Late-time decay of perturbations outside extremal charged black hole

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(Received 27 October 2015; published 27 January 2016)

We analyze the late-time decay of scalar perturbations in extremal Reissner-Nordstrom spacetime. We consider individual spherical-harmonic modes l of a test massless scalar field, restricting our attention to the initial data of compact support, with generic regular behavior across the horizon. We obtain a decay rate $\propto t^{-(2l+3)}$ (just like in Schwarzschild) for incident waves scattered by the black hole. However, for waves originating at the horizon's neighborhood, we obtain a slightly slower decay, $\propto t^{-(2l+2)}$. We discuss relations to previous works.

I. INTRODUCTION

When a black hole (BH) is perturbed by massless fields —either gravitational, electromagnetic, or scalar—at late time the perturbations typically decay as an inverse power of t (the Schwarzschild time coordinate). This phenomenon was found by Price [1,2] about four decades ago in the case of a Schwarzschild BH. The goal of this paper is to extend the large-t analysis to an extremal Reissner-Nordstrom (ERN) BH.

For simplicity and concreteness, we shall focus throughout this paper on the case of a massless, minimally coupled, test scalar field Φ with compact initial support and with generic regular behavior across the horizon. In the Schwarzschild case, Price [1] found a decay rate t^{-2l-2} when an initial static moment is present and a decay rate t^{-2l-3} when the initial data have compact support, where l denotes the mode's multipolar number. Soon afterward, Bičák [3] extended the analysis to the Reissner-Nordstrom (RN) background. In the nonextremal case, he obtained (for his choice of generic initial data) the same decay rate t^{-2l-2} as in Schwarzschild (for the initial static moment). However, in the case of ERN, Bičák found a much slower decay rate $\propto t^{-l-2}$.

To analyze the scattering problem, it is useful to introduce the tortoise radial coordinate r_* (defined below). The relevant domain then extends from the horizon ($r_* \to -\infty$) to infinity ($r \simeq r_* \to \infty$). The scattering problem involves an effective potential, a certain function $V_l(r_*)$. Bičák [3] observed that in the case of ERN, in both boundaries $r_* \to \pm\infty$ this potential has the same asymptotic behavior $V_l \simeq l(l+1)/r_*^2$ (this sharply contrasts with the nonextremal case, wherein V_l decays exponentially in r_* on approaching the horizon). Couch and Torrence [4] later showed that this symmetry applies to all values of r_* (and not only asymptotically); i.e. the function $V_l(r_*)$ is symmetric in r_*. Further discussions about this symmetry along with interesting implications of it can be found in Refs. [5,6]. This surprising mathematical symmetry connects the scattering dynamics near the horizon to that at the weak-field region.

This symmetry of the potential was later employed by Blaksley and Burko [7], who considered two special classes of initial data: (i) compact initial support which does not extend up to the horizon, and (ii) initially static multipoles that extend up to the horizon (and up to future null infinity). Using the aforementioned symmetry of $V_l(r_*)$, Blaksley and Burko obtained decay rates t^{-2l-3} and t^{-2l-2}, respectively. Note that this in itself does not conflict with Bičák's result (a decay $\propto t^{-l-2}$): Bičák obtained his result for the case of generic initial data, which extend regularly across the horizon. On the other hand, the special subclasses considered by Blaksley and Burko are nongeneric and are both "weaker" than Bičák's case (in terms of the asymptotic behavior of the initial data on approaching the horizon).

In this paper we revisit the analysis of the ERN case, for the generic class of initial data. Namely, we allow the initial data to approach (and, in fact, to cross) the horizon in a generic regular manner. We carry out the analysis entirely in the time domain. We use the following strategy: First, we use the above mentioned symmetry of $V_l(r_*)$ to transform the strong-field near-horizon scattering problem into a new, weak-field problem at large r. This allows us to employ certain approximate methods previously developed for analyzing wave dynamics in weak-field regions [1,8]. As a result, we get an approximate expression for ϕ (the inverted field), valid for evaluation points in the domain of large r and for weak-field initial data. Then, transforming the resulting ϕ back into the original problem, we obtain an approximate expression for the original field Φ valid for near-horizon evaluation points and initial data. In a subsequent paper [9] that I hope to address, we shall employ an exact result obtained by Aretakis [10], concerning the behavior of certain derivatives of Φ along the horizon, in order to verify the validity of our approximations and to obtain the exact (i.e. for generic compact initial data which crosses the horizon regularly, not necessarily confined to the near-horizon region) asymptotic expression for Φ, although it is still valid only for evaluation points in the vicinity of the horizon. We will later use the "late-time expansion" method in order to remove this restriction and

get the exact large-t asymptotic behavior of Φ at any fixed r outside the BH. Even though the details of the last two procedures are postponed to the next paper [9], we will sketch very briefly the key ingredients and main results of them in the last section of this paper.

Our final result is given in Eq. (7.6): The field generically decays at large t as t^{-2l-2}, multiplied by a certain function of r (corresponding to a static solution).

We note that in the special cases considered by Blaksley and Burko [7], their results are fully consistent with our analysis. In particular, when an ingoing wave packet is scattered off the BH (namely, no initial support at the horizon), the decay rate will be t^{-2l-3}. Our analysis is also fully consistent with the results obtained by Lucietti *et al.* [5], who numerically found decay rates $\propto t^{-2l-2}$ for $l = 0, 1, 2$.

II. FIELD EQUATION AND INITIAL-VALUE SETUP

A. Background metric

The ERN geometry is given in Schwarzschild coordinates by the line element

$$ds^2 = -(1 - M/r)^2 dt^2 + (1 - M/r)^{-2} dr^2 + r^2 d\Omega^2,$$

where M is the BH mass, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$. We focus here on the external domain, $r > M$. The tortoise coordinate $r_*(r)$ is defined by $dr/dr_* = (1 - M/r)^2$. Fixing the integration constant by setting $r_*(2M) = 0$ (for later convenience), we get

$$r_* = r - M - \frac{M^2}{r - M} + 2M \ln(r/M - 1). \quad (2.1)$$

This function diverges to $+\infty$ at $r \to \infty$ and to $-\infty$ at $r \to M$, and vanishes at $r = 2M$. We also define the two null coordinates (u, v) by setting $t = v + u$ and $r_* = v - u$, namely:

$$v = (t + r_*)/2, \qquad u = (t - r_*)/2. \quad (2.2)$$

B. Field equation

We consider a massless scalar field Φ satisfying the standard wave equation

$$\Box\Phi \equiv \Phi^{;\alpha}{}_{;\alpha} = 0. \quad (2.3)$$

We decompose Φ into spherical harmonics $Y_{lm}(\theta, \varphi)$ in the usual way,

$$\Phi = r^{-1} \sum_{lm} \psi_l(r, t) Y_{lm}(\theta, \varphi).$$

The field equation (2.3) then reduces to a certain hyperbolic equation for each l:

$$\psi_l'' - \ddot{\psi}_l = V_l(r)\psi_l, \quad (2.4)$$

where the prime and overdot, respectively, denote partial derivatives with respect to r_* and t, and the effective potential $V_l(r)$ takes the form[1]

$$V_l(r) = \left(1 - \frac{M}{r}\right)^2 \left[\frac{2M}{r^3}\left(1 - \frac{M}{r}\right) + \frac{l(l+1)}{r^2}\right]. \quad (2.5)$$

Our goal in the rest of the paper will be to analyze the asymptotic behavior of the fields ψ_l at large t, for appropriate initial conditions.

C. Characteristic initial-value problem

We now set the characteristic initial-value problem, wherein the initial value of ψ_l is specified along two intersecting radial null rays, $u = \text{const}$ and $v = \text{const}$. For convenience we choose the vertex of these two rays to be at $r_* = 0$ (but none of our conclusions should depend on the vertex location). Without loss of generality we also place the vertex at $t = 0$, placing the two initial null rays at $u = 0$ and $v = 0$. The initial data are thus composed of the two initial functions $\psi_v(v) \equiv \psi_l(u = 0, v)$ and $\psi_u(u) \equiv \psi_l(u, v = 0)$.[2]

Note that $u \to \infty$ and $v \to \infty$ correspond to the horizon and to future null infinity (FNI), respectively. Since we only consider here wave dynamics outside the BH, we shall only be concerned with the domain $0 \leq u, v < \infty$.

We shall only be interested in initial data of compact support; that is, $\psi_v(v)$ vanishes beyond a certain value of v. Furthermore, on physical grounds we shall restrict our attention to initial data which are perfectly regular at the horizon (later we shall be more specific about this aspect). Each such set of initial data may be expressed as a superposition of two components[3]:

(I) "Off-horizon compact initial data"—namely, $\psi_u(u)$ (like ψ_v) is only supported at a certain finite range of u (implying that the initial support of ψ_l is well separated from the horizon).

(II) "Horizon-based initial data"—namely, $\psi_v(v)$ vanishes, and $\psi_u(u)$ is only supported in the range $u \geq w$, for a certain $w > 0$.

Note that a localized wave packet incident from infinity (and subsequently scattered or absorbed by the BH) may be represented by initial data of type I, whereas perturbations

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initiated at the horizon's neighborhood (or during the collapse) are represented by type II. Owing to the super-position principle, it will be sufficient to consider these two scattering problems (type I and type II) separately. As discussion below, initial data of type I yield a conventional late-time behavior (as far as the large-t decay at $r > M$ is concerned), qualitatively the same as in the Schwarzschild case. On the other hand, type-II initial data lead to a slower late-time decay, and hence will generically dominate at late time—a phenomenon special to the ERN case. It is there-fore the second type—namely horizon-based initial data—that will mostly concern us here (though later we shall also comment on the generic situation, which is a superposition of I and II).

