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Lensing in the McVittie metric

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We investigate the effect of the cosmological expansion on the bending of light due to an isolated pointlike mass. We adopt the McVittie metric as the model for the geometry of the lens. Assuming a constant Hubble factor, we find an analytic expression involving the bending angle, which turns out to be increased by a contribution $1 + z_L$, where z_L is the redshift of the lens. Employing the lens equation in the thin lens approximation, we find that for Einstein ring systems, the lens mass estimation gains a correction $(1 + z_S)/(1 + z_L)$ with respect to the one based on the usual formula.

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I. INTRODUCTION

The McVittie metric [1] is a spherically symmetric solution of Einstein's equations which asymptotically tends to a Friedmann-Lemaître-Robertson-Walker (FRLW) universe. It was introduced in 1933 by McVittie in order to investigate cosmological effects on local systems, e.g., on closed orbits of planets or stars. This issue has been examined again in the past years [2–7], especially in relation to the Pioneer anomaly [8]. It is still a matter of debate whether and how much cosmological effects influence the physics of local systems.

The McVittie metric has been intensively analyzed by many authors, see, e.g., Refs. [9–15]. In particular, Nolan analyzed the mathematical properties of the McVittie solution in a series of three papers [9–11]. One of the most important results is that the McVittie metric is not a black hole solution because where one expects a horizon, there is instead a weak singularity (i.e., geodesics can be extended through it). There is an exception to this theorem: when the external, FLRW part of the McVittie solution tends to be cosmological constant dominated [12,13].

Taking advantage of the well-posedness of the McVittie metric with flat spatial hypersurfaces [9] (see also Ref. [14]), we use this solution in order to understand how the lensing phenomenon generated by a point mass is affected by the embedding of the latter in an expanding universe. This idea has been recently explored in Ref. [16], where the authors numerically show that an effect due to the Hubble constant H_0 does exist on the deflection angle.

There is ample literature on this problem, mostly specialized to the case in which a cosmological constant dominates and, for this reason, based on the Kottler (Schwarzschild-de Sitter) metric [17].

In his pioneering investigation, Islam [18] found no influence whatsoever by Λ on the bending of light. Only less than a decade ago, Ishak and Rindler [19,20], via a new definition of the bending angle, showed that an effect due to Λ indeed exists. Their work and results gave rise to many others investigations, see, e.g., Refs. [21–29]. There seems to be common agreement now that Λ indeed affects the bending of light. A debate actually exists on the entity of this influence. Among the works cited above, Refs. [23,25,26] disagree with the existence of any relevant effect caused by Λ on the lensing phenomenon. The reason is essentially the following: putting the source, the lens, and an observer in a cosmological setting, i.e., taking into account the Hubble flux, makes the Λ contribution completely negligible (but nonzero, in principle) because of how it enters the definition of the angular diameter distances and because of aberration effects due to the relative motion.

In order to better understand these points, we present here a perturbative, analytic calculation of the bending angle in the McVittie metric. We assume a constant Hubble parameter, thereby focusing on the case of a cosmological constant-dominated universe. By using the McVittie metric in the coordinates of Eq. (1), we take into account the embedding of the source, the lens, and an observer in a cosmological context, thereby potentially addressing the issues raised in Refs. [23,25,26]. We find a contribution to the bending angle proportional to $1 + z_L$, where z_L is the redshift of the lens. Afterwards, investigating the lens equation in the thin lens approximation and in the case of Einstein ring systems, we find a correction to the lens mass estimate of a factor $(1 + z_S)/(1 + z_L)$, with respect to the standard formula.

The paper is structured as follows. In Sec. II, we present the McVittie metric and tackle the lensing problem, calculating the bending angle and a new correction on it, proportional to the redshift of the lens. In Sec. III, we focus on the case of Einstein's ring systems and obtain a formula for the mass of the lens, which includes the correction in the bending angle coming from cosmology. We also determine some lenses masses, using data from the literature. Finally, Sec. IV is devoted to a discussion and the conclusion. We use natural $G = c = 1$ units throughout the paper.

