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Linearized 3D gravity with dust

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Three-dimensional gravity coupled to pressureless dust is a field theory with 1 local degree of freedom. In the canonical framework, the dust-time gauge encodes this field in the metric. We find that its dynamics, up to diffeomorphism flow, is independent of spatial derivatives and is therefore ultralocal. We study this feature further by analyzing the linearized equations of motion about flat and (anti-)de Sitter backgrounds, and show that this field may be viewed as either a traceless or a transverse mode.

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I. INTRODUCTION

Einstein gravity in three spacetime dimensions has been a subject of much study, primarily as a model for quantum gravity in a simpler setting [1,2]. In vacuum, however, the theory has only a finite number of degrees of freedom which arise either due to nontrivial topology of space [3–5], or through point particles which appear as conical defects [6]. It is therefore interesting to study three-dimensional (3D) gravitational theories that do have local field degrees of freedom. There are many such examples; prominent among them is the topologically massive theory [7].

We consider here another method for obtaining local degrees of freedom: three-dimensional Einstein gravity coupled to matter. This has been studied before; it is known, for instance, that 3D gravity with a scalar field has wave solutions [8]. We study coupling to pressureless dust in the Arnowitt-Deser-Misner (ADM) canonical framework, where spatial slices are defined by level values of a (pressureless) scalar field: the dust-time gauge. This gauge has an interesting property, namely, that the (nonvanishing) physical Hamiltonian is functionally the same as the Hamiltonian constraint [9]. A counting of degrees of freedom reveals that the theory has 1 local (configuration) degree of freedom, which in the dust-time gauge manifests itself as a metric field. In a previous work two of the present authors studied the spherically symmetric sector of this theory [10].

Here we investigate the nature of this field without imposing additional symmetries. In the next section we review the canonical framework with dust-time gauge. The main result is that in this gauge, space points decouple, giving independent and identical dynamics at each point. In Sec. III we consider the case of a vanishing cosmological constant and analyze the linearized theory in Fourier space; we solve the diffeomorphism constraint and derive the linearized canonical equations for the remaining metric function. In Sec. IV we give a similar analysis for the case of a nonvanishing cosmological constant. We conclude in Sec. V with a brief summary and implications of the result for quantum gravity.

II. ACTION AND HAMILTONIAN THEORY

The theory we study is given by the action

$$S = \frac{1}{2\pi}\int d^3x\sqrt{g}(R-2\Lambda) - \frac{1}{4\pi}\int d^3x\sqrt{g}m(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+1). \qquad (1)$$

The first integral is the usual gravitational action with a cosmological constant. The second integral is the action for the pressureless dust field, where m is a function of spacetime. Variation with respect to m constrains the dust field to have a timelike gradient $|\nabla\phi|^2=-1$.

The canonical ADM action is

$$S = \frac{1}{2\pi}\int d^3x(\tilde{\pi}^{ab}\dot{q}_{ab} + p_\phi\dot{\phi} - N\mathcal{H} - N^a\mathcal{C}_a), \qquad (2)$$

where the pairs $(q_{ab},\tilde{\pi}^{ab})$ and (ϕ,p_ϕ) are, respectively, the gravitational and dust phase-space variables. The lapse and shift functions, N and N^a, are the coefficients of the Hamiltonian and diffeomorphism constraints

$$\mathcal{H} = \mathcal{H}^G + \mathcal{H}^D, \qquad (3)$$

$$\mathcal{C}_a = \mathcal{C}_a^G + \mathcal{C}_a^D = -2D_b\tilde{\pi}^b_a + p_\phi\partial_a\phi, \qquad (4)$$

where

$$\mathcal{H}^G = \sqrt{q}\left(-R^{(2)} + \frac{1}{q}(\tilde{\pi}^{ab}\tilde{\pi}_{ab}-\tilde{\pi}^2)+2\Lambda\right), \qquad (5)$$

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$$\mathcal{H}^D = \frac{1}{2}\left(\frac{p_\phi^2}{m\sqrt{q}} + m\sqrt{q}(q^{ab}\partial_a\phi\partial_b\phi + 1)\right). \tag{6}$$

The trace of the gravitational momentum is $\tilde{\pi} = q_{ab}\tilde{\pi}^{ab}$, $R^{(2)}$ is the scalar curvature of the spatial hypersurfaces, and D_a is the covariant derivative associated with q_{ab}.