D. Asymptotic behavior at the horizon

As was already mentioned above, we consider here initial data which are perfectly regular at the horizon. For simplicity, we shall actually assume that the initial data are analytic across the horizon. This means that for any monotonic parameter $\lambda(u)$ which regularly parametrizes the null ray $v = 0$, $\psi_u(\lambda) \equiv \psi_u(u(\lambda))$ will be analytic as λ crosses the horizon (namely as $u \to \infty$). A convenient choice of such a parameter is the area coordinate r, which is a monotonically decreasing function of u along any $v =$ const ray $[dr/du = -(1 - M/r)^2 < 0]$. Furthermore, the perfect regularity of the "ingoing Eddington" coordinate system (v, r, θ, φ) at the horizon indicates the regularity of r as a parameter, on crossing the horizon.

Thus, to address the regularity of $\psi_u(u)$ at the horizon, we express it as a function of the parameter r. Analyticity of the initial function ψ_u implies that this function admits a Taylor expansion in r in the neighborhood of $r = M$. We cast this expansion in the form

$$\psi_u(u) = c_0 + c_1(r/M - 1) + c_2(r/M - 1)^2 + \cdots \quad (2.6)$$

wherein, recall, r is to be regarded as a function of u, evaluated along the ray $v = 0$ [namely $r = r(r_* = -u)$; Note that we have replaced the original Taylor's expansion parameter $r - M$ by its dimensionless counterpart $r/M - 1$ for later convenience—this merely amounts to absorbing a factor M^k in the coefficient c_k.]

Our goal in this paper is thus to analyze the late-time behavior of the field $\psi_l(u, v)$ which evolves from regular "horizon-based" initial data—namely, $\psi_v(v) = 0$, and $\psi_u(u)$ which vanishes at $u < w$, and which on approaching the horizon admits the regular expansion (2.6). To this end, we shall next introduce a transformation which mathemati-cally maps our near-horizon strong-field problem to a new, more convenient problem in weak field.

III. INVERSION TRANSFORMATION

The ERN spacetime admits a conformal transformation [4] which maps the horizon to infinity and vice versa. This

inversion transformation maps r_* to $-r_*$, which also corresponds to the transformation

$$r \to \frac{Mr}{r - M} \equiv T(r). \quad (3.1)$$

The action of this inversion on the various independent variables used below may be summarized by

$$T(r_*) = -r_*, \quad T(t) = t, \quad T(u) = v, \quad T(v) = u. \quad (3.2)$$

The crucial observation is that the effective potential is invariant under this inversion, namely $V_l(T(r)) = V_l(r)$. Therefore, given any solution $\psi_l(r, t)$ of the field equation (2.4), the inversion transformation produces a new solution $\hat{T}(\psi_l(r_*, t)) \equiv \psi_l(-r_*, t)$. In terms of other choices of the independent variables, the map $\hat{T}$ also takes the forms $\hat{T}(\psi_l(r, t)) \equiv \psi_l(T(r), t)$, and $\hat{T}(\psi_l(u, v)) \equiv \psi_l(v, u)$ (namely, the arguments u, v are simply inter-changed). Further discussions about this conformal trans-formation along with interesting implications of it can be found in Refs. [5,6].

We use this inversion to map our original problem (horizon-based perturbations) into a more familiar one, in which the perturbations are based in the large-r region. Thus, we define

$$\Psi_l(u, v) \equiv \hat{T}(\psi_l(u, v)) = \psi_l(v, u).$$

The inverted field Ψ_l satisfies the same field equation (2.4) and (2.5) as the original field ψ_l, namely

$$\Psi_l'' - \ddot{\Psi}_l = V_l(r)\Psi_l.$$

The corresponding initial functions $\Psi_v(v) \equiv \Psi_l(u = 0, v)$ and $\Psi_u(u) \equiv \Psi_l(u, v = 0)$ are immediately obtained from ψ_u, ψ_v through $\Psi_v(v = p) = \psi_u(u = p)$ and $\Psi_u(u = p) = \psi_v(v = p)$ (for any $p \geq 0$).

A. Asymptotic behavior of inverted initial data

As mentioned above, the T-inversion maps the horizon to FNI and vice versa. The asymptotic behavior of the inverted initial function Ψ_v on approaching FNI ($v \to \infty$) will thus be dictated by the horizon-regularity requirement (2.6) on ψ_u. The small near-horizon expansion parameter $r/M - 1$ is T-mapped into the small near-FNI parameter

$$\varepsilon \equiv \frac{T(r)}{M} - 1 = \frac{M}{r - M}. \quad (3.3)$$

Thus, our inverted problem involves initial data

$$\Psi_u(u) = 0 \quad (3.4)$$

and $\Psi_v(v)$ which vanishes at $v < w$, and which, on approaching FNI, behaves as

$$\Psi_v = c_0 + c_1\varepsilon + c_2\varepsilon^2 + \cdots$$

[wherein r is to be regarded as a function of v, evaluated along $u = 0$—namely, $r = r(r_* = v)$]. However, since $\varepsilon = M/r + (M/r)^2 + \cdots$ is itself a regular function of $1/r$, Ψ_v actually admits a regular expansion at FNI in powers of $1/r$:

$$\Psi_v = \hat{c}_0 + \hat{c}_1(M/r) + \hat{c}_2(M/r)^2 + \cdots$$

where $\hat{c}_j$ are dimensionless coefficients which are obtained from c_j in a straightforward manner: for example, $\hat{c}_0 = c_0$, $\hat{c}_1 = c_1$, $\hat{c}_2 = (c_2 + c_1)$, etc.

Now, it is convenient for later purposes to insert a global scale parameter into the initial function Ψ_v. In order to do so, we redefine the initial function in the following way: $\Psi_v(r) \rightarrow \Psi_v(\frac{M}{R}r)$ where here Ψ_v is regarded as a function of r, and R is the desired scale parameter. Moreover, we set the original w to be equal to M, such that the w of the new initial function is equal to R.[4] The asymptotic expansion at FNI of the new initial function takes the form

$$\Psi_v = \hat{c}_0 + \hat{c}_1(R/r) + \hat{c}_2(R/r)^2 + \cdots. \tag{3.5}$$

We deliberately chose the scale parameter to be a free parameter R rather than just the mass M, because later we shall need the freedom to scale these two parameters differently. In particular, the weak-field limit—a crucial ingredient in the analysis below—will be conveniently (and uniformly) achieved by setting $R \gg M$ (and correspondingly $w \gg M$, as noted above).

Owing to the superposition principle, it will of course be sufficient to analyze the contribution emerging from the individual inverse powers of r in Eq. (3.5). However, at this stage we still keep the entire Taylor expansion (3.5) as initial data.

Summarizing, we are considering the following initial-value problem for the inverted field Ψ_l: $\Psi_u(u)$ entirely vanishes, and $\Psi_v(v)$ is only supported at $v \geq w$ and behaves at large v as a regular Taylor expansion in R/r, as described in Eq. (3.5).

IV. WEAK-FIELD ANALYSIS

We shall now focus our attention on the case $w \gg M$,[5] wherein the initial data for the inverted field Ψ_l are

contained in the domain $r \gg M$. This will allow us to employ weak-field methods, which remarkably simplify the analysis.

Note that in terms of the original field ψ_l, this assumption amounts to assuming that the initial data along the incoming ray $v = 0$ are confined to a narrow domain in the neighborhood of the horizon, a domain characterized by $r - M \ll M$. However, throughout this and the next two sections we are concerned with the inverted field Ψ_l—for which our assumption $w \gg M$ amounts to weak-field initial data (only later, in section VII, we transform our results back to the original field ψ_l).

In the analogous Schwarzschild problem, it was noticed long time ago [1] that the inverse-power late time tails are dictated by the weak-field dynamics, namely by scatterings at $r \gg M$. We shall proceed now to extend this approach to ERN. To this end one first need to obtain the large-r asymptotic behavior of $r(r_*)$ and the effective potential V_l.[6] At $r \gg M$, the function $r(r_*)$ takes the asymptotic form

$$r = r_* - 2M\ln(r_*/M) + O(r_*^0). \tag{4.1}$$

The effective potential is dominated at large r by the centrifugal term $l(l+1)/r^2$, and hence takes the asymptotic form

$$V_l(r_*) = \frac{l(l+1)}{r_*^2} + 4Ml(l+1)\frac{\ln(r_*/M)}{r_*^3} + O(r_*^{-3}). \tag{4.2}$$

One can easily notice that this asymptotic expression, like Eq. (4.1), is independent of the BH charge; hence, it is common to the Schwarzschild and ERN cases.

In the analysis of the Schwarzschild case, Price [1] realized that the asymptotic form (4.2) of the potential is sufficient to determine the late-time decay of the scalar perturbation, regardless of the detailed form of $V_l(r)$ at smaller r values. We shall see below that this is also the case in ERN, though with one important difference: Whereas in the Schwarzschild case the curvature-induced term $\propto M\ln(r_*/M)r_*^{-3}$ is crucial for the tail formation, in our case the tails are already formed at (and thus dominated by) the leading-order, flat-space term $\propto l(l+1)r_*^{-2}$ (this results from the difference in the type of initial data along $u = 0$, namely compact versus noncompact support, as we discuss below.) In this sense, the effectiveness of the weak-field approach is much more dramatic in the ERN case.

