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Rotating (A)dS black holes in bigravity

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In this paper we explore the advantage of using the Kerr-Schild *Ansatz* in the search for analytic configurations to bigravity. It turns out that it plays a crucial role by providing means to straightforwardly calculate the square root matrix encoding the interaction terms between both gravities. In this spirit, we rederive the Babichev-Fabbri family of asymptotically flat rotating black holes with the aid of an emerging circularity theorem. Taking into account that the interaction terms contain by default two cosmological constants, we repeat our approach starting from the more natural seeds for the Kerr-Schild *Ansatz* in this context: the (A)dS spacetimes. As a result, we show that a couple of Kerr–(A)dS black holes constitute an exact solution to ghost-free bigravity. Similar to the asymptotically flat case, these black holes share the same angular momentum and (A)dS radius, but their masses are not constrained to be equal.

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I. INTRODUCTION

Almost five years ago, de Rham, Gabadadze, and Tolley proposed a ghost-free and consistent interaction potential for massive gravity [1], giving rise to what is now known as the dRGT theory. One of the distinctive characteristics of their construction of massive gravity is the need for a reference metric, due to the impossibility of constructing nonderivative self-interactions with the dynamical metric only. Later, Hassan and Rosen [2] showed that this arbitrary metric can be promoted to be dynamical too, and the resulting theory would still be free of the Boulware-Deser ghost [3], propagating a total of $5+2$ degrees of freedom; this theory became what is now called bigravity.

Thanks to overcoming the theoretical difficulties manifest in theories including massive gravitons (see e.g. [4,5] for complete reviews on the subject), the interest of the community in these models has increased in recent years, resulting in numerous studies on a wide variety of topics. For example, effort has been made to construct cosmological models that can explain the accelerated expansion of the Universe as a natural consequence of regarding the interaction of gravity through a massive boson, although no viable and stable cosmological solutions have been reported so far [6–11]. Another important branch of interest lies in finding exact configurations supporting the massive generalization of the gravitational equations. In this category, black hole solutions [12–20] play an important role since, due to the new degrees of freedom, their stability properties are notably different from those characterizing the final state of the gravitational collapse in general relativity. See Ref. [21] for a complete review of the plethora of such solutions to massive (bi)gravity.

Recently, the first rotating black hole solutions in massive gravity were presented by Babichev and Fabbri [22]; they showed the Kerr spacetime [23] is a solution to the dRGT theory with a flat reference metric if a precise relation between the coupling constants of the theory holds. This result is also true for bigravity. In this case, the solution consists of two copies of the Kerr black hole with the same angular momenta but not necessarily equal masses. It is also possible to charge the solution through a coupling between the electric field to only one of the metrics without adding undesirable degrees of freedom.

The finding by Babichev and Fabbri of rotating configurations in bigravity is a major step in the comprehension of the dynamics of this theory due to the great difficulty involved in the calculation of the ghost-free interaction terms between the involved metrics. However, the fact that the interaction terms naturally include cosmological constants for each metric makes it more expected that the resulting configurations be asymptotically (Anti-)de Sitter [(A)dS] instead of the asymptotically flat behavior unveiled by Babichev and Fabbri. In the present work, we aim to build this generalization by deriving a class of asymptotically (A)dS rotating black holes which are analogue to that originally found by Carter in [24]. We will take advantage of the generalized Kerr-Schild *Ansatz* to integrate the field equations of bigravity, making transparent how both Kerr–(A)dS black holes appear.

In Sec. II we briefly introduce our framework which corresponds to the Hassan-Rosen bigravity [2] with the coupling parameters left free up to the Fierz-Pauli limit. Next, the asymptotically flat solution of Babichev and Fabbri [22] will be rederived to illustrate our procedure. We start in Sec. III by considering one metric as a Kerr-Schild transformation and the other as proportional to a Kerr-

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Schild transformation, both starting from Minkowski spacetime and possessing different profile functions. We show how easy it is to calculate the interaction terms with the help of this *Ansatz*. As the direct integration of the involved profiles is far from straightforward, a geometrical approach proving the circularity of these stationary axisymmetric configurations is developed in Sec. IV, which will result extremely useful by fixing the angular dependence of the profiles and making straightforward the integration of the remaining bigravity equations. In Sec. V we will follow an analogue procedure but replace the Minkowski spacetime with the (A)dS one as the starting seed of the generalized Kerr-Schild transformation. Using similar circularity arguments, we prove that two Kerr–(anti)-de Sitter black holes with the same (A)dS radii and angular momenta but arbitrary masses are solutions of bigravity up to three constraints in the coupling constants of the theory, which generalize the constraints previously found by Babichev and Fabbri. A discussion of the results and the difficulties involved in their generalization is included in Sec. VI.

