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Photons without vector fields

Ibrahim Burak Ilhan and Alex Kovner

Physics Department, University of Connecticut, 2152 Hillside Road, Storrs, Connecticut 06269-3046, USA

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In this paper, we continue to pursue the question of whether gauge theories can be represented in terms of effective "scalar" degrees of freedom. We provide such a consistent representation for a free photon theory in $3+1$ dimensions. Building on results of [Phys. Rev. D 88, 125004 (2013), we construct a Lagrangian with a four-derivative kinetic term and demonstrate that, despite the seeming nonlinearity of the theory, it is equivalent to a theory of a free photon.

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I. INTRODUCTION

In a recent paper [1], we constructed an effective theory of scalar fields in $3+1$ dimensions with certain features that could potentially mimic the low-energy limit of gluodynamics. The idea was based on the known relation between confinement and Z_N symmetry in $2+1$ dimensions [2,3]. Although the main aim of this construction was to explore possible low-energy representation of non-Abelian theories, a certain limit of the construction should encompass an Abelian gauge theory and its simplest limit—the theory of a free photon [3].

The construction of Ref. [1], despite having some useful features, failed to describe this Abelian limit exactly. In this paper, we rectify this problem. We modify the Abelian limit of the model of Ref. [1] and demonstrate that the modified model is exactly equivalent to a theory of the free photon.

The Abelian limit of the model of [1] can be written as

$$\mathcal{L} = \frac{1}{4} f_{\mu\nu} f^{\mu\nu} = -\frac{1}{2}(\vec{e}^2 - \vec{b}^2), \tag{1}$$

with $e_i = f_{i0}$ and $b_i = \frac{1}{2}\epsilon_{ijk} f_{jk}$ $(f_{ij} = \epsilon_{ijk} b_k)$ and $f^{\mu\nu}$ defined as

$$f^{\mu\nu} = g\epsilon^{\mu\nu\alpha\beta}\epsilon^{abc}\phi_a \partial_\alpha \phi_b \partial_\beta \phi_c, \tag{2}$$

where the scalar fields ϕ_a, $a = 1, 2, 3$ are constrained by $\phi^a\phi^a = 1$. This model was previously discussed in Ref. [4]. There as well as in Ref. [1], it was shown that, despite some similarities, it fails to describe a free photon primarily because the magnetic field is not required to be divergenceless $\partial_i b_i \neq 0$.

To rectify this problem, we now consider the setup

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{3}$$

$$F^{\mu\nu} = g\epsilon^{\mu\nu\alpha\beta}[\epsilon^{abc}\phi_a \partial_\alpha \phi_b \partial_\beta \phi_c + (n\cdot\partial) n_\alpha \partial_\beta \Phi], \tag{4}$$

where $n = (1, 0, 0, 0)$ is a timelike unit vector and Φ is an additional scalar field [5].

The presence of an explicit timelike vector seems to render the model not Lorentz invariant. However, as we will see below, this is not quite the case. The model does possess a Lorentz-invariant superselection sector, and it is this sector that is equivalent to QED.

In this paper, we discuss how this modification changes the model in the Abelian limit. We will show that the theory has the same canonical structure as free electrodynamics. This includes the commutation relations between "electric" and "magnetic" fields as well as the Hamiltonian. The model is therefore equivalent to a theory of a free photon, even though it is not formulated in terms of the vector potential. We also discuss the action of Lorentz transformations on the basic degrees of freedom of the model. We show explicitly that the fields ϕ^i are not covariant scalar fields but rather have an "anomalous" term in their Lorentz transformation law. With this modification, we show that the model is indeed Lorentz invariant.

