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Six dimensional QCD at two loops

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We construct the six-dimensional quantum chromodynamics (QCD) Lagrangian in a linear covariant gauge and subsequently renormalize it at two loops in the modified minimal subtraction ($\overline{\text{MS}}$) scheme. The coupling constant corresponding to the gauge interaction is asymptotically free for all numbers of quark fields, N_f. Analyzing the β functions yields a rich spectrum of fixed points. For instance, the conformal window in the six-dimensional theory is at $N_f = 16$ for the $SU(3)$ color group. The critical theory structure is similar to that of an $O(N)$ scalar theory in eight dimensions. Using the large-N expansion the latter is shown to be in the same universality class as the Heisenberg ferromagnet. Similarly using the large-N_f expansion, six-dimensional QCD is shown to be in the same class as the two-dimensional non-Abelian Thirring model and four-dimensional QCD. Abelian gauge theories are also renormalized at high loops in six and eight dimensions. It is shown that the gauge parameter only appears in the electron anomalous dimension at one loop, similar to four dimensions.

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I. INTRODUCTION

In recent years there has been renewed interest in the properties of higher-dimensional quantum field theories. This has been in part due to the long established fact that quantum chromodynamics (QCD) has a nontrivial fixed point in strictly four spacetime dimensions for a range of values of the number of quarks, N_f [1]. Known as the conformal window the range is $9 \leq N_f \leq 16$ for the $SU(3)$ color group. It is due to the two-loop β function having a nontrivial fixed point when the sign of the one-loop and two-loop terms are different [1]. Termed as the Banks-Zaks fixed point the lower bound of the window is determined by the two-loop β-function coefficient. However, it is not clear whether this is the actual range of the window since the value of the coupling constant at the lower end is beyond the conventional range for perturbative reliability. Other non-Abelian gauge theories, such as those with supersymmetry, have a similar property. While the conformal window of QCD was the first to be studied, the current vision is that similar fixed points in gauge theories with other group symmetry could give insight into the theory believed to lie beyond the Standard Model. For instance, operators which are not relevant at the Gaussian fixed point could become relevant at a nontrivial fixed point and hence drive the dynamics. Recent analyses and refinements of fixed-point locations for various color groups and representations can be found, for example, in Refs. [2–10]. One key to this is the extension of our understanding of conformal field theory in two spacetime dimensions to dimensions greater than two. This is not a trivial task as the conformal group in the former dimension is infinite dimensional but finite beyond two spacetime dimensions. One notable property of two dimensions was the c-theorem [11], which carries information on the renormalization group flow of a theory. There have been attempts to find its d-dimensional generalization such as the a-theorem [12]. The aim there is to find the function, similar to two dimensions, which is positive in the renormalization group flow from ultraviolet to infrared.

Parallel to this analysis, and in parts underlying it, is the need to determine and study the explicit renormalization group functions of field theories in dimensions greater than four. There has been work in various directions recently. In particular six-dimensional $O(N)$ scalar ϕ^3 theory has received detailed attention [13–17]. These perturbative studies are complementary to the modern application of the conformal bootstrap program, originally developed in Refs. [18–24] and extended in Refs. [25–28]. Indeed one motivation in Refs. [29–31] was the connection of this cubic scalar theory with higher-spin theories which naturally emerge from AdS/CFTs. In Refs. [15,17] the conformal window was established in $d = 6 - 2\epsilon$ dimensions, extending the one-loop result of Ref. [32], and the spectrum of fixed points determined to three loops. This was later extended to four loops in Ref. [33]. The reason why the window was studied away from the critical dimension of ϕ^3 theory is that, in principle, it ought to be possible to connect the fixed points in the higher-dimensional theory with conformal field theories in lower dimensions including two. The latter is important as conformal field theories have been classified there. An example of this ambition was given in Ref. [33]. There using summation approaches, which are standard in condensed matter theory, various critical exponents derived in the ϵ expansion of the six-dimensional theory were summed to access the discrete dimensions lower than six. Central to this was the knowledge of the value of the corresponding critical exponent in the underlying two-dimensional conformal field theory. Using this as a boundary condition for the four-loop Padé approximant, estimates for the exponents were found to be competitive with strong-coupling methods for models of percolation, for instance.

This connection across the dimensions is not a novel observation as it dates from the work of Wilson and Fisher [34]. They observed that in d dimensions, where one can regard $d = D - 2\epsilon$, with D integer and the critical dimension of a theory, different quantum field theories can have the same critical exponents. This equivalence occurs at the nontrivial d-dimensional fixed point of the respective β functions which is now termed the Wilson-Fisher fixed point. This property, known as universality, is a powerful computational tool for analyzing quantum field theories. The most common example is the relation between the two-dimensional $O(N)$ nonlinear σ model and four-dimensional $O(N)$ ϕ^4 theory [34]. Each is perturbatively renormalizable in their critical dimensions but at the d-dimensional Wilson-Fisher fixed point they are in the same universality class. That they can be seen to be connected across the dimensions is possible through the large-N expansion where $1/N$ plays the role of a dimensionless coupling constant in d dimensions. Thus the apparently perturbatively nonrenormalizable nonlinear σ model is nonperturbatively renormalizable in d dimensions in the large-N expansion. To see this equivalence in depth is possible through the work of Vasiliev's group [35–37]. In Refs. [35–37] the critical exponents of the basic fields and operators were determined to the third term as functions of d. This is $O(1/N^3)$ for the matter field anomalous dimension and $O(1/N^2)$ for what would be the force or bound-state field. The exponents for the β-function slopes of the respective models are known at $O(1/N^2)$ in Refs. [36,38]. When these critical exponents are expanded using $d = D - 2\epsilon$ relative to the respective underlying theories, can one then appreciate the exact agreement with the explicit perturbative renormalization group functions. This includes, for instance, recent six-loop $\overline{\text{MS}}$ computations of the field wave-function anomalous dimension in four-dimensional $O(N)$ ϕ^4 theory [39].

What has been established more recently is the extension of this Wilson-Fisher fixed-point universality chain to six dimensions in Refs. [17,33]. Thus one natural question, which has been posed in several articles [15,17,40], concerns whether there is a tower of such theories and if so what is the algorithm to construct each in a specific spacetime dimension. Part of this article addresses this since we construct an eight-dimensional $O(N)$ scalar field theory which we will show is in the same universality class as that of $O(N)$ scalar theories. It transpires that the process to build the tower is straightforward. In essence it is in keeping with the vision of Wilson that the universal theory is an infinite set of (local) operators, obeying a symmetry such as $O(N)$, which become relevant in the renormalization-group sense in the critical dimension. Otherwise such operators are irrelevant in other critical dimensions. These remarks have to be qualified by noting that they correspond to massless theories. If mass parameters are permitted then relevant operators of theories with lower critical dimensions will be part of the universal Lagrangian. We will briefly study the massive extension of our eight-dimensional $O(N)$ theory too as it will transpire that this together with the massless version have structural similarities with the second and main thread of this article. This is the application of the above ideas to non-Abelian gauge theories with the intention of determining connections of Lagrangians of spin-1 fields across d dimensions.

In principle the construction of a similar tower of gauge theories should be feasible based on what has been found in the scalar theory case. Moreover, it should be relevant to possible directions beyond the Standard Model. For instance, for certain gauge groups, such as $SU(3) \times SU(2) \times U(1)$, there may be a flow to a nontrivial fixed point which connects with a unified theory. Also understanding the low-energy dynamics of Yang-Mills theories is currently a major goal. While the canonical QCD Lagrangian more than adequately describes high-energy quark and gluon dynamics, it lacks many features in the infrared. One notable problem is that quarks and gluons have fundamental massless propagators, which derive from the Lagrangian, but these contradict the fact that these quanta are confined and not observed in nature. In other words operators which are ultraviolet irrelevant may become infrared relevant and dominate the infrared dynamics to the extent that the quark and gluon propagators cease having their fundamental form. One such operator which has received attention at various times is the purely gluonic dimension-six operator $f^{abc} G^a_{\mu\nu} G^{b\mu\sigma} G^{c\nu}{}_\sigma$ where $G^a_{\mu\nu}$ is the gluon field strength and f^{abc} are the color group structure constants. Clearly this operator is perturbatively nonrenormalizable in four dimensions. However, based on the scalar theory picture it is possible to consider it in a renormalizable six-dimensional Lagrangian. If the fixed-point structure of the higher theory is such that the operator's coupling becomes relevant through the ϵ expansion in four dimensions at a nontrivial point then it could be part of the structure governing the infrared dynamics of gluons. While we have highlighted this specific operator, we acknowledge that there are likely to be many other operators with higher dimensions. However, it is worth considering the simplest extension of the Wilson vision for a universal gauge theory. As an aside six-dimensional gauge theories have received attention at various times [41–47]. For instance, in Ref. [41] a version of six-dimensional QCD, similar to what we will consider here, was studied at one loop but in the background field gauge. The motivation was in part to give insight into supersymmetric extensions and to provide a framework to connect with string dynamics. An approach along similar grounds but motivated by a model-building framework can be found in Ref. [42]. Partly related to these is a second area of attention which is the explicit examination of six-dimensional supersymmetric theories. While we do not consider supersymmetry explicitly here the Lagrangians of the supersymmetric gauge theories [43,47] have similarities to our nonsupersymmetric one.

Therefore, our goal will be to construct the perturbatively renormalizable six-dimensional non-Abelian gauge theory and compute its renormalization-group functions to two loops in the $\overline{\rm MS}$ scheme. We have to proceed to this order as it will be apparent that the one loop or leading order is effectively trivial from the fixed-point structure point of view. From the computational side a two-loop renormalization provides a highly nontrivial check on the explicit construction such as the issue of the gauge fixing in six dimensions. Related to this is the check that the $\overline{\rm MS}$ β functions have to be independent of the linear covariant gauge-fixing parameter. We will show this separately for each of the three three-point vertices. Concerning the aim of connecting with gluon infrared dynamics in four dimensions, it will turn out that like the eight-dimensional $O(N)$ scalar theory the six-dimensional gluon propagator will have a double-pole propagator. In four dimensions such a propagator was believed for a while [48] to be the form in the infrared which ensured a linearly rising interquark potential and hence the confining force. However, current Landau gauge lattice measurements and Schwinger-Dyson studies of the gluon propagator in four dimensions suggest otherwise in that the propagator freezes to a finite nonzero value at zero momentum. See, for instance, Refs. [49–59]. However, we will also provide modified gluon and Faddeev-Popov ghost propagators which closely resemble those developed and used in four-dimensional models of the infrared. Our approach is via corrections to scaling and is offered as a novel but alternative insight into such models rather than a justification. One further remark needs to be made in relation to the earlier scalar theory discussion. It concerns the problem of which theories lie in the tower of gauge theories. It transpires that a similar chain has been known for some time in the dimension range $2 < d < 4$. In Ref. [60] it was shown that the two-dimensional non-Abelian Thirring model and four-dimensional QCD were connected in d dimensions at their Wilson-Fisher fixed points. This was accessed via the large-N_f expansion where it is the number of quark flavors, N_f, which is the dimensionless parameter and not the number of colors. Indeed this d-dimensional equivalence was exploited in, for example, Refs. [61–63] in order to determine various large-N_f critical exponents as functions of d. While not computed to as high an order in powers of $1/N_f$ as the scalar field theories, these d-dimensional exponents will play a very useful role in connecting to and establishing the six-dimensional gauge theory as being part of the tower with the non-Abelian Thirring model as its foundation stone. Once this connection has been achieved we will be able to study the questions of the existence of a conformal window and the two-loop fixed-point structure. As a corollary and as a stepping stone beyond six dimensions we will specialize to six- and eight-dimensional Abelian gauge theories due to recent interest in these theories [64,65]. This will be at three and two loops respectively. Again we will establish the connectivity in the tower of d-dimensional Abelian gauge theories living at the Wilson-Fisher fixed point. The main motivation for this is as a prelude to analyzing an eight-dimensional non-Abelian extension. That is a more involved exercise for a later article, since one has to determine the set of independent non-Abelian operators which build the perturbatively renormalizable Lagrangian. Some work on eight-dimensional operators has been provided in Ref. [66] but this analysis was not motivated for constructing a Lagrangian. Rather it was for ascertaining the basis for the operator product expansion and QCD sum rules.

The article is organized as follows. The algorithm to construct the candidate higher-dimensional scalar theories as well as the notation we will use is discussed in Sec. II. The eight-dimensional $O(N)$ scalar theory is renormalized in the subsequent section and the fixed-point structure analyzed after showing that it is in the same universality class as the Heisenberg ferromagnet. A corollary of that computation is to consider the $Sp(N)$ version in Sec. IV. The focus then changes to gauge theories and the construction of the higher-dimensional gauge theories is discussed in Sec. V. In order to make the Wilson-Fisher fixed-point connection relevant results from the large-N_f expansion are provided in Sec. VI. The analysis of the two-loop renormalization group functions of six-dimensional QCD is given in the following section. Subsequently, we specialize to Abelian gauge theories in Sec. VIII before providing a concluding viewpoint in Sec. IX. An appendix records values for various eight-dimensional one- and two-loop integrals.

II. BACKGROUND

We begin by discussing the construction of the higher-dimensional $O(N)$ scalar quantum field theories which lie in the same universality class as the two-dimensional nonlinear σ model and ϕ^4 theory in four dimensions at the Wilson-Fisher fixed point. These theories are equivalent in $2 < d < 4$ dimensions which can be seen within the large-N expansion. In Ref. [17] the extension of the chain to six dimensions was analyzed in full and moreover gives a clue as to how to extend the sequence to eight and higher dimensions. To appreciate this it is instructive to consider the two lowest-dimensional Lagrangians which are

$$L_\phi^{(2)} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}g_1\sigma\phi^i\phi^i - \frac{1}{2}\sigma \qquad (2.1)$$

for the nonlinear σ model and

$$L_\phi^{(4)} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}g_1\sigma\phi^i\phi^i + \frac{1}{2}\sigma^2 \tag{2.2}$$

for the four-dimensional quartic theory. In Eq. (2.1) the field σ plays the role of a Lagrange multiplier field as ordinarily one would not have a linear term in a Lagrangian. The multiplier is necessary in order to restrict the $O(N)$ scalar fields to lie on the N-sphere. Choosing a coordinate system for the constraint that the length of ϕ^i is fixed to be the coupling constant would produce the nonlinear version of Eq. (2.1) which is

$$L_\phi^{(2)} = \frac{1}{2}g_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b. \tag{2.3}$$

Here $1 \leq a \leq (N-1)$ and $g_{ab}(\phi)$ is the metric of the sphere in the chosen coordinate system. Equally we have not expressed Eq. (2.2) in its canonical form as there the σ field is regarded as an auxiliary field. Eliminating it produces

$$L_\phi^{(4)} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \frac{g_1^2}{8}(\phi^i\phi^i)^2 \tag{2.4}$$

where the quartic interaction is apparent. While both Eqs. (2.3) and (2.4) are the usual formulations it is Eqs. (2.1) and (2.2) which best indicate that they both lie in the same universality class. This is because both have a common interaction. The only differences are in the terms involving σ. The key point is that the coupling constant g_1 has different canonical dimensions in each Lagrangian and this is as a result of these σ-dependent terms. They define the dimension of each coupling which can be seen if one rescales $\sigma \to \sigma/g_1$. Indeed that is the version used in the critical point large-N method of Refs. [35–37]. In other words the commonality of the $\sigma\phi^i\phi^i$ interaction is what determines the universality.

This is evident in the extension of Refs. [15,17] to six dimensions as that Lagrangian is

$$L_\phi^{(6)} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}g_1\sigma\phi^i\phi^i + \frac{1}{6}g_2\sigma^3. \tag{2.5}$$

The relation to Eqs. (2.1) and (2.2) is that it has the same common interaction as before but the σ-dependent term no longer contributes to the free Lagrangian. The reason for this new interaction is primarily to ensure the six-dimensional Lagrangian is perturbatively renormalizable. One consequence is that there is a vector of β functions and hence a rich fixed-point structure emerges [17]. However, it has been checked that the renormalization group functions at three and four loops [17,33] can be converted into critical exponents in the large-N expansion which agree precisely with the exponents computed directly in $1/N$ [35–38]. This is a nontrivial observation since the cubic σ interaction with the additional coupling constant plays a key role in ensuring consistency. For this article we will term this and similar additional interactions in other theories as the spectator interactions. This is because the connecting interaction, $\sigma\phi^i\phi^i$, is central to the universality and the spectators are dimension dependent. Moreover, when we extend the picture to gauge theories this connecting interaction actually connects the quantum of the underlying force with matter. One difference Eq. (2.5) has with the other lower-dimensional theories is that σ cannot be eliminated either as a Lagrange multiplier or an auxiliary field. Once the connectivity of Eqs. (2.1), (2.2) and (2.5) has been established it will be apparent how one extends the tower of Lagrangians to higher dimensions. There are several key ingredients. One is the connecting $\sigma\phi^i\phi^i$ interaction and the second is that the theory has to be perturbatively renormalizable in the higher dimension. In addition the field ϕ^i has, of course, to lie in the same symmetry group which is a minor observation. Given this it is straightforward to write down a candidate eight-dimensional Lagrangian for the equivalence at the Wilson-Fisher fixed point which is

$$L_\phi^{(8)} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}(\Box\sigma)^2 + \frac{1}{2}g_1\sigma\phi^i\phi^i + \frac{1}{6}g_2\sigma^2\Box\sigma + \frac{1}{24}g_3^2\sigma^4. \tag{2.6}$$

We have normalized the coupling constants in each interaction so that the Feynman rule for each is effectively unity. A similar pattern is present with the other three theories in that the σ-dependent term is extended to a quartic one as might be expected. However, contained within the perturbative renormalizability criterion is the understanding that one has a set of independent operators with which to formulate the Lagrangian. This is the reason for an interaction with a derivative coupling. On dimensional grounds there are more possible interactions with derivatives but only one is independent. They are all related by integration by parts where total derivative operators can be dropped from the Lagrangian as they can be integrated out of the action. The other major difference which first appears here is the presence of a double pole σ propagator. This is due to the fact that the canonical dimension of σ at the Wilson-Fisher fixed point is always two which is why σ has a momentum-dependent propagator in Eq. (2.5) but not in lower dimensions. It will turn out that in the gauge theory context a similar higher-order pole propagator will emerge but in a lower dimension. So Eq. (2.6) could be regarded as a simple laboratory for testing ideas in higher-dimensional QCD in much the same way that six-dimensional ϕ^3 theory was once regarded as a test bed for four-dimensional QCD [67,68], partly due to both being asymptotically free.