The momentum conjugate to the field m is zero since it appears as a Lagrange multiplier in the covariant action. However, it is still present in $\mathcal{H}^D$. The canonical action may be written in a convenient form that contains only the phase-space variables by varying the action with respect to m and substituting back the solution. This gives

$$m = \pm\frac{p_\phi}{\sqrt{q(q^{ab}\partial_a\phi\partial_b\phi + 1)}}, \tag{7}$$

which leads to

$$\mathcal{H}^D = \pm p_\phi\sqrt{q^{ab}\partial_a\phi\partial_b\phi + 1}. \tag{8}$$

It is readily verified that the constraints remain first class. We will see in the gauge fixing below how the sign is selected.

A. Time gauge fixing

We now partially reduce the theory by fixing a time gauge and solving the Hamiltonian constraint to obtain a physical Hamiltonian. We use the dust-time gauge [9,11] which equates the physical time with level values of the scalar field,

$$\lambda \equiv \phi - t \approx 0. \tag{9}$$

This has a nonzero Poisson bracket with the Hamiltonian constraint, so this pair of constraints is second class. Requiring that the gauge condition be preserved in time gives an equation for the lapse function:

$$\begin{aligned} 1 = \dot{\phi} &= \left\{\phi, \int d^3x(N\mathcal{H} + N^a\mathcal{C}_a)\right\}\Big|_{\phi=t} \\ &= N\frac{p_\phi}{m\sqrt{q}} \Rightarrow N = \frac{\sqrt{q}m}{p_\phi}. \end{aligned} \tag{10}$$

Using the relation (7) with $\phi = t$ leads to $N = \pm 1$ and implies that $\sqrt{q}m = \pm p_\phi$. The sign of the lapse function determines whether the evolution is forward ($N = +1$) or backward ($N = -1$) in time. We select the positive sign which fixes the above ambiguity in the Hamiltonian constraint, yielding $\mathcal{H}^D = +p_\phi$.

Imposing this gauge condition and solving the Hamiltonian constraint gives

$$p_\phi = -\mathcal{H}^G. \tag{11}$$

Substituting this and the gauge condition (9) into Eq. (2) gives the gauge fixed action

$$S_{\mathrm{GF}} = \frac{1}{2\pi}\int d^3x(\tilde{\pi}^{ab}\dot{q}_{ab} - \mathcal{H}^G - N^a\mathcal{C}_a^G). \tag{12}$$

This shows that the gravitational part of the Hamiltonian constraint becomes the physical Hamiltonian, and the full diffeomorphism constraint reduces to the gravitational one. This is a field theory: Three functions in q_{ab} subject to the two diffeomorphism constraints give 1 local configuration degree of freedom. This is the action we study in the remainder of the paper.

B. Equations of motion

The theory so far has been partially reduced using the dust-time gauge. The equations of motion are obtained via Poisson brackets with the Hamiltonian

$$\mathcal{H} = \frac{1}{2\pi}\int d^3x(\mathcal{H}^G + N^a\mathcal{C}_a^G). \tag{13}$$

