

Testing modified gravity and no-hair relations for the Kerr-Newman metric through quasiperiodic oscillations of galactic microquasars

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We construct multipole moments for stationary, asymptotically flat, spacetime solutions to higher-order curvature theories of gravity. The moments are defined using $3 + 1$ techniques involving timelike Killing vector constructions as in the classic papers by Geroch and Hansen. Using the fact that the Kerr-Newman metric is a vacuum solution to a particular class of $f(R)$ theories of gravity, we compute all its moments, and find that they admit recurrence relations similar to those for the Kerr solution in general relativity. It has been proposed previously that modeling the measured frequencies of quasiperiodic oscillations from galactic microquasars enables experimental tests of the no-hair theorem. We explore the possibility that, even if the no-hair relation is found to break down in the context of general relativity, there may be an $f(R)$ counterpart that is preserved. We apply the results to the microquasars GRS 1915 + 105 and GRO J1655-40 using the diskoseismology and kinematic resonance models, and constrain the spins and “charges” of their black holes.

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I. INTRODUCTION

The interpretation of parameters appearing within solutions to the theory of general relativity (GR) has a rich history [1,2]. In linear field theories, such as Newtonian gravity or Maxwell electrodynamics, it is well understood how the potential associated with a solution to the field equations can be characterized by its multipole moments [3,4]. The moments themselves generate a series expansion for the field potential, which then offers a term-by-term physical understanding [5]. Nonlinear field theories, such as GR, do not obey the principle of superposition, making multipole decomposition harder to define. Some of the better known multipole expansions in GR are due to Thorne [6], Curtis [7], Geroch [8], Hansen [9], Janis and Newman [4] and others [10–12].

One practical application of multipole moments is to characterize solutions in non-GR theories [13,14], e.g. $f(R)$ gravity [15]. Interestingly there exist vacuum solutions to higher-order curvature theories which are simultaneously nonvacuum solutions in GR. For example, the Kerr-Newman metric [16] is a *vacuum* solution to a particular class of $f(R)$ gravities [17–19]. In the GR context, astrophysical interest in the Kerr-Newman solution is limited due to the expectation that naturally formed black holes are neutral. However, in $f(R)$ gravity, the Kerr-Newman metric serves as a natural extension of the uncharged Kerr solution. In this paper, we explore the properties of this metric as a solution to a vacuum $f(R)$ theory. We introduce multipole moments for higher-order curvature theories through the Geroch-Hansen procedure, [3,8,9,20,21] compute the Kerr-Newman moments, and

find that they reduce to their Kerr counterparts found by Hansen, [9] when the appropriate limit is taken.

How can one determine if the Kerr-Newman solution in $f(R)$ gravity occurs in Nature? A contender for an experiment to answer this question is provided by quasiperiodic oscillations (QPOs) in X-ray binaries [22,23] or active galactic nuclei [24]. While the physical origin of these phenomena is still being debated [23], some models suggest that, as the accretion disk swirls around the black hole, certain fluid modes become trapped within the ergosphere [25], producing quasiperiodic patterns of electromagnetic activity via the Lense-Thirring effect [26,27]. In GR, owing to the uniqueness results of Israel, Carter, Robinson, and Hawking [28–30], the frequencies of these modes are directly related to the mass and spin of the Kerr black hole. Johannsen and Psaltis [31] have demonstrated methods by which, given an independent measurement of either the mass or spin, one can determine whether the no-hair theorem in GR is violated by monitoring QPO frequencies, or whether something else is going on, e.g. $f(R)$ gravity.

The multipole relations we find for the Kerr-Newman solution in $f(R)$ gravity obey recurrence relations similar to those of the Kerr metric in GR, in that each moment is computable from the lowest-order mass and current moments. This result indicates the presence of a no-hair-type result in $f(R)$ gravity, except with three parameters instead of two. Although it is unlikely that the Kerr-Newman metric is the unique descriptor of black holes in $f(R)$ gravity, it may form part of an extended family of solutions generated by the function f . This presents an alternative scenario to that presented by Johannsen and Psaltis [31]: if no-hair-violating data are discovered, the no-hair theorem may still be preserved, provided that GR does not apply. Hence, for any given object, an independent mass measurement

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combined with a QPO pair may not be enough to rule out no-hair relations altogether. More generally, one would need m measurements to potentially invalidate a no-hair theorem involving $m - 1$ independent parameters, but one would need only three measurements to determine whether GR (or the Kerr no-hair relation) is consistent.

Given a QPO pair, one can compare theoretical relations between the mass, spin, and quadrupole moment with measured frequencies. The Kerr-Newman metric in $f(R)$ gravity contains a parameter ρ (no longer charge). Given the measured frequencies of QPO pairs from the microquasars GRS 1915 + 105 and GRO J1655-40, we place bounds on ρ following Johannsen and Psaltis [31]. Some GR estimates place the spin of GRS 1915 + 105 as high as $a/M = 0.998$ [32], where $f(R)$ effects may be strong [13,33,34]. A similar analysis using the Kerr-Newman metric indicates that the spin of these objects may be over or underestimated depending on the sign of ρ .

