. Boyd, B. Grinstein, and R. F. Lebed, *Phys. Rev. Lett.* **74**, 4603 (1995).
- [32] M. C. Arnesen, B. Grinstein, I. Z. Rothstein, and I. W. Stewart, *Phys. Rev. Lett.* **95**, 071802 (2005).
- [33] C. Bourrely, I. Caprini, and L. Lellouch, *Phys. Rev. D* **79**, 013008 (2009).
- [34] A. Bharucha, T. Feldmann, and M. Wick, *J. High Energy Phys.* **09** (2010) 090.
- [35] R. Zhou *et al.* (Fermilab Lattice and MILC Collaborations), *Proc. Sci.*, LATTICE2011 (2011) 298 [arXiv:1111.0981].
- [36] R. Zhou *et al.* (Fermilab Lattice and MILC Collaborations), *Proc. Sci.*, LATTICE2012 (2012) 120 [arXiv:1211.1390].
- [37] R. Zhou, arXiv:1301.0666.
- [38] D. Bečirević, N. Košnik, F. Mescia, and E. Schneider, *Phys. Rev. D* **86**, 034034 (2012).
- [39] A. Ali, A. Y. Parkhomenko, and A. V. Rusov, *Phys. Rev. D* **89**, 094021 (2014).
- [40] C. Bernard, T. Burch, K. Orginos, D. Toussaint, T. A. DeGrand, C. DeTar, S. Datta, S. Gottlieb, U. M. Heller, and R. Sugar, *Phys. Rev. D* **64**, 054506 (2001).
- [41] C. Aubin, C. Bernard, C. DeTar, J. Osborn, S. Gottlieb, E. B. Gregory, D. Toussaint, U. M. Heller, J. E. Hetrick, and R. Sugar, *Phys. Rev. D* **70**, 094505 (2004).
- [42] M. Lüscher and P. Weisz, *Commun. Math. Phys.* **97**, 59 (1985).
- [43] M. Lüscher and P. Weisz, *Phys. Lett.* **158B**, 250 (1985).
- [44] Z. Hao, G. M. von Hippel, R. R. Horgan, Q. J. Mason, and H. D. Trottier, *Phys. Rev. D* **76**, 034507 (2007).
- [45] T. Blum, C. DeTar, S. Gottlieb, K. Rummukainen, U. M. Heller, J. E. Hetrick, D. Toussaint, R. L. Sugar, and M. Wingate, *Phys. Rev. D* **55**, R1133 (1997).
- [46] G. P. Lepage, *Nucl. Phys. B, Proc. Suppl.* **60A**, 267 (1998).
- [47] J. F. Lagaë and D. K. Sinclair, *Phys. Rev. D* **59**, 014511 (1998).
- [48] G. P. Lepage, *Phys. Rev. D* **59**, 074502 (1999).
- [49] K. Orginos and D. Toussaint (MILC Collaboration), *Phys. Rev. D* **59**, 014501 (1998).
- [50] K. Orginos, D. Toussaint, and R. L. Sugar (MILC Collaboration), *Phys. Rev. D* **60**, 054503 (1999).
- [51] C. Bernard, T. Burch, T. A. DeGrand, C. DeTar, S. Gottlieb, U. M. Heller, J. E. Hetrick, K. Orginos, B. Sugar, and D. Toussaint (MILC Collaboration), *Phys. Rev. D* **61**, 111502 (2000).
- [52] A. Bazavov *et al.*, *Rev. Mod. Phys.* **82**, 1349 (2010).
- [53] Y. Shamir, *Phys. Rev. D* **71**, 034509 (2005).
- [54] Y. Shamir, *Phys. Rev. D* **75**, 054503 (2007).
- [55] W.-J. Lee and S. R. Sharpe, *Phys. Rev. D* **60**, 114503 (1999).
- [56] C. Bernard, *Phys. Rev. D* **73**, 114503 (2006).
- [57] C. Bernard, M. Golterman, and Y. Shamir, *Phys. Rev. D* **77**, 074505 (2008).
- [58] S. Prelovšek, *Phys. Rev. D* **73**, 014506 (2006).
- [59] C. Bernard, C. E. DeTar, Z. Fu, and S. Prelovšek, *Phys. Rev. D* **76**, 094504 (2007).
- [60] C. Aubin, J. Laiho, and R. S. Van de Water, *Phys. Rev. D* **77**, 114501 (2008).