We have

$$\dot{q}_{ab} = \{q_{ab}, \mathcal{H}\} = \frac{2}{\sqrt{q}}(\tilde{\pi}_{ab} - \tilde{\pi}q_{ab}) + \mathcal{L}_N q_{ab}, \tag{14}$$

$$\begin{aligned} \dot{\tilde{\pi}}^{ab} = \{\tilde{\pi}^{ab}, \mathcal{H}\} &= \frac{q^{ab}}{2\sqrt{q}}[\tilde{\pi}^{cd}\tilde{\pi}_{cd} - \tilde{\pi}^2] \\ &\quad - \frac{2}{\sqrt{q}}(\tilde{\pi}^a_c\tilde{\pi}^{cb} - \tilde{\pi}\tilde{\pi}^{ab}) - \Lambda\sqrt{q}q^{ab} + \mathcal{L}_N\tilde{\pi}^{ab}, \end{aligned} \tag{15}$$

where $\mathcal{L}_N$ is a Lie derivative in the direction of the shift vector N^a. Spatial derivatives enter these equations only through the Lie derivative terms, which is a gauge variation. There are no physically meaningful spatial derivatives because the Ricci scalar term $\sqrt{q}R^{(2)}$ does not contribute to the equations of motion. This can be seen by evaluating its variation:

$$\begin{aligned} \delta(\sqrt{q}R^{(2)}) &= (\delta\sqrt{q})R^{(2)} + \sqrt{q}(R^{(2)}_{ab}\delta q^{ab}) + \nabla_c J^c \\ &= \sqrt{q}\left(R^{(2)}_{ab} - \frac{R^{(2)}}{2}q_{ab}\right)\delta q^{ab} + \nabla_c J^c \\ &= \nabla_c J^c, \end{aligned} \tag{16}$$

where $J^c = q^{ab}\delta\Gamma^c_{ab} - q^{ca}\delta\Gamma^d_{ad}$, and the last equality follows in two dimensions from the formula for the Riemann tensor $R_{abcd} = (R/2)(q_{ac}q_{bd} - q_{ad}q_{bc})$. Since the variation is a total derivative, it contributes only to a boundary term and does not affect the dynamics.

We conclude from this that the dynamics is ultralocal for any value of the cosmological constant: The evolution equations for the phase-space variables contain no spatial derivatives in the nongauge terms. This means that the degrees of freedom at each space point evolve independently of any other point, and the diffeomorphisms serve only to move the points around. We note, however, that if spatial coordinate gauges are fixed and the diffeomorphism constraint is solved, then the resulting equations of motions may turn out not to be manifestly ultralocal; this appears, for example, in the spherically symmetric reduction of the theory [10], where the coordinate gauge is such that the shift vector N^a is not zero. But if one chooses coordinate fixing conditions such that $N^a = 0$, then the dynamics remains manifestly ultralocal. Thus, whether or not one has manifest ultralocality depends on coordinate gauge.

III. LINEARIZED THEORY WITH $\Lambda = 0$

We first analyze the linearized theory of perturbations about the flat spacetime solution,

$$\Lambda = 0, \qquad q^{(0)}_{ab} = e_{ab}, \qquad \tilde{\pi}^{(0)ab} = 0, \qquad N^{(0)a} = 0, \tag{17}$$

and write the perturbed fields as

$$q_{ab} = e_{ab} + h_{ab}, \qquad \tilde{\pi}^{ab} = 0 + \tilde{p}^{ab}, \qquad N^a = 0 + n^a. \tag{18}$$

Substituting this into the diffeomorphism constraint and equations of motion gives to first order

$$\nabla_a \tilde{\pi}^{ab} \approx \partial_a \tilde{p}^{ab} = 0, \tag{19}$$

$$\dot{h}_{ab} = 2(\tilde{p}_{ab} - \tilde{p} e_{ab}) + 2\partial_{(a} n_{b)}, \tag{20}$$

$$\dot{\tilde{p}}^{ab} = 0. \tag{21}$$

We note from these equations that any solution $\tilde{p}^{ab}_s$ of the diffeomorphism constraint gives a static source for the metric perturbation, which then evolves linearly in dust time (up to the diffeomorphism term if $n^a \neq 0$).