II. EQUATIONS OF MOTION AND CANONICAL STRUCTURE

A. Equations of motion

Let us define two independent fields χ and z via $\phi_3 = z$ and $\phi_1 + i\phi_2 = \sqrt{1-z^2}e^{i\chi}$. The electromagnetic field can now be written as

$$F^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}[-2\partial_\beta\chi\partial_\alpha z + n_\alpha \partial_\beta \partial_0 \Phi] \tag{5}$$

or component by component,

$$E_i = 2\epsilon_{ijk}\partial_j z \partial_k \chi \tag{6}$$

$$B_k = [2(\partial_k\chi\partial_0 z - \partial_0\chi\partial_k z) - \partial_k\partial_0\Phi]. \tag{7}$$

The Lagrangian equations of motion that follow from the Lagrangian Eq. (3) are

$$\partial_0\partial_k[F_{ij}\epsilon^{ij0k}] = 0 = \partial_0\partial_k B_k \tag{8}$$

$$\partial_\beta \chi \partial_\alpha (F_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}) = 0 = \partial_k \chi \partial_\alpha (F_{\mu\nu}\epsilon^{\mu\nu\alpha k}) = \partial_k \chi (\partial_0 B_k + (\partial \times E)_k) \quad (9)$$

$$\partial_\beta z \partial_\alpha (F_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}) = 0 = \partial_k z \partial_\alpha (F_{\mu\nu}\epsilon^{\mu\nu\alpha k}) = \partial_k z (\partial_0 B_k + (\partial \times E)_k). \quad (10)$$

Equation (8) is a local conservation equation of a "magnetic charge density" $\partial_k B_k$. It ensures that the Hilbert space of the theory is divided into "superselection sectors" with a fixed value of the magnetic charge density. To preserve translational invariance, we limit ourselves to the sector with $\partial_k B_k = 0$. Our considerations in the rest of this paper pertain to this superselection sector alone.

Using this constraint on the magnetic field, Eqs. (9) and (10) can be inverted, with the result [6]

$$\partial_0 B_k + (\partial \times E)_k = 0. \quad (11)$$

Recall that with the field strength components given by Eq. (5) the equation

$$\partial_\mu F^{\mu\nu} = 0 \quad (12)$$

is satisfied identically. We thus have the full set of Maxwell's equations.

B. Hamiltonian

We now demonstrate that the Hamiltonian and the canonical commutation relations of the electromagnetic fields in our model are identical to those in pure QED.

The canonical momenta can be calculated from Eq. (5) as

$$p_z = \frac{\delta L}{\delta \partial_0 z} = F_{ij}\epsilon^{ij0k}\partial_k \chi = 2B_k \partial_k \chi = 2\partial_k \chi [2(\partial_k \chi \partial_0 z - \partial_0 \chi \partial_k z) - \partial_k \partial_0 \Phi] \quad (13)$$

$$p_\chi = \frac{\delta L}{\delta \partial_0 \chi} = F_{ij}\epsilon^{ijk0}\partial_k z = -2B_k \partial_k z = -2\partial_k z [2(\partial_k \chi \partial_0 z - \partial_0 \chi \partial_k z) - \partial_k \partial_0 \Phi] \quad (14)$$

$$p_\Phi = \frac{\delta L}{\delta \partial_0 \Phi} = \frac{1}{2}\partial_k (F_{ij}\epsilon^{ij0k}) = \partial_k B_k = \partial_k [2(\partial_k \chi \partial_0 z - \partial_0 \chi \partial_k z) - \partial_k \partial_0 \Phi]. \quad (15)$$

It is a straightforward matter to express the time derivative of χ and z as

$$\dot{\chi} = \frac{1}{E^2}[p_z (z\chi) + p_\chi \chi^2 + \epsilon_{ijk}\dot{\Phi}_i E_j \chi_k] \quad (16)$$

$$\dot{z} = \frac{1}{E^2}[p_z z^2 + p_\chi (z\chi) + \epsilon_{ijk}\dot{\Phi}_i E_j z_k]. \quad (17)$$

The time derivative of Φ is related to canonical momenta via

$$p_\Phi = \partial_k \left[\frac{1}{E^2}\epsilon_{klm}E_l (p_z z_m + p_\chi \chi_m) - \frac{1}{E^2}E_k E_i \dot{\Phi}_i\right] \quad (18)$$

or in terms of a "vector potential,"

$$A_k = \frac{1}{E^2}\epsilon_{klm}E_l (p_z z_m + p_\chi \chi_m), \quad (19)$$

as

$$p_\Phi = \partial_k (A_k - \hat{E}_k \hat{E}_i \dot{\Phi}_i). \quad (20)$$

The Hamiltonian is then calculated as

$$H = \int d^3x [p_z \dot{z} + p_\chi \dot{\chi} + p_\Phi \dot{\Phi} - L] = \int d^3x \frac{1}{2}(E^2 + B^2), \quad (21)$$

where we have neglected a boundary term, $\int d^3x \partial_k (B_k \dot{\Phi})$.