The paper is structured as follows. In Sec. II we revisit the definitions given by Geroch and Hansen in GR and generalize them to alternative gravities. In Sec. III we derive the equations which govern the multipole moments for spacetimes in general gravitational theories. In Sec. IV we explicitly solve these equations for the $f(R)$ Reissner-Nördstrom and Kerr-Newman black holes and use the solutions to generate the multipole moments. In Sec. V we explore the consequences for QPO modeling to illustrate how the moment calculation can be applied astrophysically. We discuss the results in Sec. VI.

II. GEROCH-HANSEN MOMENTS IN GR

A. Definitions

The Geroch and Hansen moments [8,9] are defined for asymptotically flat,¹stationary² solutions to the Einstein equations outside of a source,

$$R_{\mu\nu} = 0, \quad (1)$$

where $R_{\mu\nu}$ denotes the Ricci tensor. They are built out of potentials associated with the metric which satisfy Poisson-like equations, thereby relinearizing a significant portion of the problem. The geometric framework that facilitates the relinearization, described below, also provides a convenient recipe for generalizing from the Einstein theory to alternative gravitational theories.

We assume that the spacetime M is stationary, so that there exists a timelike vector field ξ [35] which satisfies Killing's equation³

¹We assume a 4-dimensional spacetime throughout and zero cosmological constant in GR.

²A spacetime is *stationary* if the metric tensor does not depend on any timelike coordinates.

³We consider torsionless theories of gravitation: $\nabla_\mu g_{\alpha\beta} = 0$.

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \quad (2)$$

If we define λ , the norm of ξ , by [20]

$$\lambda = \xi^\alpha \xi_\alpha, \quad (3)$$

and the twist, ω^\dagger , by [35]

$$\omega^\dagger_\alpha = \epsilon_{\alpha\beta\gamma\delta} \xi^\beta \nabla^\gamma \xi^\delta, \quad (4)$$

then the metric g may be written in the generalized Papapetrou form⁴ [10,28]

$$ds^2 = \lambda(dt + \sigma_i dx^i)^2 - \lambda^{-1} h_{ij} dx^i dx^j, \quad (5)$$

where $\omega^\dagger_i = -\lambda^{-2} \epsilon_{ijk} D^j \sigma^k$, and D forms the covariant derivative with respect to h_{ij} (see Appendix A). The spacetime then admits a 1 + 3 split, and we denote the manifold associated with the Riemannian metric h as S [9,10,36]. We assume that S is simply-connected (such as the exterior region to a black hole ergosphere), to ensure the existence of a certain scalar field (see below) [10,37].

B. Static and stationary spacetimes

Geroch initially considered static spacetimes with $\omega^\dagger = 0$ [2,3]. Starting from Einstein's equations on M , $R_{\mu\nu} = 0$, he derived the corresponding field equations over S . The fundamental variables to be computed are λ and the components h_{ij} . The equation for λ is equivalent to a conformally invariant Laplace equation for a particular scalar field ϕ^M (mass potential), for which the moments are defined. The benefit of trading λ for ϕ^M is that, when the spacetime is Minkowski ($g = \eta$), the Newtonian equation $\nabla^2 \phi^M = 0$ is recovered. In particular, the procedure uniquely pulls out the mass monopole moment for the Schwarzschild black hole, with all other moments being identically zero.

The extension to the stationary case ($\omega^\dagger \neq 0$) was given by Hansen [9]. For example, rotating solutions require an additional moment-generating field, ϕ^J [29,38]. Combining (2) and (4) with the Bianchi identities leads to

$$\nabla_\alpha \omega^\dagger_\beta - \nabla_\beta \omega^\dagger_\alpha = -\epsilon_{\alpha\beta\gamma\delta} \xi^\gamma R^\delta_{\tau\xi} R^\tau_{\xi\delta}. \quad (6)$$

Assuming the Einstein condition (1), the right-hand side of (6) vanishes, and there exists a scalar field ω^\dagger such that

$$\nabla_\alpha \omega^\dagger = \omega^\dagger_\alpha. \quad (7)$$

We explore generalizations of the quantity ω^\dagger in the next section to extend the formalism to non-Einstein theories, since the scalar ω^\dagger does not exist in general outside GR.

To ensure that the spacetime M is asymptotically flat, one requires the existence of a manifold \tilde{S} , consisting of S

⁴Throughout, Greek symbols range over spacetime indices 0,1,2,3 while Latin indices are reserved for spatial indices 1,2,3.

plus one additional point Λ , subject to certain conditions [8,20]. This point Λ is to be thought of as the ‘‘asymptotic’’ point, in the sense that we can extend the metric \mathbf{h} to the boundary of S . In particular, we require the existence of a scalar field, Ω such that $\tilde{h}_{ab} = \Omega^2 h_{ab}$ is a metric on \tilde{S} , and at Λ one has $\Omega = 0$, $\tilde{D}_a \Omega = 0$, and $\tilde{D}_a \tilde{D}_b \Omega = n \tilde{h}_{ab}$ for some n . Euclidean 3-space is asymptotically flat in the above sense with conformal factor $\Omega = r^{-2}$, where r is the distance from some origin [8,9,20].