III. EIGHT-DIMENSIONAL $O(N)$ SCALAR THEORY

While we have formulated a candidate eight-dimensional scalar theory according to certain criteria we still have to test out whether the renormalization group functions of Eq. (2.6) are consistent with the large-N critical exponents. To do this at a credible and nontrivial level requires a two-loop analysis. Therefore, we have constructed the anomalous dimensions of the two fields and the β functions to the requisite orders. Specifically we have determined the anomalous dimensions of ϕ^i and σ at three loops and the β functions of g_1 and g_2 at two loops. For the coupling of the quartic term we have computed $\beta_3(g_1, g_2, g_3)$ at one loop. The reason for the different loop orders stems partly from computational constraints. For instance, the field anomalous dimensions do not have any g_3 dependence at one loop, so that the two-loop term of $\beta_3(g_1, g_2, g_3)$ will not have any effect on the checks with the large-N exponents. However, we have evaluated the wave-function renormalization at three loops so that we can in fact check that the two-loop renormalization is consistent. This is because the triple and double poles in ϵ in the three-loop renormalization constants are determined by lower loop information. Here ϵ is the regularizing parameter in dimensional regularization which we use throughout. Hence it is possible to check that the result is consistent with this property of the renormalization group equation. One technical limitation arises in the renormalization of the vertices. In Ref. [33] we were able to exploit a property of six dimensions which was that a propagator of the form $1/(k^2)^2$, where k is the momentum, does not introduce spurious infrared infinities. Therefore, one could renormalize the three-point vertices by nullifying one external momentum. This simple infrared rearrangement meant that the vertex renormalization devolved to a problem of evaluating two-point Feynman graphs which is computationally much simpler than a full three-point function [33]. For Eq. (2.6) this nullification of the momentum on an external leg of a three-point vertex cannot be used. The main reason for this is that the σ propagator is itself now $1/(k^2)^2$. In eight dimensions this on its own does not introduce any infrared problems. However, if an external momentum is nullified then Feynman integrals will have factors such as $1/(k^2)^4$ which will produce unwanted infrared infinities which cannot be disentangled from the desired ultraviolet one. Therefore, for the three-point vertex renormalization we have chosen to evaluate the Feynman integrals for the case when none of the external momenta are nullified. Moreover, we will carry out the subtraction of infinities in the $\overline{\text{MS}}$ scheme at the fully symmetric point where the squared external momenta are all equal to $(-\tilde{\mu}^2)$ where $\tilde{\mu}$ is the mass scale introduced to ensure the coupling constants are dimensionless in d dimensions. One benefit of considering the symmetric point is that it corresponds to a nonexceptional momentum configuration. So there are no infrared issues and the poles in ϵ which emerge are purely ultraviolet. For the four-point vertex the same issues arise. One cannot nullify an external momentum to reduce the computation to a three-point one as then the momentum configuration is exceptional. Therefore, we have chosen to compute the one-loop four-point function at its fully symmetric point to ensure the result is infrared safe.

Having outlined the general method of computing the renormalization group functions we now discuss the more practical technical aspects of the process. This approach described here was also applied to the gauge theory computations presented later. Our calculations were carried out automatically using symbolic manipulation programs written in the language FORM [69,70]. The initial part of this is to generate all the Feynman diagrams electronically with the QGRAF package [71]. Once this is achieved the graphs are individually passed to an integration routine. The final stage of the process is to sum all the graphs and extract the renormalization constants. This latter part is achieved automatically by using the algorithm of Ref. [72]. In essence one computes each graph as a function of the bare parameters. These are the three coupling constants in Eq. (2.6) and in the case of the gauge theories the gauge parameter. The renormalized variables are introduced by rescaling with the respective renormalization constants corresponding to the constant of proportionality. This in effect introduces the counterterms automatically and bypasses the need to carry out subtractions on each individual graph which would be tedious for a high-loop analysis. The bulk of the work is in the integration routine and for each of the three types of Green's functions, two-, three- and four-point, we have used the Laporta algorithm [73]. This is an elegant technique which systematically creates all the relations between scalar Feynman integrals using integration by parts and then algebraically solves them in terms of a base set of integrals. This set is known as the master integrals and is ordinarily a relatively small set. They are evaluated directly if, for example, they are nested bubble graphs, or by nonintegration by parts methods. The version of the Laporta algorithm we used was REDUZE [74,75]. It creates a database of relations from which we extract the required integrals for each Green's function in FORM notation and then include the relations as a FORM module in the automatic computation. For Eq. (2.6) REDUZE is particularly appropriate for the two- and three-point functions since the higher pole σ propagator requires a larger order of integration than is ordinarily the case. The final stage is the substitution of the expressions for the master integrals. As we are interested in the structure of Eq. (2.6) we have to determine master integrals in eight dimensions. This is more straightforward than may initially seem which is due to the fact that the relevant masters are already known in lower dimensions. One can connect with these results via the Tarasov method [76,77], which allows one to relate d-dimensional Feynman integrals with integrals in $(d+2)$ dimensions. The latter have the same topology as the lower-dimensional one but with powers of the propagator which are larger than those of the original. However, such integrals can be reduced to the corresponding master in the higher dimension by application of the Laporta algorithm. Therefore, one can readily construct relations between masters in different dimensions plus lower-level masters which are already available. Therefore, if a lower-dimensional master is available it can be used to immediately determine the value of the corresponding master in two dimensions higher. This process was used to deduce the four-loop masters for the two-point functions in six dimensions in Ref. [33]. To aid an interested reader we have presented all the relevant two-loop eight-dimensional masters for the three-point function at the fully symmetric point to various orders in ϵ, where $d=8-2\epsilon$, in the Appendix as well as the one-loop four-point box at its symmetric point. The former complement the same values for the six-dimensional case which were presented in Ref. [78].

Applying this procedure we have found that the various renormalization group functions for Eq. (2.6) are

$$
\begin{aligned}
\gamma_\phi(g_1,g_2,g_3) &= -\frac{g_1^2}{12}+[801Ng_1^2+2250g_1^2-10800g_1g_2-1072g_2^2]\frac{g_1^2}{777600}\\
&\quad+[179415N^2g_1^4+89688870Ng_1^4+419904000\zeta_3g_1^4-520870500g_1^4\\
&\quad+56945160Ng_1^3g_2+87750000g_1^3g_2+3280536Ng_1^2g_2^2+186624000\zeta_3g_1^2g_2^2\\
&\quad-491324400g_1^2g_2^2-116640000g_1^2g_3^2-65550960g_1g_2^3-90720000g_1g_2g_3^2\\
&\quad-437392g_2^4-11275200g_2^2g_3^2-5370300g_3^4]\frac{g_1^2}{50388480000}+O(g_i^8),\\
\gamma_\sigma(g_1,g_2,g_3) &= [9Ng_1^2-8g_2^2]\frac{1}{1080}\\
&\quad+[-5265Ng_1^4+51840Ng_1^3g_2+4392Ng_1^2g_2^2-2344g_2^4-21600g_2^2g_3^2-12150g_3^4]\frac{1}{17496000}\\
&\quad+[-255817035N^2g_1^6-944784000\zeta_3Ng_1^6+1761646725Ng_1^6\\
&\quad-47764080N^2g_1^5g_2-66703500Ng_1^5g_2-85536N^2g_1^4g_2^2\\
&\quad-1049760000\zeta_3Ng_1^4g_2^2+3348988740Ng_1^4g_2^2+1043199000Ng_1^4g_3^2\\
&\quad+370958940Ng_1^3g_2^3+660474000Ng_1^3g_2g_3^2+19729152Ng_1^2g_2^4\\
&\quad+80578800Ng_1^2g_2^2g_3^2+28048275Ng_1^2g_3^4+36288000\zeta_3g_2^6-376270760g_2^6\\
&\quad+445780800g_2^4g_3^2+389431800g_2^2g_3^4+75451500g_3^6]\frac{1}{1133740800000}+O(g_i^8),\\
\beta_1(g_1,g_2,g_3) &= [9Ng_1^2+180g_1^2-240g_1g_2-8g_2^2]\frac{g_1}{2160}\\
&\quad+[187920Ng_1^4-516375g_1^4+65880Ng_1^3g_2+486000g_1^3g_2+2196Ng_1^2g_2^2\\
&\quad-827280g_1^2g_2^2-729000g_1^2g_3^2-36120g_1g_2^3-162000g_1g_2g_3^2-1172g_2^4\\
&\quad-10800g_2^2g_3^2-6075g_3^4]\frac{g_1}{17496000}+O(g_i^7),\\
\beta_2(g_1,g_2,g_3) &= [270Ng_1^3+27Ng_1^2g_2+76g_2^3]\frac{1}{2160}\\
&\quad+[-91125Ng_1^5+662985Ng_1^4g_2+14715Ng_1^3g_2^2+121500Ng_1^3g_3^2\\
&\quad-7083Ng_1^2g_2^3+8100Ng_1^2g_2g_3^2-43394g_2^5+43200g_2^3g_3^2\\
&\quad+109350g_2g_3^4]\frac{1}{11664000}+O(g_i^7),\\
\beta_3(g_1,g_2,g_3) &= [810Ng_1^4+27Ng_1^2g_3^2+160g_2^4+696g_2^2g_3^2+405g_3^4]\frac{1}{1620}+O(g_i^6)
\end{aligned}
\tag{3.1}
$$

where ζ_z is the Riemann zeta function and the order symbol in perturbative expressions throughout indicates any combination of couplings whose powers sum to that indicated. While the three-loop field anomalous dimensions satisfy internal consistency checks the main motivation is to ascertain whether Eq. (2.6) is in the same Wilson-Fisher universality class as Eqs. (2.1), (2.2) and (2.5) which requires computing the critical exponents in the large-N expansion. To do this we follow the same prescription and method introduced in Ref. [17]. First, we introduce rescaled coupling constants

$$g_1 = \sqrt{\frac{120\epsilon}{N}}x, \qquad g_2 = \sqrt{\frac{120\epsilon}{N}}y, \qquad g_3^2 = \frac{120\epsilon}{N}z. \tag{3.2}$$

Then solving $\beta_i(g_1, g_2, g_3) = 0$ we find

$$\begin{aligned}
x &= 1 + [-110 + 252\epsilon]\frac{1}{N} + [18150 - 48475\epsilon]\frac{1}{N^2} + [67232500 - 551223750\epsilon]\frac{1}{N^3} + O\left(\epsilon^2; \frac{1}{N^4}\right),\\
y &= -15 + [14250 - 21210\epsilon]\frac{1}{N} + [-36182250 + 89882625\epsilon]\frac{1}{N^2} + [128836402500 - 416828165250\epsilon]\frac{1}{N^3} + O\left(\epsilon^2; \frac{1}{N^4}\right),\\
z &= -60 - \frac{12000}{N} + \frac{1045416000}{N^2} - \frac{13222012800000}{N^3} + O\left(1; \frac{1}{N^4}\right)
\end{aligned} \tag{3.3}$$

where the order symbol for $1/N$ expansions indicates the truncation powers of both expansions. Using these to evaluate $\gamma_\phi(g_1, g_2, g_3)$ and $\gamma_\sigma(g_1, g_2, g_3)$ at this large-N fixed point we find that the exponents are

$$\begin{aligned}
\eta &= \left[-20\epsilon + \frac{89}{3}\epsilon^2\right]\frac{1}{N} + \left[4400\epsilon - \frac{77950}{3}\epsilon^2\right]\frac{1}{N^2} + [-968000\epsilon + 18577400\epsilon^2]\frac{1}{N^3} + O\left(\epsilon^3; \frac{1}{N^4}\right),\\
\eta + \chi &= \epsilon + [-420\epsilon + 673\epsilon^2]\frac{1}{N} + \left[428400\epsilon - \frac{4544450}{3}\epsilon^2\right]\frac{1}{N^2} + O\left(\epsilon^3; \frac{1}{N^3}\right)
\end{aligned} \tag{3.4}$$

for comparison with the exponents given in Refs. [35–37] when expanded in powers of ϵ where $d = 8 - 2\epsilon$. Here η relates to the renormalization group function $\gamma_\phi(g_1, g_2, g_3)$ and $\eta + \chi$ is the exponent underlying $\gamma_\sigma(g_1, g_2, g_3)$ in the exponent notation of Ref. [35,36]. Comparing the explicit perturbative results with the known large-N exponents there is precise agreement to $O(\epsilon^2)$. While this is not a full proof of the equivalence of Eq. (2.6) with the lower-dimensional scalar theories, it has been established in a similar way. More importantly it strongly suggests that the procedure for constructing a Lagrangian which is a partner in the d-dimensional tower is well defined. Crucial in this establishment is the spectator interactions whose effects first appear at two loops which is a reason why we constructed the wave-function renormalization group functions to this order. Given this agreement it is not difficult to write down a candidate for the next Lagrangian in the sequence. In ten dimensions following our prescription we would have

$$\begin{aligned}
L_\phi^{(10)} &= \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}(\Box\partial^\mu\sigma)(\Box\partial_\mu\sigma) + \frac{1}{2}g_1\sigma\phi^i\phi^i\\
&\quad + \frac{1}{6}g_2\sigma^2\Box^2\sigma + \frac{1}{2}g_3\sigma(\Box\sigma)^2 + \frac{1}{24}g_4^2\sigma^3\Box\sigma + \frac{1}{120}g_5^3\sigma^5
\end{aligned} \tag{3.5}$$

which is renormalizable by power counting. In constructing Eq. (3.5) we have ensured that the spectator interactions are independent. Also it shares structural similarities to Eq. (2.6) in that the σ propagator has an increased pole structure and there are more derivative couplings in addition to a pure quintic σ self-interaction.

We close this section by considering extensions to each of our scalar theories where (local) operators of dimension lower than the critical dimension for renormalizability are included. There are various reasons for this. One is that the structure of these massive Lagrangians is not unrelated to the Lagrangians with lower critical dimensions. Indeed this is an indication of the larger vision of the operators varying between being relevant and irrelevant in different dimensions. A second reason is that one can access an additional check on the equivalence of Eq. (2.6). First, the massive extension of the respective four-, six- and eight-dimensional Lagrangians are

$$\begin{aligned}
L_{\phi m}^{(4)} &= L_\phi^{(4)} + \frac{1}{2}m_1^2\phi^i\phi^i,\\
L_{\phi m}^{(6)} &= L_\phi^{(6)} + \frac{1}{2}m_1^2\phi^i\phi^i + \frac{1}{2}m_2^2\sigma^2,\\
L_{\phi m}^{(8)} &= L_\phi^{(8)} + \frac{1}{2}m_1^2\phi^i\phi^i - \frac{1}{2}m_2^2\sigma\Box\sigma + \frac{1}{2}m_3^4\sigma^2 + \frac{1}{6}m_4^2\sigma^3.
\end{aligned} \tag{3.6}$$

Common to each is a mass term for what one can regard as the matter field ϕ^i. In the context of the large-N critical point equivalence, as is well known the critical exponent of the ϕ^i mass operator is the same as σ field critical exponent [35,36]. For the six- and eight-dimensional cases there are additional lower-dimensional operators depending purely on σ including derivative couplings in the latter case. These operators are effectively the same as the interactions in theories at lower dimensions. When one determines the dimensionality of the associated coupling constant in d dimensions then it is clear why these operators are present in the massive extensions. However there is one minor caveat with this in that this also includes two-point operators which are to be regarded as part of the free Lagrangian. So, for instance, the propagators of $L^{(8)}_{\phi m}$ are

$$\langle \phi^i(p)\phi^j(-p)\rangle = \frac{\delta^{ij}}{[p^2+m_1^2]}, \qquad \langle \sigma(p)\sigma(-p)\rangle = \frac{1}{[(p^2)^2+m_2^2p^2+m_3^4]}. \tag{3.7}$$

A propagator similar to the denominator of the Stingl propagator [79] emerges for the σ field. While this may appear to be a nonstandard propagator, it transpires that this form of a propagator can arise in models of the infrared behavior of the gluon in QCD. As a further check on our equivalence we have computed the anomalous dimension of m_3 or equivalently the operator σ^2 in $L^{(8)}_{\phi m}$ at two loops. This is achieved by inserting the operator in a σ two-point function but in such a way that a momentum flows through the operator itself. The reason for this specific momentum configuration is to ensure that there are no infrared problems in extracting the associated operator renormalization constant. Therefore, at two loops we found

$$\gamma_{m_3}(g_1,g_2,g_3) = [-Ng_1^2-8g_2^2-10g_3^2]\frac{1}{120} + [14085N + 24240Ng_1^3g_2 + 3952Ng_1^2g_2^2 + 3330Ng_1^2g_3^2 + 18136g_2^4 + 25640g_2^2g_3^2 - 3150g_3^4]\frac{1}{1944000} + O(g_i^6). \tag{3.8}$$

From this if we evaluate the corresponding critical exponent in the large-N expansion using Eq. (3.4) the exponent is in precise agreement with the critical exponent ω computed in Ref. [38] at $O(1/N^2)$. In Ref. [38] ω was determined as it corresponded to the critical slope of the β function of Eq. (2.2) and was therefore of interest in accessing the higher-order perturbative structure of the $O(N)$ ϕ^4 β function. In Refs. [17,33,80] the same exponent was used to check the three- and four-loop mass anomalous dimension of the σ field in Eq. (2.5). Therefore, the same reasoning applies here and the expansion of ω in powers of $1/N$ and ϵ where $d = 8-2\epsilon$ means that it has to be consistent with the anomalous dimension of m_3 in $L^{(8)}_{\phi m}$ which is what we have found. It is possible to carry out a similar analysis for the cubic operator of $L^{(8)}_{\phi m}$ and we found

$$\gamma_{m_4}(g_1,g_2,g_3) = -[9Ng_1^2+152g_2^2+180g_3^2]\frac{1}{720} + O(g_i^4). \tag{3.9}$$

Extracting the $O(\epsilon)$ term at $O(1/N)$ of the critical anomalous dimension we find that it agrees with the exponent derived in Refs. [81–83]. To determine this anomalous dimension we inserted the cubic operator in a σ three-point function at the fully symmetric point. This was to ensure the infrared safeness of the ultraviolet renormalization.