C. Canonical structure

To show that our model is equivalent to QED, we need to make sure that the canonical commutation relations of E_i and B_i are identical in the two theories.

First of all, since all components of the electric field in our model are functions only of coordinates and not canonical momenta, they commute with each other:

$$[E_i(x), E_j(y)] = 0. \quad (22)$$

Our next goal is to calculate the commutator between the electric and magnetic fields. To do that, we set $p_\Phi = 0$, as we are only interested in this superselection sector of the theory. Then, Eq. (20) becomes

$$\frac{\partial_k A_k}{E} = \hat{E}_k \partial_k \left(\frac{\hat{E}_i \dot{\Phi}_i}{E}\right), \quad (23)$$

where we have used $\partial_k E_k = 0$.

The formal solution of this equation can be obtained as

$$\hat{E}_i \dot{\Phi}_i = E(x)\int_{-\infty}^{x} dl_C \frac{\partial_k A_k}{E}, \quad (24)$$

where the integral is along the contour C, which starts at x and goes to infinity (boundary of space). The contour is everywhere parallel to the direction of the electric field.

Using the definition, we have

$$B_k = A_k - E_k \int_{-\infty}^{x} dl_C \frac{\partial_m A_m}{E}. \quad (25)$$

As an intermediate step for the calculation of the commutator $[E, B]$, we consider

$$
\begin{aligned}
[E_i(x), A_k(y)] &= 2i\frac{E_l(y)}{E^2(y)}\epsilon_{iab}\epsilon_{klm}[\partial_a^x\delta(x-y)\chi_b(x)z_m(y) + \partial_b^x\delta(x-y)z_a(x)\chi_m(y)] \\
&= 2i\frac{E_l(y)}{E^2(y)}\epsilon_{iab}\epsilon_{klm}\partial_a^x\delta(x-y)[\chi_b(y)z_m(y) - z_b(y)\chi_m(y)] \\
&= i\hat{E}_l(y)\hat{E}_c(y)\epsilon_{iab}\epsilon_{klm}\epsilon_{cmb}\partial_a^x\delta(x-y) = i[\epsilon_{iak} - \hat{E}_b(y)\hat{E}_k(y)\epsilon_{iab}]\partial_a^x\delta(x-y). \qquad (26)
\end{aligned}
$$

Using this, we can calculate

$$
\begin{aligned}
[E_i(x), B_k(y)] &= [E_i(x), A_k(y)] - E_k(y)\int_{-\infty}^{y} dl_C \frac{\partial_m^t[E_i(x), A_m(t)]}{E(t)} \\
&= [E_i(x), A_k(y)] - E_k(y)\int_{-\infty}^{y} dl_C \frac{1}{E(t)}\partial_m^t[(\epsilon_{iam} - \hat{E}_b(t)\hat{E}_m(t)\epsilon_{iab})\partial_a^x\delta(x-t)] \\
&= [E_i(x), A_k(y)] + E_k(y)\int_{\infty}^{y} dl_C\hat{E}_m(t)\partial_m^t\left(\frac{\hat{E}_b(t)}{E(t)}\epsilon_{iab}\partial_a^x\delta(x-t)\right) \\
&= i\epsilon_{iak}\partial_a^x\delta(x-y). \qquad (27)
\end{aligned}
$$

Here we have used the fact that the integration contour C is defined to run in the direction of electric field and have assumed that the fields decrease fast enough at the boundary. The commutator Eq. (27) coincides with the corresponding commutator in QED.

