PHYSICAL REVIEW D **93,** 024036 (2016)

Entropy of nonextremal STU black holes: The F-invariant unveiled

Gábor Sárosi[1,2,*]

[1]*Department of Theoretical Physics, Institute of Physics, Budapest University of Technology, H-1521 Budapest, Hungary*

[2]*Kavli Institute for Theoretical Physics, Kohn Hall, University of California, Santa Barbara, California 93106-4030, USA*

(Received 9 September 2015; published 21 January 2016)

We find a manifestly U-duality invariant formula for the Bekenstein-Hawking entropy of the most general 4 dimensional, stationary, asymptotically flat, nonextremal STU black holes constructed recently by Chow and Compère. The expression is entirely in terms of asymptotic charges. It involves the "scalar charges" of the black hole which still need to be solved in terms of the dyonic charges and the mass. We discuss how the formula reduces to some of the known results as the Klauza-Klein black hole and the dilute gas limit of Cvetič and Larsen. We give the expected generalization to an $E_{7(7)}$ invariant in the case of maximal $\mathcal{N}=8$ supergravity.

DOI: 10.1103/PhysRevD.93.024036

I. INTRODUCTION

Recently, Chow and Compère [1,2] have constructed the most general asymptotically flat, rotating, nonextremal dyonic black hole solutions admitted by ungauged $\mathcal{N}=2$ supergravity coupled to three vector multiplets in four dimensions, or the so-called STU model. The Bekenstein-Hawking entropy of these black holes can be written in the form

$$S = 2\pi\left(\sqrt{\Delta + F} + \sqrt{-J^2 + F}\right), \tag{1}$$

where J is the angular momentum, Δ is Cayley's hyperdeterminant formed from the dyonic charges of the black hole and F is a so far unidentified U-duality invariant given in terms of auxiliary parameters.

The main result of this paper is the formula

$$F = M^4 + \frac{M^2}{12}\mathrm{Tr}K_{12} - \frac{M}{24}\mathrm{Tr}(K_{12}R) + \frac{1}{192}\left(\mathrm{Tr}(K_{11}^2) - \mathrm{Tr}(K_{11}K_{22}) - \frac{1}{2}(\mathrm{Tr}R^2)^2 + \mathrm{Tr}(R^4)\right), \tag{2}$$

where M is the mass of the black hole while R, K_{11}, K_{22}, and K_{12} are certain 6×6 matrices depending on six scalar charges, eight dyonic charges, and on the asymptotic values of the scalar fields as well. As we will explain, these matrices transform in the adjoint of the U-duality group $SL(2)_U^{\times 3}$ and therefore the expression for F is manifestly U-duality invariant. We also give the F invariant in the case of nonzero NUT charge in Eq. (86). It is important to note that while the scalar charges are functions of the mass and the dyonic charges, the asymptotic values are independent parameters and therefore the general black hole entropy depends on the scalar hair. This confirms that there is no attractor mechanism for general nonextremal STU black holes [3,4]. This observation can also be reinforced by the following argument. The dyonic charges, transforming under the U-duality group, form a prehomogeneous vector space [5]. This means that there is only one algebraically independent continuous invariant, in this case Cayley's hyperdeterminant. Since the mass and the angular momentum are U-duality invariants [6] the only way that the F-invariant can be nontrivial is that it depends on the scalar hair.

The construction of the building blocks for the F-invariant is most conveniently done using the toolbox of classifying certain entangled systems under stochastic local operations and classical communication (SLOCC) as three qubits [7], four qubits [8] and three fermions with six single particle states [9]. The idea of formulating black hole physics in supergravity in the language on quantum entanglement is not new. The entropy of extremal STU black holes is always expressed with Cayley's hyperdeterminant Δ which is a measure of tripartite entanglement for three qubits [10,11]. The timelike reduced STU model can be described in the language of four qubit entanglement, where the line element is given by a quadratic entanglement measure [12,13]. Moreover, extremal black holes with nilpotent charge vector can be classified in the language of four qubit entanglement as shown in [14]. For a review of this so-called black hole/qubit correspondence see [15].

The organization of this paper is as follows. In Sec. II we give a quick review of the conventions we use for STU supergravity and its reduction to the 3d coset model $SO(4,4)/SL(\mathbb{R})^{\times 4}$. In Sec. III we review the aspects of the black hole solution of Chow and Compère that we need. In Sec. IV we describe the action of the U-duality group $SL(2,\mathbb{R})^{\times 3}$ on the coset element putting particular emphasis on the fact that the generators for U-duality live in a different

*sarosi@phy.bme.hu

splitting of $\mathfrak{so}(4,4)$ than the one defining the 3d coset model $SO(4,4)/SL(\mathbb{R})^{\times 4}$. After this, we describe how the 16 asymptotic charges of the black hole fit into the $9 \oplus 8 \oplus 8$ representation of the U-duality group. We give the relevant covariants that can be used to generate polynomial invariants of this representation space in the simple language of fermionic entanglement theory. In Sec. V we use these covariants as building blocks to find the announced expression for the F-invariant. Section V B contains the formulas which are needed to compute the F-invariant dressed with the asymptotic values of the scalars. In Sec. VI we give three constraints among the scalar and physical charges and speculate if they are enough to determine the scalar charges uniquely. As an illustration we solve the constraints for some special cases like the Klauza-Klein black hole and the dilute gas limit of [16]. We find agreement with the literature. In Sec. VII we describe how we expect our results to generalize to an $E_{7(7)}$ invariant for maximal $\mathcal{N}=8$ supergravity.

We include two appendices containing some explicit formulas which we found too cumbersome to include in the main text. In Appendix A we set up our conventions for the Lie-algebra of $\mathfrak{so}(4,4)$ and describe the three different ways of selecting an $\mathfrak{sl}_2^{\times 4}$ subalgebra with the remaining generators transforming in the fundamental (2,2,2,2) representation. In Appendix B we give the explicit forms of the matrices R, K_{11}, K_{12}, and K_{22} appearing in expression (2) for the F-invariant.

II. $\mathcal{N}$= 2 SUPERGRAVITY

The term STU model refers to ungauged $\mathcal{N}=2$ supergravity in four dimensions coupled to 3 vector multiplets [17,18]. The model is of central importance in string theory as it can be obtained from most string theories and is related to various other supergravity theories through dualities. Also, a suitable solution of the STU model can be used [19] to generate all single centered stationary black holes of maximal $\mathcal{N}=8$ supergravity [20,21] which describes the low energy limit of M-theory compactified on T^7. The black hole solutions in the theory can be obtained from the bosonic part of the action without the hypermultiplets. There are several formulations of this action, here we simply pick one and refer to the literature on details. The action we choose is

$$S=\frac{1}{16\pi}\int \mathrm{d}x^4\sqrt{|g|}\bigg\{-\frac{R}{2}+G_{i\bar{j}}\partial_\mu\tau^i\partial^\mu\bar{\tau}^{\bar{i}} + (\mathrm{Im}\mathcal{N}_{IJ}(\mathcal{F}^I)_{\mu\nu}(\mathcal{F}^J)^{\mu\nu}+\mathrm{Re}\mathcal{N}_{IJ}(\mathcal{F}^I)_{\mu\nu}(\star\mathcal{F}^J)^{\mu\nu})\bigg\}. \tag{3}$$

Here, $\mathcal{F}^I$, $I=0,1,2,3$ are four $U(1)$ gauge field strengths and $\star\mathcal{F}^I$ is the dual of $\mathcal{F}^I$. The complex scalars τ^i, $i=1,2,3$ are coordinates of the projective special Kähler manifold $[SL(2,\mathbb{R})/SO(2)]^{\times 3}$ and $G_{i\bar{j}}$ is the Kähler metric of this space. The matrix $\mathcal{N}_{IJ}$ is a certain 4×4 matrix depending only on the scalars τ^i. As we do not need them in this paper, for explicit forms of $G_{i\bar{j}}$ and $\mathcal{N}_{IJ}$ we refer to the literature [2,13]. We will usually use the decomposition of τ^i into real in imaginary parts as

$$\tau^i = x_i + iy_i. \tag{4}$$

To describe stationary black hole solutions we proceed with the usual procedure of dimensional reduction along the time coordinate. For details of this procedure see e.g., [2,22]. The ansatz for the metric and the gauge fields is

$$\begin{aligned} \mathrm{d}s^2 &= -e^{2U}(\mathrm{d}t+\omega)^2 + e^{-2U}\mathrm{d}s^2_{(3d)}, \\ \mathcal{F}^I &= \mathrm{d}\mathcal{A}^I = \mathrm{d}(\zeta^I(\mathrm{d}t+\omega)+A^I). \end{aligned} \tag{5}$$

Here, $\mathrm{d}s^2_{(3d)} = h_{ab}\mathrm{d}x^a\mathrm{d}x^b$ is the three dimensional line element with a, b being 3d indices. The scalars U and ζ^I and the one-forms ω and A^I are considered to be three dimensional fields. One then dualizes ω and A^I to scalars σ and $\tilde{\zeta}_I$ as

$$\begin{aligned} \mathrm{d}\tilde{\zeta}_I &= \mathrm{Re}\mathcal{N}_{IJ}\mathrm{d}\zeta^J - e^{2U}\star_{(3d)}(\mathrm{d}A^J+\zeta^J\mathrm{d}\omega), \\ \mathrm{d}\sigma &= e^{4U}\star_{(3d)}\mathrm{d}\omega + \zeta^I\mathrm{d}\tilde{\zeta}_I - \tilde{\zeta}_I\mathrm{d}\zeta^I. \end{aligned} \tag{6}$$

The resulting three dimensional theory is Euclidean gravity coupled to 16 real scalars $\{U,\sigma,\tau^i,\bar{\tau}^{\bar{i}},\zeta^I,\tilde{\zeta}_I\}$ parametrizing the coset space $SO(4,4)/SL(2,\mathbb{R})^{\times 3}$. The Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{2}\sqrt{h}R[h] + g_{mn}\partial_a\Phi^m\partial^a\Phi^n, \tag{7}$$

where Φ^m denotes the scalars with $m=1,\ldots,16$. The metric g_{mn} on $SO(4,4)/SL(2,\mathbb{R})^{\times 4}$ is expressed through the line element as

$$\begin{aligned} \frac{1}{4}g_{mn}\mathrm{d}\Phi^m\mathrm{d}\Phi^n &= G_{i\bar{j}}\mathrm{d}\tau^i\mathrm{d}\bar{\tau}^{\bar{j}} + \mathrm{d}U^2 \\ &\quad + \frac{1}{4}e^{-4U}(\mathrm{d}\sigma+\tilde{\zeta}_I\mathrm{d}\zeta^I-\zeta^I\mathrm{d}\tilde{\zeta}_I)^2 \\ &\quad + \frac{1}{2}e^{-2U}[\mathrm{Im}\mathcal{N}_{IJ}\mathrm{d}\zeta^I\mathrm{d}\zeta^J \\ &\quad + (\mathrm{Im}\mathcal{N})^{-1}_{IJ}(\mathrm{d}\tilde{\zeta}_I - \mathrm{Re}\mathcal{N}_{IK}\mathrm{d}\zeta^K) \\ &\quad \times(\mathrm{d}\tilde{\zeta}_J - \mathrm{Re}\mathcal{N}_{JL}\mathrm{d}\zeta^L)]. \end{aligned} \tag{8}$$

Note that the equations of motion coming from (3) are invariant under the $SL(2,\mathbb{R})^{3\times}_U$ subgroup of the full U-duality group $E_{7(7)}$ mapping the STU model to itself. An element

$$S_1\otimes S_2\otimes S_3 \in SL(2,\mathbb{R})^{3\times}_U, \tag{9}$$

is parametrized as

$$S_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad a_i d_i - b_i c_i = 1, \quad i = 1,2,3, \tag{10}$$

and it acts on the 3 dimensional fields the following way. The scalars τ^i are transformed as

$$\tau^i \mapsto \frac{a_i \tau^i + b_i}{c_i \tau^i + d_i}, \tag{11}$$

while the electromagnetic potentials ζ^I, $\tilde{\zeta}_I$ transform in the fundamental $(2,2,2) = 8$ dimensional representation. Explicitly if one defines the three index tensor ψ_{ijk} corresponding to the amplitudes of a 3 qubit state as

$$\begin{pmatrix} \psi_{000} & \psi_{001} & \psi_{010} & \psi_{011} \\ \psi_{100} & \psi_{101} & \psi_{110} & \psi_{111} \end{pmatrix} = \begin{pmatrix} \tilde{\zeta}_4 & \zeta^3 & \zeta^2 & -\tilde{\zeta}_1 \\ \zeta^1 & -\tilde{\zeta}_2 & -\tilde{\zeta}_3 & -\zeta^4 \end{pmatrix} \tag{12}$$

the transformation rule is

$$\psi_{ijk} \mapsto (S_1)_i{}^{i'} (S_2)_j{}^{j'} (S_3)_k{}^{k'} \psi_{i'j'k'}. \tag{13}$$

In the following when we write $SL(2)$ we are referring ot $SL(2,\mathbb{R})$ unless explicitly otherwise stated.