Having established the equivalence of the renormalization group functions with lower-dimensional theories, the next task is to briefly analyze the fixed-point structure. The first issue is to see if there is a conformal window. Again we follow Refs. [15,17] and solve for the value of N where

$$\beta_1(g_1,g_2,g_3) = \beta_2(g_1,g_2,g_3) = \beta_3(g_1,g_2,g_3) = 0, \qquad \det\left(\frac{\partial\beta_i}{\partial g_j}\right) = 0 \tag{3.10}$$

where the first three equations determine the values of the couplings at the conformal window and the final equation relates to where there are zero eigenvalues of the β-function Hessian. In solving these equations we find several real solutions for N but only three are positive. These are at $N_{\rm cr} = 0.006773$, 0.043641 and 0.1097804. So in effect there is no conformal window unlike the six-dimensional case. To give a flavor of what the fixed-point structure looks like at leading order we have solved

$$\beta_1(g_1,g_2,g_3) = \beta_2(g_1,g_2,g_3) = \beta_3(g_1,g_2,g_3) = 0 \tag{3.11}$$

for the value $N = 500$. This is partly to compare with a similar analysis in the gauge theory case. Aside from the trivial solution we found the following fixed points, labeled with a subscript,

$$
\begin{aligned}
&x_{(1)} = 0.806810 + 0.262894i + O(\epsilon), \qquad y_{(1)} = -0.606634 + 11.002012i + O(\epsilon),\\
&z_{(1)} = 170.524352 - 1.949764i + O(\epsilon), \qquad x_{(2)} = 0.806810 + 0.262894i + O(\epsilon),\\
&y_{(2)} = -0.606634 + 11.002012i + O(\epsilon), \qquad z_{(2)} = 34.132430 + 10.748863i + O(\epsilon),\\
&x_{(3)} = -0.806810 + 0.262894i + O(\epsilon), \qquad y_{(3)} = 0.606634 + 11.002012i + O(\epsilon),\\
&z_{(3)} = 34.132430 - 10.748863i + O(\epsilon), \qquad x_{(4)} = -0.806810 - 0.262894i + O(\epsilon),\\
&y_{(4)} = 0.606634 - 11.002012i + O(\epsilon), \qquad z_{(4)} = 170.524352 - 1.949764i + O(\epsilon),\\
&x_{(5)} = 0.849156 + O(\epsilon), \qquad y_{(5)} = -8.093663 + O(\epsilon), \qquad z_{(5)} = -22.804535 + O(\epsilon),\\
&x_{(6)} = 0.849156 + O(\epsilon), \qquad y_{(6)} = -8.093663 + O(\epsilon), \qquad z_{(6)} = -97.139996 + O(\epsilon),\\
&x_{(7)} = O(\epsilon), \qquad y_{(7)} = 7.694838 + O(\epsilon), \qquad z_{(7)} = -21.929825 + O(\epsilon),\\
&x_{(8)} = O(\epsilon), \qquad y_{(8)} = 7.694838 + O(\epsilon), \qquad z_{(8)} = -63.157895 + O(\epsilon),\\
&x_{(9)} = O(\epsilon), \qquad y_{(9)} = O(\epsilon), \qquad z_{(9)} = 16.666667 + O(\epsilon)
\end{aligned}
\tag{3.12}
$$

in the same notation as the large-N analysis. In this list we have not included simple reflections $g_i \to -g_i$ or complex-conjugate partner solutions. For those solutions where there are real and imaginary parts for a fixed-point coupling constant the corresponding critical point anomalous dimensions are complex. So there are several cases where real anomalous dimensions for critical $\gamma_\phi(g_1, g_2, g_3)$ and $\gamma_\sigma(g_1, g_2, g_3)$ emerge. Only solution 5 is stable. Using the same labeling as for the critical points for the cases where we have real exponents we have, for example

$$
\begin{aligned}
&\gamma_\phi(g_1, g_2, g_3)|_{(5)} = -0.014421\epsilon + O(\epsilon^2), \qquad \gamma_\sigma(g_1, g_2, g_3)|_{(5)} = 0.604609\epsilon + O(\epsilon^2),\\
&\gamma_\phi(g_1, g_2, g_3)|_{(6)} = -0.014421\epsilon + O(\epsilon^2), \qquad \gamma_\sigma(g_1, g_2, g_3)|_{(6)} = 0.604609\epsilon + O(\epsilon^2),\\
&\gamma_\phi(g_1, g_2, g_3)|_{(7)} = O(\epsilon^2), \qquad \gamma_\sigma(g_1, g_2, g_3)|_{(7)} = -0.105263\epsilon + O(\epsilon^2),\\
&\gamma_\phi(g_1, g_2, g_3)|_{(8)} = O(\epsilon^2), \qquad \gamma_\sigma(g_1, g_2, g_3)|_{(8)} = -0.105263\epsilon + O(\epsilon^2),\\
&\gamma_\phi(g_1, g_2, g_3)|_{(9)} = O(\epsilon^2), \qquad \gamma_\sigma(g_1, g_2, g_3)|_{(9)} = O(\epsilon^2).
\end{aligned}
\tag{3.13}
$$

Several features emerge, which it transpires will be similar in the gauge theory case, and that is that different fixed points have the same leading-order values for the wave-function exponents. There is nothing deeply significant about this. It is mainly due to the absence of g_3 in the corresponding one-loop anomalous dimensions. Where those exponents have the same critical values the fixed points only differ in the leading-order critical value for g_3. The results for fixed points numbered 7, 8 and 9 are special cases. For these the value for the coupling at criticality means that ϕ^i is in effect a free field. Therefore, the exponents correspond to a theory which only involves the σ field in effect. For instance, solution 9 in essence is the eight-dimensional single field ϕ^4 theory when the propagator has a double pole.

IV. EIGHT-DIMENSIONAL $Sp(N)$ SCALAR THEORY

While the fixed-point structure of the $O(N)$ eight-dimensional scalar theory (2.6) does not appear as rich as the six-dimensional counterpart in that the conformal window reaches down to small N, there is a related scalar theory which does run parallel to Eq. (2.5). This is the eight-dimensional version of Eq. (2.6) but where the symmetry group is $Sp(N)$. Such a variation of the scalar theories was considered in six dimensions in Refs. [40,84]. It involves the presence of an anticommuting scalar, similar to ϕ^i, which carries the symplectic property. However, it was shown in those articles that the renormalization group functions could be simply derived from those of the $O(N)$ model by making the map $N \to -N$. Therefore, if we repeat this for the renormalization group functions of Eq. (2.6) we will be able to analyze the $Sp(N)$ version. The first step is to ascertain if there is a conformal window and again we solve Eq. (3.10) but use

$$
\tilde{x} = ix, \qquad \tilde{y} = iy, \qquad \tilde{z} = -z \tag{4.1}
$$

instead. In this instance we find a set of solutions given by

$$
\begin{aligned}
&N_{(A)} = 13563.468614 + O(\epsilon), \qquad \tilde{x}_{(A)} = 1.008162 + O(\epsilon),\\
&\tilde{y}_{(A)} = -16.322777 + O(\epsilon), \qquad \tilde{z}_{(A)} = 4.533577 + O(\epsilon),\\
&N_{(B)} = 6720.118606 + O(\epsilon), \qquad \tilde{x}_{(B)} = 1.015639 + O(\epsilon),\\
&\tilde{y}_{(B)} = -19.355633 + O(\epsilon), \qquad \tilde{z}_{(B)} = -202.850049 + O(\epsilon),\\
&N_{(C)} = 6145.191926 + O(\epsilon), \qquad \tilde{x}_{(C)} = 1.014734 + O(\epsilon),\\
&\tilde{y}_{(C)} = -22.265284 + O(\epsilon), \qquad \tilde{z}_{(C)} = -188.134273 + O(\epsilon),\\
&N_{(D)} = 6145.191926 + O(\epsilon), \qquad \tilde{x}_{(D)} = 1.014734 + O(\epsilon),\\
&\tilde{y}_{(D)} = -22.265284 + O(\epsilon), \qquad \tilde{z}_{(D)} = -446.807837 + O(\epsilon),\\
&N_{(E)} = 2.894045 + O(\epsilon), \qquad \tilde{x}_{(E)} = 0.197977i + O(\epsilon),\\
&\tilde{y}_{(E)} = -0.456225i + O(\epsilon), \qquad \tilde{z}_{(E)} = 0.215506 + O(\epsilon),\\
&N_{(F)} = 1.345536i + 6.030563 + O(\epsilon), \qquad \tilde{x}_{(F)} = 0.276745i + 0.025867 + O(\epsilon),\\
&\tilde{y}_{(F)} = -0.686337i + 0.344352 + O(\epsilon), \qquad \tilde{z}_{(F)} = 0.383205i + 0.186459 + O(\epsilon)
\end{aligned} \tag{4.2}
$$

where we have omitted the conjugate solution to F to save space. It turns out that there are several real solutions for the value of N where the number of real eigenvalues change. These are $N_{\mathrm{cr}} = 13564$, 6721, 6146, and 3.

Given the several ranges for the windows, we have analyzed representative values of N in order to see the structure of the fixed points for each sector by solving Eq. (3.11). It turns out that the behavior varies from sector to sector. Therefore, we provide a set of fixed points for various representative values of N. For instance, when $N = 15000$ we have the critical couplings

$$
\begin{aligned}
&\tilde{x}_{(1),15000} = 1.007382 + O(\epsilon), \qquad \tilde{y}_{(1),15000} = -16.164156 + O(\epsilon),\\
&\tilde{z}_{(1),15000} = 103.672328 + O(\epsilon), \qquad \tilde{x}_{(2),15000} = 1.007382 + O(\epsilon),\\
&\tilde{y}_{(2),15000} = -16.164156 + O(\epsilon), \qquad \tilde{z}_{(2),15000} = -37.868526 + O(\epsilon),\\
&\tilde{x}_{(3),15000} = 0.974832 + O(\epsilon), \qquad \tilde{y}_{(3),15000} = -47.393461 + O(\epsilon),\\
&\tilde{z}_{(3),15000} = -735.06222 + O(\epsilon), \qquad \tilde{x}_{(4),15000} = 0.974832 + O(\epsilon),\\
&\tilde{y}_{(4),15000} = -47.393461 + O(\epsilon), \qquad \tilde{z}_{(4),15000} = -2674.674316 + O(\epsilon),\\
&\tilde{x}_{(5),15000} = 0.865512 + O(\epsilon), \qquad \tilde{y}_{(5),15000} = 53.493631 + O(\epsilon),\\
&\tilde{z}_{(5),15000} = -840.729642 + O(\epsilon), \qquad \tilde{x}_{(6),15000} = 0.865514 + O(\epsilon),\\
&\tilde{y}_{(6),15000} = 53.493631 + O(\epsilon), \qquad \tilde{z}_{(6),15000} = -3827.814755 + O(\epsilon),\\
&\tilde{x}_{(7),15000} = O(\epsilon), \qquad \tilde{y}_{(7),15000} = 42.146362i + O(\epsilon), \qquad \tilde{z}_{(7),15000} = 1894.736842 + O(\epsilon),\\
&\tilde{x}_{(8),15000} = O(\epsilon), \qquad \tilde{y}_{(8),15000} = 42.146361i + O(\epsilon), \qquad \tilde{z}_{(8),15000} = 657.894737 + O(\epsilon),\\
&\tilde{x}_{(9),15000} = O(\epsilon), \qquad \tilde{y}_{(9),15000} = O(\epsilon), \qquad \tilde{z}_{(9),15000} = -500.000000.
\end{aligned} \tag{4.3}
$$

In these and subsequent fixed-point solutions we omit critical points which are related by reflections or complex conjugates. Similar features are common with the $O(N)$ theory with $N = 500$ such as solutions 7, 8 and 9 which correspond to the ϕ^i-free case. Also there are pairs with the same $\tilde{x}$ and $\tilde{y}$ values but a different value for $\tilde{z}$. The main difference is that all solutions are real when $\tilde{x} \neq 0$. By contrast examining the $N = 10000$ case we find

$$
\begin{aligned}
&\tilde{x}_{(1),10000} = 1.011031 + O(\epsilon), \qquad \tilde{y}_{(1),10000} = -17.015872 + O(\epsilon),\\
&\tilde{z}_{(1),10000} = -74.728621 + 81.472666i + O(\epsilon),\\
&\tilde{x}_{(2),10000} = 0.989137 + O(\epsilon), \qquad \tilde{y}_{(2),10000} = -36.865152 + O(\epsilon),\\
&\tilde{z}_{(2),10000} = -454.998667 + O(\epsilon),\\
&\tilde{x}_{(3),10000} = 0.989137 + O(\epsilon), \qquad \tilde{y}_{(3),10000} = -36.865152 + O(\epsilon),\\
&\tilde{z}_{(3),10000} = -1561.607482 + O(\epsilon),\\
&\tilde{x}_{(4),10000} = 0.854964 + O(\epsilon), \qquad \tilde{y}_{(4),10000} = 43.810992 + O(\epsilon),\\
&\tilde{z}_{(4),10000} = -558.723323 + O(\epsilon),\\
&\tilde{x}_{(5),10000} = 0.854964 + O(\epsilon), \qquad \tilde{y}_{(5),10000} = 43.810992 + O(\epsilon),\\
&\tilde{z}_{(5),10000} = -2585.830662 + O(\epsilon),\\
&\tilde{x}_{(6),10000} = O(\epsilon), \qquad \tilde{y}_{(6),10000} = 34.412360i + O(\epsilon),\\
&\tilde{z}_{(6),10000} = 1263.157895 + O(\epsilon),\\
&\tilde{x}_{(7),10000} = O(\epsilon), \qquad \tilde{y}_{(7),10000} = 34.412360i + O(\epsilon), \qquad \tilde{z}_{(7),10000} = 438.596491 + O(\epsilon),\\
&\tilde{x}_{(8),10000} = O(\epsilon), \qquad \tilde{y}_{(8),10000} = O(\epsilon), \qquad \tilde{z}_{(8),10000} = -333.333333 + O(\epsilon).
\end{aligned} \tag{4.4}
$$

Here there is one fully complex solution. In the next lower window the reality of all solutions is restored since, for example,

$$
\begin{aligned}
&\tilde{x}_{(1),6500} = 1.015877 + O(\epsilon), \qquad \tilde{y}_{(1),6500} = -19.862247 + O(\epsilon),\\
&\tilde{z}_{(1),6500} = -174.63918 + O(\epsilon),\\
&\tilde{x}_{(2),6500} = 1.015877 + O(\epsilon), \qquad \tilde{y}_{(2),6500} = -19.862247 + O(\epsilon),\\
&\tilde{z}_{(2),6500} = -272.79598 + O(\epsilon),\\
&\tilde{x}_{(3),6500} = 1.009679 + O(\epsilon), \qquad \tilde{y}_{(3),6500} = -25.636626 + O(\epsilon),\\
&\tilde{z}_{(3),6500} = -234.623913 + O(\epsilon),\\
&\tilde{x}_{(4),6500} = 1.009679 + O(\epsilon), \qquad \tilde{y}_{(4),6500} = -25.636626 + O(\epsilon),\\
&\tilde{z}_{(4),6500} = -669.753355 + O(\epsilon),\\
&\tilde{x}_{(5),6500} = 0.841616 + O(\epsilon), \qquad \tilde{y}_{(5),6500} = 35.419872 + O(\epsilon),\\
&\tilde{z}_{(5),6500} = -360.906735 + O(\epsilon),\\
&\tilde{x}_{(6),6500} = 0.841616 + O(\epsilon), \qquad \tilde{y}_{(6),6500} = 35.419872 + O(\epsilon),\\
&\tilde{z}_{(6),6500} = -1704.819491 + O(\epsilon),\\
&\tilde{x}_{(7),6500} = O(\epsilon), \qquad \tilde{y}_{(7),6500} = 27.744132i + O(\epsilon), \qquad \tilde{z}_{(7),6500} = 821.052632 + O(\epsilon),\\
&\tilde{x}_{(8),6500} = O(\epsilon), \qquad \tilde{y}_{(8),6500} = 27.744132i + O(\epsilon), \qquad \tilde{z}_{(8),6500} = 285.087720 + O(\epsilon),\\
&\tilde{x}_{(9),6500} = O(\epsilon), \qquad \tilde{y}_{(9),6500} = O(\epsilon), \qquad \tilde{z}_{(9),6500} = -216.666667 + O(\epsilon)
\end{aligned} \tag{4.5}
$$

when $N = 6500$. The solutions in this region in effect have the same structure as that for $N > 13563$. Dropping to the next sector two purely complex solutions emerge. For instance, when $N = 100$ we find

$$
\begin{aligned}
&\tilde{x}_{(1),100} = 0.504796 + 0.886070i + O(\epsilon), \qquad \tilde{y}_{(1),100} = 1.604579 - 6.710841i + O(\epsilon), \\
&\tilde{z}_{(1),100} = 48.95388 + 37.45947i + O(\epsilon), \\
&\tilde{x}_{(2),100} = 0.504796 + 0.886070i + O(\epsilon), \qquad \tilde{y}_{(2),100} = 1.604579 - 6.710841i + O(\epsilon), \\
&\tilde{z}_{(2),100} = 17.146965 + 5.514596i + O(\epsilon), \\
&\tilde{x}_{(3),100} = 0.567011 + O(\epsilon), \qquad \tilde{y}_{(3),100} = 3.980936 + O(\epsilon), \\
&\tilde{z}_{(3),100} = -3.101886 + O(\epsilon), \\
&\tilde{x}_{(4),100} = 0.567011 + O(\epsilon), \qquad \tilde{y}_{(4),100} = 3.980936 + O(\epsilon), \\
&\tilde{z}_{(4),100} = -25.322927 + O(\epsilon), \\
&\tilde{x}_{(5),100} = O(\epsilon), \qquad \tilde{y}_{(5),100} = 3.441236i + O(\epsilon), \qquad \tilde{z}_{(5),100} = 12.631579 + O(\epsilon), \\
&\tilde{x}_{(6),100} = O(\epsilon), \qquad \tilde{y}_{(6),100} = 3.441236i + O(\epsilon), \qquad \tilde{z}_{(6),100} = 4.385965 + O(\epsilon), \\
&\tilde{x}_{(7),100} = O(\epsilon), \qquad \tilde{y}_{(7),100} = O(\epsilon), \qquad \tilde{z}_{(7),100} = -3.333333 + O(\epsilon).
\end{aligned} \tag{4.6}
$$

Throughout each of these solutions one of the real fixed points is the one which the large-N exponents in the $Sp(N)$ version of Eq. (2.6) are connected to.