One can now express the field equations on M in terms of the variables $\{\lambda, \omega^\dagger, h_{ij}\}$. In Appendix A we show how to map these equations onto S . The equations of motion for λ and ω^\dagger are replaced by ones for $\phi^M(\lambda, \omega^\dagger)$ and $\phi^J(\lambda, \omega^\dagger)$ on S and \tilde{S} , which read⁵ [8,9]

$$\left(D^i D_i - \frac{1}{8} \mathcal{R}\right) \phi^A = \frac{15}{16} \lambda^{-2} \kappa \phi^A, \quad (8)$$

with $A = M, J$, where \mathcal{R} is the Ricci scalar on S , and $\kappa = (D^m \lambda D_m \lambda + D^m \omega D_m \omega)$. Equation (8) is solved subject to mixed Dirichlet and Neumann boundary conditions (see §3). The factor $\frac{1}{8} \mathcal{R}$ ensures that the field equations are conformally invariant with respect to ϕ^A (see Appendix B), so that transforming into the space \tilde{S} leaves the moments unaltered [10,39].

We remark in passing that one can work within the original formalism due to Geroch [8], where Eq. (8) is taken to be homogenous ($\kappa = 0$), without altering the results of this paper. The inhomogeneous terms added by Hansen (equations (2.16) in [9]) preserve the conformal properties of the Laplace equation while permitting closed form solutions ϕ^A to (8) in GR. However, this calculational advantage does not carry over to modified gravities except in special cases. When the Ricci tensor is complicated, no closed-form solutions exist in general, as there is no obvious way to write the derivatives of ω , \mathbf{Q} , or λ in terms of the 3-metric h_{ij} [see (A4)]. The homogeneous and inhomogeneous forms of the non-GR extension of equation (8) both produce the same moments, as calculated in Sec. IV B, according to Theorem 4 in [36], even though the generalized ϕ^A are different in the two cases. In this paper, we persevere with Hansen’s inhomogeneous formalism to match equation (2.16) in [9], at the cost of mildly complicating the form of Eq. (8).

In order to define the moments, we construct recursively a set of tensor fields $P^A_{a_1 \dots a_s}$ on $(\tilde{S}, \tilde{\mathbf{h}})$ by

$$P^A = \tilde{\phi}^A, \\ P^A_{a_1 \dots a_{s+1}} = \mathcal{C} \left[\tilde{D}_{a_1} P^A_{a_2 \dots a_{s+1}} - \frac{1}{2} s (2s-1) \tilde{\mathcal{R}}_{a_1 a_2} P^A_{a_3 \dots a_{s+1}} \right], \quad (9)$$

with $\tilde{\phi}^A = \Omega^{-1/2} \phi^A$, where $\mathcal{C}[J_{a \dots b}]$ denotes the symmetric, trace-free part of $J_{a \dots b}$ and $\tilde{\mathcal{R}}_{ij}$ is the Ricci tensor associated with \tilde{h}_{ij} . We now define the 2^s moment of ϕ^A to be the value of $P^A_{a_1 \dots a_s}$ at Λ and write $P^M_{a_1 \dots a_s} = M_{a_1 \dots a_s}$ and $P^J_{a_1 \dots a_s} = J_{a_1 \dots a_s}$.

For some physical intuition, consider the Newtonian gravitational potential ϕ defined on Euclidean 3-space. To perform the conformal completion we take $\Omega = r^{-2}$ and $\tilde{\phi} = \Omega^{-1/2} \phi = r\phi$. The vacuum Newtonian gravitational potential [satisfying (8)] admits an expansion of the form [9]

$$\tilde{\phi} = U + U_a x^a + \frac{1}{2!} U_{ab} x^a x^b + \dots, \quad (10)$$

where U are the multipole moments and \mathbf{x} is the position vector. It follows immediately that at the point Λ ($|\mathbf{x}| = 0$) we have

$$U = \tilde{\phi}|_\Lambda, \quad U_a = \tilde{D}_a \tilde{\phi}|_\Lambda, \quad U_{ab} = \tilde{D}_a \tilde{D}_b \tilde{\phi}|_\Lambda, \dots \quad (11)$$

Equation (11) coincides exactly with Eq. (9) when the curvature terms vanish.

In GR, one may solve (8) exactly for any stationary vacuum solution [9]. For $R_{\mu\nu} \neq 0$, factors emerge involving the Ricci tensor, such as $R_{\mu\nu} \xi^\mu \xi^\nu$, which make finding a general solution difficult. One must solve the non-GR extended version of Eq. (8), namely Eq. (15), for individual metrics in any given modified theory of gravity to define the moments using (9), as we do for the Kerr-Newman metric in Sec. IV.