We now turn to the commutator of components of magnetic fields. It is straightforward to show that $[B_i(x), B_a(y)] = 0$ as long as the curve C_x that defines $B_i(x)$ in Eq. (25) does not contain the point y and C_y does not contain x. When this condition is not met, the direct calculation of the commutator is not straightforward. Instead of attempting it, we take an indirect way. A set of relations that involve the commutator in question is easily obtained. Consider, for instance,

$$
[B_i(x)\partial_i\chi(x), B_j(y)\partial_j z(y)] = [p_z(x), p_\chi(y)] = 0. \qquad (28)
$$

Trivially,

$$
\begin{aligned}
&B_i(x)\partial_j z(y)[\partial_i\chi(x), B_j(y)] + B_j(y)\partial_i\chi(x)[B_i(x), \partial_j z(y)] + \partial_i\chi(x)\partial_j z(y)[B_i(x), B_j(y)] \\
&= \left(B_i(x)\partial_j z(y)\partial_i^{(x)}\frac{\partial A_j(y)}{\partial p_\chi(x)} - B_j(y)\partial_i\chi(x)\partial_j^{(y)}\frac{\partial A_i(x)}{\partial p_z(y)}\right) + \partial_i\chi(x)\partial_j z(y)[B_i(x), B_j(y)] \\
&= (B_i(y)\partial_i^{(x)}\delta(x-y) + B_i(x)\partial_i^{(y)}\delta(x-y)) + \partial_i\chi(x)\partial_j z(y)[B_i(x), B_j(y)] \\
&= \partial_i\chi(x)\partial_j z(y)[B_i(x), B_j(y)] = 0. \qquad (29)
\end{aligned}
$$

Here, we used the fact that $E_k z_k = E_k\chi_k = 0$ and the constraint $\partial_i B_i = 0$. Similarly,

$$
\partial_i z(x)\partial_j z(y)[B_i(x), B_j(y)] = \partial_i\chi(x)\partial_j\chi(y)[B_i(x), B_j(y)] = 0, \qquad (30)
$$

a nd by $\partial_k B_k = 0$, we have

$$
\partial_i z(x)\partial_j^y[B_i(x), B_j(y)] = \partial_i\chi(x)\partial_j^y[B_i(x), B_j(y)] = \partial_i^x\partial_j^y[B_i(x), B_j(y)] = 0. \qquad (31)
$$

Thus, the commutator matrix $M_{ij}(x,y) \equiv [B_i(x), B_j(y)]$, antisymmetric under the exchange $(i,x) \leftrightarrow (j,y)$, satisfies the set of Eqs. (29)–(31). The general solution for these equations is given by

$$
M_{ij}(x,y) = E_i(x)F_j(y) - E_j(y)F_i(x), \qquad (32)
$$

where $F_i(x)$ is an arbitrary function. However, we have already established that if x does not belong to C_y and y does not belong to C_x, then $M_{ij}(x,y) = 0$. This unambiguously fixes $F_i(x) = 0$ so that we have

$$
[B_i(x), B_j(y)] = 0 \qquad (33)
$$

for all x, y.

III. LORENTZ TRANSFORMATIONS OF THE FIELDS

The final point we address is the Lorentz transformation properties of the fields z and χ. Since the electric and magnetic fields are covariant components of the Lorentz tensor, it is clear that z and χ cannot be covariant scalar fields. The transformations of z and χ under rotations are the same as those of covariant fields, and we will not deal with those here.

Let us parametrize the infinitesimal Lorentz transformation properties of these fields in the following way:

$$\begin{aligned} z(x) &\to z(\Lambda^{-1}x) = (1+\beta\Delta)z(x) + a \\ \chi(x) &\to \chi(\Lambda^{-1}x) = (1+\beta\Delta)\chi(x) + b \\ \Theta(x) \equiv \partial_0\Phi &\to \Theta(\Lambda^{-1}x) = \Theta(x) + c. \end{aligned} \tag{34}$$

Here, β is the boost parameter, and $\Delta \equiv \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}$ with ω^{μ}_{ν}—an antisymmetric generator of Lorentz transformation. In particular, for a boost in the direction of a unit vector $\hat{n}$, $\omega^i_0 = \hat{n}_i$. The noncanonical terms a, b, and c are to be determined such that $F^{\mu\nu}$ transforms as a tensor.