One final example is of special interest. When $N = 2$ our solutions to Eq. (3.11) are

$$
\begin{aligned}
&x_{(1),2} = 0.193438i + O(\epsilon), \qquad y_{(1),2} = -0.269684i + O(\epsilon), \\
&z_{(1),2} = 0.091641 + O(\epsilon), \\
&x_{(2),2} = 0.193438i + O(\epsilon), \qquad y_{(2),2} = -0.269684i + O(\epsilon), \\
&z_{(2),2} = -0.038310 + O(\epsilon), \\
&x_{(3),2} = 0.149071i + O(\epsilon), \qquad y_{(3),2} = -0.447214i + O(\epsilon), \\
&z_{(3),2} = 0.207407 + O(\epsilon), \\
&x_{(4),2} = 0.149071i + O(\epsilon), \qquad y_{(4),2} = -0.447214i + O(\epsilon), \\
&z_{(4),2} = 0.066667 + O(\epsilon), \\
&x_{(5),2} = 0.282351 + O(\epsilon), \qquad y_{(5),2} = 0.433979 + O(\epsilon), \qquad z_{(5),2} = 0.027985 + O(\epsilon), \\
&x_{(6),2} = 0.282351 + O(\epsilon), \qquad y_{(6),2} = 0.433979 + O(\epsilon), \qquad z_{(6),2} = -0.407685 + O(\epsilon), \\
&x_{(7),2} = O(\epsilon), \qquad y_{(7),2} = 0.486664i + O(\epsilon), \qquad z_{(7),2} = 0.252632 + O(\epsilon), \\
&x_{(8),2} = O(\epsilon), \qquad y_{(8),2} = 0.486664i + O(\epsilon), \qquad z_{(8),2} = 0.087719 + O(\epsilon), \\
&x_{(9),2} = O(\epsilon), \qquad y_{(9),2} = O(\epsilon), \qquad z_{(9),2} = -0.066667 + O(\epsilon).
\end{aligned} \tag{4.7}
$$

While there are fewer purely real solutions those that are imaginary only for $\tilde{x}$ and $\tilde{y}$ will have real squares when put on the same footing as $\tilde{z}$. The main observation is that solution 4, which is a stable fixed point, has the property that

$$\tilde{y} = 3\tilde{x} + O(\epsilon), \qquad \tilde{z} = 3\tilde{x}^2 + O(\epsilon). \tag{4.8}$$

This is not an accident as a similar solution emerged in the six-dimensional $Sp(N)$ case for $N = 2$ [84], although there was no quartic interaction there. In Ref. [84] it was shown to be due to a hidden supersymmetry based on the supergroup $OSp(1|2)$. Thus it would appear that the same symmetry arises in the eight-dimensional scalar theory. One property of this supersymmetry is that the field anomalous dimension for ϕ^i and σ should be equivalent and we have checked this and found that

$$
\begin{aligned}
\gamma_\phi(g_1, g_2, g_3)|_{(4),2} &= \gamma_\sigma(g_1, g_2, g_3)|_{(4),2} \\
&= -0.111111\epsilon + O(\epsilon^2).
\end{aligned} \tag{4.9}
$$

Actually the same leading-order exponents emerge for solution 3 too but this is only due to $\gamma_\phi(g_1, g_2, g_3)$ and $\gamma_\sigma(g_1, g_2, g_3)$ not depending on g_3 at one loop. What is perhaps more intriguing is that the critical-point structure of the six-dimensional $Sp(2)$ case is given by the $q \to 0$ limit of the q-state Potts model, [85]. In Ref. [86] it was suggested that the upper critical dimension for this equivalence was six. Given the relation of Eq. (2.6) now with Eq. (2.5) at the Wilson-Fisher fixed point and the appearance of a hidden symmetry for $Sp(2)$ at a specific fixed point, similar to six dimensions [84], it would be interesting to see whether the restriction to six dimensions argued in Ref. [86] could be extended to eight dimensions. While our focus in this section has been on the $Sp(N)$ theory, which reveals a rich fixed-point spectrum on a par with that of Eq. (2.5) [17], a full analysis would require higher-order computations.

V. HIGHER-DIMENSIONAL GAUGE THEORIES

Having discussed a model scalar theory of some of the structural similarities to higher-dimensional gauge theories we now concentrate on the construction of the six-dimensional QCD Lagrangian in this section. In essence the core properties of eight-dimensional $O(N)$ ϕ^3 theory translate to the QCD case. The main difference is the presence of gauge symmetry which requires a modification of our algorithm for the completion of the higher-dimensional theory and the construction of the tower of theories which are equivalent at the Wilson-Fisher fixed point. First, we recall that the four-dimensional QCD Lagrangian is

$$L^{(4)} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 - \bar{c}^a(\partial^\mu D_\mu c)^a + i\bar{\psi}^{iI}\not{D}\psi^{iI} \tag{5.1}$$

where A^a_μ is the gluon, ψ^{iI} is the quark and c^a are the Faddeev-Popov ghost fields. Here the indices take the ranges $1 \leq a \leq N_A$, $1 \leq I \leq N_F$ and $1 \leq i \leq N_f$ where N_F and N_A are the respective dimensions of the fundamental and adjoint representations of the color group and N_f is the number of quark flavors. Also D_μ is the covariant derivative and $G^a_{\mu\nu}$ is the field strength. Throughout we choose to work with the canonical linear covariant gauge fixing whose associated gauge parameter is α. While this is a standard Lagrangian it is worth noting several features relevant to the present discussion. The Lagrangian is constructed in several stages. The first is to write down all independent local gauge-invariant operators which are built from the A^a_μ and ψ^{iI} fields and are renormalizable in the dimension of interest which is four for Eq. (5.1). For the moment we will exclude lower-dimensional operators which would introduce masses. Unlike scalar theories such gauge-invariant Lagrangians produce fields with more degrees of freedom than are present in nature and therefore a gauge fixing is required. Again this gauge fixing, which does not have to be covariant or linear as we are choosing here, has to be local, renormalizable and of dimension four. The gauge-fixing terms subsequently break gauge invariance. So one instead requires that the Lagrangian is Becchi-Rouet-Stora-Tyutin (BRST) invariant rather than gauge invariant. These considerations clearly have been satisfied in Eq. (5.1).

One ingredient from our earlier algorithm appears to have been omitted in this instance and that is the theory in two dimensions with the same symmetries which is connected via the Wilson-Fisher fixed point; in other words the base theory which is in the same universality class. This requires some care given the nature of a two-dimensional spin-1 field. It transpires that the equivalent theory is the non-Abelian Thirring model (NATM) [60], which has the Lagrangian

$$L^{(2)} = i\bar{\psi}^{iI}\not{\partial}\psi^{iI} + \frac{g}{2}(\bar{\psi}^{iI}T^a_{IJ}\gamma^\mu\psi^{iJ})^2 \tag{5.2}$$

where T^a are the color group generators. Unlike Eq. (2.1) there can be no base Lagrangian which is linear in a spin-1 field without breaking color and Lorentz symmetry. As presented the connection with Eq. (5.1) appears distant due to the absence of a field A^a_μ. However, the interaction of Eq. (5.2) may be rewritten in two dimensions in terms of an auxiliary spin-1 field to produce

$$L^{(2)} = i\bar{\psi}^{iI}\not{\partial}\psi^{iI} + g\bar{\psi}^{iI}T^a_{IJ}\gamma^\mu\psi^{iJ}A^a_\mu - \frac{g}{2}A^a_\mu A^{a\mu}. \tag{5.3}$$

As it stands this version of $L^{(2)}$ appears to be an improvement on Eq. (5.2) with regard to equivalence with Eq. (5.1) but it does not appear to be consistent with our completion argument. One objection is the apparent absence of gauge invariance and by association the gauge-fixing and ghost terms. On the contrary the equivalence with four-dimensional QCD observed in Ref. [60] has subsequently been verified computationally to several orders in the large-N_f expansion in Refs. [61–63]. The bridge is in the main twofold. First, the auxiliary field reformulation is a strictly a two-dimensional relation. Second, to proceed with the large-N_f analysis through the connecting Wilson-Fisher fixed point the key is the quark-gluon vertex which together with the quark kinetic term define the canonical dimensions of the field in the d-dimensional universality class. The sector which is purely gluonic, such as $\frac{1}{2}A^a_\mu A^{a\mu}$ in Eq. (5.3) and $G^a_{\mu\nu}G^{a\mu\nu}$ in Eq. (5.1), in essence defines the canonical dimensions of the respective coupling constants in each theory. Of course, the couplings have different dimensionalities in renormalizable theories in different spacetime dimensions. Therefore, in the large-N_f approach discussed in Ref. [63], the gauge-fixed Lagrangian at criticality has an analytically regularized gauge fixing with associated Faddeev-Popov ghost sector modifications [63]. In other words formally

$$L^{\rm NATM} = i\bar{\psi}^{iI}\not{\partial}\psi^{iI} + g\bar{\psi}^{iI}T^a_{IJ}\gamma^\mu\psi^{iJ}A^a_\mu - \bar{c}^a(\partial^\mu D_\mu c)^a - \frac{g}{2}A^a_\mu A^{a\mu} + \frac{1}{2\alpha}(\partial^\mu A^a_\mu)\frac{1}{\Box^{4-d}}(\partial^\nu A^a_\nu) \tag{5.4}$$

is used to determine the large-N_f critical exponents [63]. One immediate objection to this is that one does not have locality. Equally one also loses perturbative renormalizability for the lower-dimensional theory but at criticality these is not an issue. What one has to accept is that the critical equivalence is valid in the Landau gauge which corresponds to $\alpha = 0$. This is more subtle than it appears and is not unrelated to our algorithm extended to the gauge theory context. In two dimensions we have treated A^a_μ as an auxiliary spin-1 field. If it were a gauge field in the context of Eq. (5.1) then clearly the operator $\frac{1}{2}A^a_\mu A^{a\mu}$ is not gauge invariant. However, it is possible to write down several gauge-invariant dimension-two operators but in this instance the locality assumption has to be dropped.

For example, the operator

$$\mathcal{O}_2 = -\frac{1}{2}G^{a\mu\nu}\frac{1}{D^2}G^{a\mu\nu} \tag{5.5}$$

is dimension two and gauge invariant but clearly nonlocal. Such an operator has appeared before [87,88] in the context of three-dimensional gauge theories and studied for their relation to temperature QCD. Despite the presence of the nonlocality it is possible to localize the operator and determine its renormalization to several loop orders [89,90]. In other words this nonlocal operator can be regarded as being perturbatively renormalizable. For instance, the one-loop anomalous dimension in four dimensions is proportional to the one-loop QCD β function [89,90]. Beyond one loop this proportionality ceases. This is due to the presence of extra or ghost fields which arise in the localizing procedure and their coupling constants appear in the two-loop and higher operator anomalous dimension. Another gauge-invariant gluonic dimension-two operator is

$$\mathcal{O} \equiv \frac{1}{2}\overset{\min}{\{U\}}\int d^4x(A^{aU}_\mu)^2 \tag{5.6}$$

where A^{aU}_μ is the transport of a gauge field along a gauge orbit

$$A^U_\mu = UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \tag{5.7}$$

and U is gauge group element. By construction $\mathcal{O}$ is gauge invariant and forms the basis for a gauge fixing [91–94], which does not suffer from Gribov copy issues. There are various ways of writing $\mathcal{O}$ perturbatively in terms of other nonlocal operators [91,95]. A gauge-invariant expansion was given in Refs. [91,95] and $\mathcal{O}_2$ is in fact the first term. The three-leg operator was presented in Ref. [91] and has structural similarities to the dimension-six operator considered later. More recently an algorithm to produce the subsequent operators was given in Ref. [95]. Despite the nonlocality the one-loop renormalization of $\mathcal{O}$ was given in Ref. [96]. There it was shown that the gauge parameter was indeed absent in the anomalous dimension. While such operators address the issue of constructing a gauge-invariant dimension-two operator, which is present in theories connected at the Wilson-Fisher fixed point, there is a connection with Eq. (5.3). Although locality is sacrificed for gauge invariance to produce a nonlocal operator, both of the operators $\mathcal{O}_2$ and $\mathcal{O}$ truncate to $\frac{1}{2}A^a_\mu A^{a\mu}$ when one specifies the Landau gauge. In this gauge this operator is also BRST invariant as the ghost mass term is absent. The upshot is that as discussed in Ref. [63] when comparing our perturbative results at the Wilson-Fisher fixed point for gauge theories in the different dimensions, we can only compare critical exponents which derive from gauge-dependent renormalization group functions in the Landau gauge. For exponents based on gauge-independent renormalization group functions this point will not be relevant.

Returning to the problem of constructing a six-dimensional gauge theory the first stage is to write down the set of independent gauge-invariant dimension-six operators with which to build a Lagrangian. For the quark sector to maintain connectivity with the four-dimensional gauge theory the set includes $i\bar{\psi}^{iI}\not{D}\psi^{iI}$. In six dimensions this immediately defines the canonical dimension of the quark field to be $\frac{5}{2}$. Thus unlike two dimensions there are no quartic or higher operators which include quark fields. As such an operator would require an antiquark to ensure a Lorentz scalar term one sees that there is only one dimension-six quark operator. This is important since, for instance, when considering six-dimensional operators in four-dimensional QCD effective theories, 4-fermi operators are included in the same discussion. In the six-dimensional case they will not appear in a Lagrangian since such 4-fermi operators actually have a canonical dimension of ten and so are absent in a renormalizable Lagrangian. Such 4-fermi operators are only perturbatively renormalizable in two dimensions as is evident in Eq. (5.2) or Eq. (5.3). We now change our focus to the gluonic sector. In Refs. [41,66] such dimension-six gluonic operators were considered and it transpires that there are four potential candidates which are

$$\begin{aligned}
\mathcal{O}^{(6)}_1 &= (D_\mu G^a_{\nu\sigma})(D^\mu G^{a\nu\sigma}), & \mathcal{O}^{(6)}_2 &= (D^\mu G^a_{\mu\sigma})(D_\nu G^{a\nu\sigma}), \\
\mathcal{O}^{(6)}_3 &= (D_\mu G^a_{\nu\sigma})(D^\sigma G^{a\mu\nu}), & \mathcal{O}^{(6)}_4 &= f^{abc}G^a_{\mu\nu}G^{b\mu\sigma}G^{c\nu}_{\ \sigma}.
\end{aligned} \tag{5.8}$$

However, these are not all independent due to either integration by parts or use of the Bianchi identity

$$D_\mu G^a_{\nu\sigma} + D_\nu G^a_{\sigma\mu} + D_\sigma G^a_{\mu\nu} = 0. \tag{5.9}$$

Total derivative operators can be ignored in the Lagrangian construction due to conservation of energy-momentum.

So of the set (5.8) we are free to choose any two for our six-dimensional QCD Lagrangian $L^{(6)}$. In Ref. [41] $\mathcal{O}_1^{(6)}$ and $\mathcal{O}_2^{(6)}$ were chosen as the two independent operators but we will take a different basis which is $\mathcal{O}_1^{(6)}$ and $\mathcal{O}_4^{(6)}$. The reason for this choice rests partly in the potential connection with four dimensions as noted earlier. Thus if there are fixed points in the six-dimensional gauge theory which connect with the infrared structure of QCD in four dimensions after some sort of summation, it seems appropriate to include the key operator explicitly with its own coupling constant at the outset. Moreover, $\mathcal{O}_1^{(6)}$ is the natural extension of the gluon kinetic term which is why that is chosen for the other independent operator. Irrespective of which basis choice we make the gluon propagator will now have a double pole. However, if there is connectivity with the infrared structure of a lower-dimensional gauge theory a double-pole propagator may not be inappropriate. Another reason for taking $\mathcal{O}_1^{(6)}$ and $\mathcal{O}_4^{(6)}$ rests in the nature of the coupling constants. If one chose $\mathcal{O}_2^{(6)}$ instead of $\mathcal{O}_4^{(6)}$ then there is the problem of what relative weight to assign each term. The appropriate way to proceed is to introduce a weighting parameter such as β and include

$$\beta\mathcal{O}_1^{(6)} + (1-\beta)\mathcal{O}_2^{(6)} \tag{5.10}$$

as the two independent operators in the Lagrangian. The parameter β would not be present in the gluon propagator but would be present in the interaction terms. It is not a gauge-fixing parameter but rather represents a measure of the interpolation. Thus its renormalization would be independent of the gauge parameter in $\overline{\rm MS}$ for instance. In effect in the interaction terms the product of β with g_1 corresponds to a second coupling constant which is independent of g_1 and if one were to use this set of operators in the Lagrangian then βg_1 would be redefined as a second coupling. While this is perfectly viable as a strategy it seems more appropriate to use one two-leg operator for the kinetic term and have the second independent operator as higher leg which is why we choose $\mathcal{O}_1^{(6)}$ and $\mathcal{O}_4^{(6)}$. Equally no intermediate interpolating parameter needs to be introduced as one just couples the latter operator with the independent coupling retaining the gauge coupling, g_1, in the gluon kinetic term. Thus the gauge-invariant six-dimensional Lagrangian, $L_{\rm GI}^{(6)}$, of QCD we begin with is

$$L_{\rm GI}^{(6)} = -\frac{1}{4}(D_\mu G^a_{\nu\sigma})(D^\mu G^{a\nu\sigma}) + \frac{g_2}{6}f^{abc}G^a_{\mu\nu}G^{b\mu\sigma}G^{c\nu}{}_\sigma + i\bar{\psi}^{iI}\not{D}\psi^{iI}. \tag{5.11}$$

As we have an interaction over and above those which derive from terms involving the covariant derivative we need to be clear about the notation. Throughout when additional operators are appended to a gauge theory in higher dimensions such as here then we will use the coupling constant g_1 as that which appears in the covariant derivative, D_μ, and hence also $G^a_{\mu\nu}$. For theories with extra symmetries such as supersymmetry the second coupling, g_2, could be related to g_1. Equally if one proceeded with the choice involving β its value would be fixed by the extra symmetry. As an aside effective Lagrangians similar to Eq. (5.11) have been studied in four dimensions in various covariant and noncovariant gauges in order to explore the possible nonperturbative behavior of the gluon propagator in the infrared region [97–99].