III. NON-EINSTEIN GRAVITY

Having introduced the Geroch-Hansen procedure for GR in Sec. II B, we explore the parts that break down when the spacetime is non-Einstein.

A. Definitions

For $R_{\mu\nu} \neq 0$, the twist one-form ω^\dagger is not curl-free. We introduce a one-form \mathbf{Q} such that

$$\omega = \omega^\dagger + \mathbf{Q} \quad (12)$$

is curl-free. Then a scalar field ω exists with $\nabla_a \omega = \omega_{,a}$, provided that \mathbf{Q} satisfies

$$d\mathbf{Q} = -\star[\xi \wedge \mathbf{R}(\xi)], \quad (13)$$

where \star is the Hodge star operator and \wedge is the wedge product (see equation 2.16 of [29]). Equation (13) constitutes a first-order partial differential equation for the form \mathbf{Q} [37], which may be solved by defining a coordinate system (see Appendix C). We

⁵Scripted letters refer everywhere to curvature tensors on S .

explicitly construct \mathcal{Q} for the case of the Kerr-Newman black hole in Sec. IV.

B. Moments

In this section we derive the equations for the gravitational potentials which define the multipole moments. We

$$\begin{aligned} \left(D^m D_m - \frac{1}{8} \mathcal{R}\right) \tau^A &= \frac{1}{\sqrt{h}} \partial_i (\sqrt{h} h^{ij} \partial_j \tau^A) - \frac{1}{8} \mathcal{R} \tau^A \\ &= (D^m D_m \lambda) \tau_{,\lambda}^A + (D^m D_m \omega) \tau_{,\omega}^A + (D_i \lambda D^i \lambda) \tau_{,\lambda\lambda}^A + 2(D^j \omega D_j \lambda) \tau_{,\lambda\omega}^A + (D^i \omega D_i \omega) \tau_{,\omega\omega}^A - \frac{1}{8} \mathcal{R} \tau^A, \end{aligned} \quad (14)$$

where we have adopted the notation $f_{,\lambda\omega}^A = \frac{\partial^2 f^A}{\partial \lambda \partial \omega}$ and h forms the 3-metric as in (5). Making use of some identities derived in Appendix A [Eqs. (A4)–(A6)] we find that the vanishing of the Poisson equation

$$\left(D^m D_m - \frac{1}{8} \mathcal{R}\right) \tau^A = \frac{15}{16} \lambda^{-2} \kappa \tau^A, \quad (15)$$

is equivalent to

$$\begin{aligned} 0 &= D^i \lambda D_i \lambda \left(\frac{1}{2} \lambda^{-1} \tau_{,\lambda}^A + \tau_{,\lambda\lambda}^A - \frac{15}{16} \lambda^{-2} \tau^A \right) + D^m \omega D_m \omega \left(\tau_{,\omega\omega}^A - \lambda^{-1} \tau_{,\lambda}^A - \frac{3}{4} \lambda^{-2} \tau^A \right) + D^m \omega D_m \lambda \left(2\tau_{,\lambda\omega}^A + \frac{3}{2} \lambda^{-1} \tau_{,\omega}^A \right) \\ &\quad + D^m \omega Q_m \left(2\lambda^{-1} \tau_{,\lambda}^A - \frac{3}{8} \lambda^{-2} \tau^A \right) - \lambda^{-1} Q^m Q_m \left(\tau_{,\lambda}^A - \frac{3}{16} \lambda^{-1} \tau^A \right) + \left(D^a Q_a - \frac{3}{2} \lambda^{-1} Q_m D^m \lambda \right) \tau_{,\omega}^A - 2R^{mn} \xi_m \xi_n \tau_{,\lambda}^A \\ &\quad - \frac{R^{mn}}{8} (h_{mn} - \lambda^{-1} \xi_m \xi_n) \tau^A. \end{aligned} \quad (16)$$

For the mass potential, the above equation is to be solved subject to the Neumann conditions $\tau^M(\lambda = 1, \omega) = \frac{\omega^2}{4}$ and $\tau_{,\lambda}^M(\lambda = 1, \omega) = \frac{1}{2} - \frac{3}{16} \omega^2$. For the current potential, it is solved subject to the mixed Dirichlet $\tau^J(\lambda, \omega = 0) = 0$ and Neumann conditions $\tau'_{,\omega}(\lambda, \omega = 0) = \frac{1}{2}$ [9]. Adding constant terms to τ^A does not change the moment structure (because we have $\Omega|_{\Lambda} = 0$), so these boundary conditions are sufficient [10]. For the Einstein case $R_{\mu\nu} = 0$, equation (16) has solutions $\tau^M \rightarrow \phi^M = \frac{1}{4} \lambda^{-3/4} (\lambda^2 + \omega^2 - 1)$, and $\tau^J \rightarrow \phi^J = \frac{1}{2} \lambda^{-3/4} \omega$, which differ from Hansen's by a factor of $\lambda^{-1/4}$ due to the sign of the norm (3) (see section VII of [40]).