For simplicity, let us consider explicitly a boost transformation in the first direction, $\hat{n} = (1,0,0)$. The transformation of the components of the field strength tensor is

$$E_2(x) \to E_2(\Lambda^{-1}x) - \beta B_3(\Lambda^{-1}x); \tag{35}$$

on the other hand, writing this in terms of z, χ, and Θ, we have

$$\begin{aligned} E_2(x) &= 2[\partial_3 z(x)\partial_1\chi(x) - \partial_1 z(x)\partial_3(x)] \\ &\to 2[\partial_3 z(\Lambda^{-1}x)\partial_1\chi(\Lambda^{-1}x) - \partial_1 z(\Lambda^{-1}x)\partial_3(\Lambda^{-1}x)]. \end{aligned} \tag{36}$$

Equating the two, we obtain

$$-\beta\partial_3\Theta + 2[\partial_3 z\partial_1 b + \partial_3 a\partial_1\chi - \partial_1 z\partial_3 b - \partial_1 a\partial_3\chi] = 0. \tag{37}$$

Similarly, by considering the transformation of E_1, we obtain

$$2(\partial_2 z\partial_3 b + \partial_2 a\partial_3\chi - \partial_3 z\partial_2 b - \partial_3 a\partial_2\chi) = 0 \tag{38}$$

and for E_3

$$\beta\partial_2\Theta + 2[\partial_1 z\partial_2 b + \partial_1 a\partial_2\chi - \partial_2 z\partial_1 b - \partial_2 a\partial_1\chi] = 0. \tag{39}$$

Defining for convenience $f_i = 2(a\partial_i\chi - b\partial_i z)$ and $u_i = (0, \beta\partial_3\Theta, -\beta\partial_2\Theta)$, the above equations can be written as

$$\epsilon_{ijk}\partial_j f_k = u_i. \tag{40}$$

The general solution for f is

$$\begin{aligned} f_i &= -\frac{\epsilon_{ijk}\partial_j u_k}{\partial^2} + \partial_i\tilde{\lambda} \\ &= \beta\hat{n}_i\Theta + \partial_i\lambda, \end{aligned} \tag{41}$$

where

$$\tilde{\lambda} - \beta\frac{\hat{n}_i\partial_i}{\partial^2}\Theta = \lambda, \tag{42}$$

where the function λ still has to be determined.

We can now solve Eq. (41) for a and b by noting that Eq. (41) is identical to Eq. (7) with the substitution

$$\begin{aligned} \partial_0 z &\to a \\ \partial_0\chi &\to b \\ \partial_0\Phi &\to \lambda \\ B_k &\to \beta\Theta\hat{n}_k. \end{aligned} \tag{43}$$

Using Eqs. (16) and (17), we find

$$\begin{aligned} a &= \frac{1}{E^2}(\beta\Theta\hat{n}_i + \lambda_i)\epsilon_{ijk}E_j z_k \\ b &= \frac{1}{E^2}(\beta\Theta\hat{n}_i + \lambda_i)\epsilon_{ijk}E_j\chi_k. \end{aligned} \tag{44}$$

With this, Eq. (41) yields the equation for λ,

$$E_i(\beta\Theta\hat{n}_i + \partial_i\lambda) = 0, \tag{45}$$

from which we get

$$\lambda(x) = -\beta\int_{\infty}^{x} dl_C\hat{E}_i\hat{n}_i\Theta, \tag{46}$$

where again C is a curve in the direction of E.

To determine the remaining function c, we consider the transformation of the magnetic field. For the transformation of the magnetic field, we have

$$\begin{aligned} B_1(x) &\to B_1(\Lambda^{-1}x) \\ &= 2[\partial_1\chi(\Lambda^{-1}x)\partial_0(\Lambda^{-1}x) \\ &\quad - \partial_0\chi(\Lambda^{-1}x)\partial_1 z(\Lambda^{-1}x)] - \partial_1\Theta(\Lambda^{-1}x), \end{aligned} \tag{47}$$

which yields

$$\begin{aligned} &2[\partial_1\chi\partial_0 a + \partial_1 b\partial_0 z - \partial_0\chi\partial_1 a - \partial_0 b\partial_1 z] \\ &\quad - \partial_1 c + \beta\Delta\partial_1\Theta = 0. \end{aligned} \tag{48}$$

Similarly, the transformation of B_2 and B_3 yields

$$\begin{aligned} &2[\partial_2\chi\partial_0 a+\partial_0 z\partial_2 b-\partial_0\chi\partial_2 a-\partial_2 z\partial_0 b]\\ &\quad-\partial_2 c+\beta\Delta\partial_2\Theta=0\\ &2[\partial_3\chi\partial_0 a+\partial_0 z\partial_3 b-\partial_0\chi\partial_3 b-\partial_3 z\partial_0 b]\\ &\quad-\partial_3 c+\beta\Delta\partial_3\Theta=0. \end{aligned} \tag{49}$$