- [61] S. R. Sharpe and R. S. Van de Water, *Phys. Rev. D* **71**, 114505 (2005).
- [62] S. Dürr, *Proc. Sci.*, LAT2005 (2005) 021 [arXiv:hep-lat/0509026].
- [63] S. R. Sharpe, *Proc. Sci.*, LAT2006 (2006) 022 [arXiv:hep-lat/0610094].
- [64] A. S. Kronfeld, *Proc. Sci.*, LAT2007 (2007) 016 [arXiv:0711.0699].
- [65] G. C. Donald, C. T. H. Davies, E. Follana, and A. S. Kronfeld (HPQCD and Fermilab Lattice Collaborations), *Phys. Rev. D* **84**, 054504 (2011).
- [66] MILC Collaboration, doi: 10.15484/milc.asqtad.en06a/1178158 (2015).
- [67] MILC Collaboration, doi: 10.15484/milc.asqtad.en06b/1178159 (2015).
- [68] MILC Collaboration, doi: 10.15484/milc.asqtad.en05a/1178156 (2015).
- [69] MILC Collaboration, doi: 10.15484/milc.asqtad.en04a/1178155 (2015).
- [70] MILC Collaboration, doi: 10.15484/milc.asqtad.en15a/1178095 (2015).
- [71] MILC Collaboration, doi: 10.15484/milc.asqtad.en15b/1178096 (2015).
- [72] MILC Collaboration, doi: 10.15484/milc.asqtad.en15c/1178097 (2015).
- [73] MILC Collaboration, doi: 10.15484/milc.asqtad.en14a/1178094 (2015).
- [74] MILC Collaboration, doi: 10.15484/milc.asqtad.en13a/1178092 (2015).
- [75] MILC Collaboration, doi: 10.15484/milc.asqtad.en13b/1178093 (2015).
- [76] MILC Collaboration, doi: 10.15484/milc.asqtad.en12a/1178091 (2015).
- [77] MILC Collaboration, doi: 10.15484/milc.asqtad.en20a/1178036 (2015).
- [78] MILC Collaboration, doi: 10.15484/milc.asqtad.en20b/1178037 (2015).
- [79] MILC Collaboration, doi: 10.15484/milc.asqtad.en18a/1178033 (2015).
- [80] MILC Collaboration, doi: 10.15484/milc.asqtad.en18b/1178034 (2015).
- [81] MILC Collaboration, doi: 10.15484/milc.asqtad.en24a/1177873 (2015).
- [82] A. S. Kronfeld, *Phys. Rev. D* **62**, 014505 (2000).
- [83] M. B. Oktay and A. S. Kronfeld, *Phys. Rev. D* **78**, 014504 (2008).
- [84] J. Harada, S. Hashimoto, K.-I. Ishikawa, A. S. Kronfeld, T. Onogi, and N. Yamada, *Phys. Rev. D* **65**, 094513 (2002).
- [85] J. Harada, S. Hashimoto, A. S. Kronfeld, and T. Onogi, *Phys. Rev. D* **65**, 094514 (2002).
- [86] J. A. Bailey, Y.-C. Jang, W. Lee, and J. Leem (SWME Collaboration), *Proc. Sci.*, LATTICE2014 (2014) 389 [arXiv:1411.4227].
- [87] R. Sommer, *Nucl. Phys.* **B411**, 839 (1994).
- [88] C. Bernard, T. Burch, K. Orginos, D. Toussaint, T. A. DeGrand, C. DeTar, S. Gottlieb, U. M. Heller, J. E. Hetrick, and B. Sugar, *Phys. Rev. D* **62**, 034503 (2000).
- [89] A. Bazavov *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **85**, 114506 (2012).
- [90] R. Arthur *et al.* (RBC and UKQCD Collaborations), *Phys. Rev. D* **87**, 094514 (2013).



- [91] M. Wingate, J. Shigemitsu, C. T. H. Davies, G. P. Lepage, and H. D. Trottier, *Phys. Rev. D* **67**, 054505 (2003).
- [92] A. X. El-Khadra, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan, and J. N. Simone, *Phys. Rev. D* **64**, 014502 (2001).
- [93] S.-W. Qiu *et al.* (Fermilab Lattice and MILC Collaborations), *Proc. Sci.*, LATTICE2011 (2011) 289 [arXiv:1111.0677].
- [94] A. X. El-Khadra, E. Gámiz, A. S. Kronfeld, and M. A. Nobes, *Proc. Sci.*, LAT2007 (2007) 242 [arXiv:0710.1437].