The final aspect of our discussion centers on the form of the gauge-fixing terms which need to be present in order to carry out perturbative calculations. As in four dimensions we choose to fix in an arbitrary linear covariant gauge $\partial^\mu A^a_\mu = 0$. However, the usual four-dimensional gauge-fixing term in addition to the Faddeev-Popov ghost term which implements this condition cannot be used in six dimensions due to the fact that the canonical terms are dimension four. Instead motivated by Eq. (5.4) we use a BRST invariant dimension-six gauge fixing where the shortfall in dimensionality of the operators are made up for by spacetime derivatives. In other words our gauge-fixed six-dimensional QCD Lagrangian is

$$L^{(6)} = -\frac{1}{4}(D_\mu G^a_{\nu\sigma})(D^\mu G^{a\nu\sigma}) + \frac{g_2}{6}f^{abc}G^a_{\mu\nu}G^{b\mu\sigma}G^{c\nu}{}_\sigma - \frac{1}{2\alpha}(\partial_\mu\partial^\nu A^a_\nu)(\partial^\mu\partial^\sigma A^a_\sigma) - \bar{c}^a\Box(\partial^\mu D_\mu c)^a + i\bar{\psi}^{iI}\not{D}\psi^{iI} \tag{5.12}$$

where α is the covariant fixing parameter with the Landau gauge corresponding to $\alpha = 0$. It is straightforward to check that the Lagrangian is BRST invariant without modification of the canonical BRST transformations on the fields. The gauge-fixing term allows one to find the gluon propagator since when $\alpha \neq 0$ the quadratic part of the momentum-space Lagrangian is invertible. The gluon and ghost propagators are then

$$\langle A^a_\mu(p)A^b_\nu(-p)\rangle = -\frac{\delta^{ab}}{(p^2)^2}\left[\eta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right],$$
$$\langle c^a(p)\bar{c}^b(-p)\rangle = -\frac{\delta^{ab}}{(p^2)^2} \tag{5.13}$$

with the double-pole propagator emerging as noted earlier and similar to Refs. [97–99].

We close this section by considering the extension of the Lagrangians to lower-dimensional operators and hence mass terms. This is similar to the scalar theory case but with the constraint that additional terms have to be gauge invariant in the first instance and when the gauge is fixed they have to be BRST invariant. For a gauge theory in D dimensions where D is an integer the extra operators are no more than $(D-2)$ dimensional for the gluon and ghost sector. The upshot of this is that the structure is available from the lower-dimensional Lagrangians discussed above but with the caveat that a quark-mass operator can be included. This will be common to all gauge theories and is $(D-1)$ dimensional. We will always denote the quark mass as m_1. In four dimensions there is therefore only one dimension-two gluonic operator to be added in to $L^{(4)}$. If one requires it to be gauge invariant then one has to use $\mathcal{O}$ but weaken the locality assumption. Otherwise the only operator possible is the local BRST mass operator [100], in the massive extension of Eq. (5.1) which is

$$L_m^{(4)} = L^{(4)} + m_1 \bar{\psi}^{iI} \psi^{iI} - \frac{1}{2} m_2^2 A_\mu^a A^{a\mu} + m_2^2 \alpha \bar{c}^a c^a. \quad (5.14)$$

The pattern for six dimensions is straightforward to see and we find that the extension to Eq. (5.12) is

$$L_m^{(6)} = L^{(6)} + m_1 \bar{\psi}^{iI} \psi^{iI} - \frac{1}{4} m_2^2 G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2\alpha} m_3^2 (\partial^\mu A_\mu^a)^2 - m_3^2 \bar{c}^a (\partial^\mu D_\mu c)^a - \frac{1}{2} m_4^4 A_\mu^a A^{a\mu} + m_4^4 \alpha \bar{c}^a c^a. \quad (5.15)$$

In effect each gauge or BRST-invariant lower-dimensional operator gains a separate mass. In essence this is the coupling constant of the corresponding operator in the lower-dimensional theory and across the different dimensions these operators range from being relevant to irrelevant. While $L_m^{(4)}$ can only be extended by a BRST-invariant operator, in $L_m^{(6)}$ before gauge fixing one can have a mass associated with a gauge-invariant gluonic operator. To see the effect of such a term it is instructive to derive the propagators for $L_m^{(6)}$. We have

$$\begin{aligned}\langle A_\mu^a(p) A_\nu^b(-p) \rangle &= - \frac{\delta^{ab}}{[(p^2)^2 + m_2^2 p^2 + m_4^4]} \times \left[\eta_{\mu\nu} - \frac{[p^2 + m_3^2 - \alpha(p^2 + m_2^2)] p_\mu p_\nu}{[(p^2)^2 + m_3^2 p^2 + \alpha m_4^4]} \right], \\ \langle c^a(p) \bar{c}^b(-p) \rangle &= - \frac{\delta^{ab}}{[(p^2)^2 + m_3^2 p^2 + \alpha m_4^4]}\end{aligned} \quad (5.16)$$

for arbitrary α. Alternatively one can express the gluon propagator in terms of the respective transverse and longitudinal tensors as

$$\langle A_\mu^a(p) A_\nu^b(-p) \rangle = - \delta^{ab} \left[\frac{P_{\mu\nu}(p)}{[(p^2)^2 + m_2^2 p^2 + m_4^4]} + \frac{\alpha L_{\mu\nu}(p)}{[(p^2)^2 + m_3^2 p^2 + \alpha m_4^4]} \right] \quad (5.17)$$

where

$$P_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad L_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2}. \quad (5.18)$$

In this formulation the connection of the longitudinal part of the gluon propagator with the ghost propagator is clearer. As an aside the gluon propagator takes a simpler form in the Feynman gauge $\alpha = 1$. If in addition, for instance, it were the case that $m_2 = m_3$ then the gluon propagator would simplify further and only involve $\eta_{\mu\nu}$ similar to the completely massless theory for this specific gauge. However, this mass equality condition would require an additional symmetry in order to have this simplification. In the case when there is only a gauge-invariant dimension-four mass operator the propagators reduce to

$$\begin{aligned}\langle A_\mu^a(p) A_\nu^b(-p) \rangle |_{m_3=m_4=0} &= - \frac{\delta^{ab}}{p^2[p^2 + m_2^2]} \left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] - \alpha \delta^{ab} \frac{p_\mu p_\nu}{(p^2)^3}, \\ \langle c^a(p) \bar{c}^b(-p) \rangle |_{m_3=m_4=0} &= - \frac{\delta^{ab}}{(p^2)^2}\end{aligned} \quad (5.19)$$

so that this mass operator removes the double-pole propagator. The double pole remains in the ghost propagator to account for the corresponding pole in the longitudinal part of the gluon propagator. Another limit to consider is that of the Landau gauge as it will transpire that $\alpha = 0$ is a fixed point of the renormalization group flow. Then we have

$$\begin{aligned}\langle A_\mu^a(p) A_\nu^b(-p) \rangle |_{\alpha=0} &= - \frac{\delta^{ab}}{[(p^2)^2 + m_2^2 p^2 + m_4^4]} \left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right], \\ \langle c^a(p) \bar{c}^b(-p) \rangle |_{\alpha=0} &= - \frac{\delta^{ab}}{p^2[p^2 + m_3^2]}\end{aligned} \quad (5.20)$$

so that the gluon propagator has a denominator similar to that of a Stingl propagator [79]. The form of these massive Landau gauge propagators is interesting in respect of the current understanding of the infrared behavior of the four-dimensional gluon propagator. Briefly, lattice analyses of the gluon and Faddeev-Popov ghost propagators in the zero-momentum limit indicate that the gluon propagator freezes to a nonzero finite value while the ghost propagator behaves like $1/p^2$. This has been observed in a variety of nonperturbative studies. For instance, the present situation can be found in a representative set of articles [49–59]. This low-energy behavior has been modelled directly in four dimensions with various approaches including a modification of the Gribov Lagrangian [89,90,101]. It would be interesting to see if the lattice data could be modeled with the parametrization of Eq. (5.20). This would require appending a numerator parameter for each propagator. However, if the infrared behavior derives from a nonperturbative fixed point in four-dimensional QCD accessing it in perturbation theory will not be viable. On the contrary if a fixed point in six-dimensional QCD is in the same universality class as this infrared one in four dimensions then it may be the case that it will be computationally accessible from the higher-dimensional theory. Though it would require high-loop calculations and summation methods to quantify the qualitative behavior we have presented. Intriguingly the Schwinger-Dyson analysis of Ref. [98] produced an effective infrared QCD Lagrangian in four dimensions whose gauge-invariant part involved the two gluonic operators of Eq. (5.11) together with a mass scale necessary to balance the dimensionality. In some sense this gives weight to the idea that a perturbatively accessible fixed point of the actual six-dimensional Lagrangian of Eq. (5.11) could be in the same universality class of an infrared or nonperturbative fixed point in four-dimensional QCD. While lattice evidence [49–59], in recent years suggests a nonscaling gluon propagator in the low-momentum region, the additional freedom provided by lower-dimensional operators in Eq. (5.15), which appears to give propagators qualitatively consistent with the data, could be regarded as corrections to the scaling behavior in the neighbourhood of the fixed point. What is also apparent is the parallel relation of Eqs. (2.5) and (2.6). A toy ϕ^3 theory was examined in Refs. [67,68] as a model of QCD but equally the eight-dimensional partner has propagator structures parallel to the infrared gluon propagator behavior in Ref. [98]. Finally in comparing Eq. (5.20) with the corresponding form in the models of Ref. [101] it is interesting to contrast the nature of the operators which correspond to the masses of the gluon propagator. In Ref. [101] m_2 coupled to the dimension-two BRST-invariant gluon-mass operator, which is local in the Landau gauge, while m_4 was associated with the Landau gauge Gribov operator which is nonlocal and dimension zero.

VI. LARGE-N_f EXPANSION

As we will be using large-N_f results to compare our higher-dimensional perturbative QCD results it is worth relating relevant aspects to our Lagrangian construction. It is based on the observation of Ref. [60] that QCD and the NATM are in the same universality class. In other words the connecting interaction is the quark-gluon vertex but for the d-dimensional critical point large-N_f construction of Refs. [35,36] one has to reformulate Eq. (5.3) in a slightly different way at the outset. Beginning from Eq. (5.2) we rewrite it as

$$L^{(2)} = i\bar{\psi}^{iI}\partial\!\!\!/\psi^{iI} + \bar{\psi}^{iI}\gamma^\mu T^a_{IJ}\psi^{iJ}\tilde{A}^a_\mu - \frac{1}{2g}\tilde{A}^a_\mu\tilde{A}^{a\mu} \tag{6.1}$$

at criticality in preparation for large N_f. The main reason why the coupling constant has been rescaled into the spin-1 field is that the interaction is common to all theories in the universality class. The coupling constants have different dimensions and are themselves not universal being tied to each theory in the integer dimensions. In other words they are the couplings of different operators in the overall universal theory but their associated operator is only relevant in the critical sense in a particular spacetime dimension. A similar rescaling in $L^{(4)}$ would produce the same interaction as Eq. (6.1) but with the new coupling appearing in front of the $G^a_{\mu\nu}G^{a\mu\nu}$ term. In the following we use the same notation as Refs. [61,62]. In the limit as $N_f \to \infty$ the critical propagators behave as

$$\begin{aligned}
\langle\psi(p)\bar{\psi}(-p)\rangle &\sim \frac{A p\!\!\!/}{(p^2)^{\mu-\tilde{\alpha}}}, \\
\langle A^a_\mu(p)A^b_\nu(-p)\rangle &\sim \frac{B\delta^{ab}}{(p^2)^{\mu-\beta}}\left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right], \\
\langle c^a(p)\bar{c}^b(-p)\rangle &\sim \frac{C\delta^{ab}}{(p^2)^{\mu-\gamma}}
\end{aligned} \tag{6.2}$$

in the Landau gauge. These are the dominant scaling forms of the respective propagators. It is possible to include corrections to scaling but we omit them here [62]. The powers of the momenta in each propagator in Eq. (6.2) are the scaling dimensions of the fields and are defined as

$$\tilde{\alpha} = \mu - 1 + \frac{1}{2}\eta, \qquad \beta = 1 - \eta - \chi, \qquad \gamma = \mu - 1 + \frac{1}{2}\eta_c \tag{6.3}$$

where $d = 2\mu$ and η, χ and η_c are the critical exponents associated with the quark wave function, quark-gluon vertex operator and the Faddeev-Popov ghost wave-function renormalization group functions. On notation we use $\tilde{\alpha}$ here to avoid confusion with the gauge parameter α which was the notation for the quark dimension in the early large-N_f work [61]. The remaining parts of the scaling dimensions are the canonical dimensions of the fields as dictated by requiring that the action is dimensionless in d dimensions. Appending the ghost sector as discussed earlier there is also a ghost-gluon vertex operator anomalous dimension exponent χ_c. However, it is not independent due to the Slavnov-Taylor identity. Its manifestation at the critical point requires that [61],

$$\eta_c = \eta + \chi - \chi_c \tag{6.4}$$

is satisfied. However at leading order in $1/N_f$ there are no quark contributions to the ghost-gluon vertex and thus $\chi_{c1} = 0$. We use the notation that an exponent, such as η, is expanded as

$$\eta = \sum_{i=1}^{\infty} \frac{\eta_i}{T_F^i N_f^i}. \tag{6.5}$$

One point concerning N_f worth noting here rests in the conventions for the trace over γ matrices. Throughout we take $\mathrm{Tr} I = 4$ and retain four-dimensional γ matrices in six as well as two dimensions. This is partly because our comparison in large N_f is primarily with four-dimensional results which use this convention and the fact that we have retained that convention in our six-dimensional perturbative computations. One could of course have used higher-dimensional representations for six-dimensional γ matrices. However, that convention can be accommodated by scaling N_f itself by the appropriate factor since a closed quark loop is always associated with a γ-matrix trace. The quantities A, B and C are the associated momentum-independent amplitudes of the theory. While they can be evaluated in the large-N_f expansion they are not central to the present review.

As we will be using our results to check with the known large-N_f exponents, it is worth collecting their values for completeness here. First, the quark wave-function exponent η is given by [61],

$$\eta_1 = C_F \eta_1^{\mathrm{o}} \tag{6.6}$$

and [63],

$$\begin{aligned} \eta_2 = {} & \left[\frac{2(\mu-1)(\mu-3)}{\mu(\mu-2)} + 3\mu \left[\Theta(\mu) - \frac{1}{(\mu-1)^2} \right] \right] \frac{(\mu-1) C_F^2 \eta_1^{\mathrm{o}2}}{(\mu-2)(2\mu-1)} \\ & + \left[\frac{(12\mu^4 - 72\mu^3 + 126\mu^2 - 75\mu + 11)}{2(2\mu-1)^2(2\mu-3)(\mu-2)^2} - \frac{\mu(\mu-1)}{2(2\mu-1)(\mu-2)} [\Psi(\mu)^2 + \Phi(\mu)] \right. \\ & \left. + \frac{(8\mu^5 - 92\mu^4 + 270\mu^3 - 301\mu^2 + 124\mu - 12)\Psi(\mu)}{4(2\mu-1)^2(2\mu-3)(\mu-2)^2} \right] C_F C_A \eta_1^{\mathrm{o}2} \end{aligned} \tag{6.7}$$

where

$$\eta_1^{\mathrm{o}} = -\frac{(2\mu-1)(2-\mu)\Gamma(2\mu)}{4\mu\Gamma(2-\mu)\Gamma^3(\mu)}. \tag{6.8}$$

We have only provided the Landau-gauge expressions since that is a fixed point of the renormalization group functions and the large-N_f arbitrary gauge-dependent expression has no relation to the critical-point renormalization group functions for $\alpha \neq 0$. We have defined

$$\begin{aligned} \Theta(\mu) &= \psi'(\mu-1) - \psi'(1), \\ \Psi(\mu) &= \psi(2\mu-3) + \psi(3-\mu) - \psi(1) - \psi(\mu-1), \\ \Phi(\mu) &= \psi'(2\mu-3) - \psi'(3-\mu) - \psi'(\mu-1) + \psi'(1) \end{aligned} \tag{6.9}$$

where $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$. At leading order the gluon and ghost critical exponents are equivalent and are [61],

$$\eta + \chi = \eta_c = -\frac{C_A \eta_1^{\mathrm{o}}}{2(\mu-2) T_F N_f} + O\left(\frac{1}{T_F^2 N_f^2}\right). \tag{6.10}$$

The remaining main exponents of interest here are both gauge parameter independent but were evaluated in critical point large N_f using a scaling propagator with a nonzero gauge parameter. The first such exponent is the correction to the scaling exponent ω which is the anomalous dimension of the operator $G^a_{\mu\nu} G^{a\mu\nu}$. In other words ω relates to the β function of QCD and is the critical slope at the Wilson-Fisher fixed point. We have [62]

$$\omega = (\mu-2) - \left[(2\mu-3)(\mu-3) C_F - \frac{(4\mu^4 - 18\mu^3 + 44\mu^2 - 45\mu + 14) C_A}{4(2\mu-1)(\mu-1)} \right] \frac{\eta_1^{\mathrm{o}}}{T_F N_f} + O\left(\frac{1}{T_F^2 N_f^2}\right) \tag{6.11}$$

where the quantum electrodynamics (QED) piece was determined in Ref. [102]. Finally, the quark-mass anomalous dimension is available to two orders in large N_f and is [63],

$$\eta_{\bar{\psi}\psi 1} = -\frac{2C_F\eta_1^{\rm o}}{(\mu-2)} \tag{6.12}$$

and

$$\eta_{\bar{\psi}\psi 2} = -\frac{2\eta_2}{(\mu-2)} - \frac{2(2\mu^2-4\mu+1)C_F^2\eta_1^{\rm o2}}{(\mu-2)^3(2\mu-1)} + \frac{\mu^2(2\mu-3)^2C_FC_A\eta_1^{\rm o2}}{4(\mu-2)^3(\mu-1)(2\mu-1)} \tag{6.13}$$

where η_2 was given earlier. We note that when these exponents are expanded in $d = 4-2\epsilon$ dimensions they are in agreement with all currently available QCD renormalization group functions. This in essence is four loops but also includes the recent five-loop $\overline{\rm MS}$ quark-mass anomalous dimension of Ref. [103]. While ω corresponds to the gluonic operator of $L^{(4)}$ the exponent for the gluonic operator of $L^{(2)}$ is not independent in the Landau gauge. This is because of a Slavnov-Taylor identity [104] which means that the anomalous dimension of $\mathcal{O} = \frac{1}{2}A^a_\mu A^{a\mu}$ is the sum of the gluon and ghost anomalous dimensions. This has been verified in the Landau gauge in the large-N_f expansion [105], and in the exponent language at leading order in large N_f corresponds to

$$\eta_{\mathcal{O}1} = \eta_1 + \chi_1 - \frac{1}{2}\eta_{c1}. \tag{6.14}$$

While our focus here will mainly be on even dimensions the large-N_f exponents provide information on the odd-dimension versions of non-Abelian gauge theories. For instance, in five dimensions the above exponents evaluate to

$$\begin{aligned}
\eta &= -\frac{256C_F}{45\pi^2T_FN_f} + [200\pi^2C_A + 600\pi^2C_F - 335C_A - 8288C_F]\frac{1024C_F}{10125\pi^4T_F^2N_f^2} + O\left(\frac{1}{T_F^3N_f^3}\right), \\
\eta + \chi = \eta_c &= \frac{256C_A}{45\pi^2T_FN_f} + O\left(\frac{1}{T_F^2N_f^2}\right), \\
\omega &= \frac{1}{2} - [48C_F + 103C_A]\frac{16}{135\pi^2T_FN_f} + O\left(\frac{1}{T_F^2N_f^2}\right), \\
\eta_{\bar{\psi}\psi} &= \frac{1024C_F}{45\pi^2T_FN_f} - [600\pi^2C_A + 1800\pi^2C_F - 3005C_A - 21504C_F]\frac{4096C_F}{30375\pi^4T_F^2N_f^2} + O\left(\frac{1}{T_F^3N_f^3}\right).
\end{aligned} \tag{6.15}$$

These expressions will be of interest to any future conformal bootstrap analysis of higher-dimensional gauge theories.