We can describe the second term in the 3d Lagrangian (7) as a sigma model with target space $SO(4,4)/SL(2)^{\times 4}$. The Lie algebra $\mathfrak{so}(4,4)$ has 28 generators. To describe this coset model we have to split this as $\mathfrak{so}(4,4) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is an $\mathfrak{sl}_2^{\times 4}$ subalgebra and $\mathfrak{m}$ is its 16 dimensional fundamental representation which we denote as (2,2,2,2). There are actually three ways to perform this split. We summarize below two of them which are relevant for our purposes.

$$\begin{array}{r|l} (2,2,2,2)\{ \quad H_\Lambda, \;\; p_\Lambda = E_\Lambda + F_\Lambda & p^{Q_I} = E^{Q_I} - F^{Q_I}, \quad p^{P^I} = E^{P^I} - F^{P^I} \\ \hline \mathfrak{sl}_2^{\times 4}\{ \underbrace{\qquad k_\Lambda = E_\Lambda - F_\Lambda \qquad}_{(\mathfrak{sl}_2^{\times 4})_U} & \underbrace{k^{Q_I} = E^{Q_I} + F^{Q_I}, \quad k^{P^I} = E^{P^I} + F^{P^I}}_{(2,2,2,2)_U} \end{array} \tag{14}$$

Here, H_Λ, E_Λ, F_Λ, $\Lambda = 0,\ldots,3$ and E^{Q_I}, E^{P^I}, F^{Q_I}, F^{P^I}, $I = 1,\ldots,4$ are the 28 generators of $\mathfrak{so}(4,4)$. For an explicit definition of these generators and a detailed review of the splits, we refer to Appendix A. The subscript U refers to the splitting suited to describe the action of the U-duality group $SL(2)^{\times 3}$. The extra $SL(2)$ factor in this split is the Ehlers symmetry. The subalgebras without the subscript answer the split suited to describe the 3d fields parametrizing the coset $SO(4,4)/SL(2)^{\times 4}$.

Let us represent an element of this coset by an $SO(4,4)$ matrix as [2,12]

$$\mathcal{V} = e^{-UH_0} e^{-\frac{1}{2}\sum_i \log y_i H_i} e^{-\sum_i x_i E_i} e^{-\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I})} e^{-\frac{1}{2}\sigma E_0}. \tag{15}$$

Notice that there are no F-type generators in the above formula. This choice of gauge is called the Iwasawa gauge. The next step is to project the Mauer-Cartan 1-form to the 16 dimensional space spanned by the first line of (14) as

$$P_* = \frac{1}{2}(\mathrm{d}\mathcal{V}\mathcal{V}^{-1} + (\mathrm{d}\mathcal{V}\mathcal{V}^{-1})^{\#}). \tag{16}$$

Here, $^{\#}$ is the anti-involution defining the horizontal split of (14) [see also Eq. (A9)]. One finds that the target space metric (8) can be written as the right invariant metric on $SO(4,4)/SL(2)^{\times 4}$:

$$g_{mn}\mathrm{d}\Phi^m \mathrm{d}\Phi^n = \mathrm{Tr}(P_*^2). \tag{17}$$

The other way of writing this line element is to define the matrix

$$\mathcal{M} = \mathcal{V}^{\#}\mathcal{V}. \tag{18}$$

Then we have

$$g_{mn}\mathrm{d}\Phi^m \mathrm{d}\Phi^n = \frac{1}{4}\mathrm{Tr}((\mathcal{M}^{-1}\mathrm{d}\mathcal{M})^2). \tag{19}$$

We note that $\mathcal{M}^{-1}\mathrm{d}\mathcal{M}$, similarly to P_*, sits inside the 16 dimensional subspace which is spanned by the first line of (14).

The 16 three dimensional fields can be extracted directly from $\mathcal{M}$ (see [2] for details) so we may proceed describing the theory in terms of $\mathcal{M}$. A group element $h \in SO(4,4)$ acts naturally on $\mathcal{V}$ as

$$\mathcal{V} \mapsto q\mathcal{V}h, \tag{20}$$

where $q \in SL(2)^{\times 4}$ is a (possibly field dependent) compensator which puts $\mathcal{V}$ back to the Iwasawa gauge. We will see an example of this in Sec. IV A when we work out the action of the U-duality group on $\mathcal{V}$. The same action in terms of $\mathcal{M}$ reads as

$$\mathcal{M} \mapsto h^{\#}\mathcal{M}h. \tag{21}$$

It is then manifest that the line element and hence the 3d Lagrangian is invariant under this action of $SO(4,4)$.

III. THE MOST GENERAL NONEXTREMAL BLACK HOLE SOLUTION

Here we give a lightning review of some of the results and the solution generating technique presented in [2]. We have seen that if $\mathcal{M}$ is a solution to the equations of motion then so is $h^{\#}\mathcal{M}h$. This fact can be used to generate new solutions from a known seed. This method is used by the authors of [2] to find the most general nonextremal, rotating, asymptotically Taub-Newman-Unti-Tamburino (Taub-NUT) black hole solution with 11 independent conserved charges. These are the mass, NUT charge, angular momentum and 8 independent dyonic charges. They chose the four dimensional Kerr-Taub-NUT metric as their seed with mass m, NUT charge n and angular momentum $J = ma$ (here a is a parameter). Then they chose the group element

$$h = e^{-\sum_I \gamma_I k^{P^I}} e^{-\sum_I \delta_I k^{Q_I}}, \tag{22}$$

to charge up the solution as

$$\mathcal{M} = h^{\#}\mathcal{M}_{KTN} h. \tag{23}$$

We refer to [2] for the definition of the seed matrix $\mathcal{M}_{KTN}$. Here, we merely need how the physical charges of the black hole are expressed with the 11 parameters m, n, a, δ_I, γ_I. These charges are extracted as follows. Define the inverse radial coordinate $\rho = \frac{1}{r}$ and expand the fields around asymptotic infinity. For now, we assume the following expansion

$$\begin{aligned}
e^{2U} &= 1 - 2M\rho + O(\rho^2),\\
\zeta^I &= Q_I\rho + O(\rho^2),\\
y_i &= (1 - \Sigma_i\rho + O(\rho^2)),\\
\sigma &= -4N\rho + (4J\cos\theta + c)\rho^2 + O(\rho^3),\\
\tilde{\zeta}_I &= P^I\rho + O(\rho^2),\\
x_i &= \Xi_i\rho + O(\rho^2),
\end{aligned} \tag{24}$$

which is the same as in [2]. Here M and J are the Arnowitt-Deser-Misner (ADM) mass and angular momentum of the spacetime, Q_I and P^I are electric and magnetic charges associated to the original 4d $U(1)$ gauge fields and Ξ_i, Σ_i denote 6 scalar charges. The constant c is not playing any role in the following. The corresponding expansion for $\mathcal{M}$ is

$$\mathcal{M} = I + Q\rho + O(\rho^2), \tag{25}$$

where the charge matrix Q does not contain the angular momentum J which only enters in subleading order. Without J, these are all together 16 asymptotic charges, which can be expressed in terms of 10 seed & charge parameters, m, n, δ_I, γ_I. Therefore, the scalar charges are not independent of M, N and the dyonic charges, but we will keep them explicit until Sec. VI.

We quote the formula for the mass and the NUT charge

$$M = \mu_1 m + \mu_1 n, \qquad N = \nu_1 m + \nu_2 n, \tag{26}$$

where μ_1, μ_2, ν_1, ν_2 are functions only of δ_I, γ_I and are given explicitly as

$$\begin{aligned}
\mu_1 &= 1 + \sum_I\left(\frac{s_{\delta I}^2 + s_{\gamma I}^2}{2} - s_{\delta I}^2 s_{\gamma I}^2\right) + \frac{1}{2}\sum_{I,J} s_{\delta I}^2 s_{\gamma J}^2,\\
\mu_2 &= \sum_I s_{\delta I} c_{\delta I}\left(\frac{s_{\gamma I}}{c_{\gamma I}} c_{\gamma 1234} - \frac{c_{\gamma I}}{s_{\gamma I}} s_{\gamma 1234}\right),
\end{aligned} \tag{27}$$

$$\begin{aligned}
\nu_1 &= \sum_I s_{\gamma I} c_{\gamma I}\left(\frac{c_{\delta I}}{s_{\delta I}} s_{\delta 1234} - \frac{s_{\delta I}}{c_{\delta I}} c_{\delta 1234}\right),\\
\nu_2 &= \iota - D,
\end{aligned} \tag{28}$$

with

$$\begin{aligned}
\iota &= c_{\delta 1234} c_{\gamma 1234} + s_{\delta 1234} s_{\gamma 1234} + \sum_{I<J} c_{\delta 1234} \frac{s_{\delta IJ}}{c_{\delta IJ}} \frac{c_{\gamma IJ}}{s_{\gamma IJ}} s_{\gamma 1234},\\
D &= c_{\delta 1234} s_{\gamma 1234} + s_{\delta 1234} c_{\gamma 1234} + \sum_{I<J} c_{\delta 1234} \frac{s_{\delta IJ}}{c_{\delta IJ}} \frac{s_{\gamma IJ}}{c_{\gamma IJ}} c_{\gamma 1234}.
\end{aligned} \tag{29}$$

Here the notation is resolved as follows: $c_{\delta_I} = \cosh\delta_I$, $s_{\delta_I} = \sinh\delta_I$ and the same for γ_I. Multiple indices denote that one should take the product of the hyperbolic functions e.g., $c_{\delta IJ} = \cosh\delta_I \cosh\delta_J$. The rest of the charges can be obtained through the formulas

$$\frac{\partial 2M}{\partial\delta_I} = Q_I, \qquad \frac{\partial 2N}{\partial\delta_I} = -P^I, \tag{30}$$

$$\frac{\partial Q_I}{\partial\delta_J} = \begin{pmatrix} 2M - \Sigma_1 + \Sigma_2 + \Sigma_3 & & & \\ & 2M + \Sigma_1 - \Sigma_2 + \Sigma_3 & & \\ & & 2M + \Sigma_1 + \Sigma_2 - \Sigma_3 & \\ & & & 2M - \Sigma_1 - \Sigma_2 - \Sigma_3 \end{pmatrix}, \tag{31}$$

$$\frac{\partial P^I}{\partial\delta_J} = \begin{pmatrix} -2N & \Xi_3 & \Xi_2 & -\Xi_1 \\ \Xi_3 & -2N & \Xi_1 & -\Xi_2 \\ \Xi_2 & -\Xi_1 & -2N & -\Xi_3 \\ -\Xi_1 & -\Xi_2 & -\Xi_3 & -2N \end{pmatrix}, \tag{32}$$

$$\frac{\partial\Sigma_i}{\partial\delta_J} = \begin{pmatrix} -Q_1 & Q_2 & Q_3 & -Q_4 \\ Q_1 & -Q_2 & Q_3 & -Q_4 \\ Q_1 & Q_2 & -Q_3 & -Q_4 \end{pmatrix},$$
$$\frac{\partial\Xi_i}{\partial\delta_J} = \begin{pmatrix} -P^4 & P^3 & P^2 & -P^1 \\ P^3 & -P^4 & P^1 & -P^2 \\ P^2 & P^1 & -P^4 & -P^3 \end{pmatrix}. \tag{33}$$

Note that these identities are simple consequences of the fact that we have defined the charging up element h with the δ_I parameters being on the right. Therefore, for the charge matrix $Q = h^\# Q_{KTN} h$ one has

$$\frac{\partial Q}{\partial\delta_I} = [k^{Q_I}, Q], \tag{34}$$

and hence taking the δ_I derivative of a component of Q in some basis just amounts to multiplying with the adjoint representation of k^{Q_I} in the same basis. We have used here that h is generated by the subalgebra in the second line of (14) and hence we have $h^\# = h^{-1}$.

After reconstructing the 4d solution the Bekenstein-Hawking entropy of the black hole can be calculated. It reads as [2]

$$S = 2\pi\left(\sqrt{\Delta + F} + \sqrt{-J^2 + F}\right), \tag{35}$$

where Δ is the quartic invariant formed from the dyonic charges, and F is expressed with the seed and charge parameters as

$$F = (m^2 + n^2)(m\nu_2 - n\nu_1)^2. \tag{36}$$

To obtain asymptotically flat black holes one can cancel the NUT charge by setting $n = -m\frac{\nu_1}{\nu_2}$.

IV. THE ACTION OF THE U-DUALITY GROUP

In this section we describe in detail the transformation properties of the asymptotic charges of the black hole under U-duality. This allows us to construct polynomial invariants of these in the next section.