For any metric, the equations of motion for λ and ω may be replaced by (16), which reduces to the Laplace equation for $\mathbf{g} = \boldsymbol{\eta}$. There are also equations of motion for h_{ij} , which depend on the theory of gravity but do not affect the moments. As an example, we derive the field equations for h_{ij} for $f(R)$ gravity in Appendix A.

We explicitly solve (16) in Sec. IV for two cases of “charged” black holes, which arise as vacuum solutions in an $f(R)$ theory of gravity: the Reissner-Nördstrom and Kerr-Newman metrics. The moments for the modified gravity solution are given through (9) with ϕ^A replaced by τ^A . The process of computing \mathbf{P}^A iteratively from (9) after finding τ^A may or may not be a difficult task. Several authors [40–43] have developed alternative methods for computing the moments rather than employing the

introduce scalar fields $\tau^A(\lambda, \omega)$, generalizing the moment functions ϕ^A of Geroch and Hansen. The goal is to choose τ^A to satisfy Eq. (8) for $R_{\mu\nu} \neq 0$, with $\tau^A = \phi^A$ for $R_{\mu\nu} = 0$. Making use of the chain rule $D_j \tau^A = D_j \lambda \frac{\partial \tau^A}{\partial \lambda} + D_j \omega \frac{\partial \tau^A}{\partial \omega}$, we are able to write

definition (9) directly, which are useful for complicated metrics.

C. Static spacetimes

For $\boldsymbol{\omega}^\dagger = 0$ and $\mathcal{Q} = 0$, Eq. (16) gives

$$\begin{aligned} D^i \lambda D_i \lambda \left(\frac{1}{2} \lambda^{-1} \tau_{,\lambda}^A + \tau_{,\lambda\lambda}^A - \frac{15}{16} \lambda^{-2} \tau^A \right) \\ + \frac{1}{8} R_{mn} (h^{mn} \tau^A - \lambda^{-1} \xi^m \xi^n \tau^A - 16 \xi^m \xi^n \tau_{,\lambda}^A) = 0. \end{aligned} \quad (17)$$

Furthermore, for $R_{\mu\nu} = 0$, (17) reduces to ($D_i \lambda \neq 0$)

$$\frac{1}{2} \lambda^{-1} \tau_{,\lambda}^A + \tau_{,\lambda\lambda}^A - \frac{15}{16} \lambda^{-2} \tau^A = 0, \quad (18)$$

which has solutions $\tau^M \rightarrow \phi^M = \frac{1}{4} \lambda^{-3/4} (-1 + \lambda^2)$ and $\tau^J = 0$, where the dependence on λ differs from Geroch's result [8] due to $\kappa \neq 0$. Taking $\kappa = 0$, the appropriately modified version of Eq. (18) reads

$$\frac{1}{2} \lambda^{-1} \tau_{,\lambda}^A + \tau_{,\lambda\lambda}^A = 0, \quad (19)$$

which has the exact Geroch solutions $\tau^M \rightarrow \phi^M = -1 + \sqrt{\lambda}$ and $\tau^J = 0$ [8,40].

D. Axisymmetric spacetimes

In spacetimes that are axisymmetric, the tensor moments (9) reduce to a set of scalar moments [9]. Axisymmetry guarantees the existence of a spacelike Killing vector ξ in S , i.e. there is a second field satisfying Killing's equation (2). The conformal factor Ω can be chosen, without loss of generality, such that the axis vector $z^i = 2\epsilon^{ijk}D_j\xi_k$ satisfies

$$\tilde{z}^k\tilde{z}_k|_\Lambda = 1, \quad (20)$$

under the conformal transformation $z \rightarrow \tilde{z}$; see Eqs. (3.2)–(3.4) in Hansen [9]. Hence, at Λ , ξ defines rotations of tensors, under which the multipole moments are preserved [9,14]. Hansen showed that the only tensors at Λ invariant under the action of \tilde{z} are products of the metric and \tilde{z} itself [9]. As a result, the 2^s moment $P_{a_1\dots a_s}^A$ is necessarily a multiple of $\mathcal{C}[\tilde{z}_{i_1}\dots\tilde{z}_{i_s}]|_\Lambda$. Therefore, the only nonzero components of P^A are [14,40,43]

$$P_s^A = \frac{1}{s!}P_{a_1\dots a_s}^A z^{i_1}\dots z^{i_s}|_\Lambda. \quad (21)$$

IV. "CHARGED" BLACK HOLES

In this section we solve equation (16) for the Reissner-Nördstrom and Kerr-Newman metrics. Although originally obtained as charged black holes in GR, these black holes can also arise as vacuum solutions in modified theories of gravity, such as the $f(R)$ theory, explaining the quotation marks in the section heading. Our analysis follows various authors [13,31,44] in evaluating the multipole moments associated with a non-Einstein spacetime. The main distinction here is that the generalized Geroch-Hansen procedure allows us to directly compare the moments for exact solutions in modified gravitational theories with their Einstein counterparts at all orders.