II. GHOST-FREE BIGRAVITY

Bigravity as formulated by Hassan and Rosen [2] is a four-dimensional ghost-free theory describing two metric fields $g_{\mu\nu}$ and $f_{\mu\nu}$ interacting via a nonderivative potential. One of these fields is massive and the other is not; hence, the theory propagates a total of $5+2$ degrees of freedom. The interaction is encoded through scalar quantities computed from a matrix defined by the following quadratic relation:

$$(\gamma^2)^\mu{}_\nu = \gamma^\mu{}_\alpha \gamma^\alpha{}_\nu \equiv g^{\mu\alpha} f_{\alpha\nu}. \tag{1}$$

The bigravity defining action is

$$S[g,f] = \frac{1}{2\kappa_g}\int d^4x\sqrt{-g}R[g] + \frac{1}{2\kappa_f}\int d^4x\sqrt{-f}\mathcal{R}[f] - \frac{m^2}{\kappa}\int d^4x\sqrt{-g}\mathcal{U}[g,f], \tag{2a}$$

where $R[g]$ and $\mathcal{R}[f]$ are the Ricci scalars for each metric, κ_g and κ_f are the corresponding Einstein constants, κ is a function of κ_g and κ_f with the same dimensions, and m is the graviton mass. The interaction between the metrics is mediated by the potential

$$\mathcal{U}[g,f] = \sum_{k=0}^{4} b_k \mathcal{U}_k(\gamma), \tag{2b}$$

where b_k are coupling constants and the interaction terms are defined by

$$\begin{aligned}
\mathcal{U}_0(\gamma) &= 1,\\
\mathcal{U}_1(\gamma) &= \sum_A \lambda_A = [\gamma],\\
\mathcal{U}_2(\gamma) &= \sum_{A<B}\lambda_A\lambda_B = \frac{1}{2!}([\gamma]^2 - [\gamma^2]),\\
\mathcal{U}_3(\gamma) &= \sum_{A<B<C}\lambda_A\lambda_B\lambda_C = \frac{1}{3!}([\gamma]^3 - 3[\gamma][\gamma^2] + 2[\gamma^3]),\\
\mathcal{U}_4(\gamma) &= \lambda_0\lambda_1\lambda_2\lambda_3 = \frac{1}{4!}([\gamma]^4 - 6[\gamma]^2[\gamma^2] + 8[\gamma][\gamma^3] + 3[\gamma^2]^2 - 6[\gamma^4]).
\end{aligned} \tag{2c}$$

Here $\lambda_A (A = 0,1,2,3)$ are the eigenvalues of $\gamma^\mu{}_\nu$, and we understand the square bracket notation as $[\gamma^k] \equiv \mathrm{tr}(\gamma^k)$.

The variation of action (2) gives the bigravity field equations,

$$G^\mu{}_\nu = \frac{m^2\kappa_g}{\kappa}V^\mu{}_\nu, \qquad \mathcal{G}^\mu{}_\nu = \frac{m^2\kappa_f}{\kappa}\mathcal{V}^\mu{}_\nu, \tag{3}$$

where $G^\mu{}_\nu$ and $\mathcal{G}^\mu{}_\nu$ are the Einstein tensors for $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively, and the interaction contributions are given by

$$V^\mu{}_\nu \equiv \frac{2g^{\mu\alpha}}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{U})}{\delta g^{\alpha\nu}} = \tau^\mu{}_\nu - \mathcal{U}\delta^\mu{}_\nu, \tag{4a}$$

$$\mathcal{V}^\mu{}_\nu \equiv \frac{2f^{\mu\alpha}}{\sqrt{-f}}\frac{\delta(\sqrt{-g}\mathcal{U})}{\delta f^{\alpha\nu}} = -\frac{\sqrt{-g}}{\sqrt{-f}}\tau^\mu{}_\nu, \tag{4b}$$

with

$$\begin{aligned}
\tau^\mu{}_\nu = {}& (b_1\mathcal{U}_0 + b_2\mathcal{U}_1 + b_3\mathcal{U}_2 + b_4\mathcal{U}_3)\gamma^\mu{}_\nu\\
&- (b_2\mathcal{U}_0 + b_3\mathcal{U}_1 + b_4\mathcal{U}_2)(\gamma^2)^\mu{}_\nu\\
&+ (b_3\mathcal{U}_0 + b_4\mathcal{U}_1)(\gamma^3)^\mu{}_\nu - b_4\mathcal{U}_0(\gamma^4)^\mu{}_\nu.
\end{aligned} \tag{5}$$