- [95] G. P. Lepage and P. B. Mackenzie, *Phys. Rev. D* **48**, 2250 (1993).
- [96] Q. Mason, H. D. Trottier, C. T. H. Davies, K. Foley, A. Gray, G. P. Lepage, M. Nobes, and J. Shigemitsu (HPQCD Collaboration), *Phys. Rev. Lett.* **95**, 052002 (2005).
- [97] A. X. El-Khadra, E. Gámiz, and A. S. Kronfeld (to be published).
- [98] S. J. Brodsky, G. P. Lepage, and P. B. Mackenzie, *Phys. Rev. D* **28**, 228 (1983).
- [99] J. L. Richardson, *Phys. Lett.* **82B**, 272 (1979).
- [100] D. P. Menscher, Ph.D. thesis, University of Illinois, 2005.
- [101] C. Bernard *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **83**, 034503 (2011).
- [102] E. T. Neil *et al.* (Fermilab Lattice and MILC Collaborations), *Proc. Sci.*, LATTICE2011 (2011) 320 [arXiv:1112.3978].
- [103] G. P. Lepage, B. Clark, C. T. H. Davies, K. Hornbostel, P. B. Mackenzie, C. Morningstar, and H. Trottier, *Nucl. Phys. B, Proc. Suppl.* **106–107**, 12 (2002).
- [104] C. Morningstar, *Nucl. Phys. B, Proc. Suppl.* **109A**, 185 (2002).
- [105] C. Aubin and C. Bernard, *Phys. Rev. D* **73**, 014515 (2006).
- [106] C. Aubin and C. Bernard, *Phys. Rev. D* **76**, 014002 (2007).
- [107] J. Bijnens and I. Jemos, *Nucl. Phys.* **B840**, 54 (2010).
- [108] J. Bijnens and I. Jemos, *Nucl. Phys.* **B846**, 145 (2011).
- [109] G. Colangelo, M. Procura, L. Rothen, R. Stucki, and J. Tarrus Castella, *J. High Energy Phys.* **09** (2012) 081.
- [110] W. A. Bardeen, E. J. Eichten, and C. T. Hill, *Phys. Rev. D* **68**, 054024 (2003).
- [111] A. M. Green, J. Koponen, C. McNeile, C. Michael, and G. Thompson (UKQCD Collaboration), *Phys. Rev. D* **69**, 094505 (2004).
- [112] C. B. Lang, D. Mohler, S. Prelovsek, and R. M. Woloshyn, *Phys. Lett. B* **750**, 17 (2015).
- [113] W. Detmold, C.-J. D. Lin, and S. Meinel, *Phys. Rev. Lett.* **108**, 172003 (2012).
- [114] W. Detmold, C.-J. D. Lin, and S. Meinel, *Phys. Rev. D* **85**, 114508 (2012).
- [115] J. Flynn *et al.* (RBC and UKQCD Collaborations), arXiv:1506.06413.
- [116] J. M. Flynn *et al.* (RBC and UKQCD Collaborations), *Proc. Sci.*, LATTICE2013 (2014) 408 [arXiv:1311.2251].
- [117] F. Bernardoni, J. Bulava, M. Donnellan, and R. Sommer (ALPHA Collaboration), *Phys. Lett. B* **740**, 278 (2015).
- [118] D. Bečirević, S. Fajfer, and J. F. Kamenik, *Proc. Sci.*, LAT2007 (2007) 063 [arXiv:0710.3496].
- [119] D. Arndt and C. J. D. Lin, *Phys. Rev. D* **70**, 014503 (2004).
- [120] Y. Amhis *et al.* (Heavy Flavor Averaging Group), arXiv:1207.1158.
- [121] S. Aoki *et al.* (Flavour Lattice Averaging Group), *Eur. Phys. J. C* **74**, 2890 (2014).
- [122] T. Becher and R. J. Hill, *Phys. Lett. B* **633**, 61 (2006).
- [123] C. Bobeth, G. Hiller, D. van Dyk, and C. Wacker, *J. High Energy Phys.* **01** (2012) 107.
- [124] B. Grinstein and D. Pirjol, *Phys. Rev. D* **70**, 114005 (2004).
- [125] T. Becher, R. J. Hill, and M. Neubert, *Phys. Rev. D* **72**, 094017 (2005).
- [126] A. Ali, B. D. Pecjak, and C. Greub, *Eur. Phys. J. C* **55**, 577 (2008).