A. Iterative expansion

The large-r asymptotic form (4.2) motivates an iterative analysis of the evolution of Ψ_l. This approach was initiated

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[4]Note, however, that even though now $w = R$, we may refer to this quantity in different places along the paper as either w or R. This is mainly because in some places we want to emphasize that we use this quantity to represent the edge of the support of the initial function (along the initial ray), while in other places we want to emphasize that we use it to represent the scale parameter. Note also that the parameter w_1 (introduced below) is scaled too, similar to R.

[5]Note that $w \gg M$ is implemented by choosing the scale parameter R such that $R \gg M$.

[6]Recall that in the field equation (2.4) the effective potential at the right-hand side is to be considered as a function of r_*, through the implicit function $r(r_*)$.

by Price [1], and proved to be extremely useful in the Schwarzschild case. Specifically, we adopt here a slightly modified variant of this iterative scheme, developed by Barack's [8], and extend it to our situation of inverse-power initial data on FNI approach. In this method one first decomposes $V_l(r_*)$ into the flat-space centrifugal potential

$$V_0(r_*) = \frac{l(l+1)}{r_*^2} \qquad (4.3)$$

and a residue

$$\delta V(r_*) \equiv V_l - V_0 = 4Ml(l+1)\frac{\ln(r_*/M)}{r_*^3} + O(r_*^{-3}). \qquad (4.4)$$

We rewrite it as

$$\delta V(r_*) \cong 4\eta V_0(r_*), \qquad (4.5)$$

where

$$\eta \equiv \frac{M}{r_*}\ln\frac{r_*}{M}. \qquad (4.6)$$

Note that η is a small factor in weak field: it is of order—or smaller than—

$$\eta_w \equiv (M/w)\ln(w/M) \ll 1. \qquad (4.7)$$

Exploiting the smallness of δV, we now formally decompose Ψ_l as

$$\Psi_l = \Psi_{(0)} + \Psi_{(1)} + \Psi_{(2)} + \cdots,$$

where the terms $\Psi_{(n)}$ are defined as follows: $\Psi_{(0)}$ satisfies the flat-space l-mode wave equation

$$\Psi_{(0)}'' - \ddot{\Psi}_{(0)} - V_0(r_*)\Psi_{(0)} = 0 \qquad (4.8)$$

(with standard flat-space regularity conditions at $r_* = 0$). In turn the fields $\Psi_{(n>0)}$ satisfy a hierarchy of similar wave equations, but with a source term emerging from $\propto \delta V$:

$$\Psi_{(n)}'' - \ddot{\Psi}_{(n)} - V_0(r_*)\Psi_{(n)} = \delta V(r_*)\Psi_{(n-1)} \, (n > 0). \qquad (4.9)$$

At this point, a technical remark should be made about the above decomposition of V_l into V_0 and δV: We have treated here (and similarly in few other places later in this section) the quantity $\ln(r_*/M)$, which is indeed $> \ln(w/M)$, as a large number, thereby neglecting terms $O(r_*^{-3})$ (with no logarithmic enhancement) compared to those kept in δV. This setup is in principle fine, and it is indeed sufficient for our purposes here (as we focus later on $\Psi_{(0)}$ only). However, if one's goal is to obtain the actual

leading mass-induced corrections to the tail amplitudes of Ψ_l in ERN, this specific expansion scheme turns out to be insufficient. Instead, one has to include all the nonlogarithmic $O(r_*^{-3})$ terms as well in δV (namely, the right-hand side of Eq. (4.4) should take the form $[a\ln(r_*/M) + b]r_*^{-3}$, with the appropriate coefficients a, b). The inclusion of these nonlogarithmic $O(r_*^{-3})$ terms in δV is necessary for the following reason: When one calculates the M-induced corrections to the tails at the anticipated leading order $(M/R)\ln(R/M)$, one eventually finds that these corrections cancel out. Instead, the actual leading-order M-induced corrections to the tails only appear at nonlogarithmic order $O(M/R)$. However, in order to obtain the full tail corrections at order M/R, one must include all $O(r_*^{-3})$ terms in δV, not only the logarithmic-enhanced ones.[7]

Here, however, our goal is merely to obtain the leading-order late-time tails—which (in the ERN case, and assuming $w \gg M$) emerge from $\Psi_{(0)}$, at order $(M/R)^0$. The exposition of δV was only made here for the conceptual presentation of the iterative scheme, but we do not need to calculate $\Psi_{(1)}$—with or without the nonlogarithmic terms.

B. Initial data for the iteration fields

We still need to specify the initial data for the various $\Psi_{(n)}$ fields. We choose here the simplest approach: We adopt $\Psi_v(v)$ as the initial function for $\Psi_{(0)}$ (along $u = 0$), while setting vanishing initial data for all other $\Psi_{(n>0)}$.

At this point we must address a subtlety which has a profound effect on the mathematical nature of the initial-value problem for the iteration fields: The potential $V_0 \propto r_*^{-2}$, designed for weak-field analysis, is singular at $r_* = 0$. This is to be contrasted with the full effective potential $V_l(r_*)$, which is perfectly regular at $r_* = 0$ (and actually everywhere). Obviously the singularity of V_0 leads to a similar singularity in δV, again at $r_* = 0$. This singularity is problematic, as it may potentially cause undesired singularities in the iterative fields $\Psi_{(n)}$, an issue which must be addressed.

Consider first the evolution of $\Psi_{(0)}$. Here it turns out that the singularity problem is naturally cured, yet it has an important impact on the nature of the initial-value problem: For any initial data along (the section $v \geq 0$ of) the $u = 0$ ray, which satisfy appropriate regularity conditions as $v \to 0$, there exists a unique regular (at $r_* = 0$) solution of the field equation (4.8) throughout the domain $v \geq u \geq 0$ (namely, the portion $r_* \geq 0$ of $v \geq 0$). As a consequence, the original characteristic initial-value problem, which required initial data (for Ψ_l) along two crossing null rays $u = 0$ and $v = 0$, is actually replaced now by a mixed

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[7]It is interesting to note, however, that in the analogous problem of Schwarzschild's late-time tails, the status of those nonlogarithmic terms is different: In that case, the dominant tails emerge from $\Psi_{(1)}$ (rather than $\Psi_{(0)}$), but the nonlogarithmic $O(r_*^{-3})$ terms in δV do not contribute at leading order.

boundary/initial value problem: One should only specify initial data for $\Psi_{(0)}$ along a *single* null ray $u = 0$ (at $v \geq 0$), and in addition demand regularity along the timelike line $r_* = 0$. This will uniquely determine $\Psi_{(0)}$ in the aforementioned trianglelike domain ($v \geq u \geq 0$). Stated in other words, the initial function $\Psi_v^{(0)}(v) \equiv \Psi_{(0)}(v; u = 0)$ is sufficient, and no analogous function of u is required as initial data for $\Psi_{(0)}$.[8]

The situation with $\Psi_{(n \geq 1)}$ is more delicate. Since we approximate δV by its leading-order term $\propto r_*^{-3} \ln(r_*/M)$ (for the sake of simplifying the weak-field analysis), this expression for δV is even more singular than V_0 at $r_* \to 0$. As a consequence, the fields $\Psi_{(n \geq 1)}$ might become singular at $r_* = 0$ for a sufficiently large n. Note, however, that this divergence is merely an artifact resulting from our large-r_* approximation for δV, Eq. (4.4). This is a useful approximation in the region mostly relevant to our analysis (namely the weak-field domain), but it causes an undesired artifact at small r_*. A natural way to avoid this problem is to simply chop the expression (4.4) for δV at a certain small r_* value which we denote r_*^{shell}. This leads to the "shell model" introduced and analyzed in Ref. [8] (where r_*^{shell} was taken to be of order M). This procedure led to a well-defined expression (independent of r_*^{shell}) for the late-time tail associated with $\Psi_{(1)}$ (and an analysis regarding the smallness of the contributions coming from $\Psi_{(n>1)}$ was carried out).

As we already mentioned, the divergence of V_0 at $r_* = 0$ changes the nature of the initial-value problem for $\Psi_{(0)}$. The same applies to all iteration fields $\Psi_{(n)}$: We only need initial conditions for $\Psi_{(n)}$ along the outgoing ray $u = 0$ (which we simply take to be $\Psi_{(n>0)} = 0$)—and require regularity of $\Psi_{(n)}$ at $r_* = 0$.

C. The flat-space limit

We are focusing here on the case $w \gg M$ (i.e. $M/R \ll 1$). This limit may be achieved by either increasing w (by increasing R), or by decreasing M (fixing M or w, respectively). Both procedures are equivalent, they yield essentially the same evolving field Ψ_l. In the following discussion we shall take the view-point that the initial function $\Psi_v(v)$ is held fixed (in particular w and R are fixed), and we are considering the limit of decreasing M.

In this limiting process, $\Psi_{(0)}$ remains fixed, as both its field equation and its initial data are essentially independent of M.[9]

In order to figure out the behavior of $\Psi_{(n>0)}$ in this limiting process, we follow the lines of Ref. [8] (where the Schwarzschild case was considered). First, we notice that the value of $\Psi_{(n)}$ is determined by the values of $\delta V \Psi_{(n-1)}$ taken in the domain (i.e. the support) of the Green's function (constructed relatively to the chosen evaluation point) of the operator of the left-hand side of Eq. (4.8). The support of this Green's function is not limited only to the weak-field domain, but also reaches to the strong-field domain (and in particular reaches to $r_* = 0$), as shown and explained in detail in [8]. This means that $\Psi_{(n)}$ will be influenced by sources located at any value of r_*, and not only by large-r_* sources. However, in [8], Barack shows that (in Schwarzschild) the dominant (i.e. leading order in $M/u \lesssim M/R \ll 1$) contribution to $\Psi_{(n)}$ at FNI is only due to sources located at large r_* (i.e. $r_* \gg M$). As a result, he shows that in the case $M/R \ll 1$, (i) the iterative series converges at null infinity at late time and (ii) the dominating term of the expansion is the one of lowest possible n. Specifically, Barack shows that $\Psi_{(n+1)}/\Psi_{(n)} \propto M/R$ (potentially with logarithmic factors) at FNI.