These can be written as a single vector equation,

$$\partial_0 f_i-\partial_i f_0-\partial_i c+\beta\Delta\partial_i\Theta=0. \tag{50}$$

Using Eq. (41), this can be written as

$$\partial_i[\partial_0\lambda-f_0-c+\beta\Delta\Theta]=0, \tag{51}$$

yielding

$$\begin{aligned} c&=2(a\partial_0\chi-b\partial_0 z)-\partial_0\lambda-\beta\Delta\Theta\\ &=\beta\left[\frac{2}{E^2}\epsilon_{ijk}E_j(\partial_0\chi z_k-\partial_0 z\chi_k)\left[\Theta\hat{n}_i-\partial_i\int_\infty^x dl_C\hat{E}_l\hat{n}_l\Theta\right]\right.\\ &\quad\left.+\partial_0\int_\infty^x dl_C\hat{E}_i\hat{n}_i\Theta-\Delta\Theta\right]. \end{aligned} \tag{52}$$

Thus, we find that the fields z, χ, and Φ under a Lorentz boost transform according to Eq. (34) with a, b, and c given in Eqs. (44), (46), and (52).

IV. DISCUSSION

In this paper, we have amended the model suggested previously as a candidate for the effective description of a gauge theory. We have considered only the Abelian limit in which the model is equivalent to a theory of a free massless photon. We have proven this equivalence by considering the canonical structure of the theory. We have also shown that the basic fields of our model have interesting Lorentz transformation properties.

We note that the modification discussed here also solves a certain puzzle posed by the suggestion of Ref. [1]. Namely, the model discussed in Ref. [1] possessed a global symmetry generated by

$$C_F=\int d^3x\left[p_z\frac{\partial G[z,\chi]}{\partial\chi}-p_\chi\frac{\partial G[z,\chi]}{\partial z}\right] \tag{53}$$

for an arbitrary function of two variables $G[z,\chi]$. These transformations constitute a group of area-preserving diffeomorphisms on a sphere. The electric and magnetic fields e and b were invariant under the action of this group. That in fact suggested that the theory had some degrees of freedom in addition to the electrodynamic ones, since the action of the transformation generated by Eq. (53) on the full phase space of the theory was nontrivial. However, now the direct consequence of Eqs. (13), (14), and (15) is that the generator of this transformation vanishes,

$$C_F=0, \tag{54}$$

and thus there are no physical degrees of freedom that transform nontrivially under Eq. (53). Thus, in the present model, the group of (global) diffeomorphisms $\mathrm{Sdiff}(S^2)$ is in fact a global gauge symmetry; that is, the states are invariant under the action of C_F, and what used to be extra degrees of freedom in Ref. [1] now becomes an unphysical "gauge" coordinate. This ensures that electric and magnetic fields are the only physical degrees of freedom.

Since here we are dealing with the theory of a free photon, the charged states are not present. It should be, however, straightforward to extend this discussion to include electrically charged states. Just like in Ref. [1], we should lift the constraint of constant length of the field ϕ^a and instead allow dynamics of the modulus ϕ^2. This will regulate the energy of the charged states in the UV and will make it finite. Since the configuration space of the model is $SO(3)\times R$, and the $SO(3)$ symmetry is broken to $O(2)$, the moduli space should have nontrivial homotopy group $\Pi_2(M)=Z$, and the relevant topological charge should be identifiable with the electric charge [7].

The next set of questions to be addressed is how to move in this theory to the non-Abelian regime. According to the logic of Ref. [1], we need to find a perturbation that breaks the global symmetries of the model and through this breaking generates linear potential between the charges. The question one has to address is whether this perturbation has to preserve the $\mathrm{Sdiff}(S^2)$ gauge symmetry or if it should break it explicitly. This global gauge symmetry is a new element compared to $2+1$ dimensions [3], and we do not have any guidance from the $2+1$-dimensional models. Perhaps one should deal directly with the breaking of the generalized magnetic symmetry—the symmetry generated by the magnetic flux [8,9] in terms of its order parameter, the 't Hooft loop [2]. These questions will be addressed in future work.

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