VII. SIX-DIMENSIONAL QCD

We now turn to the renormalization of Eq. (5.12) at two loops in the $\overline{\rm MS}$ scheme. This required the renormalization of the three fields and two coupling constants. For the respective two- and three-point functions the graphs were generated by QGRAF and the numbers of Feynman diagrams for each are given in Table 1. Compared to the corresponding renormalization in four dimensions the number of graphs is similar. The main difference is in the triple-gluon vertex renormalization due to the presence of the quintic gluon vertex which first arises at two loops. The sextic gluon vertex will not be present until three loops. Unlike the parallel eight-dimensional scalar theory which mimics Eq. (5.12) in some ways, we do not have to consider four-point vertex functions to complete the full renormalization. For each of the two- and three-point functions we follow the same methodology and apply the Laporta algorithm as implemented in REDUZE. The main difference with the scalar theory is the presence of numerator scalar products and tensor integrals. For the latter we follow the projection method for the three three-point vertex renormalizations outlined in Ref. [106]. In other words we compute the three-point functions at a symmetric point where there is no nullification of external legs. It is important to be clear why we took this more involved route. In the renormalization of four-dimensional QCD the coupling-constant renormalization can be deduced from a three-point vertex by setting an external

TABLE I. Number of Feynman diagrams computed for each two- and three-point function.

Green's function	One loop	Two loop	Total
$A^a_\mu A^b_\nu$	3	18	21
$c^a\bar{c}^b$	1	6	7
$\psi\bar{\psi}$	1	6	7
$A^a_\mu A^b_\nu A^c_\sigma$	8	115	123
$c^a\bar{c}^bA^c_\sigma$	2	33	35
$\psi\bar{\psi}A^c_\sigma$	2	33	35
Total	17	211	228

momentum to zero. This is an infrared-safe procedure for this exceptional momentum configuration due to the presence of momenta in the numerator of the integrand. Moreover, this reduction of the computation to effectively a two-point function analysis means that the evaluation of the Feynman graphs can be computed relatively quickly. For Eq. (5.12) this nullification technique cannot be applied because the gluon and ghost propagators have higher-order poles. Therefore, at a nullification a Feynman integral will potentially have a $1/(k^2)^4$ factor like Eq. (2.6) but this cannot be infrared protected by numerator momenta in the six-dimensional gauge theory unlike the four-dimensional case. In other words such a nullification for Eq. (5.12) would require an infrared rearrangement. Therefore, we have proceeded by considering each three-point function at a nonexceptional momentum configuration which is infrared safe. Therefore we use the same decomposition and projection of each three-point vertex into the basis of Lorentz tensors given in d dimensions in Ref. [106]. For the quark and ghost vertices the process is similar to the four-dimensional case and does not deserve further comment. The complication occurs with the triple-gluon vertex. From Eq. (5.12) g_2 only appears in the Feynman rules for the gluon vertices. Therefore, it might be tempting to focus on the renormalization of g_2 solely from the triple-gluon vertex and assume that the renormalization constant for g_1 is deduced from one of the other two vertices. However, as an independent check on our FORM code and for completeness we have checked that the *same* $\overline{\text{MS}}$ renormalization constant for g_1 emerges from *each* of the three three-point functions. Moreover, we have performed the computation in an arbitrary linear covariant gauge and verified that α is absent in the β functions. This nontrivial check gives us confidence in the final expressions of the renormalization group functions.

The outcome of the computation is the renormalization group functions

$$
\begin{aligned}
\gamma_A(g_1,g_2,\alpha) &= [20\alpha C_A - 199C_A - 16N_fT_F]\frac{g_1^2}{60} + [130\alpha^2C_A^2g_1^3 + 1095\alpha C_A^2g_1^3 - 81412C_A^2g_1^3 + 2178C_A^2g_1^2g_2 + 5658C_A^2g_1g_2^2 \\
&\quad - 630C_A^2g_2^3 - 1568C_AN_fT_Fg_1^3 - 1248C_AN_fT_Fg_1^2g_2 + 192C_AN_fT_Fg_1g_2^2 - 6080C_FN_fT_Fg_1^3]\frac{g_1}{4320} + O(g_i^6), \\
\gamma_c(g_1,g_2,\alpha) &= C_A[\alpha-5]\frac{g_1^2}{12} + [-55\alpha^2C_Ag_1^2 + 60\alpha C_Ag_1^2 - 19952C_Ag_1^2 + 2700C_Ag_1g_2 + 600C_Ag_2^2 \\
&\quad - 1088N_fT_Fg_1^2]\frac{C_Ag_1^2}{8640} + O(g_i^6), \\
\gamma_\psi(g_1,g_2,\alpha) &= C_F[3\alpha+5]\frac{g_1^2}{6} + [75\alpha^2C_Ag_1^2 + 1830\alpha C_Ag_1^2 + 43617C_Ag_1^2 - 600C_Ag_2^2 - 8000C_Fg_1^2 \\
&\quad + 2048N_fT_Fg_1^2]\frac{C_Fg_1^2}{4320} + O(g_i^6)
\end{aligned} \tag{7.1}
$$

for the wave-function renormalization. In our convention the nonrenormalization of α manifests itself in the relation

$$\gamma_A(g_1,g_2,\alpha) + \gamma_\alpha(g_1,g_2,\alpha) = 0 \tag{7.2}$$

which we have checked is satisfied at two loops. The β functions are

$$
\begin{aligned}
\beta_1(g_1,g_2) &= [-249C_A - 16N_fT_F]\frac{g_1^3}{120} + [-50682C_A^2g_1^3 + 2439C_A^2g_1^2g_2 + 3129C_A^2g_1g_2^2 - 315C_A^2g_2^3 - 1328C_AN_fT_Fg_1^3 \\
&\quad - 624C_AN_fT_Fg_1^2g_2 + 96C_AN_fT_Fg_1g_2^2 - 3040C_FN_fT_Fg_1^3]\frac{g_1^2}{4320} + O(g_i^7), \\
\beta_2(g_1,g_2) &= [81C_Ag_1^3 - 552C_Ag_1^2g_2 + 135C_Ag_1g_2^2 - 15C_Ag_2^3 + 104N_fT_Fg_1^3 - 48N_fT_Fg_1^2g_2]\frac{1}{120} \\
&\quad + [10212C_A^2g_1^5 - 417024C_A^2g_1^4g_2 + 142617C_A^2g_1^3g_2^2 - 1014C_A^2g_1^2g_2^3 - 4725C_A^2g_1g_2^4 \\
&\quad + 450C_A^2g_2^5 - 7052N_fT_FC_Ag_1^5 - 20296N_fT_FC_Ag_1^4g_2 + 8868C_AN_fT_Fg_1^3g_2^2 \\
&\quad - 1056C_AN_fT_Fg_1^2g_2^3 + 61600N_fT_FC_Fg_1^5 - 30400N_fT_FC_Fg_1^4g_2]\frac{1}{14400} + O(g_i^7).
\end{aligned} \tag{7.3}
$$

The one-loop term of $\beta_1(g_1,g_2)$ is clearly negative for all N_f unlike four dimensions and therefore in six dimensions the quark-gluon coupling is asymptotically free. The corresponding one-loop result in six-dimensional QED was recently given in Ref. [65] with which we agree.

In order to provide more checks on the connection of Eq. (5.12) with lower-dimensional gauge theories at the Wilson-Fisher fixed point we have also computed the quark-mass operator anomalous dimension at two loops. To do this we inserted the mass operator $\bar{\psi}\psi$ in a quark two-point function but such that there is a momentum flowing into the operator itself similar to the parallel scalar theory calculation. At one and two loops there are 1 and 13 graphs respectively. For $\alpha \neq 0$ we find a gauge-parameter-independent $\overline{\rm MS}$ expression for the quark-mass operator anomalous dimension since

$$\gamma_{\bar{\psi}\psi}(g_1,g_2) = -\frac{5}{3}C_F g_1^2 + [-11301 C_A g_1^2 + 300 C_A g_2^2 - 200 C_F g_1^2 - 544 N_f T_F g_1^2]\frac{C_F g_1^2}{1080} + O(g_i^6). \tag{7.4}$$

This expression was derived from the massless version of six-dimensional QCD [Eq. (5.12)]. As it is possible to include lower-dimensional operators with associated masses, we have also determined the renormalization of m_i in Eq. (5.15) at one loop in the Landau gauge. This choice of gauge is motivated by the potential connection with the infrared structure in four dimensions. In this instance the presence of four mass terms means that we have to determine the mixing matrix of mass anomalous dimensions which limits this analysis to the leading order. However, that is sufficient to form a picture of how the masses relate under renormalization. If we formally label the operators by the label of the associated mass as given in Eq. (5.15) then we find that the mixing matrix, $\gamma_{ij}(g_1,g_2,\alpha)$, is sparse at one loop and the only nonzero elements are

$$\begin{aligned}
\gamma_{11}(g_1,g_2,0) &= -\frac{5}{3}C_F g_1^2 + O(g_i^4),\\
\gamma_{21}(g_1,g_2,0) &= \frac{4}{3}T_F N_f g_1^2 + O(g_i^4),\\
\gamma_{22}(g_1,g_2,0) &= -\frac{2}{3}C_A g_2^2 - 2C_A g_1 g_2 - \frac{4}{15}T_F N_f g_1^2 \\
&\quad + \frac{281}{60}C_A g_1^2 + O(g_i^4),\\
\gamma_{44}(g_1,g_2,0) &= -\frac{2}{15}T_F N_f g_1^2 - \frac{28}{15}C_A g_1^2 + O(g_i^4).
\end{aligned} \tag{7.5}$$

One feature of the result is that $\gamma_{44}(g_1,g_2,0)$ satisfies

$$\gamma_{44}(g_1,g_2,0) = \frac{1}{2}[\gamma_A(g_1,g_2,0) + \gamma_c(g_1,g_2,0)] + O(g_i^4) \tag{7.6}$$

parallel to the corresponding four-dimensional relation. This six-dimensional result is consistent with the large-N_f exponent.

One of the motivations for studying Eq. (5.12) is to establish the connection of four-dimensional QCD with a higher-dimensional gauge theory in the Wilson-Fisher chain. To access the large-N_f exponents of the previous section we set $d = 6 - 2\epsilon$ and follow the algorithm of Ref. [17]. First, we define scaled couplings by

$$g_1 = \frac{i}{2}\sqrt{\frac{15\epsilon}{T_F N_f}}x, \qquad g_2 = \frac{i}{2}\sqrt{\frac{15\epsilon}{T_F N_f}}y \tag{7.7}$$

and solve $\beta_i(g_1,g_2) = 0$ for the critical values of x and y to $O(\epsilon^2)$. We find

$$\begin{aligned}
x &= 1 + \left[-\frac{249}{32}C_A + \left[\frac{475}{48}C_F + \frac{5855}{768}C_A\right]\epsilon\right]\frac{1}{T_F N_f}\\
&\quad + \left[\frac{186003}{2048}C_A^2 + \left[-\frac{197125}{512}C_A C_F - \frac{7530655}{32768}C_A^2\right]\epsilon\right]\frac{1}{T_F^2 N_f^2} + O\left(\epsilon^2;\frac{1}{T_F^3 N_f^3}\right),\\
y &= \frac{13}{4} + \left[-\frac{51327}{2048}C_A + \left[\frac{2325}{64}C_F + \frac{62385}{4096}C_A\right]\epsilon\right]\frac{1}{T_F N_f} + O\left(\epsilon^2;\frac{1}{T_F^2 N_f^2}\right).
\end{aligned} \tag{7.8}$$

Equipped with these we have expanded out the other renormalization group functions (7.1), to the same orders as the available exponents in the Landau gauge in both ϵ and $1/N_f$ and found full agreement. Another check derives from $\gamma_{22}(g_1,g_2,0)$ of Eq. (7.6) which corresponds to the renormalization of the mass associated with the field-strength operator in Eq. (5.12). In four dimensions this operator would be the gluon kinetic term and its large-N_f critical exponent, ω, relates to the critical slope of the four-dimensional QCD β function. Expanding ω_1 in Eq. (6.11) to $O(\epsilon)$ near six dimensions we get precise agreement. For this element of Eq. (7.6) at one loop it will be an eigen-anomalous dimension as there are no other entries at this order in the matrix. At higher order mixing with this operator would require diagonalizing Eq. (7.6). The agreement of the perturbative results with the large-N_f exponents is important for various reasons. For instance, it demonstrates that the role of the spectator operator with coupling g_2 is crucial in getting agreement. For instance, the presence of g_2 in $\gamma_{22}(g_1, g_2, 0)$ is necessary for the check with ω to work at leading order in ϵ. This spectator operator is present to ensure renormalizability in six dimensions but would be irrelevant in lower dimensions at the Gaussian fixed point. That the exponents can be derived in the large-N_f expansion form a critical theory with only a quark-gluon interaction is remarkable in some sense. Moreover it substantiates the point of view of Ref. [60] that the triple- and higher-leg gluon interactions derive from three-point and higher Green's functions with only quark loops and no gluon interactions. That this picture extends to six dimensions establishes the same point of view for quintic gluon interactions in Eq. (5.12).

Having established the connection with a lower-dimensional gauge theory we now turn to the analysis of the six-dimensional renormalization group functions in their own right. One of the interests in higher-dimensional cubic scalar theories was to ascertain where the conformal window existed if present at all. In four-dimensional QCD this equates to the range of N_f for which a Banks-Zaks fixed point is present [1]. Therefore we proceed by solving

$$\beta_1(g_1, g_2) = \beta_2(g_1, g_2) = 0, \qquad \frac{\partial\beta_1}{\partial g_1}\frac{\partial\beta_2}{\partial g_2} - \frac{\partial\beta_1}{\partial g_2}\frac{\partial\beta_2}{\partial g_1} = 0. \tag{7.9}$$

The first two determine the location of zeros of the β functions while the third is the condition for a zero eigenvalue in the matrix of β-function slopes. Like Ref. [17] we find three solutions one of which is a real solution and two other two are complex conjugates. The real solution is

$$\begin{aligned} N_{f(A)} &= 2.797566\frac{C_A}{T_F} + [2.198165C_F - 3.432003C_A]\frac{\epsilon}{T_F} + O(\epsilon^2), \\ x_{(A)} &= 0.390349 + [0.162047C_F - 0.064751C_A]\frac{\epsilon}{C_A} + O(\epsilon^2), \\ y_{(A)} &= 0.965498 + [0.412927C_A + 0.185332C_F]\frac{\epsilon}{C_A} + O(\epsilon^2) \end{aligned} \tag{7.10}$$

and the other two are

$$\begin{aligned} N_{(B)} &= [3.283595 + 0.660678i]\frac{C_A}{T_F} + [[-2.089275 - 3.907327i]C_A + [3.737235 + 1.869896i]C_F]\frac{\epsilon}{T_F} + O(\epsilon^2), \\ x_{(B)} &= 0.0344173i + 0.420036 + [[-0.039289 - 0.150826i]C_A + [0.244905 + 0.074815i]C_F]\frac{\epsilon}{C_A} + O(\epsilon^2), \\ y_{(B)} &= 1.467391 - 0.100116i + [[-0.079478 - 0.487643i]C_A + [1.077389 - 0.083226i]C_F]\frac{\epsilon}{C_A} + O(\epsilon^2) \end{aligned} \tag{7.11}$$

and its complex conjugate denoted by C. For reference the real solution for $SU(3)$ is

$$\begin{aligned} N_{f(A)}|_{SU(3)} &= 16.785398 - 14.730246\epsilon + O(\epsilon^2), \\ x_{(A)}|_{SU(3)} &= 0.390349 + 0.007270\epsilon + O(\epsilon^2), \\ y_{(A)}|_{SU(3)} &= 0.965498 + 0.495297\epsilon + O(\epsilon^2). \end{aligned} \tag{7.12}$$

Interestingly the location of the conformal window in purely six dimensions is between $N_f = 16$ and 17 similar to four-dimensional QCD. However, in Ref. [17] the higher-dimensional theory and the ϵ expansion were used to estimate the boundary of the window in a lower dimension by summation. If we consider that approach the two-loop correction to the real solution, $N_{f(A)}$, is comparable to the one-loop part. This suggests that perturbation theory may not be reliable. However, using a simple Padé approximant, which is possible due to the negative correction, in four dimensions we find $N_{f(A)} = 8.939991$. This is lower than the leading order, and similar to the situation in scalar $O(N)$ ϕ^3 theory. It would be interesting to see what effect the three-loop corrections would have on this critical N_f value.