A. Reissner-Nördstrom metric

A static, spherically symmetric metric on M may be written in Boyer-Lindquist (t, r, θ, ϕ) coordinates as (see e.g. [35])

$$ds_M^2 = A(r)dt^2 - B(r)dr^2 - r^2d\Omega^2. \quad (22)$$

Static solutions have $\omega^\dagger = 0$, and $\lambda = A(r)$. From (5), the line element on S is

$$d\sigma_S^2 = B(r)A(r)dr^2 + r^2A(r)d\Omega^2. \quad (23)$$

The Einstein-Maxwell field equations read

$$R^{\mu\nu} - \frac{R}{2}g^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\alpha}F_\alpha^\nu - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \right), \quad (24)$$

where $F^{\alpha\beta}$ is the Faraday tensor (e.g. [29]). Choosing a point source to generate F yields the Reissner-Nördstrom solution (e.g. [45])

$$A(r) = 1 - \frac{2M}{r} + \frac{\rho}{r^2} = B(r)^{-1}. \quad (25)$$

The parameter ρ , for the Einstein-Maxwell case, is defined as the square of the total electromagnetic charge [45],

$$\rho = q^2. \quad (26)$$

We avoid writing ρ as a square, because ρ can be interpreted as a negative number in modified gravity. To the authors' knowledge, there is no neat physical interpretation for ρ presently available in an $f(R)$ gravity where there is zero electromagnetic field. In the vacuum theory it appears as an integration constant, and seems to have the effect of mass renormalization (see Sec. IV B).

The Geroch-Hansen moments in GR are not able to describe the multipolar decomposition of this solution⁶ since $R_{\mu\nu} \neq 0$. However, the Reissner-Nördstrom metric is in fact a solution to any vacuum $f(R)$ gravity with $f(0) = f'(0) = 0$, such as the $f(R) = R^{1+\delta}$ models studied by Clifton and Barrow [49]. In this paper we consider models with $\delta > 0$. Clifton and Barrow found that perihelion precession observations of Mercury place the tight bounds $0 \leq \delta < 7.2 \times 10^{-19}$ [49].

The field equations for $f(R)$ gravity read (e.g. [15])

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)f'(R) = 0. \quad (27)$$

Noting that the Reissner-Nördstrom spacetime has $R = 0$, setting $f(0) = f'(0) = 0$ automatically guarantees that (22) with (25) is a solution to (27), since $\nabla_\mu f'(R) = R_{,\mu}f''(R)$ and R is constant.

Using our expression for the 3-metric (23), Eq. (17) for the Reissner-Nördstrom solution becomes

$$0 = D^i\lambda D_i\lambda \left(\frac{1}{2}\lambda^{-1}\tau_{,\lambda}^M + \tau_{,\lambda\lambda}^M - \frac{15}{16}\lambda^{-2}\tau^M \right) - \frac{(\lambda-1)^4\rho\{8M^2p(\lambda)^2 + (\lambda-1)^2\rho^2 + 4M[p(\lambda)^2 + M^2]p(\lambda)\}}{8\lambda[M+p(\lambda)]^8} \times [(\lambda-1)\tau^M + 16\lambda^2\tau_{,\lambda}^M] = 0, \quad (28)$$

⁶However one can define separate moments for the Schwarzschild component and electromagnetic 4-potential A ; see [11,46–48].

with $p(\lambda) = [M^2 + (\lambda - 1)\rho]^{1/2}$. We have $\tau^J = 0$ by the boundary conditions, as expected physically. Explicitly taking the covariant derivatives of λ , i.e. $D_i\lambda = D_iA(r) = \delta_i^r A'(r) = \delta_i^r A'(A^{-1}(\lambda))$, we find

$$0 = [(\lambda - 1)\rho + 30\lambda^{-1}p(\lambda)^2]\tau^M - 16[p(\lambda)^2 - \lambda^2\rho]\tau_{,\lambda}^M - 32\lambda p(\lambda)^2\tau_{,\lambda\lambda}^M. \quad (29)$$

Equation (29) with boundary conditions $\tau^M(\lambda = 1) = 0$, $\tau_{,\lambda}^M(\lambda = 1) = 1/2$ has a lengthy solution involving confluent hypergeometric functions [50], which is plotted in Fig. 1 for a variety of values of ρ .

We now compute the moments following the recipe in Sec. II and Sec. III. Following Hansen [9], we adopt a new coordinate system $\{t, \bar{R}, \theta, \phi\}$ which simplifies the calculation of Ω considerably, where the new radial coordinate \bar{R} is defined via

$$\bar{R} = \exp \left[\int dr \sqrt{\frac{h_{rr}}{h_{\theta\theta}}} \right]. \quad (30)$$

Upon inversion for the Reissner-Nördstrom spacetime, we find

$$r = \frac{1}{\bar{R}} \left(1 + M\bar{R} + \frac{1}{4}(M^2 - \rho)\bar{R}^2 \right). \quad (31)$$