Our goal is to fix coordinate gauges, solve the diffeomorphism constraint, and study the linearized equations of motion for an appropriate set of physical variables. It is easiest to work in k-space, as done, for example, in [12], using the Fourier transformed fields

$$\bar{h}_{ab}(t,k) = \frac{1}{2\pi}\int d^2x e^{-ik_c x^c} h_{ab}(t,x), \tag{22}$$

$$\bar{p}^{ab}(t,k) = \frac{1}{2\pi}\int d^2x e^{-ik_c x^c} \tilde{p}^{ab}(t,x), \tag{23}$$

$$\bar{n}^{a}(t,k) = \frac{1}{2\pi}\int d^2x e^{-ik_c x^c} n^{a}(t,x). \tag{24}$$

The transform of the symplectic term in the canonical action is

$$\begin{aligned}
&\int d^2x \dot{h}_{ab}(t,x)\tilde{p}^{ab}(t,x) \\
&\quad = \frac{1}{(2\pi)^2}\int d^2x d^2k d^2\bar{k} e^{ix^c(k_c+\bar{k}_c)} \dot{\bar{h}}_{ab}(t,k)\bar{p}^{ab}(t,\bar{k}), \\
&\quad = \int d^2k d^2\bar{k} \delta^2(k+\bar{k}) \dot{\bar{h}}_{ab}(t,k)\bar{p}^{ab}(t,\bar{k}), \\
&\quad = \int d^2k \dot{\bar{h}}_{ab}(t,k)\bar{p}^{ab}(t,-k).
\end{aligned} \tag{25}$$

The linearized equations of motion in k-space become

$$\dot{\bar{h}}_{ab} = 2(\bar{p}_{ab} - \bar{p}\delta_{ab}) + 2ik_{(a}\bar{n}_{b)}, \tag{26}$$

$$\dot{\bar{p}}^{ab} = 0. \tag{27}$$

The perturbations are still subject to the k-space diffeomorphism constraint,

$$\bar{C}^G(n) \equiv \bar{n}_a k_b \bar{p}^{ab} = 0, \tag{28}$$

which contributes the linear term in $\bar{n}^a$ to the $\dot{\bar{h}}_{ab}$ equation.

To fix coordinate gauges and solve the diffeomorphism constraint, it is convenient to expand the symmetric k-space tensors $(\bar{h}_{ab}, \bar{p}^{ab})$ in a suitable matrix basis A^I, $I = 1, 2, 3$, as

$$\bar{h}_{ab} = h_I A^I_{ab}, \qquad \bar{p}^{ab} = p^I A^{ab}_I. \tag{29}$$

A convenient choice of basis is

$$\begin{aligned}
(A^1)_{ab} &\coloneqq \frac{1}{\sqrt{2}}\delta_{ab}, \\
(A^2)_{ab} &\coloneqq \frac{\sqrt{2}}{|k|^2}k_a k_b - \frac{1}{\sqrt{2}}\delta_{ab}, \\
(A^3)_{ab} &\coloneqq \frac{1}{\sqrt{2}|k|^2}(\epsilon_{ac}k_b k^c + \epsilon_{bc}k_a k^c),
\end{aligned} \tag{30}$$

where ϵ_{ab} is the Levi-Civita symbol. These are defined in analogy with a similar basis used in $3+1$ gravity: The first two correspond to scalar degrees of freedom, and the third to a "vector" degree of freedom. This basis is orthogonal and normalized with the inner product

$$(A^I, A^J) \coloneqq (A^I)_{ab}(A^J)_{cd}\delta^{ac}\delta^{bd} = \delta^{IJ}. \tag{31}$$

The symplectic term becomes

$$\dot{\bar{h}}_{ab}\bar{p}^{ab} = h_I A^I_{ab} p^J A^{ab}_J = h_I p^I, \tag{32}$$

so the expansion coefficients h_I, p^I are canonically conjugate. The basis also satisfies

$$k^a A^1_{ab} = k^a A^2_{ab} = \frac{1}{\sqrt{2}} k_b,$$
$$k^a A^3_{ab} = \frac{1}{\sqrt{2}} \epsilon_{bc} k^c,$$
$$\delta^{ab} A^2_{ab} = \delta^{ab} A^3_{ab} = 0. \tag{33}$$