A. Action on fields

Recall that the coset element is parametrized in Iwasawa gauge as

$$\mathcal{V} = e^{-UH_0} e^{-\frac{1}{2}\sum_i \log y_i H_i} e^{-\sum_i x_i E_i} e^{-\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I})} e^{-\frac{1}{2}\sigma E_0}. \tag{37}$$

Notice that the gauge is such that the scalar fields are in an "upper triangular" form in the sense that the F type generators are not present. We claim that the U-duality group is generated by the generators H_i, E_i, F_i, $i = 1, 2, 3$ of (14) and the action is a simple right action on $\mathcal{V}$ followed by the left action of a local compensator $q \in SO(2)^{\times 3} \subset SL(2)_U^{\times 3}$ restoring the Iwasawa gauge:

$$\mathcal{V} \mapsto q\mathcal{V}g, \qquad g \in SL(2)_U^{\times 3}. \tag{38}$$

Indeed, we may write this as

$$q\mathcal{V}g = e^{-UH_0}(q e^{-\frac{1}{2}\sum_i \log y_i H_i} e^{-\sum_i x_i E_i} g) \times (g^{-1} e^{-\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I})} g) e^{-\frac{1}{2}\sigma E_0}, \tag{39}$$

as the generators E_0 and H_0 commute with $SL(2)_U^{\times 3}$. We see that the part with the potentials transform simply with the adjoint action. It is convenient to describe the 16 dimensional representation $(2,2,2,2)_U$ and its element $\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I})$ with the amplitudes ψ_{ijkl} of a four qubit state through (A7). Since there are no F^{P^I} and F^{Q_I} generators present this four qubit state has $\psi_{1jkl} \equiv 0$ and therefore it is actually a three qubit state. We can think of $\psi_{1jkl} \equiv 0$ as the gauge fixing condition. Then, under the U-duality transformation

$$\sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I}) \mapsto g^{-1} \sum_I (\zeta^I E^{Q_I} + \tilde{\zeta}_I E^{P^I}) g, \quad g \in SL(2)_U^{\times 3}, \tag{40}$$

this associated state transform as

$$\psi_{ijkl} \mapsto (S_1)_j{}^{j'} (S_2)_k{}^{k'} (S_3)_l{}^{l'} \psi_{ij'k'l'}, \quad S_1 \otimes S_2 \otimes S_3 \in SL(2)_U^{\times 3}, \tag{41}$$

where, according to (A7), $S_1 \otimes S_2 \otimes S_3$ is the $SL(2)_U^{\times 3}$ element associated to g^{-1}. We see that the gauge condition $\psi_{1jkl} \equiv 0$ on the coset element does not change. Therefore, this gives the required action of the U-duality group on the potentials given in (12) and no compensator is needed. We only need to worry about the gauge condition on the scalars. Multiplying the scalar term from the right with g spoils the gauge we chose and hence we need a local compensator. Using the fact that each scalar parametrizes the coset space $SL(2)/SO(2)$ we expect the local compensator to be from $SO(2)^{\times 3}$ [23] and therefore to have the form

$$q = e^{\sum_{i=1}^{3} \alpha_i k_i} = e^{\sum_{i=1}^{3} \alpha_i (E_i - F_i)}. \quad (42)$$

Now it is easy to verify that if we let g to be the $SL(2)_U^{\times 3}$ element corresponding to

$$\begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \otimes \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix} \otimes \begin{pmatrix} d_3 & -b_3 \\ -c_3 & a_3 \end{pmatrix}, \quad (43)$$

in the standard representation, then in order to restore the Iwasawa gauge the α_i of the compensator have to be chosen as

$$\tan \alpha_i = \frac{c_i y_i}{c_i x_i + a_i}. \quad (44)$$

Then, we have

$$q e^{-\frac{1}{2}\sum_i \log y_i H_i} e^{-\sum_i x_i E_i} g = e^{-\frac{1}{2}\sum_i \log y_i' H_i} e^{-\sum_i x_i' E_i}, \quad (45)$$

with the primed scalars being

$$x_i' = \frac{(d_i + c_i x_i)(b_i + a_i x_i) + a_i c_i (x_i^2 + y_i^2)}{(d_i + c_i x_i)^2 + c_i^2 y_i^2}, \\ y_i' = \frac{y_i}{(d_i + c_i x_i)^2 + c_i^2 y_i^2}, \quad (46)$$

which just corresponds to the usual action of the U-duality group on the scalars

$$\tau_i' = \frac{a_i \tau_i + b_i}{c_i \tau_i + d_i}, \quad (47)$$

with $\tau_i = x_i + i y_i$.

B. Action on asymptotic charges

Now recall that the asymptotic values of the fields are conveniently encoded in a charge matrix Q [see (25)] defined from the series expansion of $\mathcal{M} = \mathcal{V}^\# \mathcal{V}$ around asymptotic infinity. Now we need to be slightly more general than in [2] and cover the moduli space by letting $\mathcal{M}(\rho = 0) \neq I$. To define this generalized Q first expand $\mathcal{M}$ around asymptotic infinity

$$\mathcal{M} = \mathcal{M}^{(0)} + \mathcal{M}^{(1)} \rho + O(\rho^2). \quad (48)$$

Then note that $\mathcal{M} = \mathcal{V}^\# \mathcal{V}$ is a proper element of $SO(4,4)$ for every value of the coordinates. It follows that $\mathcal{M}^{(0)}$ and $(\mathcal{M}^{(0)})^{-1}\mathcal{M}$ are all good $SO(4,4)$ elements. Expanding this latter yields

$$(\mathcal{M}^{(0)})^{-1}\mathcal{M} = I + Q\rho + O(\rho^2), \quad (49)$$

where we have defined the "dressed" charge matrix [19]

$$Q = (\mathcal{M}^{(0)})^{-1}\mathcal{M}^{(1)}. \quad (50)$$

As this is an expansion of an element of $SO(4,4)$ around the identity, Q is an element of the Lie algebra $\mathfrak{so}(4,4)$. This is not true for $\mathcal{M}^{(1)}$ alone. Now we slightly generalize the expansion (24) of the fields around asymptotic infinity by letting

$$y_i = Y_i(1 - \Sigma_i \rho + O(\rho^2)), \qquad x_i = X_i + \Xi_i \rho + O(\rho^2), \quad (51)$$

with the rest being the same as in (24), but now with the possibility of having arbitrary values for the scalar fields at infinity. We give this general Q matrix in Sec. V B and for now, go back to use $X_i = 0$ and $Y_i = 1$ as in [2]. However, we stress that X_i and Y_i do transform under U-duality. One should not worry about this too much as everything we do in the following relies only on the fact that Q is an element of $\mathfrak{so}(4,4)$ which is independent of this choice and hence straightforwardly generalize to arbitrary X_i and Y_i. Setting $X_i = 0$ and $Y_i = 1$ results in the simple form of Q

$$Q = 2MH_0 + \sum_i \Sigma_i H_i + 2N p_0 - \sum_i \Xi_i p_i \\ - \sum_I (Q_I p^{Q_I} + P^I p^{P^I}). \quad (52)$$

We see that in this case Q lives in the 16 dimensional subspace (2,2,2,2) spanned by the first line of (14).

From the previous subsection we conclude that $\mathcal{M}$ transforms under U-duality as $\mathcal{M} \mapsto g^\# \mathcal{M} g$, and hence the charge matrix simply transforms with the adjoint action

$$Q \mapsto g^{-1} Q g, \qquad g \in SL(2)_U^{\times 3}. \quad (53)$$

Now from (52) we easily see that under the decomposition (14) of $\mathfrak{so}(4,4)$ suitable for U-duality, Q has components both in $\mathfrak{sl}_2^{\times 4}{}_U$ and $(2,2,2,2)_U$. As the splitting ensures that these components do not mix under the adjoint action of $\mathfrak{sl}_2^{\times 4}{}_U$ we may consider these parts separately

$$Q = Q_- + Q_+, \quad (54)$$

where

$$Q_- = 2MH_0 + \sum_i \Sigma_i H_i + 2N p_0 - \sum_i \Xi_i p_i \in \mathfrak{sl}_2^{\times 4}{}_U, \\ Q_+ = -\sum_I (Q_I p^{Q_I} + P^I p^{P^I}) \in (2,2,2,2)_U. \quad (55)$$

Let us first consider Q_-. Clearly, M and N are invariant under the adjoint action of $SL(2)_U^{\times 3}$ as expected. The six scalar charges Σ_i, Ξ_i parametrize an element $\mathfrak{sl}_2^{\times 3}{}_U$ and hence they transform in the *adjoint representation* of $SL(2)_U^{\times 3}$. We define the matrices

$$R_i = \begin{pmatrix} \Sigma_i & -\Xi_i \\ -\Xi_i & -\Sigma_i \end{pmatrix}, \tag{56}$$

transforming under U-duality as

$$R_i \mapsto \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} R_i \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}^{-1}, \tag{57}$$

with

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2).$$

Note that the symmetricity of R_i is spoiled by this transformation but this is simply a consequence of fixing the asymptotic values of the scalars which, again, are not invariant under U-duality (see Sec. V B for the general form of R).

The element Q_+ transforms as a four qubit state ψ_{ijkl} under the full $SL(2)_U^{\times 4}$ and it decomposes into a pair of three qubit states $(\psi_1)_{jkl} \equiv \psi_{0jkl}$ and $(\psi_2)_{jkl} \equiv \psi_{1jkl}$ when just the U-duality group $SL(2)_U^{\times 3}$ is used. The explicit amplitudes can be read off using (A7) and are given as

$$\left(\begin{array}{cccc} \psi_{0000} & \psi_{0001} & \psi_{0010} & \psi_{0011} \\ \psi_{0100} & \psi_{0101} & \psi_{0110} & \psi_{0111} \\ \hline \psi_{1000} & \psi_{1001} & \psi_{1010} & \psi_{1011} \\ \psi_{1100} & \psi_{1101} & \psi_{1110} & \psi_{1111} \end{array}\right) = \left(\begin{array}{cccc} -P^4 & -Q_3 & -Q_2 & P^1 \\ -Q_1 & P^2 & P^3 & Q_4 \\ \hline -Q_4 & P^3 & P^2 & Q_1 \\ P^1 & Q_2 & Q_3 & -P^4 \end{array}\right). \tag{58}$$

Note that this pair is related through

$$|\psi_2\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} |\psi_1\rangle. \tag{59}$$

For the corresponding pair of three qubit states dressed with nontrivial scalar asymptotics, see again Sec. V B.

The index corresponding to the first qubit transforms as a doublet under the extra Ehlers $SL(2)$. Note that the scalar charges are singlets under the Ehlers symmetry: the adjoint 28 of $SO(4,4)$, where Q lives in, decomposes under the maximal subgroup $SL(2)_U^{\times 4} = SL(2)_{\rm Ehlers} \times (SL(2)_U^{\times 3})$ as $28 = (3,1) \oplus (1,9) \oplus (2,8)$.

C. The algebra of covariants

We have seen that the asymptotic charges transform under U-duality as a (not general) vector in $9 \oplus 8 \oplus 8$, where 9 refers to the adjoint representation, while 8 is the fundamental corresponding to a three qubit state. If we dress up the charge matrix with the asymptotic values of the scalars, then Q actually fills out the representation $9 \oplus 8 \oplus 8$. In order to be able to write up invariants we first need to construct covariants with indices transforming the same way. Luckily, there exists a construction, called the moment map, which allows one to associate an element transforming in 9 to a *pair* of vectors in 8. Unfortunately, this construction will result in an unnecessarily large covariant algebra. We can significantly reduce this by incorporating "triality" symmetry of the STU model: the symmetry under permutation of the three $SL(2)$ factors. This leads us to consider the embedding ${\mathfrak{sl}_2^{\times 3}}_U \subset \mathfrak{sl}_6$ and to construct the moment map in the $SL(6)$ covariant language of three fermions with six single particle states [9,24,25].

Consider fermionic creation an annihilation operators p^a and n_a, $a = 1, \ldots, 6$ satisfying

$$\{p^a, n_b\} = \delta^a{}_b, \quad \{p^a, p^b\} = 0, \quad \{n_a, n_b\} = 0. \tag{60}$$

An unnormalized three fermion state can be written as

$$|P\rangle = \frac{1}{3!} P_{abc} p^a p^b p^c |0\rangle \in \wedge^3(\mathbb{C}^6), \tag{61}$$

with the antisymmetric tensor P_{abc} having 20 independent components. The so-called SLOCC group of this system is $SL(6,\mathbb{C})$ acting locally on the amplitudes as

$$P_{abc} \mapsto S_a{}^{a'} S_b{}^{b'} S_c{}^{c'} P_{a'b'c'}, \qquad S \in SL(6,\mathbb{C}). \tag{62}$$

The moment map associates an $\mathfrak{sl}(6,\mathbb{C})$ element to $|P\rangle$. This element reads as

$$(K_P)^a{}_b = \frac{1}{2!3!} \epsilon^{ac_1c_2c_3c_4c_5} P_{bc_1c_2} P_{c_3c_4c_5}. \tag{63}$$

It is clear that if we transform the state as in (62) this covariant transforms as

$$K_P \mapsto (S^T)^{-1} K_P S^T, \tag{64}$$

hence the powers of its trace are continuous invariants. It turns out that the action of $SL(6,\mathbb{C})$ on 20 admits a single independent continuous invariant, quartic in the amplitudes, given by

$$\mathcal{D}(P) = \frac{1}{6} \mathrm{Tr} K_P^2. \tag{65}$$

Note that this quantity is a measure of tripartite entanglement for the fermions [9,26]. The situation is different if we have two states $|P\rangle$ and $|Q\rangle$ at our disposal. In this case we can define the following covariants

$$(K_P)^a{}_b = \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5}P_{bc_1c_2}P_{c_3c_4c_5},$$
$$(K_Q)^a{}_b = \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5}Q_{bc_1c_2}Q_{c_3c_4c_5},$$
$$(K_{PQ})^a{}_b = \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5}P_{bc_1c_2}Q_{c_3c_4c_5},$$
$$(K_{QP})^a{}_b = \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5}Q_{bc_1c_2}P_{c_3c_4c_5}. \tag{66}$$

Traces of products of these define invariants. In particular there is now a nonzero invariant bilinear product

$$(P,Q) = \frac{1}{3}\mathrm{Tr}K_{PQ} = -\frac{1}{3}\mathrm{Tr}K_{QP} = \frac{1}{3!3!}\epsilon^{c_1c_2c_3c_4c_5c_6}P_{c_1c_2c_3}Q_{c_4c_5c_6}, \tag{67}$$

which also shows that the covariants K_{PQ} and K_{QP} are now *not* elements of the Lie algebra $\mathfrak{sl}_6$.