Having found the region where there is a conformal window it is worth analyzing the renormalization group functions within this for specified values of N_f. We take $N_f = 3$, 12 and 16. These choices are motivated by values in four dimensions. For instance, the first is because it corresponds to the number of light quarks. The value of 16 is chosen since it is the largest within the six-dimensional conformal window. Finally, we consider $N_f = 12$ since there is interest in four-dimensional theories with this value due to trying to understand the Banks-Zaks fixed point nonperturbatively on the lattice. For each of the cases there are four solutions to the equations

$$\beta_1(g_1, g_2) = 0, \qquad \beta_2(g_1, g_2) = 0 \tag{7.13}$$

for a particular value of N_f excluding the trivial one. In this counting we ignore solutions which are obtained from these by reflections $g_i \to -g_i$. For each value of N_f we give the location of the fixed point in terms of x and y and the renormalization group functions evaluated at each fixed point in the Landau gauge. Included in this are the eigen-critical exponents $\omega_\pm$ which are the eigenvalues of the matrix $\frac{\partial\beta_i}{\partial g_j}$. The signs of these exponents determine the stability or otherwise of the fixed point. In our labeling of the four nontrivial solutions for each N_f value, solutions 1 and 3 are stable while 2 and 4 are saddle points. For solutions 1, 2 and 3 the first term of the same exponent in each solution is the same. This is because the one-loop term of the corresponding renormalization group function only depends on g_1 and there is no g_2 dependence in the one-loop term of $\beta_1(g_1, g_2)$. The values for the exponents begin to differ at $O(\epsilon^2)$ due to g_2 appearing in the two-loop expressions. This is the main reason for a two-loop renormalization as a one-loop analysis would not reveal distinctive differences. The solution labeled 4 is somewhat different in that it corresponds to $g_1 = 0$. In effect there are no quarks or Faddeev-Popov ghosts in the corresponding Lagrangian but only the three-leg gauge-invariant operator. Also the kinetic term for the gluon derives from the free part of $\mathcal{O}_1^{(6)}$. In some sense this solution is not interesting as all the critical exponents are zero except $\omega_\pm$ which take their canonical values of 2ϵ and $-\epsilon$ respectively. Therefore, solution 4 appears to be in effect a free-field solution. So we do not explicitly record any exponent values for solution 4.

More specifically our results for $N_f = 3$ are

$$\begin{aligned}
x_{(1)} &= 0.176432 - 0.003730\epsilon + O(\epsilon^2), \\
y_{(1)} &= 0.936586 + 0.548703\epsilon + O(\epsilon^2), \\
\gamma_A(g_1, g_2, 0)|_{(1)} &= 0.805447\epsilon - 0.127340\epsilon^2 + O(\epsilon^2), \\
\gamma_c(g_1, g_2, 0)|_{(1)} &= 0.097276\epsilon + 0.063670\epsilon^2 + O(\epsilon^2), \\
\gamma_\psi(g_1, g_2, 0)|_{(1)} &= -0.086468\epsilon + 0.139232\epsilon^2 + O(\epsilon^2), \\
\gamma_{\bar\psi\psi}(g_1, g_2, 0)|_{(1)} &= 0.172936\epsilon - 0.079266\epsilon^2 + O(\epsilon^2), \\
\omega_+|_{(1)} &= 2.000000\epsilon + 0.084558\epsilon^2 + O(\epsilon^3), \\
\omega_-|_{(1)} &= 0.598467\epsilon + 0.971180\epsilon^2 + O(\epsilon^3), \\
x_{(2)} &= 0.176432 + 0.052245\epsilon + O(\epsilon^2), \\
y_{(2)} &= 0.558153 + 0.113962\epsilon + O(\epsilon^2), \\
\gamma_A(g_1, g_2, 0)|_{(2)} &= 0.805447\epsilon - 0.040157\epsilon^2 + O(\epsilon^2), \\
\gamma_c(g_1, g_2, 0)|_{(2)} &= 0.097276\epsilon + 0.020079\epsilon^2 + O(\epsilon^2), \\
\gamma_\psi(g_1, g_2, 0)|_{(2)} &= -0.086468\epsilon + 0.145505\epsilon^2 + O(\epsilon^2), \\
\gamma_{\bar\psi\psi}(g_1, g_2, 0)|_{(2)} &= 0.172936\epsilon - 0.091812\epsilon^2 + O(\epsilon^2), \\
\omega_+|_{(2)} &= 2.000000\epsilon - 1.184488\epsilon^2 + O(\epsilon^3), \\
\omega_-|_{(2)} &= -0.329946\epsilon - 0.055233\epsilon^2 + O(\epsilon^3), \\
x_{(3)} &= 0.176432 + 0.111433\epsilon + O(\epsilon^2), \\
y_{(3)} &= 0.093152 + 0.204520\epsilon + O(\epsilon^2), \\
\gamma_A(g_1, g_2, 0)|_{(3)} &= 0.805447\epsilon - 0.007256\epsilon^2 + O(\epsilon^2), \\
\gamma_c(g_1, g_2, 0)|_{(3)} &= 0.097276\epsilon + 0.003628\epsilon^2 + O(\epsilon^2), \\
\gamma_\psi(g_1, g_2, 0)|_{(3)} &= -0.086468\epsilon + 0.120224\epsilon^2 + O(\epsilon^2), \\
\gamma_{\bar\psi\psi}(g_1, g_2, 0)|_{(3)} &= 0.172936\epsilon - 0.041251\epsilon^2 + O(\epsilon^2), \\
\omega_+|_{(3)} &= 2.000000\epsilon - 2.526371\epsilon^2 + O(\epsilon^3), \\
\omega_-|_{(3)} &= 0.735370\epsilon - 0.731156\epsilon^2 + O(\epsilon^3), \\
x_{(4)} &= O(\epsilon^2), \\
y_{(4)} &= 0.730297 - 0.365148\epsilon + O(\epsilon^2).
\end{aligned} \tag{7.14}$$

When $N_f = 12$ we have

$$\begin{aligned}
x_{(1)} &= 0.337460 + 0.040482\epsilon + O(\epsilon^2), \\
y_{(1)} &= 1.540384 + 1.051213\epsilon + O(\epsilon^2), \\
\gamma_A(g_1, g_2, 0)|_{(1)} &= 0.822064\epsilon - 0.071166\epsilon^2 + O(\epsilon^2), \\
\gamma_c(g_1, g_2, 0)|_{(1)} &= 0.088968\epsilon + 0.035583\epsilon^2 + O(\epsilon^2), \\
\gamma_\psi(g_1, g_2, 0)|_{(1)} &= -0.079083\epsilon + 0.129510\epsilon^2 + O(\epsilon^2), \\
\gamma_{\bar\psi\psi}(g_1, g_2, 0)|_{(1)} &= 0.158165\epsilon - 0.078886\epsilon^2 + O(\epsilon^2), \\
\omega_+|_{(1)} &= 2.00000\epsilon - 0.479845\epsilon^2 + O(\epsilon^3), \\
\omega_-|_{(1)} &= 0.256816\epsilon + 0.215915\epsilon^2 + O(\epsilon^3), \\
x_{(2)} &= 0.337460 + 0.108330\epsilon + O(\epsilon^2), \\
y_{(2)} &= 1.030159 + 0.255942\epsilon + O(\epsilon^2), \\
\gamma_A(g_1, g_2, 0)|_{(2)} &= 0.822064\epsilon - 0.026704\epsilon^2 + O(\epsilon^2), \\
\gamma_c(g_1, g_2, 0)|_{(2)} &= 0.088968\epsilon + 0.013352\epsilon^2 + O(\epsilon^2), \\
\gamma_\psi(g_1, g_2, 0)|_{(2)} &= -0.079083\epsilon + 0.130122\epsilon^2 + O(\epsilon^2), \\
\gamma_{\bar\psi\psi}(g_1, g_2, 0)|_{(2)} &= 0.158165\epsilon - 0.080112\epsilon^2 + O(\epsilon^2), \\
\omega_+|_{(2)} &= 2.000000\epsilon - 1.284069\epsilon^2 + O(\epsilon^3), \\
\omega_-|_{(2)} &= -0.134787\epsilon + 0.145784\epsilon^2 + O(\epsilon^3), \\
x_{(3)} &= 0.337460 + 0.174074\epsilon + O(\epsilon^2), \\
y_{(3)} &= 0.466593 + 1.074306\epsilon + O(\epsilon^2),
\end{aligned}$$

$$
\begin{aligned}
\gamma_A(g_1,g_2,0)|_{(3)} &= 0.822064\epsilon - 0.001544\epsilon^2 + O(\epsilon^2),\\
\gamma_c(g_1,g_2,0)|_{(3)} &= 0.088968\epsilon + 0.000772\epsilon^2 + O(\epsilon^2),\\
\gamma_\psi(g_1,g_2,0)|_{(3)} &= -0.079083\epsilon + 0.120155\epsilon^2 + O(\epsilon^2),\\
\gamma_{\bar\psi\psi}(g_1,g_2,0)|_{(3)} &= 0.158165\epsilon - 0.060177\epsilon^2 + O(\epsilon^2),\\
\omega_+|_{(3)} &= 2.000000\epsilon - 2.063346\epsilon^2 + O(\epsilon^3),\\
\omega_-|_{(3)} &= 0.283665\epsilon - 0.844111\epsilon^2 + O(\epsilon^3),\\
x_{(4)} &= O(\epsilon^2),\\
y_{(4)} &= 1.460593 - 0.730297\epsilon + O(\epsilon^2).
\end{aligned}
\tag{7.15}
$$

Finally,

$$
\begin{aligned}
x_{(1)} &= 0.382473 + 0.070883\epsilon + O(\epsilon^2),\\
y_{(1)} &= 1.584443 + 0.868230\epsilon + O(\epsilon^2),\\
\gamma_A(g_1,g_2,0)|_{(1)} &= 0.828571\epsilon - 0.050107\epsilon^2 + O(\epsilon^2),\\
\gamma_c(g_1,g_2,0)|_{(1)} &= 0.085714\epsilon + 0.025053\epsilon^2 + O(\epsilon^2),\\
\gamma_\psi(g_1,g_2,0)|_{(1)} &= -0.076190\epsilon + 0.125124\epsilon^2 + O(\epsilon^2),\\
\gamma_{\bar\psi\psi}(g_1,g_2,0)|_{(1)} &= 0.152381\epsilon - 0.077478\epsilon^2 + O(\epsilon^2),\\
\omega_+|_{(1)} &= 2.000000\epsilon - 0.741312\epsilon^2 + O(\epsilon^3),\\
\omega_-|_{(1)} &= 0.144292\epsilon - 0.115028\epsilon^2 + O(\epsilon^3),\\
x_{(2)} &= 0.382473 + 0.135377\epsilon + O(\epsilon^2),\\
y_{(2)} &= 1.067822 - 0.219836\epsilon + O(\epsilon^2),\\
\gamma_A(g_1,g_2,0)|_{(2)} &= 0.828571\epsilon - 0.017142\epsilon^2 + O(\epsilon^2),\\
\gamma_c(g_1,g_2,0)|_{(2)} &= 0.085714\epsilon + 0.008571\epsilon^2 + O(\epsilon^2),\\
\gamma_\psi(g_1,g_2,0)|_{(2)} &= -0.076190\epsilon + 0.123897\epsilon^2 + O(\epsilon^2),\\
\gamma_{\bar\psi\psi}(g_1,g_2,0)|_{(2)} &= 0.152381\epsilon - 0.075023\epsilon^2 + O(\epsilon^2),\\
\omega_+|_{(2)} &= 2.000000\epsilon - 1.415811\epsilon^2 + O(\epsilon^3),\\
\omega_-|_{(2)} &= -0.050461\epsilon + 0.434552\epsilon^2 + O(\epsilon^3),\\
x_{(3)} &= 0.382473 + 0.166952\epsilon + O(\epsilon^2),\\
y_{(3)} &= 0.789993 + 2.368779\epsilon + O(\epsilon^2),\\
\gamma_A(g_1,g_2,0)|_{(3)} &= 0.828571\epsilon - 0.005495\epsilon^2 + O(\epsilon^2),\\
\gamma_c(g_1,g_2,0)|_{(3)} &= 0.085714\epsilon + 0.002748\epsilon^2 + O(\epsilon^2),\\
\gamma_\psi(g_1,g_2,0)|_{(3)} &= -0.076190\epsilon + 0.120535\epsilon^2 + O(\epsilon^2),\\
\gamma_{\bar\psi\psi}(g_1,g_2,0)|_{(3)} &= 0.152381\epsilon - 0.068298\epsilon^2 + O(\epsilon^2),\\
\omega_+|_{(3)} &= 2.000000\epsilon - 1.746028\epsilon^2 + O(\epsilon^3),\\
\omega_-|_{(3)} &= 0.077597\epsilon - 0.989589\epsilon^2 + O(\epsilon^3),\\
x_{(4)} &= O(\epsilon^2),\\
y_{(4)} &= 1.686548 - 0.843274\epsilon + O(\epsilon^2)
\end{aligned}
\tag{7.16}
$$

for $N_f = 16$. For $N_f > 16$ there are two real solutions and two complex-conjugate solutions ignoring the reflection symmetry. For the real solutions one is stable while the other is a saddle point. The former has a nonzero value for g_1 at criticality and is the solution which in effect corresponds to the large-N_f solution. The other real solution is the effective free-field solution as it corresponds to $g_1 = 0$.

VIII. HIGHER-DIMENSIONAL QED

Having concentrated for the most part on non-Abelian gauge theories we devote the remainder of our analysis to Abelian theories in six and higher dimensions. One of the reasons for this is that the analysis is more straightforward due to fewer interactions and also because of recent activity in this area [64,65]. The easier calculability has allowed the authors of Refs. [64,65] to extract interesting features of the F-theorem in higher-dimensional Abelian gauge theories which may be shared with non-Abelian ones. Based on our earlier considerations the six-dimensional QED Lagrangian is [65],

$$
\begin{aligned}
L^{(6)}|_{\mathrm{QED}} = &-\frac{1}{4}(\partial_\mu F_{\nu\sigma})(\partial^\mu F^{\nu\sigma})\\
&-\frac{1}{2\alpha}(\partial_\mu\partial^\nu A_\nu)(\partial^\mu\partial^\sigma A_\sigma) + i\bar\psi^i \not{D}\psi^i.
\end{aligned}
\tag{8.1}
$$

The main differences are the absence of the three-point operator with coupling g_2 which was proportional to the color group structure functions and the replacement of the covariant derivative in the gauge field kinetic term by the partial derivative. The gauge-fixing term is similar to QCD but in a linear covariant gauge there are no Faddeev-Popov ghosts. The upshot is that we have renormalized Eq. (8.1) to *three* loops in the $\overline{\mathrm{MS}}$ scheme. We find

$$
\begin{aligned}
\gamma_A(g_1,\alpha) = &-\frac{4}{15}N_f g_1^2 - \frac{38}{27}N_f g_1^4\\
&+ 17N_f[200 - 111N_f]\frac{g_1^6}{6075} + O(g_1^8),\\
\gamma_\psi(g_1,\alpha) = &\,[3\alpha+5]\frac{g_1^2}{6} + 2[32N_f - 125]\frac{g_1^4}{135}\\
&+ [2864N_f^2 - 648000\zeta_3 N_f + 730375N_f\\
&+ 1944000\zeta_3 - 1033000]\frac{g_1^6}{243000} + O(g_1^8),\\
\beta_1(g_1) = &-\frac{2}{15}N_f g_1^3 - \frac{19}{27}N_f g_1^5\\
&+ 17N_f[200 - 111N_f]\frac{g_1^7}{12150} + O(g_1^9)
\end{aligned}
\tag{8.2}
$$

where we confirm the one-loop asymptotically free β function of Ref. [65]. To derive these expressions we have independently renormalized the photon two-point function and the electron-photon vertex separately so that the Ward-Takahashi identity

$$\beta_1(g_1) = \frac{g_1}{2}\gamma_A(g_1,\alpha) \tag{8.3}$$

emerges naturally and plays the role of a computational check. One advantage of considering the Abelian theory is that there are no triple or quartic photon vertices. Thus we can access the vertex renormalization by the method discussed in Ref. [33]. There six-dimensional ϕ^3 theory was renormalized to four loops by considering only two-point functions. The three-point Green's functions with zero momentum insertions were generated by expanding the massless propagator with the appropriate Feynman rule for the insertion. Also for Eq. (8.1) nullification does not involve a vertex with three photons. So no infrared problems arise which prevented us from using this approach in QCD. Overall this reduces the number of graphs to be evaluated. In addition we have also determined the $\overline{\text{MS}}$ electron-mass anomalous dimension which is

$$\gamma_{\bar\psi\psi}(g_1) = -\frac{5g_1^2}{3} + [-68N_f - 25]\frac{g_1^4}{135} + [13456N_f^2 + 648000\zeta_3 N_f - 818575N_f + 1215000\zeta_3 - 726875]\frac{g_1^6}{121500} + O(g_1^8). \tag{8.4}$$

From the three-loop results, there are several interesting features. First, we have computed all renormalization group functions in terms of a nonzero α. However, from Eq. (8.2) the only place where α appears in these $\overline{\text{MS}}$ results is in the one-loop term of the electron wave-function anomalous dimension. Clearly in $\overline{\text{MS}}$ the β function will be α independent which by the Ward-Takahashi identity means that the photon anomalous dimension is independent of the gauge parameter. The absence of α beyond one loop in $\gamma_\psi(g_1,\alpha)$ might be surprising if it was not in fact completely parallel to the situation in four dimensions. Indeed from explicit four-loop computations α is absent after one loop [107]. In Refs. [108,109] an argument was given which suggested that to all orders α appears only in the one-loop term. The fact that such a property seems to be present in six dimensions suggests that the result is independent of dimension. The next observation concerns the β function which is that each term is negative for $N_f > 1$. When $N_f = 1$ there is a pseudo-Banks-Zaks fixed point at $g_1 = 2.415479$. We have attributed it as a nonstandard Banks-Zaks fixed point as it derives from an imbalance of the signs of the first three terms rather than the one- and two-loop terms in the QCD case. As such it may not survive in a four-loop analysis. As a check on our three-loop results we have verified that the critical exponents determined at the Wilson-Fisher fixed point agree precisely with the corresponding ones determined at various orders in the large-N_f expansion [61,62,102] with those exponents being expanded in the neighborhood of six dimensions. This is another reason why we considered the Abelian theory at one loop order higher than the non-Abelian extension. It was important to put a six-dimensional gauge theory on the same footing as scalar ϕ^3 theory. In other words there is a tower of gauge theories driven by a common interaction and underlying symmetry.