Similarly, in ERN (which has the same asymptotic structure as Schwarzschild at $r_* \gg M$) we will regard the fields $\Psi_{(n)}$ evaluated in the vicinity of FNI (late times and $r_* \gg M$) as originating from sources located at large r_*. As a result, $\Psi_{(1)}$ decreases in the limit $M/R \ll 1$, as we regard its source as coming from the weak-field domain and as this source term is proportional to δV, which in turn is $\propto \eta \lesssim \eta_w \ll 1$, cf. Eqs. (4.9), (4.5), and (4.7). This argument applies to higher-order iterations as well (like in Schwarzschild), suggesting that as M decreases, $\Psi_{(n)}$ should decrease as $(\eta_w)^n$ (or roughly as $(M/R)^n$). We may therefore naturally expect that at the limit $M \ll R$, the late-time tails will be dominated by the lowest possible n—namely by the flat-space field $\Psi_{(0)}$ (remember, though, that this approximation for Ψ_l is only valid for evaluation points in the vicinity of FNI, i.e. at late times and in the domain $r_* \gg M$).

At this point it is instructive to compare our present situation to the standard problem of late-time tails in Schwarzschild. In that case too, for weak-field initial data, the late-time tails are dominated by the minimal possible iteration order n. However, in the Schwarzschild case, considering compact initial support (the case of greatest physical motivation), it turns out that $\Psi_{(0)}$ yields no contribution to the late-time field. This is directly related to the compact support of the initial data.[10] In our problem, however—the inverted field Ψ_l on the ERN background—the initial data is noncompactly supported, resulting in a

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[8]Recall that this behavior of $\Psi_{(0)}$—namely, the existence of unique regular solutions, and also the need for characteristic initial data only on a single null ray—is a well-known feature of the standard flat-space wave equation for a multipole l of massless scalar field. [To verify the connection, one should just view the spatial variable "r_*" in Eq. (4.8) as the standard radial coordinate "r" in a fiducial flat space.]

[9]The dependence of $v(r)$ (along the ray $u = 0$) on M fades out as $M \to 0$.

[10]In the other case of initial static moment, the initial data are not compactly supported. Nevertheless, for the specific initial function characterizing the initial static moment, it turns out that again $\Psi_{(0)}$ has no contribution to the late-time field. This situation changes when the initial function Ψ_v is noncompactly supported and decays as v^{-k} (for a range of k values), as we discuss.

nonvanishing tail contribution already for $n = 0$, as we soon show. Correspondingly, in our problem it will be $\Psi_{(0)}$ (rater than $\Psi_{(1)}$) which dominates the late-time tails. For this reason we shall devote the next two sections to study the evolution of $\Psi_{(0)}$ and its late-time tail.

It should be pointed out already at this stage that there is a potential loophole in the above reasoning, concerning the dominance of $\Psi_{(0)}$ in the late-time tail: It is not difficult to imagine a situation in which $\Psi_{(1)}$ is indeed suppressed globally as M/w (or η_w), and yet it may decay slower than $\Psi_{(0)}$ at large t. However, there is evidence that this is not the case, which we discuss in Sec. VII. The first evidence is that $\Psi_{(1)}$ decays with the same inverse power of t as $\Psi_{(0)}$, yet its overall amplitude is suppressed by the small parameter $\sim M/w$. The second and much stronger evidence emerges from the horizon's exact conserved quantities (discussed in the summary in Sec. VII) and suggests that the mere effect of the higher-order iteration fields $\Psi_{(n>0)}$ is to modify the value of the constant prefactor in the resulting expression for $\Psi_{(0)}$. We address this issue further in Sec. VII.

Therefore, in the case $w \gg M$ considered here, it is indeed possible to obtain the leading-order (in $\sim M/w$) of the late-time tails from the flat-space field $\Psi_{(0)}$. This is the strategy that we shall use in the rest of this paper.

V. ANALYSIS OF THE FLAT-SPACE FIELD $\Psi_{(0)}$

We turn now to analyze the evolution of $\Psi_{(0)}$, governed by Eq. (4.8). In several occasions it will be useful to interpret this equation as that of a standard scalar field $\tilde{\Phi}$, related to $\Psi_{(0)}$ through

$$\tilde{\Phi} \equiv \Psi_{(0)} Y_{lm}(\theta, \varphi)/r_*, \qquad (5.1)$$

which lives in a fiducial $3 + 1$ flat background with standard Minkowski spherical coordinates $(t, r_*, \theta, \varphi)$. (Here r_* serves as the standard radial coordinate in this fiducial flat spacetime; note that in Minkowski $r_* = r$.) The standard wave equation $\Box\tilde{\Phi} = 0$ then reduces to Eq. (4.8).

For later convenience we also express the wave equation (4.8) in terms of the null coordinates u, v:

$$\Psi_{(0),uv} = -\frac{l(l+1)}{(v-u)^2}\Psi_{(0)}. \qquad (5.2)$$

A. The general solution

The general solution of this equation is known explicitly. It involves two arbitrary functions $g(u)$ and $h(v)$, and it takes the form

$$\Psi_0(u, v) = \sum_{n=0}^{l} A_n^l \frac{g^{(n)}(u) + (-1)^n h^{(n)}(v)}{(v-u)^{l-n}}, \qquad (5.3)$$

where the upper index "(n)" denotes nth-order derivative of the function with respect to its argument, and where A_n^l are coefficients given by

$$A_n^l = \frac{(2l-n)!}{n!(l-n)!}. \qquad (5.4)$$

If characteristic initial data for Ψ_0 were given in a local "diamond" which does not extend to $r_* = 0$, the two functions $g(u)$ and $h(v)$ could be chosen arbitrarily and independently. However, when the domain of dependence contains the line $r_* = 0$ (which is the case in our problem), for an arbitrary choice of $g(u)$ and $h(v)$ the solution (5.3) generically diverges at $r_* = 0$, owing to the factor $(v - u)^{l-n}$ in the denominator.[11] The most singular contribution, proportional to $(v - u)^{-l}$, emerges solely from the $n = 0$ term. In that term the numerator reads $g(u) + h(v)$. The only way to avoid this divergence is to have the two functions $g(u)$ and $h(v)$ cancel each other along the line $v = u$. This cancellation occurs if and only if $g(u)$ and $h(v)$ have the same functional form, that is, $g(x) = -h(x) \equiv f(x)$. Thus, the general solution regular at $r_* = 0$ takes the form

$$\Psi_0(u, v) = \sum_{n=0}^{l} A_n^l \frac{f^{(n)}(u) - (-1)^n f^{(n)}(v)}{(v-u)^{l-n}}. \qquad (5.5)$$

We shall refer to the function $f(x)$, which underlies the expression in the right-hand side (through substitutions $x \to u$ and $x \to v$), as the "generating function." Also, we shall denote the differential operator at the right-hand side, acting on the function f, by $\Delta(f)$.

Since the form of the general solution for $\Psi_0(u, v)$ is known, the remaining challenge is to find the function f which corresponds to the specified initial function $\Psi_v(v)$. Namely, we seek the function f satisfying

$$\Delta(f)(u = 0, v) = \Psi_v(v). \qquad (5.6)$$

In the following subsections we shall proceed to construct the solution(s) of this equation. To this end, we shall first discuss several properties of the generating function f and of the differential operator Δ.

B. General properties of the generating function f

We would first like to explore the space of homogeneous solutions of Eq. (5.6), namely the functions f satisfying $\Delta(f)(u = 0, v) = 0$. Note that $\Psi_v(v) = 0$ implies the vanishing of $\Psi_0(u, v)$ due to uniqueness (and vice versa —the vanishing of $\Psi_0(u, v)$ trivially implies $\Psi_v(v) = 0$); hence, these are the functions f generating a trivial field $\Psi_0(u, v) = 0$.

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[11]In the special case $l = 0$, Ψ_0 itself is finite, yet $\tilde{\Phi} = \Psi_{(0)}/r_*$ generically diverges.

We shall therefore focus on the projection of Eq. (5.5) to $u = 0$:

$$\Psi_v(v) = \sum_{n=0}^{l} A_n^l \frac{f^{(n)}(0) - (-1)^n f^{(n)}(v)}{v^{l-n}} \equiv \Delta_0(f(v)),$$

(5.7)

which defines the differential operator Δ_0 acting on functions $f(v)$. Here $f^{(n)}(0)$ means $f^{(n)}(v)$ evaluated at $v = 0$.

Yet another relevant differential operator is the restriction of Δ_0 to its local piece:

$$D_0(f(v)) \equiv \sum_{n=0}^{l} (-1)^{n+1} A_n^l v^{n-l} f^{(n)}(v),$$

(5.8)

such that

$$\Psi_v(v) = \Delta_0(f(v)) = D_0(f(v)) + \sum_{n=0}^{l} A_n^l v^{n-l} f^{(n)}(v = 0).$$

(5.9)

Recall the difference between these three linear operators: Δ is a partial differential operator, Δ_0 is an ordinary differential operator, but yet a nonlocal one (it involves the values of $f^{(n)}$ at $v = 0$), but D_0 is a more standard, local, ordinary differential operator.