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II. THE MCVITTIE METRIC AND LENSING

The McVittie metric [1] has the following form:

$$ds^2 = -\left(\frac{1-\mu}{1+\mu}\right)^2 dt^2 + (1+\mu)^4 a(t)^2 (d\rho^2 + \rho^2 d\Omega^2), \quad (1)$$

where $a(t)$ is the scale factor and

$$\mu \equiv \frac{M}{2a(t)\rho}, \quad (2)$$

where M is the mass of the pointlike lens. When $\mu \ll 1$, the metric (1) can be approximated by

$$ds^2 = -(1-4\mu)dt^2 + (1+4\mu)a(t)^2(d\rho^2 + \rho^2 d\Omega^2), \quad (3)$$

which is the usual perturbed FLRW metric in the Newtonian gauge and 2μ is what is usually called the gravitational potential.

We adopt the same formalism Dodelson uses in Chap. 10 of his textbook [30]. We use as a time variable the background comoving distance χ (as if it there was no pointlike mass) from us to the plane where the photon is at a certain time t. See Fig. 1.

The relation between χ and the background expansion is the usual one for the FLRW metric:

$$\frac{d\chi}{dt} = -\frac{1}{a}, \quad (4)$$

and the comoving distances of the source and of the lens, χ_S and χ_L, respectively, do not change.

The null geodesics equation for the transversal displacement l^i of the photon is the following:

$$\frac{a}{p}\frac{1-\mu}{1+\mu}\frac{d}{d\chi}\left(\frac{p}{a}\frac{1+\mu}{1-\mu}\frac{dl^i}{d\chi}\right) = \frac{2(1-\mu)}{(1+\mu)^7}\delta^{il}\partial_l\mu + 2Ha\left[1 + \frac{2\partial_t\mu}{(1+\mu)H}\right]\frac{dl^i}{d\chi} - \frac{2}{1+\mu}(\delta^i_j\partial_k\mu + \delta^i_k\partial_j\mu - \delta_{jk}\delta^{il}\partial_l\mu)\frac{dl^j}{d\chi}\frac{dl^k}{d\chi}, \quad (5)$$

where $i = 1, 2$; $H \equiv \partial_t a/a$ is the Hubble factor, and p is the photon proper momentum. Since the McVittie metric is spherically symmetric, the two equations for $i = 1, 2$ are identical. The equation describing the evolution of the proper momentum p is the following:

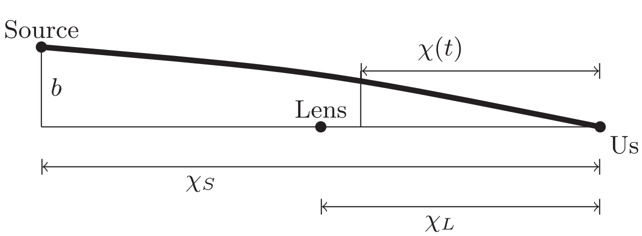

FIG. 1. Scheme of lensing.

$$\frac{1}{p}\frac{dp}{dt} = -H - \frac{2}{1+\mu}\partial_t\mu + 2\frac{P^i\partial_i\mu}{p(1+\mu)^2}. \quad (6)$$

The H term gives the cosmological contribution to the redshift of the photon, whereas the other two terms provide a gravitational redshift contribution, due to the gravitational field of the pointlike mass.

The above Eqs. (5) and (6) reproduce the known results of the cosmological perturbation theory for shear-free scalar perturbations. To show this, one has to consider the limit $\mu \ll 1$ and, defining $\Psi = -2\mu$, one can obtain the corresponding formulas of Ref. [30].

Now we do consider μ small and investigate how a point mass in the expanding Universe affects the trajectory of a light ray. In order to do this, in Eq. (5), we consider $\mu \ll 1$ and small displacements $l^i \ll \chi$.