One observation which has been made in the context of the tower of theories in d dimensions is that at the Wilson-Fisher fixed point one of the two connecting theories is asymptotically free while the other is nonasymptotically free [65]. In other words there is a type of ultraviolet/infrared duality across the dimensions such that the infrared fixed point of one is an ultraviolet fixed point of the other. In the $O(N)$ scalar theory case the nonlinear σ model is asymptotically free in two dimensions whereas ϕ^4 theory is not in four dimensions. The six-dimensional partner is asymptotically free although it is not immediately clear if this is the case in the eight-dimensional cousin (2.6). This is because asymptotic freedom usually refers to the theory with a single scalar field and no $O(N)$ symmetry. In the theories in six and lower dimensions they all have a single coupling in that instance. In the eight-dimensional scalar theory case in the absence of the $O(N)$ symmetry there are two couplings. Setting $g_1 = 0$ in Eq. (3.1) both one-loop terms of $\beta_2(g_1,g_2,g_3)$ and $\beta_3(g_1,g_2,g_3)$ are positive. So in this instance it appears that the base eight-dimensional theory is not asymptotically free. As noted in Ref. [65] a similar picture is present in the QED tower. In four dimensions QED is not asymptotically free whereas in six dimensions it is [65]. As we have considered the eight-dimensional $O(N)$ scalar theory extension it is worthwhile repeating the exercise for QED in eight dimensions. To write down the Lagrangian one has to follow our earlier prescription which requires extra interactions akin to the situation for Eq. (2.6). We have

$$L^{(8)}|_{\text{QED}} = -\frac{1}{4}(\partial_\mu\partial_\nu F_{\sigma\rho})(\partial^\mu\partial^\nu F^{\sigma\rho}) - \frac{1}{2\alpha}(\partial_\mu\partial^\nu A_\nu)(\partial^\mu\partial^\sigma A_\sigma) + i\bar\psi^i \not{D}\psi^i + \frac{g_2^2}{32}F_{\mu\nu}F^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} + \frac{g_3^2}{8}F_{\mu\nu}F^{\mu\sigma}F_{\nu\rho}F^{\sigma\rho}. \tag{8.5}$$

The corresponding QCD Lagrangian would be much more involved. For instance, dimension-eight and -ten gluonic operators were considered for $SU(N_c)$ gauge theories in Ref. [66]. Equipped with Eq. (8.5) we have found that the renormalization group functions to a similar order as Eq. (2.6) are

$$
\begin{aligned}
\gamma_A(g_1,g_2,g_3,\alpha) &= \frac{N_f g_1^2}{35} + \frac{11 N_f g_1^4}{120} + O(g_i^6), \\
\gamma_\psi(g_1,g_2,g_3,\alpha) &= [2\alpha+7]\frac{g_1^2}{12} + [-964N_f - 13475]\frac{g_1^4}{33600} + O(g_i^6), \\
\beta_1(g_1,g_2,g_3) &= \frac{N_f g_1^3}{70} + \frac{11 N_f g_1^5}{240} + O(g_i^7), \\
\beta_2(g_1,g_2,g_3) &= [1120 g_1^4 N_f + 72 g_1^2 g_2^2 N_f - 861 g_2^4 - 1659 g_2^2 g_3^2 - 609 g_3^4]\frac{1}{1260} + O(g_i^6), \\
\beta_3(g_1,g_2,g_3) &= [-1568 g_1^4 N_f + 144 g_1^2 g_3^2 N_f - 21 g_2^4 - 294 g_2^2 g_3^2 - 1029 g_3^4]\frac{1}{2520} + O(g_i^6), \\
\gamma_{\bar\psi\psi}(g_1,g_2,g_3) &= -\frac{7g_1^2}{12} + [2052 N_f - 1225]\frac{g_1^4}{100800} + O(g_i^6).
\end{aligned} \tag{8.6}
$$

The structure of these functions is different from those of Eq. (2.6). The absence of a triple photon vertex means that at one loop there are no g_2 or g_3 couplings in $\beta_1(g_1,g_2,g_3)$. That this persists at two loops is somewhat surprising given that there is one topology which involves a quartic photon vertex in the electron-photon vertex function. It transpires that the graph is finite. Equally the two-loop photonic sunset graph in the photon two-point function is also finite which ensures the Ward-Takahashi identity is not violated. The absence of g_2 and g_3 dependence at least to two loops in $\beta_1(g_1,g_2,g_3)$ exposes the nonasymptotic freedom of eight-dimensional QED. Next we note that at least at two loops the electron anomalous dimension has no gauge parameter dependence in the two-loop term. While this is not inconsistent with the lower-dimensional observations it again lends some weight to the one-loop α dependence being dimension independent. The final comment on Eq. (8.6) is that we have again verified that the critical exponents at the Wilson-Fisher fixed point are in exact agreement with the exponents from the large-N_f expansion when evaluated near eight dimensions.

In light of these latter remarks it is worth making a few brief comments about what lies beyond eight dimensions for QED. For instance, one can try and address the issue of asymptotic freedom in higher dimensions by exploiting properties of the gauge theory which are not present in a scalar theory. One feature in QED is that the β function of the gauge coupling to matter can be deduced from the photon two-point function. At one loop the graph does not involve photon propagators. This is under the assumption that there are no triple photon vertices. If such a three-point vertex is present then the following argument will be invalid. However, if the only three-point vertex is the electron-photon one then from the photon two-point function the one-loop β function is

$$\beta_1^{(D)}(g_1,g_2,\ldots) = \frac{2(-1)^{D/2}\Gamma(\frac{1}{2}D)N_f g_1^3}{(D-1)\Gamma(D-2)} + O(g_i^5) \tag{8.7}$$

in $d = D - 2\epsilon$ where D is an even integer with $D > 2$. The expression tallies with the known results up to eight dimensions. Under the assumption we have made it is evident that QED yoyos between being asymptotically free and not being asymptotically free. The origin of the varying sign is the residue of the simple poles in the Euler Γ-function when one expands around the appropriate simple pole in ϵ to determine the photon wave-function renormalization constant and via the Ward-Takahashi identity the β function of g_1. In using Eq. (8.7) it is important to realize that it is only valid for even integers larger than two. It cannot be used in the intervening continuous dimensions and expressed in terms of a regularizing parameter which has already been used to determine Eq. (8.7) in the $\overline{\text{MS}}$ scheme.

IX. DISCUSSION

We make some closing observations. First, we have achieved one of the main goals which was to construct and establish higher-dimensional field theories which lie in the same universality class as already well-established theories at the Wilson-Fisher fixed point. The process is based on a common interaction which underpins each Lagrangian in a chain as well as renormalizability. Aside from the fields being in the same symmetry groups, one consequence is the appearance of extra interactions over and above the core one connecting all candidates. These spectator interactions play a key role in ensuring d-dimensional equivalences. In their critical dimension the extra coupling constants produce a rich spectrum of fixed points and if analyzed in d dimensions several of these may be connected to nontrivial and perhaps nonperturbative fixed points in the companion lower-dimensional model. One hint of this, for example, may be in the infrared behavior of the four-dimensional gluon propagator. In the Landau gauge it has been shown to freeze to a finite nonzero value at zero momentum in lattice analyses over recent years [49–59]. Such behavior for the gluon and Faddeev-Popov ghost propagators can be mimicked from the six-dimensional gauge theory if one allows for the presence of lower-dimensional operators in the Lagrangian with associated masses. While this approach should be regarded as a model it may be indicative that higher-dimensional operators, including the six-dimensional ones considered here, could become relevant in the critical sense and be the dominant operators driving the gluon propagator infrared behavior. There is evidence from a Schwinger-Dyson analysis to support this [98]. On a related issue we have established that six-dimensional QCD has asymptotic freedom. So this theory is potentially another where the issues of color confinement could be investigated especially as its Abelian partner is also asymptotically free but probably does not have confinement. At this stage it is still perhaps premature to think that links with lower-dimensional nonperturbative fixed points have been fully established. This is primarily because in the gauge theory we only performed the renormalization to two loops. This was mainly to demonstrate the viability of the approach. A one-loop computation would not really have been sufficient since in the critical dimension the effect of the spectator interaction coupling constant does not appear in anomalous dimensions until two loops. Going beyond two loops is possible but not as straightforward as for a four-dimensional gauge theory due to the technical issues surrounding infrared problems if vertices are computed at exceptional momenta configurations. However, using a nonexceptional setup would require the three-loop three-point masters which are not yet known even in four dimensions. With the two-loop renormalization of Eq. (5.12) it should be possible to extend the F-theorem studies in six-dimensional QED [65], to the non-Abelian case. Moreover, it would be interesting to compare the perturbative picture with a gauge theory conformal bootstrap analysis.

Our final remarks are aimed at trying to give an overall perspective. Several interesting features emerged in six-dimensional gauge theories. For instance, properties of four-dimensional gauge theories appear to have parallels in higher dimensions. One, which is not surprising, is that the Landau-gauge anomalous dimension of the dimension-two local gluon-mass operator is the sum of the gluon and ghost anomalous dimensions. This follows purely as a consequence of the universal structure of BRST invariance and the dimension-independent proof of Ref. [104]. What was less apparent was the result for the electron wave-function anomalous dimension. The gauge parameter dependence arose only in the one-loop term in the six- and eight-dimensional cases to the various orders we computed. This may give some insight into the reasoning behind the four-dimensional argument of Refs. [108,109]. From another point of view it might be better to examine the connection of the field theories in different dimensions at a more fundamental level. A clue to this may be in the way we had to carry out our higher dimension renormalization. For instance, the underlying master integrals were deduced using Tarasov's method [76,77], which connects masters in d dimensions to those in $(d+2)$ dimensions. While this is at a Feynman-integral level there is a hint that there is a Lagrangian field theory connection which may be quantifiable using path-integral methods. An indication of this here may be seen in the operators in various Lagrangians. For instance, in Eq. (5.20) the mass parameters m_2 and m_4 are associated with $G^a_{\mu\nu}G^{a\mu\nu}$ and the Landau-gauge operator $\mathcal{O}$. In the corresponding four-dimensional propagator [89,90,101], the respective operators are $\mathcal{O}_2$ and the Gribov operator, $\mathcal{O}_\gamma$, which is

$$\mathcal{O}_\gamma = \frac{1}{2} A^a_\mu \left(\frac{1}{\partial^\nu D_\nu}\right)^{ab} A^b_\mu \tag{9.1}$$

which is also nonlocal. The anomalous dimension of the latter is formally the same as $\mathcal{O}$ in that it is the sum of the gluon and ghost anomalous dimensions. However, comparing the structure of the respective operators between four and six dimensions they are essentially equivalent when one recognizes that the nonlocality accounts for the differing dimensionalities. This may be an indication that nonlocal problems in lower dimensions could be studied in a local higher-dimensional context and give insight into effective field theories.

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APPENDIX: EIGHT-DIMENSIONAL MASTER INTEGRALS

In this appendix we record the values of the various one- and two-loop three-point master integrals at the fully symmetric point needed to carry out the renormalization of Eqs. (2.6) and (8.5) in eight dimensions. They are constructed from lower-dimensional masters using Tarasov's method [76,77]. For ease of comparison and definition we use the same labeling of the integrals as that given in the four-dimensional summary of Ref. [110]. First, the one-loop triangle integral is

$$\begin{aligned}\mathcal{M}^{(1)}_{31} = \Bigg[& -\frac{1}{8\epsilon} - \frac{61}{144} - \frac{2}{81}\pi^2 + \frac{1}{27}\psi'\left(\frac{1}{3}\right) \\ & + \left[-\frac{895}{864} - \frac{23}{864}\pi^2 - \frac{2}{3}s_3\left(\frac{\pi}{6}\right) + \frac{1}{18}\psi'\left(\frac{1}{3}\right) + \frac{35}{5832}\sqrt{3}\pi^3 + \frac{1}{216}\sqrt{3}\ln^2(3)\pi\right]\epsilon + O(\epsilon^2)\Bigg](-\tilde{\mu}^2).\end{aligned} \tag{A1}$$

At two loops we have

$$\mathcal{M}^{(2)}_{42} = \left[\frac{1}{50400\epsilon^2} + \frac{323}{4233600\epsilon} + \left[2234400\psi'\left(\frac{1}{3}\right) - 1754200\pi^2 + 6391701\right]\frac{1}{80015040000}\right.$$
$$+ \left[81496800\sqrt{3}\ln^2(3)\pi - 133358400\sqrt{3}\ln(3)\pi + 79301600\sqrt{3}\pi^3 + 3775312800\psi'\left(\frac{1}{3}\right)\right.$$
$$+ 4800902400 s_2\left(\frac{\pi}{6}\right) - 9601804800 s_2\left(\frac{\pi}{2}\right) - 18136742400 s_3\left(\frac{\pi}{6}\right) + 6401203200 s_3\left(\frac{\pi}{2}\right)$$
$$\left.\left. - 2773272600\pi^2 - 2667168000\zeta_3 - 11102348079\right]\frac{\epsilon}{20163790080000} + O(\epsilon^2)\right]\tilde{\mu}^8,$$
$$\mathcal{M}^{(2)}_{43} = \left[\frac{1}{2880\epsilon^2} + \frac{401}{172800\epsilon} + \left[-2400\psi'\left(\frac{1}{3}\right) - 3800\pi^2 + 937449\right]\frac{1}{93312000}\right.$$
$$+ \left[108000\sqrt{3}\ln^2(3)\pi - 1944000\sqrt{3}\ln(3)\pi - 244000\sqrt{3}\pi^3 + 4600800\psi'\left(\frac{1}{3}\right)\right.$$
$$+ 69984000 s_2\left(\frac{\pi}{6}\right) - 139968000 s_2\left(\frac{\pi}{2}\right) - 108864000 s_3\left(\frac{\pi}{6}\right) + 93312000 s_3\left(\frac{\pi}{2}\right)$$
$$\left.\left. - 9563400\pi^2 - 38880000\zeta_3 + 611480367\right]\frac{\epsilon}{16796160000} + O(\epsilon^2)\right]\tilde{\mu}^6,$$
$$\mathcal{M}^{(2)}_{52} = \left[\frac{1}{960\epsilon^2} + \frac{2371}{345600\epsilon} + \left[-38400\psi'\left(\frac{1}{3}\right) - 6800\pi^2 + 5299929\right]\frac{1}{186624000} + O(\epsilon)\right]\tilde{\mu}^6,$$
$$\mathcal{M}^{(2)}_{61} = \left[\frac{1}{240\epsilon^2} + \frac{329}{9600\epsilon} + \left[16200\sqrt{3}\ln^2(3)\pi - 194400\sqrt{3}\ln(3)\pi - 17400\sqrt{3}\pi^3 + 628800\psi'\left(\frac{1}{3}\right) + 6998400 s_2\left(\frac{\pi}{6}\right)\right.\right.$$
$$- 13996800 s_2\left(\frac{\pi}{2}\right) - 11664000 s_3\left(\frac{\pi}{6}\right) + 9331200 s_3\left(\frac{\pi}{2}\right) - 494800\pi^2$$
$$\left.\left. - 1166400\zeta_3 + 19175391\right]\frac{1}{108864000} + O(\epsilon)\right]\tilde{\mu}^4 \tag{A2}$$

where

$$s_n(z) = \frac{1}{\sqrt{3}}\Im\left[\mathrm{Li}_n\left(\frac{e^{iz}}{\sqrt{3}}\right)\right] \tag{A3}$$

and $\mathrm{Li}_n(z)$ is the polylogarithm function. We have used the notation of Ref. [110] but it is worth noting that they are related to cyclotomic polynomials [111]. We have not included values for the two-loop masters $\mathcal{M}^{(1)}_{21}$, $\mathcal{M}^{(2)}_{31}$, $\mathcal{M}^{(2)}_{41}$ and $\mathcal{M}^{(2)}_{51}$ in the notation of Ref. [109] as they are products of one-loop masters or two-loop two-point integrals.

Finally we record the value of the eight-dimensional one-loop four-point box integral at the fully symmetric point. This was required for the renormalization of Eq. (2.6). In Ref. [112] the four-dimensional version was derived but again we have used Refs. [76,77] for our purposes. Using the same notation as Ref. [112] the corresponding $d = 8 - 2\epsilon$-dimensional value is

$$D^{(1)}\left(-\tilde{\mu}^2, -\tilde{\mu}^2, -\tilde{\mu}^2, -\tilde{\mu}^2, -\frac{4}{3}\tilde{\mu}^2, -\frac{4}{3}\tilde{\mu}^2\right) = \frac{1}{6\epsilon} + \frac{11}{18} - \frac{1}{24}\ln\left(\frac{4}{3}\right) + \frac{25}{192}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) - \frac{29}{96}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) + O(\epsilon) \tag{A4}$$

where [112],

$$\Phi_1(x, y) = \frac{1}{\lambda}\left[2\mathrm{Li}_2(-\rho x) + 2\mathrm{Li}_2(-\rho y) + \ln\left(\frac{y}{x}\right)\ln\left(\frac{(1+\rho y)}{(1+\rho x)}\right) + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\right] \tag{A5}$$

and

$$\lambda(x,y) = \sqrt{\Delta_G}, \qquad \rho(x,y) = \frac{2}{[1 - x - y + \lambda(x,y)]} \tag{A6}$$

with

$$\Delta_G(x,y) = x^2 - 2xy + y^2 - 2x - 2y + 1. \tag{A7}$$

We note that the finite piece can also be expressed in terms of the Clausen function $\mathrm{Cl}_2(\theta)$ via [113],

$$\begin{aligned} \Phi_1\left(\frac{3}{4},\frac{3}{4}\right) &= \sqrt{2}\left[2\mathrm{Cl}_2\left(2\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) + \mathrm{Cl}_2\left(2\cos^{-1}\left(\frac{1}{3}\right)\right)\right], \\ \Phi_1\left(\frac{9}{16},\frac{9}{16}\right) &= \frac{4}{\sqrt{5}}\left[2\mathrm{Cl}_2\left(2\cos^{-1}\left(\frac{2}{3}\right)\right) + \mathrm{Cl}_2\left(2\cos^{-1}\left(\frac{1}{9}\right)\right)\right]. \end{aligned} \tag{A8}$$

The finite part has been provided for the reader interested in the Tarasov approach.

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