Now let us describe how to embed three qubit states

$$|\psi\rangle = \sum_{i,j,k=0}^{1}\psi_{ijk}|ijk\rangle, \tag{68}$$

into our fermionic vector space. The standard thing to do is the following:

$$|\psi\rangle \mapsto |P_\psi\rangle = \sum_{i,j,k=0}^{1}\psi_{ijk}p^{i+1}p^{j+3}p^{k+5}|0\rangle. \tag{69}$$

Then, it is easy to see that a three qubit SLOCC transformation

$$\psi_{ijk} \mapsto (S_1)_i{}^{i'}(S_2)_j{}^{j'}(S_3)_k{}^{k'}\psi_{i'j'k'},$$
$$S_1\otimes S_2\otimes S_3 \in SL(2)_U^{\times 3}, \tag{70}$$

can be implemented in the language of three fermions (62) by choosing

$$S = \begin{pmatrix} S_1 & & \\ & S_2 & \\ & & S_3 \end{pmatrix} \in SL(6,\mathbb{C}). \tag{71}$$

Finally, we can associate covariants transforming in the adjoint of $SL(2)_U^{\times 3}$ to the pair of three qubit states given in (58) using (66) and (69) as

$$(K_{11})^a{}_b = \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5}(P_{\psi_1})_{bc_1c_2}(P_{\psi_1})_{c_3c_4c_5},$$
$$(K_{22})^a{}_b = \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5}(P_{\psi_2})_{bc_1c_2}(P_{\psi_2})_{c_3c_4c_5},$$
$$(K_{12})^a{}_b = \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5}(P_{\psi_1})_{bc_1c_2}(P_{\psi_2})_{c_3c_4c_5},$$
$$(K_{21})^a{}_b = \frac{1}{2!3!}\epsilon^{ac_1c_2c_3c_4c_5}(P_{\psi_2})_{bc_1c_2}(P_{\psi_1})_{c_3c_4c_5}. \tag{72}$$

For an explicit form of these matrices see Appendix B. Notice the crucial fact that these four matrices transform exactly the same way under $SL(2)_U^{\times 3}$ as the matrix

$$R = \begin{pmatrix} R_1^T & & \\ & R_2^T & \\ & & R_3^T \end{pmatrix} = \begin{pmatrix} \Sigma_1 & -\Xi_1 & 0 & 0 & 0 & 0 \\ -\Xi_1 & -\Sigma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_2 & -\Xi_2 & 0 & 0 \\ 0 & 0 & -\Xi_2 & -\Sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Sigma_3 & -\Xi_3 \\ 0 & 0 & 0 & 0 & -\Xi_3 & -\Sigma_3 \end{pmatrix}, \tag{73}$$

formed by the scalar charges [see Eq. (56)].

The four matrices of (72) can be grouped into a 2×2 block matrix K_{ab} transforming under the Ehlers $SL(2)$ as $2\times 2 = 1\oplus 3$ in its ab indices. We note that the singlet part satisfies the relation

$$K_{21} - K_{12} = \left(\sum_I (Q_I^2 + (P^I)^2)\right)I, \tag{74}$$

and hence only three of the K_{ab}s give independent covariants.

V. THE F-INVARIANT

A. Construction of the invariant

Let us begin by listing the independent primitive invariants of homogeneous degree less than or equal to four in the asymptotic charges, that can be formed by our covariants.

(i) Degree 1 invariants:

$$M, \qquad N. \tag{75}$$

(ii) Degree 2 invariants:

$$\mathrm{Tr}(K_{12}), \qquad \mathrm{Tr}(R^2). \tag{76}$$

(iii) Degree 3 invariants:

$$\mathrm{Tr}(K_{12}R), \qquad \mathrm{Tr}(K_{11}R). \tag{77}$$

(iv) Degree 4 invariants:

$$\mathrm{Tr}K_{11}^2, \quad \mathrm{Tr}K_{12}^2, \quad \mathrm{Tr}(K_{11}K_{22}), \quad \mathrm{Tr}(R^4). \tag{78}$$

To reduce the set of independent invariants we have used the identities

$$\begin{aligned}\mathrm{Tr}(K_{11}R) &= -\mathrm{Tr}(K_{22}R),\\ \mathrm{Tr}(K_{11}^2) &= \mathrm{Tr}(K_{22}^2),\\ \mathrm{Tr}(K_{11}R^2) &= \mathrm{Tr}(K_{22}R^2) = 0,\\ \mathrm{Tr}(K_{12}K_{11}) &= -\mathrm{Tr}(K_{12}K_{22}),\\ \mathrm{Tr}(K_{12}R^2) &= \frac{1}{6}(\mathrm{Tr}K_{12})(\mathrm{Tr}R^2),\\ 3\mathrm{Tr}(K_{11}K_{22}) + (\mathrm{Tr}K_{12})^2 - 3\mathrm{Tr}(K_{12}^2) &= 0.\end{aligned} \tag{79}$$

Note that we can write some well-known U-duality invariants in this language. First of all, Cayley's hyperdeterminant

$$\Delta = \frac{1}{16}\Bigg(4(Q_1Q_2Q_3Q_4 + P^1P^2P^3P^4) + 2\sum_{J<K} Q_JQ_KP^JP^K - \sum_J (Q_J)^2(P^J)^2\Bigg) \tag{80}$$

is just given as

$$\Delta = -\frac{1}{96}\mathrm{Tr}K_{11}^2 = -\frac{1}{96}\mathrm{Tr}K_{22}^2. \tag{81}$$

The asymptotic value of the quadratic symplectic invariant I_2 is

$$I_2^\infty = \frac{1}{4}\sum_I (Q_I^2 + (P^I)^2) = -\frac{1}{12}\mathrm{Tr}K_{12}. \tag{82}$$

The quadratic invariant

$$S_2^\infty = \frac{1}{4}G_{i\bar{j}}\partial_r\tau^i\partial_r\bar{\tau}^{\bar{i}}|_{r\to\infty} = \frac{1}{4}\sum_i(\Xi_i^2 + \Sigma_i^2), \tag{83}$$

can be expressed as

$$S_2^\infty = \frac{1}{8}\mathrm{Tr}(R^2). \tag{84}$$

Now recall the formula for the F invariant in terms of the charge-up parameters δ_I, γ_I and seed variables m and n:

$$F = (m^2 + n^2)(m\nu_2 - n\nu_1)^2, \tag{85}$$

where ν_1 and ν_2 are the functions given in Eq. (28) and they do not scale with the charges. We see that F is of homogeneous degree 4 in the charges and we expect it to be U-duality invariant. From the 10 invariants that we have identified at the beginning of this section we can form 22 monomials of homogeneous degree four. We can form a linear combination of these, equate it to F and try to solve for the coefficients. The simplest way to do this is to generate various random sets of parameters m, n, δ_I, and γ_I and try to solve the resulting numerical, linear equations simultaneously. If this works for a considerably higher number of equations than the number of variables, which is 22, we can probably trust our coefficients. We did this procedure for 600 equations and we have found a single solution. The obtained numerical coefficients have been rationalized and the result was tested analytically with a computer algebra system. The result is

$$\begin{aligned}F = {}& M^4 + M^2N^2 + \frac{M^2}{12}\mathrm{Tr}K_{12} - \frac{M}{24}\mathrm{Tr}(K_{12}R)\\ &+ \frac{N^2}{24}\mathrm{Tr}(R^2) - \frac{N}{24}\mathrm{Tr}(K_{11}R) + \frac{1}{192}\Big(\mathrm{Tr}(K_{11}^2)\\ &- \mathrm{Tr}(K_{11}K_{22}) - \frac{1}{2}(\mathrm{Tr}R^2)^2 + \mathrm{Tr}(R^4)\Big).\end{aligned} \tag{86}$$

In the asymptotically flat case one sets $N = 0$ (or $n = -m\frac{\nu_1}{\nu_2}$). In this case the F invariant reads as

$$\begin{aligned}F &= m^4\frac{(\nu_1^2 + \nu_2^2)^3}{\nu_2^4}\\ &= M^4 - M^2I_2^\infty - \frac{M}{24}\mathrm{Tr}(K_{12}R) - \frac{1}{2}\Delta\\ &\quad - \frac{1}{192}\mathrm{Tr}(K_{11}K_{22}) - \frac{1}{6}(S_2^\infty)^2 + \frac{1}{192}\mathrm{Tr}(R^4),\end{aligned} \tag{87}$$

where we have reintroduced the familiar U-duality invariants where it is possible. We stress that for general scalar asymptotics one should use the R and K_{ab} matrices as given in Sec. V B. It is useful to write the F invariant without an explicit reference to the auxiliary 6 dimensional representation that we have introduced. We may employ the invariant bilinear product of (67) to write

$$\begin{aligned}\mathrm{Tr}K_{12} &= 3(P_{\psi_1}, P_{\psi_2}),\\ \mathrm{Tr}(K_{12}R) &= -(P_{\psi_1}, R_*P_{\psi_2}),\\ \mathrm{Tr}(K_{11}K_{22}) &= -(P_{\psi_1}, (K_{22})_*P_{\psi_1}),\\ \mathrm{Tr}(K_{11}^2) &= -(P_{\psi_1}, (K_{11})_*P_{\psi_1}),\end{aligned} \tag{88}$$

where we have defined the action of a Lie algebra element $t \in \mathfrak{sl}(6)$ on $P \in \wedge^3\mathbb{C}$ as

$$(t_*P)_{abc} = t^d{}_aP_{dbc} + t^d{}_bP_{adc} + t^d{}_cP_{abd}. \tag{89}$$

Also, let us define the 8×8 matrix $\hat{R}$ corresponding to R in the fundamental representation of the U-duality group:

$$\hat{R} = R_1^T \otimes I \otimes I + I \otimes R_2^T \otimes I + I \otimes I \otimes R_3^T, \quad (90)$$

see (56) for the definition of R_is. Then we have

$$\mathrm{Tr}R^4 - \frac{1}{2}(\mathrm{Tr}R^2)^2 = -\frac{1}{8}\left(\mathrm{Tr}\hat{R}^4 - \frac{1}{8}(\mathrm{Tr}\hat{R}^2)^2\right). \quad (91)$$

We may then rewrite the F invariant as

$$\begin{aligned} F = {} & M^4 + M^2N^2 + \frac{M^2}{4}(P_{\psi_1}, P_{\psi_2}) + \frac{M}{24}(P_{\psi_1}, R_* P_{\psi_2}) \\ & + \frac{N^2}{96}\mathrm{Tr}(\tilde{R}^2) + \frac{N}{24}(P_{\psi_1}, R_* P_{\psi_1}) \\ & - \frac{1}{192}(P_{\psi_1}, (K_{11} - K_{22})_* P_{\psi_1}) \\ & + \frac{1}{1536}\left(\frac{1}{8}(\mathrm{Tr}\hat{R}^2)^2 - \mathrm{Tr}(\hat{R}^4)\right), \end{aligned} \quad (92)$$

which will be well suited for generalization to the E_7 invariant case.

As a final remark, we note that a single centered STU back hole parametrized by six moduli and eight dyonic charges is expected to have five independent U-duality invariants [6]. The reason that we have more than this in (76)–(78) is that we treat the scalar charges as independent variables. This allowed us to turn F into a polynomial invariant.

B. General scalar asymptotics

Here we consider explicitly the computation of F for general asymptotic values X_i and Y_i of the moduli. We expand the "dressed" charge matrix defined in (50) using (51) as

$$\begin{aligned} Q = {} & q_{H_\Lambda} H_\Lambda + q_{E_\Lambda} E_\Lambda + q_{F_\Lambda} F_\Lambda + q_{E^{Q_I}} E^{Q_I} \\ & + q_{E^{P^I}} E^{P^I} + q_{F^{Q_I}} F^{Q_I} + q_{F^{P^I}} F^{P^I}, \end{aligned} \quad (93)$$

The part in ${\mathfrak{sl}_2^{\times 4}}_U$ reads as

$$\begin{aligned} Q_- = {} & 2MH_0 + 2Np_0 + \sum_{i=1}^{3}\left[\left(\Sigma_i - \frac{\Xi_i X_i}{Y_i^2}\right)H_i \right. \\ & \left. + \left(-\Xi_i - 2\Sigma_i X_i + \frac{\Xi_i X_i^2}{Y_i^2}\right)E_i - \frac{\Xi_i}{Y_i^2}F_i\right], \end{aligned} \quad (94)$$

and hence the R_i matrices of (56) obtain the following dressing

$$R_i = \begin{pmatrix} \Sigma_i - \frac{\Xi_i X_i}{Y_i^2} & -\Xi_i - 2\Sigma_i X_i + \frac{\Xi_i X_i^2}{Y_i^2} \\ -\frac{\Xi_i}{Y_i^2} & -\Sigma_i + \frac{\Xi_i X_i}{Y_i^2} \end{pmatrix}. \quad (95)$$

The four qubit state in $(2,2,2,2)_U$ is

$$Q_+ = q_{E^{Q_I}} E^{Q_I} + q_{E^{P^I}} E^{P^I} + q_{F^{Q_I}} F^{Q_I} + q_{F^{P^I}} F^{P^I}. \quad (96)$$