Thus $I = 2, 3$ are the traceless modes, and the linear combination $A^2 - A^1$ is transverse. There is no transverse and traceless mode in this model since there are no metric perturbations satisfying both of these conditions: The only solution to the two equations

$$\delta^{ab} h_I A^I_{ab} = 0 = k^a h_I A^I_{ab} \tag{34}$$

is $h_I = 0 \ \forall \ I$.

We now write the linearized k-space equations in the basis (30). The diffeomorphism constraint takes a useful form: Decomposing the first order shift vector in components parallel and perpendicular to k^a,

$$\bar{n}^a = n_\parallel \frac{k^a}{|k|} + n_\perp \epsilon^{ab} \frac{k_b}{|k|}, \tag{35}$$

this constraint (28) may be written as

$$\bar{\mathcal{C}}^G = \bar{\mathcal{C}}^G_\parallel + \bar{\mathcal{C}}^G_\perp = n_\parallel \frac{|k|}{\sqrt{2}} (p_1 + p_2) + n_\perp \frac{|k|}{\sqrt{2}} p_3. \tag{36}$$

This gives two separate constraints on the three momenta:

$$p_1 + p_2 \approx 0, \qquad p_3 \approx 0. \tag{37}$$

The equations of motion are

$$\dot{h}_1 = -2p_1 + \sqrt{2}|k| n_\parallel, \tag{38}$$

$$\dot{h}_2 = 2p_2 + \sqrt{2}|k| n_\parallel, \tag{39}$$

$$\dot{h}_3 = 2p_3 + \sqrt{2}|k| n_\perp, \tag{40}$$

$$\dot{p}_I = 0 \quad \forall \ I, \tag{41}$$

where we have redefined $n_\parallel$ and $n_\perp$ to absorb the factor of i in (26). This shows the advantage of the chosen basis—the equations of motion are decoupled, and the diffeomorphism constraint breaks neatly into components parallel and perpendicular to k^a.

A. Gauge fixing and physical degrees of freedom

The fully reduced theory of physical degrees of freedom is obtained by imposing two gauge conditions and solving the two diffeomorphism constraints (37). This will give a Hamiltonian theory of one pair of phase-space variables. There is more than one way to do this, but we will focus on the choices that are natural in our chosen decomposition.

Given the form of the diffeomorphism constraint, it is natural to set the gauge $h_3 = 0$. This is second class with the constraint $p_3 \approx 0$. Requiring that the gauge condition be preserved in time means $\dot{h}_3 = 0$, which gives the condition $n_\perp = 0$ on the linear shift. These steps constitute a partial gauge fixing of the theory, leaving the pair (h_1, p_1) and (h_2, p_2), and the remaining diffeomorphism constraint

$$p_1 + p_2 \approx 0. \tag{42}$$

The next step is to fix the remaining gauge freedom. There are multiple ways of doing this, and in the following we present a few cases.

1. Traceless gauge

Let us recall from (33) that only the traceless mode remains if we set the gauge $h_1 = 0$. This is second class with the reduced diffeomorphism constraint (42), which is solved by setting $p_1 = -p_2$. Dynamical preservation of this gauge requires $\dot{h}_1 = 0$, which gives

$$n_\parallel = -\frac{\sqrt{2}}{|k|} p_2. \tag{43}$$

Thus the fully gauge fixed theory has only the traceless mode (h_2, p_2), satisfying the equations of motion

$$\dot{h}_2 = \dot{p}_2 = 0. \tag{44}$$

Notice the linear-order shift vector $n_\parallel$ works to cancel out the right-hand side of the equation of motion for h_2.