An alternative way to write the interaction potential is in terms of the matrix $\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \gamma^\mu{}_\nu$, giving

$$\mathcal{U}[g,f] = \sum_{k=0}^{4} c_k \mathcal{U}_k(\mathcal{K}), \tag{6}$$

where the interaction terms $\mathcal{U}_k(\mathcal{K})$ are again defined as in (2c) after the replacement $\gamma \to \mathcal{K}$, and both sets of coupling constants b_k and c_k are linearly related. It is desirable to exactly reproduce the Fierz-Pauli mass term in the weak field limit, which is expressed by means of the matrix $\mathcal{K}$ as

$$+\frac{m_{FP}^2}{2}([\mathcal{K}]^2 - [\mathcal{K}^2]). \tag{7}$$

Hence, it can be tracked down directly from the quadratic contribution of the potential in the $\mathcal{K}$ formulation just by choosing $c_2 = -1$ and $\kappa = \kappa_g$, which makes the parameter

m^2 in action (2a) become precisely the Fierz-Pauli mass in flat spacetime. In other words, one of the coupling constants is not free since it is chosen *a priori* as the Fierz-Pauli mass. Returning to the γ formulation, this implies

$$-1 = c_2 = b_2 + 2b_3 + b_4 \Rightarrow b_2 = -1 - 2b_3 - b_4. \quad (8)$$

We will use this normalization of the couplings throughout our work, so the constant b_2 will not appear as it is replaced in favor of b_3 and b_4.

III. KERR-SCHILD *ANSATZ* IN BIGRAVITY

The first nontrivial rotating solution in bigravity was found by Babichev and Fabbri [22]; it consists of a pair of Kerr black holes with different masses but rotating with the same angular momenta. In the following section, we present a rederivation of this solution following a different approach. It is done to illustrate the procedure that will prove useful later in deducing the rotating solutions with the cosmological constant.

The nonlinearity of Einstein field equations makes difficult any attempt to find general solutions to a theory of gravity. However, there are many strategies that can be followed to reduce the level of complexity, and one of them is to start with an educated *Ansatz*. For instance, almost all known black holes can be written as so-called Kerr-Schild transformations from the spacetime defining their asymptotic behavior [25]. The simplification of this *Ansatz* in standard gravity lies in the fact that the field equations become linearized exactly, i.e., without approximations.

Because in bigravity we have two sets of Einstein field equations (3) coupled by an interaction potential (2b), we can expect a similar simplification for such kinds of *Ansätze*. Concretely, we will assume the first metric as a Kerr-Schild transformation from Minkowski spacetime and the second one as being proportional to a different Kerr-Schild transformation also from Minkowski spacetime,

$$ds_g^2 = g_{\mu\nu}dx^\mu dx^\nu = ds_M^2 + 2S_1 l \otimes l, \quad (9a)$$

$$ds_f^2 = f_{\mu\nu}dx^\mu dx^\nu = C^2(ds_M^2 + 2S_2 l \otimes l), \quad (9b)$$

where ds_M^2 is the Minkowski metric, l is the tangent vector to a null, geodesic, and shear-free congruence on Minkowski spacetime, S_1 and S_2 are a pair of scalar profiles, and C is a dimensionless proportionality constant. We are interested in describing stationary and axisymmetric spacetimes with these *Ansätze*. Hence, the above ingredients must be compatible with these symmetries, and this is best realized in the so-called "*ellipsoidal coordinates*" [26], where the Minkowski metric is written as

$$ds_M^2 = -dt^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{\Sigma}{r^2 + a^2}dr^2 + \Sigma d\theta^2, \quad (10)$$

with $\Sigma = r^2 + a^2\cos^2\theta$. These coordinates are understood if the Cartesian spatial slices of Minkowski spacetime are foliated by ellipsoids of revolution,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1, \quad (11)$$

rather than standard spheres. Notice that the spheres are recovered for $a = 0$; consequently, the parameter a denotes the departure from sphericity of the ellipsoids. The coordinate r labels each ellipsoid and the angular coordinates θ and ϕ parametrize the ellipsoids as is easily inferred from Eq. (11). The stationary and axisymmetric isometries are represented in this coordinates by the Killing vectors $k = \partial_t$ and $m = \partial_\phi$, respectively. The relevance of these coordinates lies in the fact that it is possible to prove that in Minkowski spacetime there exists only one congruence of shearfree null geodesics that is at the same time stationary and axisymmetric [27]; these coordinates define a parametrization where the related tangent vector can be expressed in closed form as

$$l = dt - a\sin^2\theta d\phi + \frac{\Sigma}{r^2 + a^2}dr. \quad (12)$$

Finally, in order to respect the stationary and axisymmetric isometries, the profiles must be independent of the coordinates t and ϕ, i.e., $S_1 = S_1(r,\theta)$ and $S_2 = S_2(r,\theta)$. All these ingredients completely determine a stationary and axisymmetric Kerr-Schild transformation from flat spacetime.