The amplitudes corresponding to the first three qubit state [see (A7)] are unchanged

$$\begin{aligned} & \begin{pmatrix} \psi_{0000} & \psi_{0001} & \psi_{0010} & \psi_{0011} \\ \psi_{0100} & \psi_{0101} & \psi_{0110} & \psi_{0111} \end{pmatrix} \\ & \equiv \begin{pmatrix} q_{E^{P^4}} & q_{E^{Q_3}} & q_{E^{Q_2}} & -q_{E^{P^1}} \\ q_{E^{Q_1}} & -q_{E^{P^2}} & -q_{E^{P^3}} & -q_{E^{Q_4}} \end{pmatrix} \\ & = \begin{pmatrix} -P^4 & -Q_3 & -Q_2 & P^1 \\ -Q_1 & P^2 & P^3 & Q_4 \end{pmatrix}. \end{aligned} \quad (97)$$

On the other hand, the second three qubit state

$$\begin{aligned} & \begin{pmatrix} \psi_{1000} & \psi_{1001} & \psi_{1010} & \psi_{1011} \\ \psi_{1100} & \psi_{1101} & \psi_{1110} & \psi_{1111} \end{pmatrix} \\ & = \begin{pmatrix} -q_{F^{Q_4}} & q_{F^{P^3}} & q_{F^{P^2}} & q_{F^{Q_1}} \\ q_{F^{P^1}} & q_{F^{Q_2}} & q_{F^{Q_3}} & -q_{F^{P^4}} \end{pmatrix} \end{aligned} \quad (98)$$

has the following dressing

$$\psi_{1ijk} = D_{ii'}^{(1)} D_{jj'}^{(2)} D_{kk'}^{(3)} \psi_{0i'j'k'}, \quad (99)$$

or equivalently,

$$|\psi_2\rangle = (D^{(1)} \otimes D^{(2)} \otimes D^{(3)})|\psi_1\rangle, \quad (100)$$

with

$$D^{(i)} = \frac{1}{Y_i}\begin{pmatrix} X_i & -X_i^2 - Y_i^2 \\ 1 & -X_i \end{pmatrix}. \quad (101)$$

This is the generalization of the relation (59) and shows that the pair of three qubit states are related by a moduli dependent $SL(2)_U^{3\times}$ transformation. The formula (86) for F is then valid for arbitrary scalar moduli provided that we use the dressed R_i matrices of (95) and the K_{ab} matrices calculated from (97) and (99) through (72).

Now we can also relate the charge matrix of an arbitrary solution through U-duality to an auxiliary charge matrix with trivial scalar asymptotics. Consider the duality transformation

$$J_1 \otimes J_2 \otimes J_3 \in SL(2)_U^{\times 3}, \quad J_i = \frac{1}{\sqrt{Y_i}}\begin{pmatrix} 1 & -X_i \\ 0 & Y_i \end{pmatrix}, \quad (102)$$

and denote the corresponding $SO(4,4)$ element with $\mathcal{J}$. One may check with explicit computation that the asymptotic value of the coset element (48) satisfies

$$\mathcal{J}^\# \mathcal{J} = \mathcal{M}^{(0)}. \quad (103)$$

As a consequence, for any general charge matrix Q we may define a duality transformed one

$$\tilde{Q} = \mathcal{J} Q \mathcal{J}^{-1}, \tag{104}$$

which corresponds to a black hole with trivial moduli. Now we use the relations

$$D^{(i)} = J_i^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} J_i,$$
$$R_i = J_i^{-1} \begin{pmatrix} \Sigma_i & -\frac{\Xi_i}{Y_i} \\ -\frac{\Xi_i}{Y_i} & -\Sigma_i \end{pmatrix} J_i \tag{105}$$

to relate the auxiliary charges of $\tilde{Q}$ to the physical ones of Q. Following IV we deduce that the dyonic charge vectors $|\tilde{\psi}_{1,2}\rangle$ of $\tilde{Q}$ are expressed with the physical charges $|\psi_1\rangle$ as

$$\begin{aligned} |\tilde{\psi}_1\rangle &= (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle, \\ |\tilde{\psi}_2\rangle &= (J_1 \otimes J_2 \otimes J_3)(D^{(1)} \otimes D^{(2)} \otimes D^{(3)})|\psi_1\rangle \\ &\equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &\quad \times (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle. \end{aligned} \tag{106}$$

We see that the vectors $|\tilde{\psi}_{1,2}\rangle$ are indeed related as in (59) which is valid only for the canonical moduli. For the scalar charges one has

$$\tilde{R}_i = J_i R_i J_i^{-1} \equiv \begin{pmatrix} \Sigma_i & -\frac{\Xi_i}{Y_i} \\ -\frac{\Xi_i}{Y_i} & -\Sigma_i \end{pmatrix}. \tag{107}$$

Since the F-invariant is blind to the transformation (104) we obtained the result that for any asymptotics we may calculate F by just using the formula for canonical moduli with replacing the dyonic charges by $(J_1 \otimes J_1 \otimes J_3)|\psi_1\rangle$ and scaling the scalar charges as $\Xi_i \mapsto \frac{\Xi_i}{Y_i}$, i.e.,

$$\begin{aligned} &F(X_i, Y_i, |\psi_1\rangle, \Sigma_i, \Xi_i) \\ &\quad = F\left(X_i = 0, Y_i = 1, (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle, \Sigma_i, \frac{\Xi_i}{Y_i}\right). \end{aligned} \tag{108}$$

VI. RELATIONS BETWEEN SCALAR CHARGES AND PHYSICAL CHARGES

Our formula is entirely in terms of the asymptotic charges of the black hole and is manifestly invariant under U-duality and permutation of scalars. However, it does contain explicitly the scalar charges Σ_i and Ξ_i which are not independent of M, N, Q_I, and P^I. A formula entirely in terms of the physical charges would require solving for the functions $\Sigma_i(M, N, Q, P)$, $\Xi_i(M, N, Q, P)$. In this section, we provide constraints that these functions must satisfy and solve them for some special cases. We illustrate the example of the four electric charge Cvetič-Youm black hole that in general it is not possible to give the F-invariant in terms of radicals of the physical charges.

We start by describing the constraint equations. Recall, that the charge matrix Q of (50) transforms as a four qubit state [see (A6)] under the action of the $SL(2)^{4\times}$ spanned by the second line of (14). The charge-up matrix h of Eq. (22) is an element of this $SL(2)^{4\times}$. It is known [8,13] that this 16 dimensional representation admits four algebraically independent continuous invariants. All of these can be checked to be proportional to some power of the combination $(m^2 + n^2)$ of the seed parameters. It follows that they provide 3 independent polynomial equations among the 16 asymptotic charges in Q. Let us describe a convenient and simple way of obtaining these equations. Consider the characteristic polynomial of the charge matrix Q:

$$p(\lambda) = \det(\lambda I - Q). \tag{109}$$

It is clear that the characteristic polynomial is invariant under the charge up operation and hence we have

$$p(\lambda) = p_0(\lambda), \tag{110}$$

where the characteristic polynomial for the seed solution is

$$\begin{aligned} p_0(\lambda) &= \det(\lambda I - Q_{KTN}) \\ &= \lambda^4(\lambda^2 - 4m^2 - 4n^2)^2 \\ &= \lambda^4(\lambda^2 - \mathrm{Tr}Q^2)^2, \end{aligned} \tag{111}$$

where we have used the invariance of $\mathrm{Tr}Q^2$ to express $m^2 + n^2$ in terms of asymptotic charges. The two polynomials agree iff all of their coefficients agree hence we have the following 9 polynomial equations

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}(p(\lambda) - p_0(\lambda))|_{\lambda=0} = 0, \quad k = 0, \ldots, 8. \tag{112}$$

One can check that for $k = 1, 3, 5, 6, 7, 8$ these are trivially satisfied and hence we are left with 3 equations. These 3 equations have linearly independent gradients in the scalar charges which indicates a three dimensional solution set. However, this does not tell us anything about the set of real solutions. One can easily check that as one approaches the seed solution by taking $Q_I \to 0$ and $P^I \to 0$, there is only one real root satisfying the consistency requirement $\Sigma_i \to 0$. This shows that it is possible that the three equations are enough to determine the scalar charges uniquely. To say something more precise about this one would need to determine at least the real dimension of the semialgebraic set defined by these equations, but to our knowledge, there is no method to do this to date. Instead, we provide solutions to (112) for some special charge vectors, where the F-invariant is known explicitly, and hence we can compare our results with the existing literature.

Before doing so, we comment on what happens with these constraints when one considers black holes with nontrivial asymptotic moduli. In this case one can just replace Q in (109) with the auxiliary charge vector $\tilde{Q}$, as readily seen from (104). This shows that whatever expressions $\Sigma_i = f_i(M,|\psi_1\rangle)$, $\Xi_i = g_i(M,|\psi_1\rangle)$ we find for trivial moduli by solving (112), we can safely use them for nontrivial moduli as $\Sigma_i = f_i(M,(J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle)$ and $\Xi_i = Y_i g_i(M,(J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle)$. Combine this with (108) to get the expected result

$$F(X_i, Y_i, |\psi_1\rangle) = F(X_i = 0, Y_i = 1, (J_1 \otimes J_2 \otimes J_3)|\psi_1\rangle), \tag{113}$$

which is then guaranteed to be valid as long as we can use (112) to solve for the scalar charges. This is the case for the first two of the following examples.

A. Klauza-Klein black hole

The Klauza-Klein black hole [27] is obtained by setting $Q_2 = Q_3 = Q_4 = 0$ and $P^2 = P^3 = P^4 = 0$ and $N = 0$. From the parametrization (30)–(33) we can deduce that $\Xi_i = 0$ and $\Sigma_2 = \Sigma_3 = -\Sigma_1$ in this case. We can put this into (112) as an ansatz and observe that it automatically solves two equations. The third equation reads as

$$\Sigma_1(8M^2 + (P^1)^2 + Q_1^2 - 2\Sigma_1^2) = 2M((P^1)^2 - Q_1^2). \tag{114}$$

As expected, there is only one root that vanishes for zero charges. For this root we have found numerical agreement between our formula

$$F = \left(M^2 - \frac{1}{4}(P^1)^2\right)\left(M^2 - \frac{1}{4}Q_1^2\right) + \frac{1}{8}M\Sigma_1((P^1)^2 - Q_1^2) - \frac{1}{16}\Sigma_1^4, \tag{115}$$

for the F-invariant and the complicated expression presented in [2] for the Klauza-Klein black hole. We may further specialize by setting $P^1 = 0$. In this case the above equation factorizes as

$$(2M + \Sigma_1)(Q_1^2 + 4M\Sigma_1 - 2\Sigma_1^2) = 0. \tag{116}$$

The physical root is $\Sigma_1 = M - \sqrt{M^2 + \frac{1}{2}Q_1^2}$. Upon substituting this into the formula for the F-invariant we get

$$F = \frac{1}{64}\left(32M^4 - 40M^2Q_1^2 - Q_1^4 + 4M(4M^2 + 2Q_1^2)^{\frac{3}{2}}\right), \tag{117}$$

in complete agreement with [2].

B. $-iX^0X^1$ supergravity black hole

Now let us consider the axion-dilaton black hole of [28]. We set the electric and magnetic charges pairwise equal $Q_1 = Q_4$, $Q_2 = Q_3$, $P^1 = P^4$ and $P^2 = P^3$. We also set the NUT charge to zero. From (30)–(33) we observe that in this case we have $\Sigma_2 = \Sigma_3 = \Xi_2 = \Xi_3 = 0$. Using this as an ansatz in (112) we are left with a single equation

$$\begin{aligned}2(P^1)^2(2M\Sigma_1 + (P^2)^2 - Q_1^2 - Q_2^2) + 2Q_2(4M\Xi_1(P^2) + 2MQ_2\Sigma_1 + Q_1^2Q_2) \\ = 4M^2\Xi_1^2 + 4M^2\Sigma_1^2 + (P^1)(8M\Xi_1Q_1 - 8(P^2)Q_1Q_2) + 2(P^2)^2(2M\Sigma_1 + Q_1^2 + Q_2^2) + 4MQ_1^2\Sigma_1 \\ + (P^1)^4 + (P^2)^4 + Q_1^4 + Q_2^4,\end{aligned} \tag{118}$$

which admits a single real solution

$$\Sigma_1 = \frac{(P^1)^2 - (P^2)^2 - Q_1^2 + Q_2^2}{2M}, \quad \Xi_1 = \frac{P^2Q_2 - P^1Q_1}{M}. \tag{119}$$

This agrees with the axion-dilaton charge obtained in [28]

$$\Upsilon = i(\Xi_1 - i\Sigma_1) = -\frac{(Q_1 + iP^1)^2 + (-P^2 + iQ_2)^2}{2M}. \tag{120}$$

Upon inserting this into our formula (86) for the F-invariant we obtain

$$F = \frac{1}{16}(4M^2 - (P^1 - P^2)^2 - (Q_1 - Q_2)^2) \times (4M^2 - (P^1 + P^2)^2 - (Q_1 + Q_2)^2), \tag{121}$$

in complete agreement with [2,28]. We note here that when the single modulus of this model is turned on we have to replace the charges according to (113). Explicitly, this leads to the following replacement rule

$$\begin{pmatrix} P^1 \\ Q_1 \end{pmatrix} \mapsto \frac{1}{\sqrt{Y_1}} \begin{pmatrix} 1 & -X_1 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} P^1 \\ Q_1 \end{pmatrix},$$
$$\begin{pmatrix} Q_2 \\ -P^2 \end{pmatrix} \mapsto \frac{1}{\sqrt{Y_1}} \begin{pmatrix} 1 & -X_1 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} Q_2 \\ -P^2 \end{pmatrix}, \tag{122}$$

which, again, agrees with [28].