The space of homogeneous solutions of these differential operators was analyzed in Ref. [8]. Here we merely mention the results:

(I) The basis of homogeneous solutions of D_0 is

$$v^j, \qquad l+1 \le j \le 2l.$$

(5.10)

(II) The basis of homogeneous solutions of Δ_0 (or, equivalently, Δ) is larger:

$$v^j, \qquad 0 \le j \le 2l.$$

(5.11)

Recall that D_0 is a rather standard (i.e. local) lth-order differential operator; hence, it must admit l-independent homogeneous solutions, as in Eq. (5.10). It is a bit surprising that the set (5.11) is larger, including $2l + 1$ independent functions, even though Δ_0 too is an lth-order differential operator. This abnormally large basis is only possible due to the nonlocal nature of the operator Δ_0.

This somewhat confusing structure of the space of homogeneous solutions associated with Δ_0 and D_0 is most easily illustrated in the $l = 0$ case. In that case, $\Psi_0(u, v) = f(u) - f(v) \equiv \Delta(f)$, then $\Psi_v(v) = f(v = 0) - f(v) \equiv \Delta_0(f)$, and $D_0(f) = -f(v)$ is a trivial (i.e. zero-order) differential operator. Obviously

D_0 has no (nonvanishing) homogeneous solutions, yet Δ_0 does have one nonvanishing homogeneous solution: $f(v) = \text{const}$. For this choice, $\Psi_0(u, v) = f(u) - f(v)$ indeed vanishes.

Note also that every homogeneous solution of D_0 must also be a homogeneous solution of Δ_0, as can be seen by comparing the two bases (5.10) and (5.11). This must indeed be the case, by virtue of Eq. (5.9), because for any member of the smaller basis (5.10), all functions $f^{(n)}(v)$ with $0 \le n \le l$ vanish at $v = 0$.

1. Representative generating function

For a given initial function $\Psi_v(v)$, the general solution of the inhomogeneous equation $\Delta_0(f(v)) = \Psi_v(v)$, for $f(v)$, is any specific inhomogeneous solution, plus any homogeneous solution. This is an infinite set, a $2l + 1$-parameter family of solutions. We shall prescribe here the construction of a single solution $f(v)$, for a given function $\Psi_v(v)$—which we shall call the "representative solution."

We shall assume that $\Psi_v(v)$ vanishes as $v \to 0$. This is trivially satisfied by the initial functions considered here, because they were chosen to vanish at $v < w$. We note, however, that even without this choice of restricted support, $\Psi_v(v \to 0)$ should vanish in order for $\tilde{\Phi} \propto \Psi_{(0)}/r_*$ to be bounded at $u = v = 0$.

We define our representative solution $f(v)$ as follows: It is the solution of the lth order ODE,

$$D_0(f(v)) = \Psi_v(v),$$

(5.12)

subject to the initial conditions

$$f^{(n)}(v = 0) = 0, \qquad 0 \le n \le l-1.$$

(5.13)

Notice that by virtue of these two equations, $f^{(l)}$ vanishes at $v = 0$. Therefore, each of the terms in the sum in the right-hand side of Eq. (5.9) vanishes, yielding the required relation $\Delta_0(f(v)) = \Psi_v(v)$. We shall denote this representative generating function by $f_r(v)$.

For the class of initial functions $\Psi_v(v)$ considered here, which vanish at $v < w$, $f_r(v)$ too will vanish at $v < w$. Therefore, the representative solution will take the form

$$f_r(v) = \Theta(v - w) f_w(v),$$

(5.14)

where Θ denotes the standard step function, and $f_w(v)$ is defined to be the solution of the ODE (5.12) with the appropriate initial conditions at $v = w$:

$$f^{(n)}(v = w) = 0, \qquad 0 \le n \le l-1.$$

(5.15)

2. Compactly supported initial data

Although in our (inverted) ERN case the function $\Psi_v(v)$ is of noncompact support, it will be useful to consider here the other case, in which the initial function $\Psi_v(v)$ is of

compact support. Let us assume thus that the support of $\Psi_v(v)$ is restricted to the range[12]

$$w < v < v_{\max}. \tag{5.16}$$

Consider now the expression (5.14) for $f_r(v)$. Since Ψ_v vanishes at $v > v_{\max}$, throughout this domain $f_r(v) = f_w(v)$ will be a certain homogeneous solution, which we denote $f_{\hom}(v)$; namely, it satisfies $D_0(f_{\hom}(v)) = 0$. We can therefore express $f_r(v)$ as

$$f_r(v) = \Theta(v_{\max} - v)[\Theta(v - w)f_w(v)] + \Theta(v - v_{\max})f_{\hom}(v). \tag{5.17}$$

By virtue of Eq. (5.10), $f_{\hom}$ must be a superposition of the form

$$f_{\hom}(v) = \sum_{j=l+1}^{2l} a_j v^j \tag{5.18}$$

with certain coefficients a_j (whose values will not concern us). From Eq. (5.11), such a superposition also satisfies $\Delta(f_{\hom})(u, v) = 0$.

Consider now the behavior of Ψ_0 in the portion $u > v_{\max}$ of $r_* > 0$. Any point in this domain satisfies $v_{\max} < u < v$. Therefore, in the right-hand side of Eq. (5.5), the functions $f^{(n)}(u)$ and $f^{(n)}(v)$ may all be replaced, respectively, by $f_{\hom}^{(n)}(u)$ and $f_{\hom}^{(n)}(v)$, implying that

$$\Psi_0(u, v) = \Delta(f_{\hom})(u, v) = 0 \qquad (v > u > v_{\max}). \tag{5.19}$$

We conclude that if the initial function $\Psi_v(v)$ is of compact support, restricted to $v \leq v_{\max}$, then Ψ_0 strictly vanishes throughout the domain $u > v_{\max}$ (of $r_* > 0$). In particular, in this case Ψ_0 will have no contribution whatsoever to the late-time tails—either at future null infinity or along $r = \text{const}$.

Note that this is precisely the reason why in the usual Schwarzschild problem, with compactly-supported initial data, no late-time contribution emerges from Ψ_0, hence the dominant contribution to the late-time tails emerges from $\Psi_{(1)}$ (however, in our inverted ERN problem, the situation is different due to the noncompactness of the initial support).

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[12]The restriction $v > w$ can be relaxed if we assume the regularity condition $\Psi_v(v \to 0) = 0$.

VI. CONSTRUCTION OF $\boldsymbol{\Psi_0}$ FOR INVERSE-POWER INITIAL DATA

A. Simplifying the initial-value setup

The initial conditions for Ψ_0, the function $\Psi_v(v)$, was given in Eq. (3.5). We shall now proceed to simplify our initial-value setup, as we now describe.

Recall that we have assumed analyticity of $\psi_u(r)$ across the horizon, which in turn implies that the inverted field Ψ_v is analytic in $1/r$ for a sufficiently large r. In particular, the series (3.5) converges at a sufficiently large r. However, we have also assumed that $\Psi_v(v)$ vanishes at $v \leq w$ (for some $w > 0$). In fact this implies that $\Psi_v(v)$ is not everywhere analytic (neither in v nor in r). But nevertheless we assume that it is everywhere $C^{(\infty)}$.

Thus, our initial function $\Psi_v(v)$ presumably admits the following properties: (i) it is $C^{(\infty)}$ everywhere (in r and, hence, in v too), (ii) it vanishes at $v \leq w$, and (iii) throughout $v \geq w_1$ (for a certain $w_1 \geq w$), it is given by the convergent series (3.5).

To simplify the initial-value setup, we first note that owing to the superposition principle, it will be sufficient to consider the contributions emerging from individual terms $\hat{c}_k(R/r)^k$ separately. Thus, we shall be concerned with an initial function which at $v \geq w_1$ takes the form

$$\Psi_v^k = \hat{c}_k(R/r)^k. \tag{6.1}$$

In addition, Ψ_v^k is assumed to vanish at $v \leq w$; this implies that in the domain $w \leq v \leq w_1$, Ψ_v^k will inevitably deviate from Eq. (6.1); i.e., it is not analytic. We shall assume that Ψ_v^k is $C^{(\infty)}$.[13]

Next, to further simplify the analysis, we shall approximate r in Eq. (6.1) by r_*. This simplification is justified by our weak-field approximation. Recall that the relative difference between r and r_* is $\propto M/r$ (setting aside logarithmic corrections), which is negligible in the relevant domain ($r \gtrsim R \gg M$).