Equation (5) can be then simplified as follows:

$$\frac{d^2 l^i}{d\chi^2} = 4\partial_i\mu. \quad (7)$$

Using Eq. (2) in the equation above, one gets:

$$\frac{d^2 l}{d\chi^2} = -\frac{2Ml}{a(\chi)[(\chi-\chi_L)^2 + l^2]^{3/2}}, \quad (8)$$

where we dropped the index i denoting the transversal direction, thanks to a spherical symmetry. With the following definitions:

$$x \equiv \frac{\chi}{\chi_L}, \qquad \alpha \equiv \frac{2M}{\chi_L}, \qquad y \equiv \frac{l}{\chi_L}, \quad (9)$$

Eq. (8) becomes:

$$\frac{d^2 y}{dx^2} = -\alpha\frac{y}{a(x)[(x-1)^2 + y^2]^{3/2}}. \quad (10)$$

Note that a vanishing α implies that $a(x)$ has no effect on the trajectory. This is a sort of "casting out nines", since indeed we do not expect lensing caused by cosmology only.

We suppose the source to be at a comoving distance χ_S and solve the above equation considering a small α, via the following expansion of the solution:

$$y = y^{(0)} + \alpha y^{(1)} + \alpha^2 y^{(2)} + \cdots, \quad (11)$$

and retaining the first order only in α. As initial conditions, we choose:

$$y(x_S) = y_S, \qquad y(0) = 0, \quad (12)$$

which mean that the light ray starts from the source with an impact parameter $b \equiv y_S \chi_L$, see Fig. 1, and it must arrive to us in order to be detected; thus, $y(0) = 0$.

The zero-order solution is trivial, i.e., a straight line:

$$y^{(0)} = C_1 x + C_2. \tag{13}$$

We choose the two integration constants so that $y^{(0)} = y_S$, i.e., the trajectory is a straight, horizontal line, see Fig. 2.

The first-order solution is then given by the following equation:

$$\frac{d^2 y^{(1)}}{dx^2} = -\frac{y_S}{a(x)[(x-1)^2 + y_S^2]^{3/2}}, \tag{14}$$

for which we must choose the following initial conditions:

$$y^{(1)}(x_S) = 0, \qquad y^{(1)}(0) = -y_S/\alpha, \tag{15}$$

in order to respect those in Eq. (12) for the full solution.

Let's now consider the contribution of $a(x)$. For a constant Hubble factor $H = H_0$, one can easily determine the scale factor as a function of the comoving distance:

$$\chi = \int_0^z \frac{dz'}{H(z')} = \frac{z}{H_0} \equiv \frac{1}{H_0}\left(\frac{1}{a} - 1\right), \tag{16}$$

where we solved the integral by introducing the redshift z, defined as in the last equality of the above equation. This is, in principle, incorrect because one should also take into account the gravitational redshift caused by the point mass, according to Eq. (6). However, that would produce a second order contribution in α in Eq. (10), so we neglect it.

Using Eq. (16), Eq. (14) becomes:

$$\frac{d^2 y^{(1)}}{dx^2} = -\frac{y_S(1 + H_0 \chi_L x)}{[(x-1)^2 + y_S^2]^{3/2}}. \tag{17}$$

In the limit $y_S \ll 1$, the deviation angle

$$\delta \equiv \left.\frac{dy}{dx}\right|_{x=0} - \left.\frac{dy}{dx}\right|_{x=x_S}, \tag{18}$$

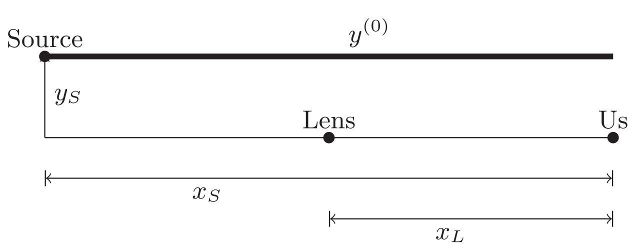

FIG. 2. Zero-order trajectory.

derived by using the solution of Eq. (17) is the following:

$$\delta = \frac{2\alpha(1 + \chi_L H_0)}{y_S} + \mathcal{O}(y_S). \tag{19}$$

Note that Eq. (17) can be solved exactly, but its solution is quite cumbersome so we do not write it down here explicitly.