C. Dilute gas limit

We can recover the dilute gas limit of [16] as well. In this limit, we have the following constraint among the magnetic charges

$$P^1 + P^2 + P^3 + P^4 = 0. \tag{123}$$

Provided that this is true, all three equations of (112) can be solved *exactly* by setting

$$\Sigma_1 = -2M + Q_2 + Q_3, \qquad \Sigma_2 = 2M - Q_2 - Q_4,$$
$$\Sigma_3 = 2M - Q_3 - Q_4, \qquad \Xi_1 = P^2 + P^3,$$
$$\Xi_2 = -P^2 - P^4, \qquad \Xi_3 = -P^3 - P^4. \tag{124}$$

These roots cannot be physical for all values of the charges as for vanishing charges we must have $\Sigma_i = 0$. However, in the dilute gas limit, the charges are large and hence this requirement is outside of the region of validity. Define the excitation energy as $\delta M = M - M_{BPS} = M - \frac{1}{4}(Q_1 + Q_2 + Q_3 + Q_4)$. We obtain the dilute gas limit of F by substituting (124) into (86) and scaling $Q_i \to \mu^2 Q_i$, $i = 2, 3, 4$ and $P^I \to \mu P^I$, while keeping δM fixed. Upon $\mu \to \infty$ the leading order μ^8 in F vanishes. The next to leading order contribution is the coefficient of μ^6 which is

$$F_0 = \frac{1}{2}\delta M Q_2 Q_3 Q_4, \tag{125}$$

which agrees with the result of [16].

D. Four charge Cvetič-Youm black hole

Here we set all the magnetic charges and the NUT charge to zero but allow for arbitrary electric charges [29]. We have all $\Xi_i = 0$ but we still need to solve for all the Σ_i. Instead of trying to solve the constraint equations we can simply use that as all $\gamma_I = 0$, the equations for the δ_I derivatives (30)–(33) give the full Jacobian for the change of variables from boost parameters to asymptotic charges. The best thing to do is to write differential equations for the functions $Q_I(2M, \Sigma_i)$ because these are remarkably easily solved. The solution is such that

$$\sqrt{4Q_I^2 + C_I} = \frac{\partial Q_I}{\partial \delta_I}, \tag{126}$$

where the right-hand side is understood to be given through (31). The C_I are constants of integration. Comparing with the actual parametrization reveals that $C_I = 4m$, $I = 1, \ldots, 4$. The solution for the scalar charges is then easily obtained to be

$$\Sigma_1 = -\frac{1}{2}\sqrt{m^2 + Q_1^2} + \frac{1}{2}\sqrt{m^2 + Q_2^2} + \frac{1}{2}\sqrt{m^2 + Q_3^2} - \frac{1}{2}\sqrt{m^2 + Q_4^2},$$
$$\Sigma_2 = \frac{1}{2}\sqrt{m^2 + Q_1^2} - \frac{1}{2}\sqrt{m^2 + Q_2^2} + \frac{1}{2}\sqrt{m^2 + Q_3^2} - \frac{1}{2}\sqrt{m^2 + Q_4^2},$$
$$\Sigma_3 = \frac{1}{2}\sqrt{m^2 + Q_1^2} + \frac{1}{2}\sqrt{m^2 + Q_2^2} - \frac{1}{2}\sqrt{m^2 + Q_3^2} - \frac{1}{2}\sqrt{m^2 + Q_4^2}. \tag{127}$$

Plugging these expressions into the formula (86) for the F-invariant we recover the expression given in [2]. This expression still depends on the seed parameter m. A novel result here is that m is determined in terms of the physical charges through the equation

$$M = \frac{1}{4}\left(\sqrt{m^2 + Q_1^2} + \sqrt{m^2 + Q_2^2} + \sqrt{m^2 + Q_3^2} + \sqrt{m^2 + Q_4^2}\right). \tag{128}$$

We happily acknowledge that the right-hand side is greater than $\frac{1}{4}(Q_1 + Q_2 + Q_3 + Q_4)$, and hence the requirement of solvability is $M \geq M_{BPS}$. Note that this example illustrates that it is in general not possible to express the F-invariant in terms of radicals of the physical charges. Indeed, one may rewrite (128) as a system of five polynomial equations as $M = \frac{1}{4}\sum_{I=1}^4 x_I$ and $x_I^2 = m^2 + Q_I^2$ for the five variables m^2 and x_I. Then one may use some algorithm to cast this system into regular chains. The first element of the chain can be chosen to depend only on m^2 and then to acquire m we need to consider only this equation and forget about the others.[1] We do not present this equation here due to its length but it is a general, fifth order polynomial equation for m^2. Then, due to the Abel-Ruffini theorem, one cannot have an expression for m in terms of radicals of the coefficients.

VII. GENERALIZATION FOR $\mathcal{N} = 8$ SUPERGRAVITY

We have seen that in the STU case the charge matrix is an element of $\mathfrak{so}(4,4)$ and the U-duality group $SL(2)_U^{\times 3} \subset SO(4,4)$ acts on it through the adjoint representation of $SO(4,4)$. This representation decomposes as $28 = 1 \oplus 1 \oplus 1 \oplus 9 \oplus 8 \oplus 8$. There is a general way of constructing invariants on $9 \oplus 8 \oplus 8$ which allowed us to identify the F-invariant. In the $\mathcal{N} = 8$ case the 3d coset

[1] Note that not all of the roots of this equation are solutions to (128).

model is $E_{8(8)}/SO^*(16)$ and the U-duality group is $E_{7(7)}$. We expect that in this case the asymptotic charges of the black hole parametrize a Lie-algebra element $Q \in \mathfrak{e}_8$ and the U-duality group $E_{7(7)} \subset E_{8(8)}$ just acts by the adjoint action of $E_{8(8)}$. This representation decomposes as $248 = 1 \oplus 1 \oplus 1 \oplus 133 \oplus 56 \oplus 56$ and hence the relevant representation space is $133 \oplus 56 \oplus 56$ with 56 replacing three qubit states containing dyonic charges and 133 replacing 9 containing the 70 scalar charges.[2] The moment map from pairs of 56 to 133 can be formulated. The construction goes as follows. There is an $E_{7(7)}$ invariant antisymmetric bilinear form on 56, let us denote this by $\langle .,. \rangle$. We may define an $\mathfrak{e}_7$ element $T_{\Psi_1\Psi_2}$ associated to the pair $\Psi_1, \Psi_2 \in 56$ by demanding

$$\kappa(T, T_{\Psi_1\Psi_2}) = \langle \Psi_1, T\Psi_2 \rangle, \quad \forall\, T \in \mathfrak{e}_7. \tag{129}$$

Here, κ is the Killing form on $\mathfrak{e}_7$. Using $56 \cong \wedge^2\mathbb{C}^8 \oplus \wedge^2(\mathbb{C}^8)^*$ we may parametrize Ψ_a, $a = 1, 2$ with a pair of antisymmetric 8×8 matrixes:

$$\Psi_a = ((x^{(a)})^{ij}, y^{(a)}_{ij}), \quad (x^{(a)})^{ij} = -(x^{(a)})^{ji}, \quad y^{(a)}_{ij} = -y^{(a)}_{ji} \tag{130}$$

and using $\mathfrak{e}_7 \cong \mathfrak{sl}_8 \oplus \wedge^4\mathbb{C}^8$ we can parametrize the generators as

$$T = (\Lambda^i{}_j, \Sigma_{ijkl}), \qquad \Lambda^i{}_i = 0, \tag{131}$$

and Σ_{ijkl} totaly antisymmetric. We refer to the appendix of [20] for the commutation relations, the action of $\mathfrak{e}_7$ on 56 and the Killing form in terms of this parametrization. The invariant bilinear product reads as

$$\langle \Psi_1, \Psi_2 \rangle = (x^{(1)})^{ij} y^{(2)}_{ij} - (x^{(2)})^{ij} y^{(1)}_{ij}. \tag{132}$$

Then, using the definition (129), a short exercise reveals the explicit form of the moment map to be

$$\begin{aligned} T_{\Psi_a\Psi_b} &= ((\Lambda_{(ab)})^i{}_j, (\Sigma_{(ab)})_{ijkl}), \\ (\Lambda_{(ab)})^i{}_j &= -\frac{1}{6}((x^{(a)})^{in} y^{(b)}_{jn} + (x^{(b)})^{in} y^{(a)}_{jn}) \\ &\quad + \frac{1}{48}((x^{(a)})^{nm} y^{(b)}_{nm} + (x^{(b)})^{nm} y^{(a)}_{nm})\delta^i{}_j, \\ (\Sigma_{(ab)})_{ijkl} &= \frac{1}{48}(\epsilon_{ijklmnop}(x^{(a)})^{mn}(x^{(b)})^{op} - y^{(a)}_{[ij} y^{(b)}_{kl]}). \end{aligned} \tag{133}$$

Note that in this formalism we have the Cartan-Cremmer-Julia invariant expressed as

$$\begin{aligned} I_4 &\equiv \frac{1}{2}\langle \Psi, T_{\Psi\Psi}\Psi \rangle \\ &= x^{ij}x^{kl}y_{ik}y_{jl} - \frac{1}{4}(x^{ij}y_{ij})^2 + \frac{1}{96}(\epsilon_{ijklmnop}x^{ij}x^{kl}x^{mn}x^{op} \\ &\quad + \epsilon^{ijklmnop}y_{ij}y_{kl}y_{mn}y_{op}). \end{aligned} \tag{134}$$

The conventions of [2] are such that $I_4 = 4\diamond$ and $\diamond$ is the one that reduces to Δ of (80) for the STU duality frame. The dyonic charges parametrize $\Psi_1, \Psi_2 \in 56$ generalizing ψ_1, ψ_2. The STU charges (58) sit inside this Ψ_1 and Ψ_2 as

$$\begin{aligned} &\left(\begin{array}{cccc} y^{(2)}_{12} & (x^{(2)})^{34} & (x^{(2)})^{56} & y^{(2)}_{78} \\ (x^{(2)})^{78} & y^{(2)}_{56} & y^{(2)}_{34} & (x^{(2)})^{12} \\ \hline y^{(1)}_{12} & (x^{(1)})^{34} & (x^{(1)})^{56} & y^{(1)}_{78} \\ (x^{(1)})^{78} & y^{(1)}_{56} & y^{(1)}_{34} & (x^{(1)})^{12} \end{array}\right) \\ &= \frac{1}{\sqrt{2}}\left(\begin{array}{cccc} \psi_{0000} & \psi_{0001} & \psi_{0010} & \psi_{0011} \\ \psi_{0100} & \psi_{0101} & \psi_{0110} & \psi_{0111} \\ \hline \psi_{1000} & \psi_{1001} & \psi_{1010} & \psi_{1011} \\ \psi_{1100} & \psi_{1101} & \psi_{1110} & \psi_{1111} \end{array}\right) \\ &= \frac{1}{\sqrt{2}}\left(\begin{array}{cccc} -P^4 & -Q_3 & -Q_2 & P^1 \\ -Q_1 & P^2 & P^3 & Q_4 \\ \hline -Q_4 & P^3 & P^2 & Q_1 \\ P^1 & Q_2 & Q_3 & -P^4 \end{array}\right), \end{aligned} \tag{135}$$

with all the remaining $(x^{(a)})^{ij}$ and $y^{(a)}_{ij}$ vanishing. See [30,31] for details. One immediately verifies that the relation to the fermionic inner product of Eq. (67) is simply

$$\langle \Psi_1, \Psi_2 \rangle = (P_{\psi_1}, P_{\psi_2}), \tag{136}$$

and that $\diamond$ reduces to Δ. As the next step, we may parametrize an element $R \in \mathfrak{e}_7$ with 70 scalar charges. Denote the corresponding 56×56 matrix in the adjoint representation with $\mathcal{R}$. In the STU duality frame $\mathcal{R}$ should reduce to $\hat{R}$ of (90) on the eight dimensional subspace of 56, where the pair of (135) lives, and zero everywhere else. Then, the F invariant (92) can be written in a manifestly E_7 invariant way as:

$$\begin{aligned} F &= M^4 + M^2N^2 + \frac{M^2}{4}\langle \Psi_1, \Psi_2 \rangle + \frac{M}{24}\langle \Psi_1, \mathcal{R}\Psi_2 \rangle \\ &\quad + \frac{N}{24}\langle \Psi_1, \mathcal{R}\Psi_1 \rangle + \frac{N^2}{96}\mathrm{Tr}(\mathcal{R}^2) \\ &\quad - \frac{1}{192}\langle \Psi_1, (T_{\Psi_1\Psi_2} - T_{\Psi_2\Psi_2})\Psi_1 \rangle \\ &\quad + \frac{1}{1536}\left(\frac{1}{8}(\mathrm{Tr}\mathcal{R}^2)^2 - \mathrm{Tr}(\mathcal{R}^4)\right). \end{aligned} \tag{137}$$

[2] Recall that in the STU case 9 contained 6 scalar charges, this is just an artifact of fixing the scalar asymptotics.