Inspection of the resulting line element shows that a suitable Ω can be taken as

$$\Omega = \frac{\bar{R}^2}{1 - \frac{1}{4}(M^2 - \rho)\bar{R}^2}, \quad (32)$$

to yield

$$d\tilde{\sigma}_S^2 = d\bar{R}^2 + \bar{R}^2 d\theta^2 + \bar{R}^2 \sin^2 \theta d\phi^2, \quad (33)$$

satisfying $\tilde{\mathbf{h}} = \Omega^2 \mathbf{h}$. These choices reduce to those used by Geroch and Hansen [8,9] when $\rho \rightarrow 0$. Upon substituting $\tilde{\tau}^M = \Omega^{-1/2} \tau^M$ into (21), we find that the only nonzero

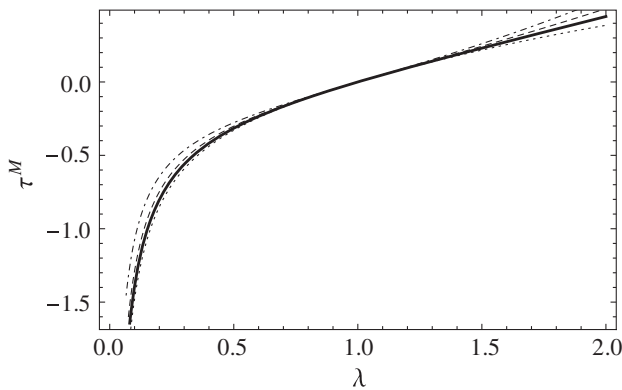


FIG. 1. Solutions $\tau^M(\lambda)$ to the Poisson equation (29) with $\rho = -0.4M^2$ (dotted), $\rho = 0.0$ (solid), $\rho = 0.4M^2$ (dashed), and $\rho = 0.99M^2$ (dash-dotted).

multipole moment for the $f(R)$ Reissner-Nördstrom spacetime is

$$M_0 = -(M^2 - \rho)^{1/2}. \quad (34)$$

Equation (34) agrees with Geroch's result $M_0 = -M$ as a special case for the Schwarzschild solution ($\rho \rightarrow 0$).

B. Kerr-Newman solution

The Kerr-Newman metric in Boyer-Lindquist-type (t, r, θ, ϕ) coordinates takes the form (e.g. [28])

$$\begin{aligned} g_{tt} &= 1 + \frac{\rho - 2Mr}{r^2 + a^2 \cos^2 \theta}, \\ g_{rr} &= -\frac{r^2 + a^2 \cos^2 \theta}{a^2 - 2Mr + r^2 + \rho}, \\ g_{t\phi} &= \frac{a(2Mr - \rho) \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}, \\ g_{\theta\theta} &= -r^2 - a^2 \cos^2 \theta, \\ g_{\phi\phi} &= -\left(r^2 + a^2 - \frac{a^2(\rho - 2Mr) \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \right) \sin^2 \theta. \end{aligned} \quad (35)$$

The parameter $a = J/M$ measures the spin, and ρ is once again the total charge as in (26) for the Einstein-Maxwell theory (but not in modified gravity).

The Kerr-Newman metric, which has $R = 0$, is a vacuum solution to any $f(R)$ theory⁷ with $f(0) = f'(0) = 0$ by the same argument as in Sec. IV A. The Ricci tensor has a nonvanishing off-diagonal component

$$R_t^\phi = \frac{2a\rho}{(r^2 + a^2 \cos^2 \theta)^3}, \quad (36)$$

so the generalized twist must be constructed. Solving Eq. (13) using the equivalent system (C2)–(C4) we find \mathbf{Q} has nonzero components

$$Q_r = -\frac{2a\rho \cos \theta}{(r^2 + a^2 \cos^2 \theta)^2}, \quad (37)$$

and

$$Q_\theta = -\frac{2a\rho \sin \theta}{(r^2 + a^2 \cos^2 \theta)^2}, \quad (38)$$

and hence

$$\omega = \frac{2aM \cos \theta}{r^2 + a^2 \cos^2 \theta}, \quad (39)$$

from (6) and (12). Equation (16) may now be written down in its entirety. We do not record the result here as it is

⁷This solution also appears as a vacuum solution in the Randall-Sundrum braneworld class of theories [51].

lengthy. We use the relation $\lambda = g_{tt}$ and Eq. (39) to express the coordinates r and θ as well the components of \mathbf{Q} in terms of $r = r(\lambda, \omega)$, $\theta = \theta(\lambda, \omega)$, and $Q_\alpha = Q_\alpha(\lambda, \omega)$. All terms involving the Ricci tensor may then be expressed in terms of λ and ω .