We are thus led to consider the initial functions $\hat{\Psi}_v^k(v)$ (for all integers $k \geq 0$), which are presumably $C^{(\infty)}$, supported at $v \geq w$ only, and which satisfy $\hat{\Psi}_v^k = \hat{c}_k(R/r_*)^k$ at $v \geq w_1$, namely

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[13]It is not difficult to construct such a $C^{(\infty)}$ extension of (6.1) to the domain $v \leq w_1$, as follows: First, choose a parameter $w \leq \tilde{w} < w_1$ such that the sum (3.5) still converges to Ψ_v at $w = \tilde{w}$. Let $p(r)$ be any smooth transition function, that is, a monotonically increasing $C^{(\infty)}$ function satisfying $p = 0$ at $v \leq \tilde{w}$ and $p = 1$ at $v \geq w_1$. Let us define $\tilde{\Psi}_v = \sum_k \hat{c}_k(Rp/r)^k$ and $\Delta\Psi_v = \Psi_v - \tilde{\Psi}_v$. Notice that $\Delta\Psi_v$ is $C^{(\infty)}$ and is only supported at $w \leq v \leq w_1$. We now define $\Psi_v^k = \hat{c}_k(Rp/r)^k + \tilde{c}_k\Delta\Psi_v$, where $\tilde{c}_k = \hat{c}_k(R/r_1)^k/\Psi_v(v = w_1)$ and $r_1 \equiv r(v = w_1) = r(r_* = w_1)$. Note that $\sum_k \tilde{c}_k = 1$. It is not difficult to show that (i) Ψ_v^k vanishes at $v \leq w$, (ii) it coincides with Eq. (6.1) at $v \geq w_1$; (iii) it is $C^{(\infty)}$ everywhere, and (iv) $\sum_k \Psi_v^k = \Psi_v$.

$$\hat{\Psi}_v^k(v) = \hat{c}_k (R/v)^k \qquad (6.2)$$

at $v \geq w_1$.[14]

B. Generating function for inverse-power initial data

In order to analyze the $\Psi_{(0)}$ field evolving from the above initial function $\hat{\Psi}_v^k(v)$, we first construct the corresponding representative generating function $f_r(v)$, which satisfies the inhomogeneous ODE $D_0(f_r(v)) = \hat{\Psi}_v^k(v)$ with the initial conditions (5.13). Since the space of homogeneous solutions is known, Eq. (5.10), all we need at this stage is to construct a single inhomogeneous solution, which we denote $f_{\text{inh}}(v)$. As we explain in the next subsection, the late-time behavior of the field $\Psi_{(0)}$ depends only on the values that $f_r(v)$ takes in the range $v \geq w_1$. Therefore, we shall focus on finding $f_{\text{inh}}(v)$ [and $f_r(v)$] in that range (where $\hat{\Psi}_v^k(v) = \hat{c}_k (R/v)^k$). The structure (5.8) of the operator D_0 immediately suggests the existence of a specific solution of the form $f_{\text{inh}}(v) = \text{const} \cdot v^{l-k}$ (in $v \geq w_1$). A direct substitution yields

$$f_{\text{inh}}(v) = (\alpha \hat{c}_k R^k) v^{l-k} \qquad (v \geq w_1), \qquad (6.3)$$

where $\alpha \equiv \alpha(l, k)$ is defined by

$$1/\alpha = \sum_{n=0}^{l} (-1)^{n+1} A_n^l \prod_{j=0}^{n-1} (l - k - j)$$

$$= -\sum_{n=0}^{l} A_n^l \prod_{j=0}^{n-1} (j + k - l).$$

Using MATHEMATICA, one can easily find the more explicit expression

$$\alpha = -\frac{k!}{(k+l)!}. \qquad (6.4)$$

Once the specific inhomogeneous solution is known, the general solution of the ODE (5.12) is obtained by adding a general homogeneous solution of the form (5.18). Therefore, the representative generating function (in the range $v \geq w_1$) takes the form

$$f_r(v) = (\alpha \hat{c}_k R^k) v^{l-k} + \sum_{j=1}^{l} c_j v^{l+j} \qquad (v \geq w_1), \quad (6.5)$$

with certain coefficients c_j. These coefficients are in principle determined by integrating the ODE (5.12) with

the initial conditions (5.13), hence they depend on the specific form of $\hat{\Psi}_v^k(v)$ at $v < w_1$. However, for the analysis below we shall not need these coefficients.

C. Late-time behavior of Ψ_0

1. General expression

Consider now the behavior of Ψ_0 in the range $v > u \geq w_1$. From the form (5.5) of the general solution, it is obvious that only the behavior of the generating function at $v \geq w_1$ is relevant. Since the homogeneous piece $\sum c_j v^{l+j}$ yields no contribution to Ψ_0 [cf. Eq. (5.11)], the late-time behavior of Ψ_0 is (precisely)

$$\Psi_0(u, v) = (\alpha \hat{c}_k R^k) \Delta(x^{l-k}) \qquad (v > u \geq w_1), \quad (6.6)$$

where, recall,

$$\Delta(x^{l-k})$$
$$\equiv \sum_{n=0}^{l} A_n^l \left[\frac{d^n}{du^n} (u^{l-k}) - (-1)^n \frac{d^n}{dv^n} (v^{l-k}) \right] (v - u)^{n-l}. \qquad (6.7)$$

As was discussed in the previous section, $\Delta(x^j)$ vanishes for all $0 \leq j \leq 2l$, cf. Eq. (5.11). It immediately follows that at late time (namely, $u \geq w_1$), Ψ_0 vanishes for any $k \leq l$. We only need to consider the contributions from $k > l$.

This last result—the strict vanishing of late-time $\Psi_0(u, v)$ for all $k \leq l$—involves a subtlety: If Ψ_0 vanishes, why is $\hat{\Psi}_v^k(v)$ nonvanishing? To address this question we must distinguish between two different situations: (i) In the situation described above, the initial function $\hat{\Psi}_v^k(v)$ is $\propto v^{-k}$ only at $v \geq w_1$ (for some $w_1 > 0$). In that case, $\Psi_0(u, v)$ vanishes at $v > u \geq w_1$, but *not* necessarily in the domain $w_1 > u > 0$. This allows continuity of Ψ_0 on approaching $u = 0$. (ii) The more subtle case is the one in which $w_1 \to 0$, namely, $\hat{\Psi}_v^k \propto v^{-k}$ on the entire (portion $v > 0$ of the) ray $u = 0$. In that case, the above result would mean that $\Psi_0(u, v) = 0$ on the entire (portion $r_* > 0$ of the) domain $u \geq 0$—which would conflict with the nonvanishing value of $\hat{\Psi}_v^k$ at $u = 0$. The resolution of this conflict is a bit tricky: In this situation of $\hat{\Psi}_v^k \propto v^{-k}$ extending all the way to $v = 0$, $\hat{\Psi}_v^k = \Psi_0(u = 0)$ diverges at the origin ($u = v = 0$). This divergence violates the regularity condition that we have imposed along the central worldline $r_* = 0$. Thus, in this case there is a conflict between (I) the form of the initial function at $v \to 0$, (II) the assumed regularity condition at $r_* = 0$, and (III) the assumption of continuity of $\Psi_0(u, v)$ on approaching $u \to 0$. The above construction of $\Psi_0(u, v)$ was based on Eq. (5.5), which in turn was based on the assumption of regularity at $r_* = 0$, hence continuity at $u \to 0$ has been sacrificed.

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[14]The difference between $\hat{\Psi}_v^k \propto (R/r_*)^k$ and the "true" initial data $\Psi_v^k \propto (R/r)^k$, being proportional to M/R (aside from logarithmic corrections), should naturally be considered as a seed for $\Psi_{(1)}$. Here, however, we are only concerned with $\Psi_{(0)}$; hence, this difference between $\hat{\Psi}_v^k$ and Ψ_v^k will not concern us.

2. Decay rate along future null infinity

Consider next the behavior of Ψ_0 at the limit $v \to \infty$ (at fixed $u \geq w_1$), which characterizes the approach to future null infinity (FNI). In the v-derivatives term in the squared brackets in Eq. (6.7), for each n the dominant contribution (in terms of expansion in $1/v$) is $\propto v^{-k}$ and will hence vanish at FNI (recall that the relevant values of k are $k > l$). In the u-derivatives term, the dominant contribution of the nth term is $\propto v^{n-l}$, which vanishes for all $n < l$. The only nonvanishing contribution to $\Delta(x^{l-k})$ at FNI is thus the $n = l$ term, which—since $A_l^l = 1$—is just $(d/du)^l(u^{l-k})$. It immediately follows that $\Delta(x^{l-k})$ vanishes for any $0 < k \leq l$. Calculation yields

$$\Delta(x^{l-k}) = \frac{d^l}{du^l}(u^{l-k}) = p_k u^{-k} \qquad (v \to \infty, u \geq w_1),$$
(6.8)

where

$$p_k = (-1)^l \prod_{j=0}^{l-1}(j + k - l) = (-1)^l \prod_{j=1}^{l}(k - j). \quad (6.9)$$

The last expression shows at once that p_k vanishes for any $0 < k \leq l$. This again demonstrates that—at least at FNI—Ψ_0 vanishes throughout $u \geq w_1$ for any $0 < k \leq l$. It confirms the more general observation made above, that the late-time contributions to Ψ_0 emerge only from $k > l$ terms.[15] Noting this range of k values, it will be more convenient to express p_k as

$$p_k = (-1)^l \frac{(k-1)!}{(k-l-1)!} \qquad (k > l). \quad (6.10)$$

Substituting back in Eqs. (6.8) and (6.6), we find (for $k > l$ only)

$$\Psi_0(u, v) = (\beta \hat{c}_k R^k)u^{-k} \qquad (v \to \infty, u \geq w_1), \quad (6.11)$$

where $\beta \equiv \beta(l, k)$ is given by

$$\beta = (-1)^{l+1} \frac{(k-1)!}{(k-l-1)!} \frac{k!}{(k+l)!}$$

$$= (-1)^{l+1} \frac{k[(k-1)!]^2}{(k-l-1)!(k+l)!}.$$

While the specific value of the coefficient β is not so crucial, the important observation is that for any $k > l$ this coefficient is nonvanishing. Recalling the vanishing of all $k \leq l$ contributions, we conclude that the late-time field at

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[15]The $k = 0$ case requires a special treatment. (Nevertheless, the vanishing of the $k = 0$ contribution at $u \geq w_1$ follows from the more general observation which we have just mentioned.)