Let us compute now the square of the matrix γ according to its definition (1) by taking the product

$$\begin{aligned}(\gamma^2)^\mu{}_\nu &= (\eta^{\mu\alpha} - 2S_1 l^\mu l^\alpha)C^2(\eta_{\alpha\nu} + 2S_2 l_\alpha l_\nu) \\ &= C^2[\delta^\mu{}_\nu - 2(S_1 - S_2)l^\mu l_\nu]. \end{aligned} \quad (13)$$

It is precisely here where the utility of the Kerr-Schild *Ansatz* becomes manifest, since, independently of the seed metric, the null character of the involved vector field makes its contribution to the matrices nilpotent, which leads to a truncation of the matrix power expansion for the square root. Hence, it is straightforward to find a closed form for the square root matrix using the Kerr-Schild *Ansätze:*

$$\gamma^\mu{}_\nu = C[\delta^\mu{}_\nu - (S_1 - S_2)l^\mu l_\nu]. \quad (14)$$

Now we may proceed to write down the interaction terms (4), supported by the previously mentioned nilpotent property which allows us to build any power of the square root matrix as

$$(\gamma^n)^\mu{}_\nu = C^n[\delta^\mu{}_\nu - n(S_1 - S_2)l^\mu l_\nu], \tag{15}$$

arriving at these particularly simple expressions,

$$V^\mu{}_\nu = P_1\delta^\mu{}_\nu - CP_0(S_1 - S_2)l^\mu l_\nu, \tag{16a}$$

$$\mathcal{V}^\mu{}_\nu = \frac{P_2}{C^3}\delta^\mu{}_\nu - \frac{P_0}{C^3}(S_1 - S_2)l^\mu l_\nu, \tag{16b}$$

where the coefficients are linear combinations of the coupling constants:

$$P_0 \equiv -2Cb_4 + C(C-4)b_3 + b_1 - 2C, \tag{17a}$$

$$P_1 \equiv 3C^2 b_4 - C^2(C-6)b_3 - 3Cb_1 - b_0 + 3C^2, \tag{17b}$$

$$P_2 \equiv -C(C^2-3)b_4 - 3C(C-2)b_3 - b_1 + 3C. \tag{17c}$$

In the following section, we use these expressions to prove a circularity theorem, which is the basis of the integration procedure allowing us to obtain rotating solutions from the Kerr-Schild *Ansatz* [27].

IV. A CIRCULARITY THEOREM

We start by recalling that for any stationary axisymmetric spacetime with commuting Killing vector fields $k = \partial_t$ and $m = \partial_\phi$, the following geometrical identities hold [28]:

$$C_k \equiv d*(k\wedge m\wedge dk) - 2*(k\wedge m\wedge R(k)) = 0, \tag{18a}$$

$$C_m \equiv d*(k\wedge m\wedge dm) - 2*(k\wedge m\wedge R(m)) = 0. \tag{18b}$$

Here the Killing fields are understood as one-forms, $k = g_{\mu\nu}k^\nu dx^\mu$ and $m = g_{\mu\nu}m^\nu dx^\nu$, while the Ricci one-forms amount to $R(k) = R_{\mu\nu}k^\nu dx^\mu$ and $R(m) = R_{\mu\nu}m^\nu dx^\mu$. These identities are the basis of the so-called "circularity theorem" in general relativity; in vacuum they imply that the functions under the differential are constants, which in turn must vanish at the symmetry axis where $m = 0$; consequently,

$$k\wedge m\wedge dk = 0 = k\wedge m\wedge dm. \tag{19}$$

These are the Frobenius integrability conditions defining circularity; i.e., the planes orthogonal to the Killing vectors at any point are integrable to surfaces orthogonal to the Killing fields in the whole spacetime. Choosing coordinates along Killing fields and on their orthogonal surfaces, the circular metric becomes block diagonal; the iconic example are the well-known Boyer-Lindquist coordinates [29] of the Kerr black hole [23].