VIII. CONCLUSIONS

In this work we managed to express the F-invariant of Chow and Compère, and hence the Bekenstein-Hawking entropy of general asymptoticaly flat (or Taub-NUT), nonextremal black holes admitted by the STU model in terms of 16 asymptotic charges. These are not all independent: six scalar charges are functions of the mass, NUT charge, and eight dyonic charges. However, the expression for F with the scalar charges being explicit makes the U-duality invariance manifest and allowed us to conjecture the generalization (137) of the F-invariant to the $E_{7(7)}$ invariant case. We have argued that a formula in terms of only the physical charges requires one to find the real solutions of a system of polynomial equations. We have solved these equations for some known special cases and recovered the expected expressions.

An important question left open is whether Eqs. (112), together with the condition of reality, are enough to determine the scalar charges uniquely or one needs some additional constraints. Also, it would be very interesting to see a microscopic origin of this entropy formula. There are several results along these lines including near-extremal black holes [32–35], neutral, nonextremal ones [36] and the recently constructed dilute gas limit of the general nonextreme STU black holes [16]. Another interesting problem would be to find the uplift of the formula for 5 dimensional finite temperature black holes and black rings written as a qubic invariant of $E_{6(6)}$ [37]. A way of finding this formula would probably be to recast our expression for the F-invariant in the language of Freudenthal triple systems [38] and write it in terms of elements of the corresponding cubic Jordan algebra [39].

Finally, as adding a possible new twist to the black hole/qubit correspondence, we note that the difference of the inner and outer horizon radius is [2]

$$
\begin{aligned}
r_+ - r_- &= 2\sqrt{m^2+n^2-a^2} \\
&= \frac{1}{2}\sqrt{\mathrm{Tr}Q^2\left(1-\frac{J^2}{F}\right)}, \qquad (138)
\end{aligned}
$$

which measures extremality, and is U-duality invariant as expected. It is known that the quantity $\mathrm{Tr}Q^2$ can be reinterpreted though (52) and (A15) as the quadratic four qubit entanglement measure [13]. Then, so called nilpotent states with $\mathrm{Tr}Q^2 = 0$ always correspond to extremal black holes and they can be classified in the language of four qubit entanglement [14]. However, we see that there are extremal black holes corresponding to semisimple charge vectors: these are the extremal, fast rotating black holes [2] with $F = J^2$. It would be interesting to see if these black holes relate to the entanglement properties of semisimple four qubit states.

ACKNOWLEDGMENTS

I am grateful for Alejandra Castro for bringing my attention to the problem. I would like to thank Alejandra Castro, Gary Horowitz, Péter Lévay, and Joe Polchinski for reading the manuscript and providing valuable feedback. I am grateful for Geoffrey Compére, Victor Lekeu, and Mario Trigiante for useful correspondence. I would like to thank the Gordon and Betty Moore Foundation for financial support. This research was supported in part by the National Science Foundation under Grant No. NSF PHY11-25915.

APPENDIX A: DIFFERENT WAYS OF SPLITTING $\mathfrak{so}(4,4)$ AS $\mathfrak{sl}_2^{\times 4} \oplus (2,2,2,2)$

We define the group $SO(4,4)$ as the set of 8×8 matrices O keeping the bilinear form

$$
G = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad (A1)
$$

i.e., $OGO^T = G$. Here, I is the 4×4 identity matrix. The Lie algebra $\mathfrak{so}(4,4)$ is spanned by matrices T satisfying $TG + GT^T = 0$. We use the same parametrization of these matrices as in [2]. We denote by E_{ij} the 8×8 matrix with 1 in the (i,j) component and zeros everywhere else. The Cartan generators are given by

$$
\begin{aligned}
H_0 &= E_{33}+E_{44}-E_{77}-E_{88}, \\
H_1 &= E_{33}-E_{44}-E_{77}+E_{88}, \\
H_2 &= E_{11}+E_{22}-E_{55}-E_{66}, \\
H_3 &= E_{11}-E_{22}-E_{55}+E_{66}, \qquad (A2)
\end{aligned}
$$

while the roots are parametrized as

$$
\begin{aligned}
E_0 &= E_{47}-E_{38}, & E_1 &= E_{87}-E_{34}, \\
E_2 &= E_{25}-E_{16}, & E_3 &= E_{65}-E_{12}, \\
E^{Q_1} &= E_{45}-E_{18}, & E^{Q_2} &= E_{32}-E_{67}, \\
E^{Q_3} &= E_{36}-E_{27}, & E^{Q_4} &= E_{41}-E_{58}, \\
E^{P^1} &= E_{57}-E_{31}, & E^{P^2} &= E_{46}-E_{28}, \\
E^{P^3} &= E_{42}-E_{68}, & E^{P^4} &= E_{17}-E_{35}, \\
F_0 &= E_{74}-E_{83}, & F_1 &= E_{78}-E_{43}, \\
F_2 &= E_{52}-E_{61}, & F_3 &= E_{56}-E_{21}, \\
F^{Q_1} &= E_{54}-E_{81}, & F^{Q_2} &= E_{23}-E_{76}, \\
F^{Q_3} &= E_{63}-E_{72}, & F^{Q_4} &= E_{14}-E_{85}, \\
F^{P^1} &= E_{75}-E_{13}, & F^{P^2} &= E_{64}-E_{82}, \\
F^{P^3} &= E_{24}-E_{86}, & F^{P^4} &= E_{71}-E_{53}. \qquad (A3)
\end{aligned}
$$

1. U-duality split

It is easy to see that the generators H_Λ, E_Λ, F_Λ, $\Lambda = 0, 1, 2, 3$ form four commuting $\mathfrak{sl}_2$ algebras:

$$[H_\Lambda, E_\Lambda] = 2E_\Lambda, \qquad [H_\Lambda, F_\Lambda] = -2F_\Lambda, \quad [E_\Lambda, F_\Lambda] = H_\Lambda. \tag{A4}$$

The remaining 16 generators E^{P^I}, E^{Q_I}, F^{P^I}, F^{Q_I} form the fundamental $(2,2,2,2)_U$ representation of this $(\mathfrak{sl}_2^{\times 4})_U$ algebra under the adjoint action.

A vector of this representation can nicely be described by the 16 amplitudes ψ_{ijkl} of a four qubit state

$$|\psi\rangle = \sum_{i,j,k,l=0}^{1} \psi_{ijkl}|ijkl\rangle, \tag{A5}$$

transforming as

$$\psi_{ijkl} \mapsto (S_0)_i{}^{i'}(S_1)_j{}^{j'}(S_2)_k{}^{k'}(S_3)_l{}^{l'}\psi_{i'j'k'l'}, \quad S_0 \otimes S_1 \otimes S_2 \otimes S_3 \in SL(2)^{\times 4}, \tag{A6}$$

under the action of the group $SL(2)^{\times 4}$ generated by the algebra (A4). In terms of the Lie algebra generators one writes this vector as

$$\begin{aligned}\Psi = {}& \psi_{0000}E^{P^4} + \psi_{0001}E^{Q_3} + \psi_{0010}E^{Q_2} - \psi_{0011}E^{P^1} \\ & + \psi_{0100}E^{Q_1} - \psi_{0101}E^{P^2} - \psi_{0110}E^{P^3} - \psi_{0111}E^{Q_4} \\ & - \psi_{1000}F^{Q_1} + \psi_{1001}F^{P^3} + \psi_{1010}F^{P^2} + \psi_{1011}F^{Q_1} \\ & + \psi_{1100}F^{P^1} + \psi_{1101}F^{Q_2} + \psi_{1110}F^{Q_3} - \psi_{1111}F^{P_4},\end{aligned} \tag{A7}$$

transforming as (A6) under $\Psi \mapsto g\Psi g^{-1}$.

2. Sigma model split

We can realize the split $\mathfrak{sl}_2^{\times 4} \oplus (2,2,2,2)$ in a different way suited to writing the timelike reduced STU action (7) as a sigma model on $SO(4,4)/SL(2)^{\times 4}$. We may introduce the symmetric bilinear form

$$\eta = \mathrm{diag}(-1,-1,1,1,-1,-1,1,1), \tag{A8}$$

and look for the subgroup $SO(2,2) \times SO(2,2) \cong SL(2)^{\times 4}$ keeping this fixed. This subgroup is generated by the -1 eigenspace of the involution

$$T^\# = \eta T^T \eta. \tag{A9}$$

This eigenspace is spanned by the 12 generators

$$k_\Lambda = E_\Lambda - F_\Lambda, \quad k^{Q_I} = E^{Q_I} + F^{Q_I}, \quad k^{P^I} = E^{P^I} + F^{P^I}, \tag{A10}$$

while the $+1$ eigenspace is spanned by

$$\begin{aligned} & H_\Lambda, \qquad p_\Lambda = E_\Lambda + F_\Lambda, \\ & p^{Q_I} = E^{Q_I} - F^{Q_I}, \qquad p^{P^I} = E^{P^I} - F^{P^I}. \end{aligned} \tag{A11}$$

To see that this indeed realizes a $\mathfrak{sl}_2^{\times 4} \oplus (2,2,2,2)$ split define [40,41]

$$\begin{aligned}
\tilde{H}_1 &= 1/2(-k^{Q_4} - k^{Q_1} - k^{Q_2} - k^{Q_3}), \\
\tilde{H}_2 &= 1/2(k^{Q_4} + k^{Q_1} - k^{Q_2} - k^{Q_3}), \\
\tilde{H}_3 &= 1/2(k^{Q_4} - k^{Q_1} + k^{Q_2} - k^{Q_3}), \\
\tilde{H}_4 &= 1/2(k^{Q_4} - k^{Q_1} - k^{Q_2} + k^{Q_3}), \\
\tilde{E}_1 &= 1/4(-k_0 + k_1 + k_2 + k_3 + k^{P^1} + k^{P^2} + k^{P^3} + k^{P^4}), \\
\tilde{E}_2 &= 1/4(k_0 - k_1 + k_2 + k_3 + k^{P^1} - k^{P^2} - k^{P^3} + k^{P^4}), \\
\tilde{E}_3 &= 1/4(k_0 + k_1 - k_2 + k_3 - k^{P^1} + k^{P^2} - k^{P^3} + k^{P^4}), \\
\tilde{E}_4 &= 1/4(k_0 + k_1 + k_2 - k_3 - k^{P^1} - k^{P^2} + k^{P^3} + k^{P^4}), \\
\tilde{F}_1 &= 1/4(k_0 - k_1 - k_2 - k_3 + k^{P^1} + k^{P^2} + k^{P^3} + k^{P^4}), \\
\tilde{F}_2 &= 1/4(-k_0 + k_1 - k_2 - k_3 + k^{P^1} - k^{P^2} - k^{P^3} + k^{P^4}), \\
\tilde{F}_3 &= 1/4(-k_0 - k_1 + k_2 - k_3 - k^{P^1} + k^{P^2} - k^{P^3} + k^{P^4}), \\
\tilde{F}_4 &= 1/4(-k_0 - k_1 - k_2 + k_3 - k^{P^1} - k^{P^2} + k^{P^3} + k^{P^4}).
\end{aligned} \tag{A12}$$

One easily verifies that

$$[\tilde{H}_J, \tilde{E}_J] = 2\tilde{E}_J, \qquad [\tilde{H}_J, \tilde{F}_J] = -2\tilde{F}_J, \qquad [\tilde{E}_J, \tilde{F}_J] = \tilde{H}_J, \tag{A13}$$

with $J = 1, \ldots, 4$ and all other commutators vanishing. We can write an element of the $+1$ eigenspace of $^\#$ in terms of four qubit amplitudes ψ_{ijkl} transforming as in (A6) under this new $SL(2)^{\times 4}$. It reads explicitly as

$$\tilde{\Psi} = \tilde{\Psi}_{H_\Lambda}H_\Lambda + \tilde{\Psi}_{p_\Lambda}p_\Lambda + \tilde{\Psi}_{Q_I}p^{Q_I} + \tilde{\Psi}_{P^I}p^{P^I}, \tag{A14}$$

where

$$\begin{aligned}
\tilde{\Psi}_{H_0} &= \psi_{0001} + \psi_{0010} + \psi_{0100} - \psi_{0111} - \psi_{1000} + \psi_{1011} + \psi_{1101} + \psi_{1110},\\
\tilde{\Psi}_{H_1} &= \psi_{0001} + \psi_{0010} - \psi_{0100} + \psi_{0111} + \psi_{1000} - \psi_{1011} + \psi_{1101} + \psi_{1110},\\
\tilde{\Psi}_{H_2} &= \psi_{0001} - \psi_{0010} + \psi_{0100} + \psi_{0111} + \psi_{1000} + \psi_{1011} - \psi_{1101} + \psi_{1110},\\
\tilde{\Psi}_{H_3} &= -\psi_{0001} + \psi_{0010} + \psi_{0100} + \psi_{0111} + \psi_{1000} + \psi_{1011} + \psi_{1101} - \psi_{1110},\\
\tilde{\Psi}_{p_0} &= -\psi_{0000} + \psi_{0011} + \psi_{0101} + \psi_{0110} - \psi_{1001} - \psi_{1010} - \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{p_1} &= -\psi_{0000} + \psi_{0011} - \psi_{0101} - \psi_{0110} + \psi_{1001} + \psi_{1010} - \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{p_2} &= -\psi_{0000} - \psi_{0011} + \psi_{0101} - \psi_{0110} + \psi_{1001} - \psi_{1010} + \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{p_3} &= -\psi_{0000} - \psi_{0011} - \psi_{0101} + \psi_{0110} - \psi_{1001} + \psi_{1010} + \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{Q_1} &= 2(\psi_{0100} - \psi_{1011}),\\
\tilde{\Psi}_{Q_2} &= 2(\psi_{0010} - \psi_{1101}),\\
\tilde{\Psi}_{Q_3} &= 2(\psi_{0001} - \psi_{1110}),\\
\tilde{\Psi}_{Q_4} &= 2(\psi_{1000} - \psi_{0111}),\\
\tilde{\Psi}_{P^1} &= \psi_{0000} + \psi_{0011} - \psi_{0101} - \psi_{0110} - \psi_{1001} - \psi_{1010} + \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{P^2} &= \psi_{0000} - \psi_{0011} + \psi_{0101} - \psi_{0110} - \psi_{1001} + \psi_{1010} - \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{P^3} &= \psi_{0000} - \psi_{0011} - \psi_{0101} + \psi_{0110} + \psi_{1001} - \psi_{1010} - \psi_{1100} + \psi_{1111},\\
\tilde{\Psi}_{P^4} &= -\psi_{0000} - \psi_{0011} - \psi_{0101} - \psi_{0110} - \psi_{1001} - \psi_{1010} - \psi_{1100} - \psi_{1111}.
\end{aligned} \tag{A15}$$

Note that the $SL(2)^{\times 4}$ invariant $\mathrm{Tr}\tilde{\Psi}^2$ is a quadratic measure of four qubit entanglement [8].