Since the remaining degrees of freedom are static, the general solution is given by arbitrary functions of k:

$$h_2 = \alpha(k), \qquad p_2 = \beta(k). \tag{45}$$

2. Transverse gauge

To leave the transverse mode as the remaining phase-space pair, we use the gauge condition $h_1 + h_2 = 0$. This is second class with $\mathcal{C}^G_\parallel \approx 0$, which is again solved by setting $p_1 = -p_2$. Preserving this gauge dynamically requires $\dot{h}_1 + \dot{h}_2 = 0$, which implies

$$n_\parallel = \frac{\sqrt{2}}{|k|} p_1. \tag{46}$$

We can write the fully reduced theory using the variables (h_1, p_1). This gives the same equations of motion as found in the traceless gauge:

$$\dot{h}_1 = \dot{p}_1 = 0, \tag{47}$$

due to a cancellation coming from the solution to $n_\parallel$.

The solution to the equations of motion is again given in terms of arbitrary functions that are constant in time:

$$h_1 = \alpha(k), \qquad p_1 = \beta(k). \tag{48}$$

3. Gauges with time dependence

The two gauges used to fix diffeomorphism invariance discussed above give rise to the simplest of evolution equations. It is interesting to look at other gauge choices, most of which give nontrivial dynamical equations. Let us consider a more general gauge

$$h_1 = f(t, k, h_2, p_1, p_2), \tag{49}$$

where f is an arbitrary function. This is second class with (42), which gives $p_1 = -p_2$. Dynamical preservation of this gauge now requires $\dot{h}_1 - \dot{f} = 0$. Using the equations of motion, this fixes the lapse to be

$$n_\parallel = -\frac{\sqrt{2}p_2}{k} + \frac{1}{\sqrt{2}k}\frac{\partial f}{\partial t}\left(1 - \frac{\partial f}{\partial h_2}\right)^{-1}. \tag{50}$$

The resulting equations of motion for the physical degrees of freedom (h_2, p_2) are

$$\dot{h}_2 = \frac{\partial f}{\partial t}\left(1 - \frac{\partial f}{\partial h_2}\right)^{-1}, \tag{51}$$

$$\dot{p}_2 = 0. \tag{52}$$

The special case $f = \mu t p_2$, $\mu =$ constant, reduces the first equation to that of a free particle $\dot{h}_2 = \mu p_2$.

That different gauges give rise to very different dynamics is of course expected for a generally covariant theory. The interesting feature is that the choice of dust time, together with either the transverse or traceless gauges for the diffeomorphism constraint, leads to the simplest solution.

4. Spacetime fields

The solutions presented above are given in terms of (h_I, p_I), which are matrix expansion coefficients in k-space. The first order ADM variables are obtained from inverse Fourier transforms. For example, the metric perturbation in the traceless gauge is

$$h_{ab}(x) = -\frac{1}{\sqrt{2}\pi}\partial_a\partial_b \int d^2k e^{ik_c x^c}\frac{\alpha(k)}{|k|^2} - \frac{e_{ab}}{2\sqrt{2}\pi}\int d^2k e^{ik_c x^c}\alpha(k). \tag{53}$$

The other Fourier transforms take a similar form, and since each contains the arbitrary function f for the general gauge, one obtains a large class of spacetimes for the linearized theory.

IV. LINEARIZED THEORY WITH $\Lambda \neq 0$

We now give an analysis of the linearized theory with a nonzero cosmological constant. As the background solution we take the (anti-)de Sitter solution, which in the dust-time gauge is

$$q_{ab} = a(t)^2\delta_{ab}, \qquad \tilde{\pi}^{ab} = b(t)\delta^{ab}, \qquad N^a = 0, \tag{54}$$

where $a(t)$ and $b(t)$ are

$$a(t) = \begin{cases} a_0\cosh(\sqrt{|\Lambda|}t) & \Lambda > 0 \\ a_0\cos(\sqrt{|\Lambda|}t) & \Lambda < 0, \end{cases}$$
$$b(t) = \begin{cases} -\sqrt{\Lambda}\tanh(\sqrt{|\Lambda|}t) & \Lambda > 0 \\ \sqrt{\Lambda}\tan(\sqrt{|\Lambda|}t) & \Lambda < 0, \end{cases} \tag{55}$$

for $a(0) = a_0$. It is readily verified that these solutions satisfy the diffeomorphism constraint and equations of motion (14), (15).