FNI is dominated by the $k = l + 1$ tail, and it hence decays as u^{-l-1}.

To summarize this subsection, we want to write the dominant late-time tail ($k = l + 1$) at FNI explicitly. In order to do so, we first calculate the relevant β,

$$\beta_{l+1} \equiv \beta(l, k = l + 1) = (-1)^{l+1} \frac{(l+1)(l!)^2}{(2l+1)!}$$

$$= (-1)^{l+1} \frac{2[(l+1)!]^2}{(2l+2)!},$$

and then obtain [using Eq. (6.11)]

$$\Psi_0^{k=l+1}(u, v) = \hat{c}_{l+1}R^{l+1}(-1)^{l+1} \frac{2[(l+1)!]^2}{(2l+2)!} u^{-(l+1)}$$

$$(v \to \infty, u \geq w_1). \quad (6.12)$$

3. Decay rate at fixed r_*

Once the behavior of Ψ_0 along FNI is known, we can analyze $\Psi_0(u, v)$ throughout the domain $u \geq w_1$ and obtain its decay rate along lines of constant $r_* > 0$.

This problem is simplified by the fact that throughout the relevant domain $v > u \geq w_1$, the only relevant piece of the initial null ray $u = 0$ is $v \geq w_1$, wherein $\hat{\Psi}_v^k(v)$ is an inverse power. Recalling Eqs. (6.6) and (6.7), it is straightforward to show that $\Psi_0(u, v)$ must take the form $u^{-k}F_{lk}(u/v)$, where F_{lk} is a certain function of its argument. The problem of determining $\Psi_0(u, v)$ then reduces to that of obtaining the function $F_{lk}(u/v)$. This, in turn, is achieved by recalling that this function must satisfy a certain ODE, obtained by substituting the above expression for $\Psi_0(u, v)$ in the field equation. Solving this ODE, and matching to a regularity condition at $r_* = 0$ (namely $u/v = 1$) as well as to the known boundary condition at FNI (i.e. $u/v = 0$), will fully determine F_{lk} and hence also $\Psi_0(u, v)$.

The aforementioned ODE is simplified if we change its independent variable from u/v to $1 - u/v \equiv y$. Thus, we write

$$\Psi_0(u, v) = u^{-k}G(y) \qquad (v > u \geq w_1), \quad (6.13)$$

where G is a yet unknown function of its argument. The field equation (5.2) then reduces to the ODE:

$$G''(y) = \frac{1-k}{1-y}G'(y) + \frac{l(l+1)}{(1-y)y^2}G(y). \quad (6.14)$$

This equation is solvable for each l, k. However, we shall not need the explicit solution here, and a general discussion of the properties of $G(y)$ will suffice (below, however, we shall consider in more detail the explicit solution in the important case $k = l + 1$).

The limit $y \to 0$ corresponds to $r_* = 0$. The asymptotic behavior at this limit can be easily analyzed. The right-hand side of the above ODE reduces to

$$(1 - k)G'(y) + \frac{l(l+1)}{y^2}G(y).$$

The term $\propto G'(y)$ becomes unimportant at this limit, and we obtain the two usual spherical-harmonic asymptotic solutions: a regular solution $G \simeq y^{l+1}$ and a singular solution $G \simeq y^{-l}$.[16] The regularity condition at $r_* = 0$ then selects the regular solution, yielding

$$G(y) = g_0 y^{l+1} + O(y^{l+2}) \qquad (y \ll 1). \qquad (6.15)$$

The coefficient g_0 is to be determined by matching to FNI, but we shall not need it here.

Consider now the behavior of Ψ_0 along a line of constant r_*, at very late time ($t \gg r_*$). Since $t/r_* = 2/y - 1$, the inequality $t \gg r_*$ corresponds to $y \ll 1$; hence,

$$\Psi_0(u, v) = g_0 u^{-k}[y^{l+1} + O(y^{l+2})] \qquad (t \gg r_*).$$

Noting that $y = 2r_*/(r_* + t) \cong 2r_*/t$, we obtain

$$\Psi_0 = \tilde{g}_0 r_*^{l+1} t^{-(k+l+1)}[1 + O(r_*/t)] \quad (t \gg r_*), \quad (6.16)$$

where $\tilde{g}_0 \equiv 2^{k+l+1} g_0$.

We see that the asymptotic behavior at large t is $\propto t^{-l-k-1}$ and is hence dominated by the smallest possible k. However, we already know that for any $k \leq l$ the field Ψ_0 vanishes throughout the domain—namely, at any fixed $r_* > 0$ it vanishes for sufficiently large t (stated in other words, the prefactor g_0 vanishes for any $k \leq l$). We therefore conclude that the late-time decay of Ψ_0 along lines of fixed r_* is dominated (like that at FNI) by the $k = l + 1$ term, and decays as $\propto t^{-(2l+2)}$.

Let us consider in more detail the dominant contribution $k = l + 1$. In this case, the regular solution of Eq. (6.14) is exactly $G(y) = g_0^{(l+1)} y^{l+1}$ (where $g_0^{(l+1)}$ is the constant g_0 for the case $k = l + 1$); hence,

$$\Psi_0^{k=l+1}(u, v) = g_0^{(l+1)}(y/u)^{l+1} = g_0^{(l+1)}\left(\frac{1}{u} - \frac{1}{v}\right)^{l+1}$$

$$(v > u \geq w_1). \qquad (6.17)$$

Note the regularity of this expression at $r_* = 0$. Also, at FNI it becomes $g_0^{(l+1)} u^{-(l+1)}$, which allows us to obtain $g_0^{(l+1)}$ by matching to Eq. (6.12):

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[16]In both solutions, the additional $\propto G'(y)$ term, and also the factor $1 - y$ in the denominators in Eq. (6.14), only affect higher-order terms in the expansion in y.

$$g_0^{(l+1)} = \hat{c}_{l+1} R^{l+1}\beta_{l+1} = \hat{c}_{l+1} R^{l+1}(-1)^{l+1}\frac{2[(l+1)!]^2}{(2l+2)!}.$$

To find the asymptotic behavior at $t \gg r_*$, recall that

$$\frac{1}{u} - \frac{1}{v} = \frac{r_*}{uv} \cong \frac{4r_*}{t^2};$$

hence,

$$\Psi_0^{k=l+1} \cong g_0^{(l+1)}(4r_*)^{l+1} t^{-(2l+2)}[1 + O(r_*/t)] \quad (t \gg r_*). \tag{6.18}$$

Note that this result [Eq. (6.18)] is in complete correspondence with the one obtained above for general k [Eq. (6.16)].

VII. BACK TO THE ORIGINAL FIELD ψ_l AND SUMMARY

So far we have analyzed the late-time tail associated with $\Psi_{(0)}$. One may wonder what would be the contribution to the tail from the higher-order moments $\Psi_{(n>0)}$. For example, as noted in the final part of subsection IVC, one may worry that some $\Psi_{(n>0)}$ would decay at late times slower than $\Psi_{(0)}$ (even though its amplitude is smaller when $R \gg M$). To this end one may extend the procedure of [8] (the "shell toy-model") from the Schwarzschild case to our problem. Within that context, it was found by A. Ori [11] that the tail contribution from $\Psi_{(n=1)}$ (in the domain $r_* \gg M$) again decays as $\Psi_{(0)}$, though with a coefficient which scales as M/R. There is evidence (emerging from the horizon's exact conserved quantities discussed below) suggesting that the mere effect of the $n > 0$ term would be to replace the coefficient $g_0^{(l+1)}$ in Eqs. (6.18) and (6.17) by another coefficient $\hat{g}$. Note that $\hat{g} \cong g_0^{(l+1)}$ as long as $R \gg M$, but in the case of "strong initial data," namely $R \sim M$, one expects that $\hat{g}$ will involve significant contributions from $n > 0$. We shall probably treat this issue further in a subsequent paper [9] that I hope to address.

Noting that (i) in the case $R \gg M$ considered here and for evaluation points in the vicinity of FNI (i.e. at late times and in the domain $r_* \gg M$), Ψ_l should be well approximated by Ψ_0 and that (ii) the late-time decay of Ψ_0 along lines of fixed r_* is dominated (like that at FNI) by the $k = l + 1$ term, we can write the following expression for the field Ψ_l [using Eq. (6.18)]:

$$\Psi_l \cong \hat{g}(4r_*)^{l+1} t^{-(2l+2)}[1 + O(r_*/t)] \quad (t \gg r_* \gtrsim R \gg M), \tag{7.1}$$

or [using Eq. (6.17)]

$$\Psi_l \cong \hat{g}\left(\frac{1}{u} - \frac{1}{v}\right)^{l+1} \qquad (v \gg u \gg w_1), \tag{7.2}$$

where $\hat{g} \cong g_0^{(l+1)}$ (presumably with corrections of order M/R).

We can now reinvert Ψ_l back to the original ψ_l, using $\psi_l = \hat{T}(\Psi_l)$, recovering our original problem of a horizon-based initial perturbation. Recall that in this transformation—which switches between the horizon and FNI—t is preserved, r_* changes its sign, and u, v interchange. The initial-data support is now restricted to the domain $u > w$ along the ingoing initial ray ($v = 0$). This corresponds to $-r_* > w$ and in terms of r to $r \leq M + \delta R$, where δR is given by $\delta R = r(r_* = -w) - M$ and at the limit $R \gg M$ by $\delta R = M^2/R$. The weak-field condition $R \gg M$ is now mapped into $\delta R \ll M$, to which we may refer as the "inverted weak-field" condition. Therefore, inverting Eqs. (7.1) and (7.2), we get

$$\psi_l \cong \hat{g}(-4r_*)^{l+1}t^{-(2l+2)}[1+O(r_*/t)] \quad (t \gg -r_* \gtrsim R \gg M),$$

(7.3)

and

$$\psi_l \cong \hat{g}\left(\frac{1}{v} - \frac{1}{u}\right)^{l+1} = (-1)^{l+1}\hat{g}\left(\frac{1}{u} - \frac{1}{v}\right)^{l+1}$$

(7.4)

$$(u \gg v \gg w_1).$$

Note that it is easy to see from Eq. (7.4) that along the horizon ($u \to \infty$), ψ_l decays as v^{-l-1}.