3. The third split

The previous two splits are related by triality of $\mathfrak{so}(4,4)$ and hence there must be one more inequivalent splitting of the algebra. Indeed this split is given by the ± 1 eigenspaces of the involution

$$T \mapsto \eta_2 T^T \eta_2, \tag{A16}$$

where

$$\eta_2 = \begin{pmatrix} \epsilon \otimes \epsilon & 0 \\ 0 & \epsilon \otimes \epsilon \end{pmatrix}, \tag{A17}$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The -1 eigenspace is 12 dimensional and spans $\mathfrak{sl}_2^{\times 4}$ algebra. The $+1$ eigenspace forms the representation (2,2,2,2) under this. We do not need this split in the following hence we omit the explicit form of the generators.

APPENDIX B: EXPLICIT COVARIANTS FOR STU BLACK HOLES

Here we list the explicit forms of the covariants needed to calculate the F-invariant when the scalar asymptotics are set such that $X_i = 0$ and $Y_i = 1$.

$$R = \begin{pmatrix} \Sigma_1 & -\Xi_1 & 0 & 0 & 0 & 0 \\ -\Xi_1 & -\Sigma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_2 & -\Xi_2 & 0 & 0 \\ 0 & 0 & -\Xi_2 & -\Sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Sigma_3 & -\Xi_3 \\ 0 & 0 & 0 & 0 & -\Xi_3 & -\Sigma_3 \end{pmatrix}, \tag{B1}$$

$$K_{12} = \begin{pmatrix} K_{12}^{(1)} & 0 & 0 \\ 0 & K_{12}^{(2)} & 0 \\ 0 & 0 & K_{12}^{(3)} \end{pmatrix}, \tag{B2}$$

where

$$K_{12}^{(1)} = \begin{pmatrix} -Q_2^2 - Q_3^2 - (P^1)^2 - (P^4)^2 & -Q_4P^1 + Q_3P^2 + Q_2P^3 - Q_1P^4 \\ -Q_4P^1 + Q_3P^2 + Q_2P^3 - Q_1P^4 & -Q_1^2 - Q_4^2 - (P^2)^2 - (P^3)^2 \end{pmatrix},$$
$$K_{12}^{(2)} = \begin{pmatrix} -Q_1^2 - Q_3^2 - (P^2)^2 - (P^4)^2 & Q_3P^1 - Q_4P^2 + Q_1P^3 - Q_2P^4 \\ Q_3P^1 - Q_4P^2 + Q_1P^3 - Q_2P^4 & -Q_2^2 - Q_4^2 - (P^1)^2 - (P^3)^2 \end{pmatrix},$$
$$K_{12}^{(3)} = \begin{pmatrix} -Q_1^2 - Q_2^2 - (P^3)^2 - (P^4)^2 & Q_2P^1 + Q_1P^2 - Q_4P^3 - Q_3P^4 \\ Q_2P^1 + Q_1P^2 - Q_4P^3 - Q_3P^4 & -Q_3^2 - Q_4^2 - (P^1)^2 - (P^2)^2 \end{pmatrix} \tag{B3}$$

$$K_{11} = \begin{pmatrix} K_{11}^{(1)} & 0 & 0 \\ 0 & K_{11}^{(2)} & 0 \\ 0 & 0 & K_{11}^{(3)} \end{pmatrix}, \tag{B4}$$

where

$$K_{11}^{(1)} = \begin{pmatrix} Q_1P^1 - Q_2P^2 - Q_3P^3 + Q_4P^4 & 2(Q_1Q_4 + P^2P^3) \\ -2(Q_2Q_3 + P^1P^4) & -Q_1P^1 + Q_2P^2 + Q_3P^3 - Q_4P^4 \end{pmatrix},$$
$$K_{11}^{(2)} = \begin{pmatrix} -Q_1P^1 + Q_2P^2 - Q_3P^3 + Q_4P^4 & 2(Q_2Q_4 + P^1P^3) \\ -2(Q_1Q_3 + P^2P^4) & Q_1P^1 - Q_2P^2 + Q_3P^3 - Q_4P^4 \end{pmatrix},$$
$$K_{11}^{(3)} = \begin{pmatrix} -Q_1P^1 - Q_2P^2 + Q_3P^3 + Q_4P^4 & 2(Q_3Q_4 + P^1P^2) \\ -2(Q_1Q_2 + P^3P^4) & Q_1P^1 + Q_2P^2 - Q_3P^3 - Q_4P^4 \end{pmatrix}, \tag{B5}$$

finally

$$K_{22} = \begin{pmatrix} K_{22}^{(1)} & 0 & 0 \\ 0 & K_{22}^{(2)} & 0 \\ 0 & 0 & K_{22}^{(3)} \end{pmatrix}, \tag{B6}$$

where

$$K_{22}^{(1)} = \begin{pmatrix} -Q_1P^1 + Q_2P^2 + Q_3P^3 - Q_4P^4 & 2(Q_2Q_3 + P^1P^4) \\ -2(Q_1Q_4 + P^2P^3) & Q_1P^1 - Q_2P^2 - Q_3P^3 + Q_4P^4 \end{pmatrix},$$
$$K_{22}^{(2)} = \begin{pmatrix} Q_1P^1 - Q_2P^2 + Q_3P^3 - Q_4P^4 & 2(Q_1Q_3 + P^2P^4) \\ -2(Q_2Q_4 + P^1P^3) & -Q_1P^1 + Q_2P^2 - Q_3P^3 + Q_4P^4 \end{pmatrix},$$
$$K_{22}^{(3)} = \begin{pmatrix} Q_1P^1 + Q_2P^2 - Q_3P^3 - Q_4P^4 & 2(Q_1Q_2 + P^3P^4) \\ -2(Q_3Q_4 + P^1P^2) & -Q_1P^1 - Q_2P^2 + Q_3P^3 + Q_4P^4 \end{pmatrix}. \tag{B7}$$

[1] D. D. Chow and G. Compère, Seed for general rotating non-extremal black holes of $\mathcal{N} = 8$ supergravity, Classical Quantum Gravity 31, 022001 (2014).

[2] D. D. Chow and G. Compère, Black holes in $n = 8$ supergravity from so (4, 4) hidden symmetries, Phys. Rev. D 90, 025029 (2014).

[3] R. Kallosh, N. Sivanandam, and M. Soroush, The non-bps black hole attractor equation, J. High Energy Phys. 03 (2006) 060.

[4] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, Non-supersymmetric attractors, Phys. Rev. D 72, 124021 (2005).

[5] M. Sato, T. Kimura *et al.*, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. **65**, 1 (1977).
[6] L. Andrianopoli, A. Gallerati, and M. Trigiante, On extremal limits and duality orbits of stationary black holes, J. High Energy Phys. 01 (2014) 053.
[7] W. Dür, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A **62**, 062314 (2000).
[8] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Four qubits can be entangled in nine different ways, Phys. Rev. A **65**, 052112 (2002).
[9] P. Lévay and P. Vrana, Three fermions with six single-particle states can be entangled in two inequivalent ways, Phys. Rev. A **78**, 022329 (2008).
[10] M. J. Duff, String triality, black hole entropy, and cayley's hyperdeterminant, Phys. Rev. D **76**, 025017 (2007).
[11] V. Coffman, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A **61**, 052306 (2000).
[12] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante, and T. Van Riet, Generating geodesic flows and supergravity solutions, Nucl. Phys. **B812**, 343 (2009).
[13] P. Lévay, S t u black holes as four-qubit systems, Phys. Rev. D **82**, 026003 (2010).
[14] L. Borsten, D. Dahanayake, M. J. Duff, A. Marrani, and W. Rubens, Four-qubit entanglement classification from string theory, Phys. Rev. Lett. **105**, 100507 (2010).
[15] L. Borsten, M. Duff, and P. Levay, The black-hole/qubit correspondence: An up-to-date review, Classical Quantum Gravity **29**, 224008 (2012).
[16] M. Cvetič and F. Larsen, Black holes with intrinsic spin, J. High Energy Phys. 11 (2014) 033.
[17] E. Cremmer, C. Kounnas, A. Van Proeyen, J. Derendinger, S. Ferrara, B. De Wit, and L. Girardello, Vector multiplets coupled to $n = 2$ supergravity: Super-Higgs effect, flat potentials and geometric structure, Nucl. Phys. **B250**, 385 (1985).
[18] M. J. Duff, J. T. Liu, and J. Rahmfeld, Four-dimensional string/string/string triality, Nucl. Phys. **B459**, 125 (1996).
[19] M. Cvetič and C. M. Hull, Black holes and u-duality, Nucl. Phys. **B480**, 296 (1996).
[20] E. Cremmer and B. Julia, The so (8) supergravity, Nucl. Phys. **B159**, 141 (1979).
[21] E. Cremmer and B. Julia, The $n = 8$ supergravity theory. i. the Lagrangian, Phys. Lett. **80B**, 48 (1978).
[22] P. Breitenlohner, D. Maison, and G. Gibbons, 4-dimensional black holes from Kaluza-Klein theories, Commun. Math. Phys. **120**, 295 (1988).
[23] L. Carbone, S. Murray, and H. Sati, Integral group actions on symmetric spaces and discrete duality symmetries of supergravity theories, J. Math. Phys. (N.Y.) **56**, 103501 (2015).
[24] G. Sárosi and P. Lévay, Entanglement in fermionic Fock space, J. Phys. A **47**, 115304 (2014).
[25] G. Sárosi and P. Lévay, Entanglement classification of three fermions with up to nine single-particle states, Phys. Rev. A **89**, 042310 (2014).
[26] G. Sárosi and P. Lévay, Coffman-Kundu-Wootters inequality for fermions, Phys. Rev. A **90**, 052303 (2014).
[27] D. Rasheed, The rotating dyonic black holes of Kaluza-Klein theory, Nucl. Phys. **B454**, 379 (1995).
[28] E. Lozano-Tellechea and T. Ortın, The general, duality-invariant family of non-bps black-hole solutions of $n = 4$, $d = 4$ supergravity, Nucl. Phys. **B569**, 435 (2000).
[29] M. Cvetič and D. Youm, Entropy of nonextreme charged rotating black holes in string theory, Phys. Rev. D **54**, 2612 (1996).
[30] R. Kallosh and A. Linde, Strings, black holes, and quantum information, Phys. Rev. D **73**, 104033 (2006).
[31] M. J. Duff and S. Ferrara, E 7 and the tripartite entanglement of seven qubits, Phys. Rev. D **76**, 025018 (2007).
[32] J. Maldacena, A. Strominger, and E. Witten, Black hole entropy in m-theory, J. High Energy Phys. 12 (1997) 002.
[33] G. T. Horowitz, D. A. Lowe, and J. M. Maldacena, Statistical Entropy of Nonextremal Four-Dimensional Black Holes and u Duality, Phys. Rev. Lett. **77**, 430 (1996).
[34] R. Emparan and G. T. Horowitz, Microstates of a Neutral Black Hole in m Theory, Phys. Rev. Lett. **97**, 141601 (2006).
[35] G. T. Horowitz and M. M. Roberts, Counting the Microstates of a Kerr Black Hole in m Theory, Phys. Rev. Lett. **99**, 221601 (2007).
[36] A. Castro, A. Maloney, and A. Strominger, Hidden conformal symmetry of the Kerr black hole, Phys. Rev. D **82**, 024008 (2010).
[37] D. Gaiotto, A. Strominger, and X. Yin, New connections between 4d and 5d black holes, J. High Energy Phys. 02 (2006) 024.
[38] S. Krutelevich, Jordan algebras, exceptional groups, and Bhargava composition, Journal of algebra **314**, 924 (2007).
[39] L. Borsten, D. Dahanayake, M. Duff, and W. Rubens, Black holes admitting a Freudenthal dual, Phys. Rev. D **80**, 026003 (2009).
[40] D. Katsimpouri, A. Kleinschmidt, and A. Virmani, An inverse scattering formalism for STU supergravity, J. High Energy Phys. 03 (2014) 101.
[41] S. Banerjee, B. D. Chowdhury, B. Vercnocke, and A. Virmani, Non-supersymmetric microstates of the MSW system, J. High Energy Phys. 05 (2014) 011.