The standard circularity argument of general relativity for stationary and axisymmetric configurations cannot be straightforwardly extended to bigravity due to the nontrivial interaction terms (4). However, for the Kerr-Schild *Ansätze* (9), which are not circular by construction, we will establish a circularity theorem. This result will fix the angular dependencies of the involved profiles, imposing at the same time a constraint between the coupling constants. This theorem reduces each Einstein equation to a single independent equation that we easily integrate in the rest of the section.

We start by calculating the following quantities, using the Kerr-Schild *Ansatz* (9a) in both the definition of the Killing one-forms and the interaction term (16a),

$$\frac{*(k\wedge m\wedge dk)}{l_t} = \frac{*(k\wedge m\wedge dm)}{l_\phi} = -\frac{2l_t\sin\theta}{\Sigma}\partial_\theta(\Sigma S_1), \tag{20}$$

$$\begin{aligned}\frac{*(k\wedge m\wedge V(k))}{l_t} &= \frac{*(k\wedge m\wedge V(m))}{l_\phi}\\ &= -CP_0(S_1 - S_2)*(k\wedge m\wedge l),\end{aligned} \tag{21}$$

where the interaction one-forms are defined analogously to the Ricci one-forms as $V(k) = V_{\mu\nu}k^\nu dx^\mu$ and $V(m) = V_{\mu\nu}m^\nu dx^\mu$. Using Einstein equations for the metric $g_{\mu\nu}$ in the identities (18) and taking into account the explicit expressions of the above quantities, we arrive at the following identity,

$$\frac{l_\phi}{l_t}C_k - C_m = -*(k\wedge m\wedge dk)d\left(\frac{l_\phi}{l_t}\right) = 0, \tag{22}$$

which implies the circularity conditions (19). Using the explicit expressions (20), the circularity automatically fixes the angular dependence of the profile. Additionally, taking into account the circularity in the identities (18) together with the Einstein equations necessarily implies that the expressions (21) must also be identically zero. This imposes a constraint between the coupling constants for the nontrivial case of the different profiles. Exactly the same can be concluded for the second metric; we can repeat the same arguments for the equations analogous to (18)–(22) built from $f_{\mu\nu}$. Hence, we establish a circularity theorem for both metrics which has as its consequences

$$S_i(r,\theta) = \frac{rM_i(r)}{\Sigma}, \qquad P_0 = 0, \qquad i = 1,2. \tag{23}$$

Now the integration of the remaining Einstein equations is straightforward. Bearing in mind the circularity restrictions, the interaction terms (16) reduce to the diagonal form

$$V^\mu{}_\nu = P_1\delta^\mu{}_\nu, \qquad \mathcal{V}^\mu{}_\nu = \frac{P_2}{C^3}\delta^\mu{}_\nu, \tag{24}$$

and the only independent equations for each Einstein set are the following combinations:

$$\frac{2rM_1}{\Delta_1}\left(G^r{}_t+\frac{a}{r^2+a^2}G^r{}_\phi\right)-\left(G^r{}_r-\frac{m^2\kappa_g}{\kappa}V^r{}_r\right) \\ =\frac{2r^2}{\Sigma^2}M_1'+\frac{m^2\kappa_g}{\kappa}P_1=0, \tag{25a}$$

$$\frac{2C^2rM_2}{\Delta_2}\left(\mathcal{G}^r{}_t+\frac{a}{r^2+a^2}\mathcal{G}^r{}_\phi\right)-C^2\left(\mathcal{G}^r{}_r-\frac{m^2\kappa_f}{\kappa}\mathcal{V}^r{}_r\right) \\ =\frac{2r^2}{\Sigma^2}M_2'+\frac{m^2\kappa_f}{\kappa}\frac{P_2}{C}=0, \tag{25b}$$

with $\Delta_i=r^2+a^2-2rM_i(r)$, $i=1, 2$. Because Σ carries the only θ dependence on the right-hand side, the only way to fulfill these equations is if each term independently vanishes, implying $M_1(r)=m_1$, $M_2(r)=m_2$, and $P_1=0=P_2$, with m_1 and m_2 being independent integration constants. The rest of the Einstein equations are automatically satisfied. Finally, the most general family of stationary axisymmetric Kerr-Schild transformations from flat spacetime solving the bigravity equations is