- [127] C. Bobeth, G. Hiller, and D. van Dyk, *J. High Energy Phys.* **07** (2010) 098.
- [128] M. Beylich, G. Buchalla, and T. Feldmann, *Eur. Phys. J. C* **71**, 1635 (2011).
- [129] W.-S. Hou, M. Kohda, and F. Xu, *Phys. Rev. D* **90**, 013002 (2014).
- [130] G. Burdman, Z. Ligeti, M. Neubert, and Y. Nir, *Phys. Rev. D* **49**, 2331 (1994).
- [131] M. B. Wise, *Phys. Rev. D* **45**, R2188 (1992).
- [132] *The Discovery Potential of a Super B Factory*, edited by J. L. Hewett and D. G. Hitlin (SLAC, Menlo Park, 2005).
- [133] N. Isgur and M. B. Wise, *Phys. Rev. D* **42**, 2388 (1990).
- [134] B. Colquhoun, C. T. H. Davies, J. Kettle, J. Koponen, A. T. Lytle, R. J. Dowdall, and G. P. Lepage (HPQCD Collaboration), *Phys. Rev. D* **91**, 114509 (2015).
- [135] J. Charles, A. Le Yaouanc, L. Oliver, O. Pene, and J. C. Raynal, *Phys. Rev. D* **60**, 014001 (1999).
- [136] M. Beneke and T. Feldmann, *Nucl. Phys.* **B592**, 3 (2001).
- [137] P. Ball and R. Zwicky, *J. High Energy Phys.* **04** (2006) 046.
- [138] V. M. Braun, D. Y. Ivanov, and G. P. Korchemsky, *Phys. Rev. D* **69**, 034014 (2004).
- [139] A. Khodjamirian, T. Mannel, and N. Offen, *Phys. Lett. B* **620**, 52 (2005).
- [140] S. J. Lee and M. Neubert, *Phys. Rev. D* **72**, 094028 (2005).
- [141] R. Arthur, P. A. Boyle, D. Brömmel, M. A. Donnellan, J. M. Flynn, A. Jüttner, T. D. Rae, and C. T. C. Sachrajda (RBC and UKQCD Collaborations), *Phys. Rev. D* **83**, 074505 (2011).
- [142] K. Chetyrkin, J. H. Kuhn, and M. Steinhauser, *Comput. Phys. Commun.* **133**, 43 (2000).
- [143] G. Bell, M. Beneke, T. Huber, and X.-Q. Li, *Nucl. Phys.* **B843**, 143 (2011).
- [144] A. Khodjamirian, T. Mannel, A. A. Pivovarov, and Y.-M. Wang, *J. High Energy Phys.* **09** (2010) 089.
- [145] S. L. Glashow, D. Guadagnoli, and K. Lane, *Phys. Rev. Lett.* **114**, 091801 (2015).
- [146] D. Du, A. X. El-Khadra, S. Gottlieb, A. S. Kronfeld, J. Laiho, E. Lunghi, R. S. Van de Water, and R. Zhou, arXiv:1510.02349 [Phys. Rev. D (to be published)].
- [147] A. Bazavov *et al.* (MILC Collaboration), *Phys. Rev. D* **82**, 074501 (2010).
- [148] A. Bazavov *et al.* (MILC Collaboration), *Phys. Rev. D* **87**, 054505 (2013).
- [149] E. Follana, Q. Mason, C. Davies, K. Hornbostel, G. P. Lepage, J. Shigemitsu, H. Trottier, and K. Wong (HPQCD Collaboration), *Phys. Rev. D* **75**, 054502 (2007).
- [150] A. Bazavov *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. Lett.* **112**, 112001 (2014).

- [151] A. Bazavov *et al.* (Fermilab Lattice and MILC Collaborations), *Phys. Rev. D* **90**, 074509 (2014).
- [152] D. Bečirević, S. Prelovšek, and J. Zupan, *Phys. Rev. D* **67**, 054010 (2003).
- [153] D. Bečirević, S. Prelovšek, and J. Zupan, *Phys. Rev. D* **68**, 074003 (2003).
- [154] D. Bečirević, S. Fajfer, and J. F. Kamenik, *J. High Energy Phys.* **06** (2007) 003.
- [155] S. Di Vita *et al.*, *Proc. Sci.*, LAT2010 (2011) 301[arXiv:1104.0869].
- [156] C. Aubin and C. Bernard, *Phys. Rev. D* **68**, 034014 (2003).