For perturbations about this background we write

$$\begin{aligned} q_{ab} &= a(t)^2\delta_{ab} + h_{ab}, \\ \tilde{\pi}^{ab} &= b(t)\delta^{ab} + \tilde{p}^{ab}, \\ N^a &= 0 + n^a. \end{aligned} \tag{56}$$

Calculating the diffeomorphism constraint and equations of motion to first order gives

$$\nabla_a\tilde{\pi}^{ab} \approx \partial_a\tilde{p}^{ab} + a^2 b\left(\partial_a h^{ab} - \frac{1}{2}\partial^b h\right) = 0, \tag{57}$$

$$\dot{h}_{ab} = 2(a^{-2}\tilde{p}_{ab} - \tilde{p}\delta_{ab}) - a^2 bh\delta_{ab} + 2\partial_{(a}n_{b)}, \tag{58}$$

$$\dot{\tilde{p}}^{ab} = \left(a^2h^{ab} - \frac{1}{2}h\delta^{ab}\right)(\Lambda - b^2) + a^{-2}bp\delta^{ab} - 2a^2b\partial^{(a}n^{b)}. \tag{59}$$

These equations are significantly more complicated than those written above (for $\Lambda = 0$), particularly the equation of motion for the momentum perturbation, which is no longer identically zero. The is because the background momentum is nonvanishing.

We now proceed as before by going to Fourier space. Since the background metric is spatially constant, the same Fourier transform as used in the last section is suitable; see e.g. [12] for a similar case. The outcome is the same; the transformed fields $\bar{h}_{ab}(t,k)$ and $\bar{p}^{ab}(t,k)$ are conjugate variables, and the equations of motion and diffeomorphism constraint can again be expressed in terms of these variables and the shift perturbation $\bar{n}^a(t,k)$.

With the equations written in k-space, we again expand in a basis for symmetric matrices. For the given background solution, the following basis is suitable:

$$(A^1)_{ab} \coloneqq \frac{1}{\sqrt{2}} a^2 \delta_{ab},$$
$$(A^2)_{ab} \coloneqq \frac{\sqrt{2}}{|k|^2} k_a k_b - \frac{1}{\sqrt{2}} a^2 \delta_{ab},$$
$$(A^3)_{ab} \coloneqq \frac{1}{\sqrt{2}|k|^2} (\epsilon_{ac} k_b k^c + \epsilon_{bc} k_a k^c). \quad (60)$$

It reduces to the $\Lambda = 0$ basis for $a(t) = 1$. The metric perturbation is expanded as $\bar{h}_{ab} = h_I A^I_{ab}$ and $\bar{p}^{ab} = p^I A_I^{ab}$, taking care to raise and lower indices with the background metric $a^2 \delta_{ab}$ and its inverse $a^{-2} \delta^{ab}$.

The diffeomorphism constraint again splits into components parallel and perpendicular to k^a. One finds that the following two conditions must hold separately:

$$p_1 + p_2 + a^2 b \left(h_2 - \frac{1}{2} h_1 \right) \approx 0,$$
$$p_3 + a^2 b h_3 \approx 0. \quad (61)$$

The pair (h_3, p_3) are again constrained separately from the other phase-space degrees of freedom, but the coefficient p_3 is no longer required to vanish.

At this point we can explore different gauge choices for the coefficients (h_I, p^I). Although the equations are more complicated to work with, the qualitative features remain the same. Again we have a natural choice of $h_3 = 0$ which sets $p_3 = 0$ and fixes $n_\perp = 0$. It then remains to impose a single condition on the remaining phase-space degrees of freedom. The result is a completely reduced theory which can be expressed in terms of either (h_1, p_1) or (h_2, p_2) and the corresponding equations of motion.