Recalling that $\alpha \equiv 2M/\chi_L$ and $y_S = b/\chi_L$, we can write the above formula as:

$$\delta = \frac{4M(1 + \chi_L H_0)}{b} + \mathcal{O}(b/\chi_L). \tag{20}$$

This result is similar to the one in the Schwarzschild case, except for the fact that the mass seems to be increased by a relative amount of $H_0 \chi_L$, and b is not the proper closest approach distance to the lens, but it is the comoving transversal position of the source.

From Eq. (16), we know that $H_0 \chi_L = z_L$, the redshift of the lens. Therefore, the farther the lens is, the larger is the effect of the cosmology on the deflection angle, as one intuitively would expect. Considering the standard ΛCDM model Friedmann equation

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \Omega_{\rm m}(1+z)^3, \tag{21}$$

H is approximately constant only as long as $\Omega_\Lambda \gg \Omega_{\rm m}(1+z)^3$. Using the observed values for the density parameters, approximately $\Omega_\Lambda = 0.7$ and $\Omega_{\rm m} = 0.3$, the above condition amounts to a state that $z \ll 0.3$. Therefore, the correction we found in Eq. (20) for the bending angle is reliable only when the redshifts involved are very small, much less than 0.3.

Let us consider now the lens equation in the thin lens approximation:

$$\theta \approx \beta + \delta \frac{D_{LS}}{D_S}, \tag{22}$$

where θ is the angular apparent position of the source, β is the actual angular position of the source, $D_{\rm LS}$ is the angular distance between the lens and the source, and D_S is the angular distance between us and the source. Note that, being the metric (1) in an isotropic form, coordinate angles are equal to physical angles. This can be checked, for example, using the definition introduced in Ref. [19] or via the construction used in Ref. [21].

Using Eq. (16), the angular diameter distance between the source and us can be expressed as

$$D_S = \frac{1}{H_0}(1 - a_S) = \frac{1}{H_0}\frac{z_S}{1 + z_S}. \tag{23}$$

The angular diameter distance between the lens and us has a similar form:

$$D_L = \frac{1}{H_0}(1 - a_L) = \frac{1}{H_0}\frac{z_L}{1+z_L}. \qquad (24)$$

On the other hand, $D_{LS} \neq D_S - D_L$, but, see, e.g., Ref. [31]:

$$D_{LS} = a_S(\chi_S - \chi_L) = \frac{1}{H_0}\frac{z_S - z_L}{1+z_S}. \qquad (25)$$

Using Eq. (20), Eq. (22) can be written as follows:

$$\theta \approx \beta + \frac{4M(1+z_L)}{b}\frac{D_{LS}}{D_S}. \qquad (26)$$

For comparison, let us consider the standard lens equation, which is based on the union of results coming from the Schwarzschild metric and from the FLRW metric, see, e.g., Refs. [31,32]:

$$\theta \approx \beta + \frac{4M}{r_0}\frac{D_{LS}}{D_S}. \qquad (27)$$

Here, r_0 is the closest approach distance to the lens. In a cosmological setting, it is considered as a proper distance and, therefore, expressed as $r_0 = \theta D_L$. In our case, b is the comoving transversal position of the lens; thus, it has a different interpretation which we will make clearer in the next section. Note the correction factor $1 + z_L$ from Eq. (27) to Eq. (26), which indeed suggests a correction to the mass coming from the cosmological setting of the problem. In Ref. [23], the author performs a very similar analysis as the one presented in this section, but using physical distances instead of comoving ones. He finds no relevant contribution to the bending angle, i.e., he reproduces Eq. (27) with $r_0 = \beta D_L$. This result has been contested by Ref. [20], see also Ref. [27], because it is missing an important term proportional to Λ. Note that the author of Ref. [23] seems to have acknowledged the existence of such term. For more detail on this debate, see the above cited references.