$$ds_g^2=ds_M^2+\frac{2m_1r}{\Sigma}l\otimes l, \tag{26a}$$

$$ds_f^2=C^2\left(ds_M^2+\frac{2m_2r}{\Sigma}l\otimes l\right), \tag{26b}$$

$$P_0=P_1=P_2=0, \tag{26c}$$

where the Minkowski metric and the null vector l written in ellipsoidal coordinates are given by (10) and (12), respectively. Additionally, the constraints between the coupling constants (26c) are read from definitions (17) and are not satisfied for $C=1$, which justify the use of the proportionality constant. This solution corresponds to a pair of Kerr black holes [23] with the same angular momenta but different masses. It was originally obtained in [22] by direct substitution. The link between the coupling constants used in their work and ours is the following,

$$b_0=-\frac{3\alpha+\beta-\Lambda_g+3}{1-\bar\kappa\Lambda_f},\qquad b_1=\frac{2\alpha+\beta+1}{1-\bar\kappa\Lambda_f},$$
$$b_3=\frac{\beta}{1-\bar\kappa\Lambda_f},\qquad b_4=\frac{\alpha-\beta+\bar\kappa\Lambda_f-1}{1-\bar\kappa\Lambda_f},$$
$$\frac{\kappa_g}{\kappa}m^2=(1-\bar\kappa\Lambda_f)\bar m^2, \tag{27a}$$

where in Ref. [22], $\bar m$ is the mass, $\bar\kappa$ is the ratio of Einstein constants for both metrics, and $\Lambda_{g,f}$ are the dimensionless cosmological constants.

It is worth noting that although the circularity theorem applies to both metrics, one could find the Boyer-Lindquist coordinates to block-diagonalize one of them, but since the masses are not equal, those coordinates are not suitable to diagonalize simultaneously the second metric, a fact already noticed by Babichev and Fabbri [22].

V. KERR–(A)DS BLACK HOLES IN BIGRAVITY

The rotating solutions found by Babichev and Fabbri in [22] are without doubt an interesting and nontrivial result extending the scope of the physics of black holes that can be understood under a ghost-free dynamics. At the same time, it is a little surprising to find just asymptotically flat configurations since the coupling constants b_0 and b_4 in action (2) play the role of cosmological constants for each metric. This suggests the possibility that rotating configurations can be generalized to include asymptotic behaviors with nontrivial constant curvature. These spacetimes are very well known in general relativity in the presence of a cosmological constant and correspond to the Kerr–(A)dS black hole originally discovered by Carter in [24]. The possibility of their inclusion within the bigravity vacua was also discussed in Ref. [22]; they use rotating generalizations of the Eddington-Finkelstein null coordinates that they intend to generalize in the presence of a cosmological constant. As we identify in the previous section, the success in the absence of a cosmological constant is due more to the underling Kerr-Schild structure. Fortunately, Carter himself presented the Kerr–(A)dS black holes in a generalized Kerr-Schild form starting from the (anti)–de Sitter spacetime [30], and this form has even been amenable to higher-dimensional extensions [26]. This will be precisely our starting point in the search for a generalization of the solutions [22]. We use the fact that the Kerr-Schild *Ansätze* (9) can be generalized by taking as the seed the (A)dS spacetime instead of the flat one,

$$ds_g^2=g_{\mu\nu}dx^\mu dx^\nu=ds_0^2+2S_1l\otimes l, \tag{28a}$$

$$ds_f^2=f_{\mu\nu}dx^\mu dx^\nu=C^2(ds_0^2+2S_2l\otimes l), \tag{28b}$$

where, in the conventions of [26], the (A)dS metric in ellipsoidal coordinates is

$$ds_0^2=-\frac{(1-\lambda r^2)\Delta_\theta}{1+\lambda a^2}dt^2+\frac{(r^2+a^2)\sin^2\theta}{1+\lambda a^2}d\phi^2 \\ +\frac{\Sigma}{(1-\lambda r^2)(r^2+a^2)}dr^2+\frac{\Sigma}{\Delta_\theta}d\theta^2. \tag{29}$$

Here, $\Delta_\theta=1+\lambda a^2\cos^2\theta$, again $\Sigma=r^2+a^2\cos^2\theta$, and the constant curvature is fixed by the scalar $R=12\lambda$, which determines the effective cosmological constant $\Lambda_{\rm eff}=3\lambda$, with $\lambda=\pm1/\ell^2$ being the inverse of the square (A)dS radius. The null, geodesic, and shear-free vector field on (A)dS is [26]

$$l = \frac{\Delta_\theta}{1+\lambda a^2}dt - \frac{a\sin^2\theta}{1+\lambda a^2}d\phi + \frac{\Sigma}{(1-\lambda r^2)(r^2+a^2)}dr. \tag{30}$$

In these coordinates, again the stationary and axisymmetric isometries are manifest if the scalar profiles are also invariant along the Killing fields $k=\partial_t$ and $m=\partial_\phi$ by choosing $S_1 = S_1(r,\theta)$ and $S_2 = S_2(r,\theta)$. The Minkowski limit of Sec. III is consistently recovered for the vanishing curvature, $\lambda = 0$.