So far, our analysis of ψ_l was restricted to evaluation points in the domain of large $-r_*/M$ (and large r_*/M for the inverted field Ψ_l) and to the case of weak field initial data (for Ψ_l. In the case of ψ_l, the condition is the "inverted weak-field" one), i.e. $R \gg M$ (or $\delta R \ll M$ for ψ_l). In the next paper [9], we would relax this restrictions and derive an expression for the late-time tail of the field ψ_l to the leading order in $|r_*|/t$, at any r_* and for general horizon-based initial data (finite $\delta R/M$). In what follows, we only sketch very briefly the key ingredients and main results and discuss the case of general initial data and relations to previous works.

We begin with the "inverted weak-field" restriction $\delta R \ll M$. We would like to relax this assumption and to achieve better control on the contribution of higher-order terms in the iteration scheme (which presumably yields contributions $\propto (\delta R/M)^n$ to $\hat{g}$). As was mentioned above, an analysis of the $n = 1$ term was already carried out (in the Ψ_l context), indicating that the effect of this term is to merely modify the coefficient $g_0^{(l+1)}$, by an amount $\propto M/R$. We would like in addition (i) to verify that this behavior indeed extends to higher-n terms as well and (ii) to obtain the explicit expression for $\hat{g}$, for finite $\delta R/M$, which incorporates the overall contributions of all n.

To achieve these goals we exploit the conserved quantities found by Aretakis [10]. He analyzed the evolution of certain derivatives of ψ_l along the horizon of extremal RN, and obtained interesting exact results, which are easiest to

express in ingoing-Eddington (v, r) coordinates. Following are the results which are most relevant to our analysis: (i) The quantity $A_l = d_l(\psi_l)$, where d_l is the differential operator $(\partial/\partial r)^{l+1} + (l/M)(\partial/\partial r)^l$ evaluated at the horizon, is exactly conserved (i.e. independent of v), and (ii) for all $j \leq l$, $(\partial/\partial r)^j\psi_l$ decays to zero as $v \to \infty$. (Note that (ii) also implies that $(\partial/\partial r)^{l+1}\psi_l$ asymptotically approaches A_l at $v \to \infty$.)

By applying the operator d_l to the original initial data (2.6) (i.e. before the inversion and the insertion of the scale parameter R) and to Eq. (7.4), one verifies the consistency of the latter asymptotic expression, and furthermore obtains the exact coefficient $\hat{g}$:

$$\hat{g} = M^{l+1}(-1)^{l+1}\frac{2[(l+1)!]^2}{(2l+2)!}\left(c_{l+1} + \frac{l}{l+1}c_l\right)$$

$$= M^{l+1}\beta_{l+1}\left(c_{l+1} + \frac{l}{l+1}c_l\right).$$

(7.5)

Note that this is an exact expression, not restricted to the "inverted weak-field" case.

We now turn to discuss the second restriction, according to which our calculation of ψ_l is a good approximation only for evaluation points in the domain of large $-r_*/M$. One might suspect that once $-r_*$ increase and becomes comparable to (or smaller than) M—and obviously when r_* becomes positive—this approximation would break down. In the subsequent paper [9] we shall show, however, that in fact the validity of the late-time behavior $t^{-(2l+2)}$ is not restricted to the domain $M \ll -r_*$. Rather, it holds throughout the domain $t \gg |r_*|$. The only thing which changes when $-r_*$ becomes comparable to (or smaller than) M is the dependence on r_*: The power law $(-r_*)^{l+1}$ is replaced by a static solution of the field equation.

In order to obtain this result, we employ the so-called late-time expansion, presented for example in Refs. [12,13]. In this procedure, we assume that $\psi_l(r, t)$ admits a large-t asymptotic behavior of the form $\sum_{j=0}^{\infty} H_j(r)(t/M)^{-m-j}$, with a certain leading power m and certain functions $H_j(r)$. Substituting this expression in the field equation, one first finds that $H_{j=0}(r)$ must be a static solution (regular at $r \to \infty$), namely $\alpha(r/M)(r/M - 1)^{-l-1}$, where α is a constant. In addition, one obtains a hierarchy of ODEs for the various functions $H_{j>0}(r_*)$ (whose forms will not concern us here). To obtain the leading power m (along with α), one matches the near-horizon asymptotic form of the dominant term $j = 0$—namely $\alpha(-r_*/M)^{l+1}(t/M)^{-m}$—to the large-$t$ asymptotic form of ψ_l obtained in Eq. (7.3). The matching zone is $t \gg -r_* \gg M$. One then easily finds that $m = 2l + 2$ and $\alpha = 4^{l+1}M^{-(l+1)}\hat{g}$. Therefore, the final result is

$$\psi_l(r, t) \cong 4^{l+1}\hat{g}M^{l+1}(r/M)(r/M - 1)^{-l-1}t^{-2l-2}$$

(7.6)

$$(t \gg |r_*|).$$

Note that $\hat{g}$ contains a factor of M^{l+1} and is therefore dimensionful [see Eq. (7.5)].

So far we have addressed the case of horizon-based initial data. In the other case of off-horizon compact initial data (case I in the classification above), all coefficients c_k vanish, and so does $\hat{g}$. The decay rate will then be faster than t^{-2l-2}. A prototype of that case is the situation of a (compact-support) pulse incident from large r towards the BH. In this case the classic analysis by Price [1] should apply. This analysis (like subsequent variants [8,13]) demonstrates that the Schwarzschild's large-t tail is predominantly a weak-field phenomenon—that is, a scattering of a certain ingoing field off a weak-field curvature potential. The latter is $\propto M$ and is insensitive to Q (which only affects the curvature at the next-to-leading order). Thus, in the case of off-horizon compact initial data we expect a large-t tail $\propto t^{-2l-3}$—and an FNI tail $\propto u^{-l-2}$—just like in Schwarzschild. Note, however, that since ingoing perturbations will usually cross the horizon, due to the presence of centrifugal-like potential there, a scattering dynamics will take place near the horizon as well (similar to that at large r), and will lead to an inverse-power decay along the horizon, as v^{-l-2}. This also follows quite immediately from the fact that the notion of "off-horizon compact initial data" is invariant under T-inversion, and the latter maps the standard $\propto u^{-l-2}$ decay at FNI to a $\propto v^{-l-2}$ decay at the horizon.

As was mentioned above, a generic compact-support initial data may be represented as a superposition of the two types, namely "off-horizon" and "horizon-based" initial-data sets. The late-time decay will then be dominated by the above "horizon-based" contribution: Decay rates like u^{-l-1} at FNI, like v^{-l-1} along the horizon, and like t^{-2l-2} at large t (namely $t \gg |r_*|$), according to Eq. (7.6), along with (7.5) and (7.4).

In Ref. [3], Bičák obtained a large-t asymptotic behavior which involves the same function of r as in (7.6), but with a different inverse power t^{-l-2}. For $l \neq 0$, his result is inconsistent with our analysis. Furthermore, one can use Bičák's expression as a seed of a large-t expansion, constructing the corresponding ψ_l (with $m = l + 2$ rather

than $2l + 2$) [9], and when d_l is applied to this ψ_l, one finds that A_l actually diverges, which is inconsistent with regular initial data. The numerical results [5] for $l = 1, 2$ also excludes the inverse power t^{-l-2} and support our analytical result t^{-2l-2}. It remains unclear where for $l \neq 0$ the problem in [3] arose.

More generally, by combining the large-t expansion and the conservation of A_l, one can show that for generic regular initial data, the only consistent inverse power is $m = 2l + 2$ [9]: A_l diverges for any $m \leq 2l + 1$, and vanishes for any $m \geq 2l + 3$ (which would be inconsistent with generic regular initial data, wherein $\hat{g} \neq 0$).

In Ref. [7], Blaksley and Burko investigated two special cases of initial data: (i) initial static moment that extends up to the horizon, and (ii) the case of "off-horizon compact initial support" (in our terminology). In both cases, all the coefficients $c_{k<l}$ in the initial-data horizon expansion (2.6) vanish. Obviously, analyzing such a subclass of initial data is inadequate for testing Bičák's claim, especially because a-priori one might naturally expect that the smallest-k terms would dominate the late-time tail. When restricted to that subclass, our results are consistent with those of Blaksley and Burko. In particular, for an initial static moment at the horizon, one finds that $c_{l+1}, c_l \neq 0$ (and all other c_k vanish), yielding $\hat{g} \neq 0$ and hence a late-time decay $\propto t^{-2l-2}$.

It may be interesting to extend this analysis to coupled gravitational and electromagnetic perturbations on extremal RN. In Ref. [14], Bičák employed his results from [15] and showed that the scalar-field perturbations serve as a prototype for these coupled perturbations; he then used his results from Ref. [3] to deduce the late-time behavior of them. It will therefore be interesting to revisit his analysis in [14] and obtain the late-time behavior of coupled perturbations (on ERN) using the results of the present paper.

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