III. IN CASE OF ALIGNMENT

The solution we have exploited in the previous section does not work in the case of an alignment among the source, the lens and an observer. See Fig. 3.

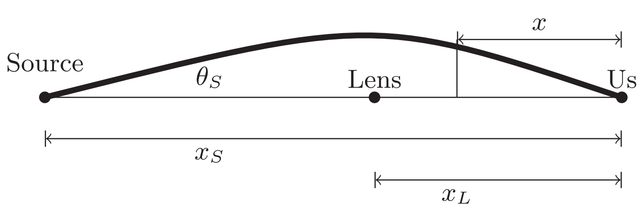

FIG. 3. Scheme of lensing in case of alignment.

In this case, the zero order solution cannot be a horizontal trajectory, because it would never reach us. The zero order trajectory is now

$$y^{(0)} = \theta_S(x_S - x), \qquad (28)$$

where $\theta_S \ll 1$. In order for the trajectory to reach us, we must choose the initial condition $y^{(1)}(0) = -\theta_S x_S/\alpha$.

Computing again the deflection angle, we get:

$$\delta = \frac{4M(1+\chi_L H_0)}{\theta_S(\chi_S - \chi_L)} + \frac{\chi_L}{2(\chi_S - \chi_L)} + \mathcal{O}(\theta_S). \qquad (29)$$

In order to deal with the denominator of the first term on the right-hand side of Eq. (29), we use Eq. (25), i.e.:

$$\theta_S(\chi_S - \chi_L) = \theta_S(1+z_S)D_{LS} = (1+z_S)\theta_E D_L, \qquad (30)$$

where the last equality holds true since the proper distances $\theta_S D_{LS}$ and $\theta_E D_L$ are, indeed, the same and equal to the closest approach proper distance. Now, using the lens Eq. (22) with $\beta = 0$, we obtain the following formula for the Einstein ring angle:

$$\theta_E = \sqrt{4M\frac{1+z_L}{1+z_S}\frac{D_{LS}}{D_L D_S}}. \qquad (31)$$

Comparing this result with the usual one coming from Eq. (27) applied to the Einstein ring case, i.e.:

$$\theta_E = \sqrt{4M\frac{D_{LS}}{D_L D_S}}, \qquad (32)$$

one infers that the correction induced when determining the mass of the lens, or the value of H_0, is $(1+z_S)/(1+z_L)$.

Using Eqs. (23), (24), and (25) in order to write the angular diameter distances of Eq. (31) as functions of the redshift, one obtains:

$$\theta_E = \sqrt{4MH_0\frac{(1+z_L)^2(z_S - z_L)}{(1+z_S)z_S z_L}}. \qquad (33)$$

We apply formula (33) for some Einstein ring systems observed by the CASTLES Survey [33] and calculate the mass estimations of the lenses in Table I, assuming $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$.

The correction $(1+z_S)/(1+z_L)$ for the Einstein ring systems considered varies from a 1.2 to a 3.1 factor. In particular, since $z_S > z_L$, the new mass estimate is always larger than the usual one, based on Eq. (32).

Two comments are in order here. First, the values computed in Table I are rough estimates, based on the assumption of a pointlike lens. In reality, lenses are not pointlike, and models for their matter density distribution

TABLE I. Mass estimates. In the last column, the $(1+z_S)/(1+z_L)$ contribution is the one coming purely from the bending angle correction. The angle θ_E is given in arcseconds.