The key to success in the Kerr-Schild transformations is that the form of the square root matrix (14) is unaltered by the replacement in the *Ansatz* described before; this implies that the interaction terms remain the same as in (16), so the procedure will closely resemble the one where the Minkowski spacetime is chosen as the seed. In fact, the circularity theorem of Sec. IV applies unchanged since Eqs. (20)–(22) look exactly the same for $\lambda \neq 0$. As a consequence, one arrives again at the circular profiles and the same constraint (23). As we have seen, the circularity implies the diagonalization of the interaction terms (16), and the fact that there is only one independent equation for each Einstein system,

$$\frac{2rM_1}{\Delta_1^r}\left(\frac{1}{1-\lambda r^2}G^r{}_t + \frac{a}{r^2+a^2}G^r{}_\phi\right) - \left(G^r{}_r - \frac{m^2\kappa_g}{\kappa}V^r{}_r\right)$$
$$= \frac{2r^2M_1'}{\Sigma^2} + \frac{m^2\kappa_g P_1}{\kappa} + 3\lambda = 0, \tag{31a}$$

$$\frac{2C^2rM_2}{\Delta_2^r}\left(\frac{1}{1-\lambda r^2}\mathcal{G}^r{}_t + \frac{a}{r^2+a^2}\mathcal{G}^r{}_\phi\right)$$
$$- C^2\left(\mathcal{G}^r{}_r - \frac{m^2\kappa_f}{\kappa}\mathcal{V}^r{}_r\right) = \frac{2r^2M_2'}{\Sigma^2} + \frac{m^2\kappa_f P_2}{\kappa C} + 3\lambda = 0, \tag{31b}$$

where $\Delta_i^r = r^2 + a^2 - 2rM_i(r) - \lambda r^2(r^2+a^2)$. Again the dependence in θ is fixed, and the above equations are only satisfied if each functionally independent term on θ vanishes separately, which means $M_1(r) = m_1$ and $M_2(r) = m_2$, with m_1 and m_2 arbitrary and independent integration constants, and new constraints on the coupling constants. Hence, we can write the new solution as

$$ds_g^2 = ds_0^2 + \frac{2m_1 r}{\Sigma} l \otimes l, \tag{32a}$$

$$ds_f^2 = C^2\left(ds_0^2 + \frac{2m_2 r}{\Sigma} l\otimes l\right), \tag{32b}$$

$$P_0 = 0, \qquad \kappa_g P_1 = \frac{\kappa_f P_2}{C} = -\frac{3\lambda\kappa}{m^2}, \tag{32c}$$

where the (A)dS metric, ds_0^2, in ellipsoidal coordinates is given in (29), their null vector l is (30), and definitions (17) determine the new constraints (32c) for the coupling constants, allowing the single effective cosmological constant $\Lambda_{\rm eff} = 3\lambda$. This solution corresponds to a family of stationary-axisymmetric Kerr-Schild transformations from (anti–)de Sitter spacetime which describes two Kerr–(A)dS black holes with the same angular momenta and (A)dS radii but independently defined masses.

Now it is possible to have a trivial proportionality constant, $C = 1$, but at the cost of constraining the effective cosmological constant as $\Lambda_{\rm eff} = 3\lambda = -(\kappa_f/\kappa)m^2$ (notice that m^2 is no longer the mass square in this context; hence, it is just a no necessarily positive coupling constant). Additionally, just as in the asymptotically flat problem, both metrics are circular, but it is not possible to diagonalize them together through the same Boyer-Lindquist-like transformation.

VI. DISCUSSION

In this paper we have explored the consequences of using Kerr-Schild transformations for the dynamics of ghost-free bigravity. The first case studied corresponded to rotating asymptotically flat spacetimes producing the solution of two Kerr black holes with different masses already reported by Babichev and Fabbri [22]. We managed to advance a little further in the comprehension of how this configuration appears since our *Ansätze* were not the Kerr black holes themselves but rather the most general stationary-axisymmetric Kerr-Schild transformations from flat spacetime [27]. It is particulary simple to calculate the interaction terms in this case; the null character of the Kerr-Schild vector gives rise to a nilpotent contribution; consequently, any matrix power series is necessarily truncated, especially the one defining the square root matrix encoding the interactions. This allows us to find that the dynamics of the theory imposes a circularity theorem applicable to both metrics. As a consequence, the resulting profiles necessarily have to be those of the Kerr spacetime, but for independent masses. No other rotating solution of massive (bi)gravity with a flat asymptotic can be constructed in this manner.