V. DISCUSSION

We studied three-dimensional gravity coupled to dust in the dust-time gauge. The structure of the canonical theory is similar to that in four dimensions, where the physical Hamiltonian is the same expression as the Hamiltonian constraint. We found that the theory is ultralocal, a result peculiar to three spacetime dimensions due to the evolution equation for π^{ab}. For an understanding of the local metric degree of freedom in this time gauge, we analyzed the linearized theory about the flat and (anti-)de Sitter spacetimes in k-space. For the $\Lambda = 0$ case, this gives a curious result for the transverse or traceless gauge: The linearized evolution equations are trivial, which means that any initial perturbations do not evolve, and remain frozen. For $\Lambda \neq 0$ the dynamics are more complicated but the qualitative features remain the same since the remaining physical degrees of freedom can be expressed as a transverse or traceless mode.

This result means that the fully gauge fixed linearized theory in dust-time gauge, in the chosen k-space basis, is such that the physical Hamiltonian of the physical modes is exactly zero. The simplicity of this solution is due to the choice of time gauge, the matrix basis, and the transverse or traceless coordinate gauge. The situation is analogous to the classical mechanics problem of finding the time-dependent canonical transformation that maps a nontrivial Hamiltonian to the trivial one. Had we used another time gauge at the outset, the physical Hamiltonian and linearized equations of motion would have been very different, and nontrivial, because the solution of the Hamiltonian constraint would not be as simple as the one in dust-time gauge. We see this also in the dust-time gauge, if the remaining gauge choices are made in a more general way, as in (49). It remains to explore the full nonlinear ultralocal equations (14) and (15).

Given the ultralocality and trivial evolution of the flat space perturbations in the transverse or the traceless gauge, the corresponding quantum theory is effectively solved. For each k, one defines creation and annihilation operators in either case as $a^\pm \coloneqq h \pm ip$. The quantum states are the vacuum defined by $a|0\rangle = 0$, and the excited states $a^+ \ldots a^+|0\rangle = |n\rangle$. Since the physical Hamiltonian is zero, there is no time-dependent Schrodinger equation to write down, and no evolution of these states.

In spacetime dimensions higher than three, little is known about the canonical quantization of a matter–gravity system. It is natural to apply the procedure used here to the linearized theory for $3 + 1$ gravity coupled to dust, a work presently in progress. It is of interest to see whether the simplifications from the dust-time gauge and choice of matrix basis in k-space produce new insights to this difficult problem. Another direction for further work is the inclusion of matter fields in addition to dust; the physical Hamiltonian for such cases is a simple sum of the gravity Hamiltonian $\mathcal{H}_G$ and the standard Hamiltonian for the matter field [9].

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[1] S. Carlip and J. Nelson, Phys. Rev. D **51**, 5643 (1995).
[2] S. Carlip, *Quantum Gravity in 2+1 dimensions* (Cambridge University Press, Cambridge, England, 1998).
[3] A. Ashtekar, V. Husain, C. Rovelli, J. Samuel, and L. Smolin, Classical Quantum Gravity **6**, L185 (1989).
[4] E. Witten, Nucl. Phys. **B311**, 46 (1988).
[5] V. Moncrief, J. Math. Phys. **31**, 2978 (1990).
[6] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).
[7] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982); **281**, 409 (2000).
[8] V. Husain, Phys. Rev. D **50**, R2361 (1994).
[9] V. Husain and T. Pawlowski, Phys. Rev. Lett. **108**, 141301 (2012).
[10] V. Husain and J. Ziprick, Phys. Rev. D **91**, 124074 (2015).
[11] J. Świeżewski, Classical Quantum Gravity **30**, 237001 (2013).
[12] D. Langois, Classical Quantum Gravity **11**, 389 (1994).