Object	References	z_S	z_L	θ_E [arcsec]	$M/M_\odot$ from Eq. (33)	$(1+z_S)/(1+z_L)$
Q0047-2808	[34]	3.60	0.48	1.35	1.1×10^{12}	3.1
PMNJ0134-0931	[35]	2.216	0.77	0.365	8.2×10^{10}	1.8
B0218 + 357	[36]	0.96	0.68	0.17	2.4×10^{10}	1.2
CFRS03.1077	[37]	2.941	0.938	1.05	8.1×10^{11}	2.0
MG0751 + 2716	[38]	3.20	0.35	0.35	5.7×10^{10}	3.1
HST15433 + 5352	[39]	2.092	0.497	0.59	1.6×10^{11}	2.1
MG1549 + 3047	[40]	1.17	0.11	0.9	8.8×10^{10}	2.0
MG1654 + 1346	[41]	1.74	0.25	1.05	2.9×10^{11}	2.2
PKS1830-211	[42]	2.51	0.89	0.5	1.7×10^{11}	1.9
B1938 + 666	[43]	2.059	0.881	0.5	1.7×10^{11}	1.6

are necessary in order to provide a more precise, and reliable, value for the mass. For example, the authors of Ref. [28] calculated mass estimates for three objects of Table I assuming a cored density profile, and they found different values, even if of the same order of magnitude, from those presented here. Since our purpose is to stress the existence of the new contribution correcting the bending angle, viz., Eqs. (20) and (29), we do not discuss the details of modeling the density profile of the lens, leaving it as a future investigation.

Second, the correction factors reported in the last column of Table I are based on the new bending angle computed in Eqs. (20) and (29). These formulas, as we already commented in the previous section, are reliable only when our Universe is Λ dominated, i.e., only for $z \ll 0.3$, which is never the case for the objects in Table I. Therefore, those factors ranging from 1.2 to 3.1 are not to be taken too seriously. What is perhaps to be taken seriously is that the bending angle and the lens equation calculated in the McVittie metric are different from the usual Schwarzschild + FLRW case, even if the McVittie metric should, in some sense, be a merging of the Schwarzschild and FLRW metrics. This fact may deserve further and more profound investigation.

Finally, we check the approximation $\mu \ll 1$, on which our calculations are based. The maximum value of μ is when $y = y(x_L)$, i.e., at the lens. In this case, we can write:

$$\mu_{\rm max} \simeq \frac{M}{\theta_E D_L} = \frac{H_0 M(1+z_L)}{\theta_E z_L}. \tag{34}$$

From Table I, the lowest θ_E is of the order 0.1 arcsec. The product $H_0 M$ is the ratio between the gravitational radius of the lens and the Hubble radius. For a typical $M \simeq 10^{11}\, M_\odot$ and $H_0 \simeq 10^{-18}$ s, one obtains $H_0 M \simeq 10^{-12}$. Therefore, $H_0 M/\theta_E \simeq 10^{-9}$, which is very small indeed. The redshift term $(1+z_L)/z_L$ is of the order $\mathcal{O}(z_L)$, so it does not change our estimate. The other approximation used, in Eq. (29), is that θ_S is very small. From Eq. (25):

$$\theta_S \simeq \frac{\theta_E D_L}{D_{LS}} = \theta_E \frac{z_L(1+z_S)}{z_S - z_L}. \tag{35}$$

The largest θ_E from Table I is 1.35 arcsec, corresponding to $\simeq 2\times 10^{-2}$ rad. The redshift factor is about 0.1; therefore, $\theta_S \simeq 10^{-3}$.

IV. DISCUSSION AND CONCLUSIONS

We have analyzed the issue of whether cosmology affects local phenomena such as the bending of light by a compact mass. To this purpose, we adopted the McVittie metric, which describes the geometry of a point mass (the lens, in our picture) in the expanding Universe. We assume a constant Hubble factor, thereby assuming a cosmological constant-dominated FLRW universe surrounding the point mass. We found an important correction to the bending angle, proportional to $1+z_L$, where z_L is the redshift of the lens. In the lens equation, within the thin-lens approximation, we found that the mass estimation has to be corrected by a factor of $(1+z_S)/(1+z_L)$. We exploit data and analysis performed by the CASTLES Survey and calculate mass estimations for some Einstein ring systems. Future work, based on a numerical solution of Eq. (10), shall be necessary in order to determine a more reliable correction to the bending angle, based on the current standard cosmological model, where an important fraction of matter is present and therefore, the Hubble parameter is not constant.

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