Of course, the existence of more general rotating solutions, whose integration is tackled by different procedures, remains an open problem. For example, an interesting question is if it is possible to have solutions not only with different masses but also with different angular momenta. This question was already posed in Ref. [22], where no definite answer was given due to the manifest difficulty of the involved calculations. A starting point to explore this question is to consider the slight modification of the Kerr-Schild *Ansätze* (9), in which the seed flat spacetimes of each metric are foliated by revolution ellipsoids (11) with different ellipticity parameters a_1 and a_2. The shear-free null geodesics of each differently foliated flat spacetime will be accordingly parametrized, and unfortunately the nilpotent property giving rise to the simple expression (14) for the square root matrix will no longer apply. However, in this case, the square root matrix can still be calculated by going to the tetrad formalism, where the chosen tetrads must satisfy the so-called symmetrization condition which warrants their equivalence to the metric formulation [31,32]. It is possible to show that the symmetrization condition necessarily requires that $a_1 = a_2$, which returns us to the setting analyzed in our paper. This points to a possible obstruction to the existence of two ghost-free interacting metrics with different angular momenta. Interestingly, this would have the advantage that both metrics become singular exactly at the same spacetime locus: the famous ringlike singularity of the Kerr black hole [corresponding to $\Sigma = 0$, or, equivalently, the interception of the plane $z = 0$ with the $r = 0$ revolution ellipsoid (11)]. In other words, two different angular momenta would suppose the existence of two ringlike singularities, making even more complex the causal structure of a rotating bigravity spacetime.

Another generalization also discussed in Ref. [22] is the possibility of having asymptotically (A)dS rotating spacetimes. Their strategy, based on the use of rotating generalizations of the Eddington-Finkelstein null coordinates, ends up being too difficult to apply in the presence of a cosmological constant. However, once one identifies that the fundamental underlying null structure is the one associated with a Kerr-Schild transformation, the generalization to include an asymptotic cosmological constant is straightforward. This is the focus of the second case under study, where we manage to repeat the arguments and find a new rotating solution to the bigravity equations consisting of two Kerr-(A)dS black holes sharing the same angular momentum and (A)dS radius but having independent masses. Three constraints between the coupling constants and the proportionality constant defining the second metric are needed in order to allow this configuration. The first one is a result of the circularity of these backgrounds and is independent of the existence of an effective cosmological constant. The other two constraints define the same effective cosmological constant for each metric. Thanks to the nontrivial character of this effective cosmological constant, the bigravity interaction potential is no longer vanishing, which means these black holes are not decoupled as in the asymptotically flat case. Seemingly, one could think of generalizing this result by considering two different (A)dS radii λ_1 and λ_2 for each metric in the *Ansätze* (28). This consideration would spoil, of course, the simplifications in finding the square root matrix brought by the Kerr-Schild *Ansatz*, but we could switch to the tetrad formalism and go forward. What we found is that even letting the angular momenta and cosmological constants be unconstrained, the same symmetrization condition as in the asymptotically flat case requires both angular momenta and both cosmological constants to be equal, making our consideration in this sense the most general.

Finally, with our approach, it is very easy to charge one of the two asymptotically (A)dS black holes by coupling the Maxwell field to the corresponding metric, let us say $g_{\mu\nu}$. This was first shown in the asymptotically flat case by Babichev and Fabbri [22]. The Kerr-Schild *Ansatz* is extended to the charged case by choosing the vector potential as proportional to the null vector l, which in (A)dS is (30). It is an easy task to solve the Maxwell equations guided by the circularity that must satisfy a stationary axisymmetric electromagnetic field [28]; the result is $A = qrl/\Sigma$ where q is the electric charge. Consequently, the profiles are again the circular ones and only the first of them becomes modified since now $M_1(r) = m_1 - q^2/2r$, giving a Kerr-Newmann-(A)dS black hole coupled to a Kerr-(A)dS one with the same properties and constraints as before. One can also ask what kind of rotating configurations could lead to coupling the Maxwell field nonminimally via the effective metric that has been intensely analyzed recently in the literature [33]. It is possible to show that our approach is incompatible with this nonminimal coupling and, therefore, to favor only the minimal coupling to a single metric, as we just described.

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