Defining a new radial coordinate \bar{R} through the implicit relationship

$$r = \frac{1}{\bar{R}} \left[1 + M\bar{R} + \frac{1}{4}(M^2 - a^2 - \rho)\bar{R}^2 \right], \quad (40)$$

we find that

$$\Omega = \frac{\bar{R}^2}{\left\{ \left[1 - \frac{1}{4}(M^2 - a^2 - \rho)\bar{R}^2 \right]^2 - a^2\bar{R}^2 \sin^2\theta \right\}^{1/2}}, \quad (41)$$

yields the desired form of the line element

$$d\tilde{\sigma}_3^2 = d\bar{R}^2 + \bar{R}^2 d\theta^2 + \frac{\bar{R}^2 \sin^2\theta d\phi^2}{1 - a^2\bar{R}^2 \sin^2\theta \left[1 - \frac{1}{4}(M^2 - a^2 - \rho)\bar{R}^2 \right]^{-2}}, \quad (42)$$

with $\tilde{\mathbf{h}} = \Omega^2 \mathbf{h}$. A quick check shows that (42) does satisfy the constraints at the point Λ ($\bar{R} = 0$).

Solving (16) and iteratively evaluating the moments $\mathbf{P}^A|_\Lambda$ through equation (21) we obtain the multipole moments for the Kerr-Newman solution in $f(R)$ gravity:

$$M_{2s} = (-1)^{s+1} (M^2 - \rho)^{1/2} a^{2s}, \quad (43)$$

$$M_{2s+1} = 0, \quad (44)$$

$$J_{2s+1} = (-1)^s (M^2 - \rho)^{1/2} a^{2s+1}, \quad (45)$$

$$J_{2s} = 0. \quad (46)$$

Again, these results agree with those found by Hansen [9] for the Kerr solution in the limit $\rho \rightarrow 0$. It is worth pointing out that the moments may each be written in terms of M_0 and J_1 (instead of M , ρ , and a) to recover the same expressions as in Kerr, i.e.

$$M_{2s} = (-1)^s M_0 a^{2s}, \quad (47)$$

and

$$J_{2s+1} = (-1)^s J_1 a^{2s}. \quad (48)$$

Therefore, in this sense, the (Kerr) no-hair relation is not violated. Moreover, this allows one to interpret ρ as an $f(R)$ renormalization of the mass term in the Kerr metric; if the Kerr mass M is replaced by the $f(R)$ term $\sqrt{M^2 - \rho}$, then the Kerr moments are indistinguishable from the $f(R)$ Kerr-Newman moments.

V. QPOS IN MICROQUASARS

To illustrate how the above results can be applied to practical problems, we consider X-ray observations of QPOs in accreting compact objects, especially microquasars [31,52]. There are several models in the literature for the origin of QPOs (e.g. [53]). One possibility is that QPOs arise from normal modes of oscillation trapped due to Lense-Thirring precession in the accretion disks surrounding black holes [26,54]. We explore two simplified models along these lines: the diskoseismology model (e.g. [55]) in Sec. V A and the kinematic resonance model (e.g. [56]) in Sec. V B. In the diskoseismology model, QPO frequencies are attributed to the lowest order gravity (g -) and corrugation (c -) modes [57]. In the kinematic resonance model, the QPOs arise from resonances between the radial epicyclic frequencies of disk particles and the corresponding Keplerian frequencies [56]. Assuming particular resonances for the objects GRS 1915 + 105 and GRO J1655-40, we explore the range of spin a and ‘‘charge’’ ρ values compatible with the QPO models above and independent mass measurements. We stick to using the parameter ρ throughout this section, although we can easily solve for (say) the mass quadrupole moment M_2 defined in (43) from

$$\rho = M^2 - M_2^2/a^4. \quad (49)$$

There are large systematic uncertainties in determining black hole spins from QPOs, because there are many viable models through which the data can be interpreted. The purpose of this section is to point out that there may also be uncertainties in the gravitational theory, which compound those that enter through QPO modeling.

A. Diskoseismology model

For a massive particle falling freely in a stationary, axisymmetric metric with four-momentum

$$p^\alpha = \mu \frac{dx^\alpha}{d\tau}, \quad (50)$$

we have three conserved quantities: rest mass μ (which we set to 1 without loss of generality), energy $E = -p_t$, and axial angular momentum $L_z = p_\phi$ [28]. Raising the indices on these momenta we obtain the relations [13]

$$p^t = -\frac{g_{\phi\phi}E + g_{t\phi}L_z}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}, \quad (51)$$

$$p^\phi = \frac{g_{t\phi}E + g_{tt}L_z}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}. \quad (52)$$

If we specialize to motion in the equatorial plane ($p^\theta = 0$), appropriate for a thin, unwarped accretion disk, the remaining independent component of the geodesic equations turns into

$$\frac{1}{2}g_{rr}(p^r)^2 = \frac{1}{2}[-g_{tt}(p^t)^2 - 2g_{t\phi}p^t p^\phi - g_{\phi\phi}(p^\phi)^2 - 1]. \quad (53)$$

The right-hand side of (53), written V_{eff} , is called the effective potential. Circular orbits are then characterized by the additional constraints $p^r = 0$ and $\frac{dp^r}{dr} = 0$ [28,31]. Specializing to this case, we have from Eq. (53)

$$V_{\text{eff}} = 0, \quad (54)$$

and

$$\frac{d}{dr} \left(\frac{V_{\text{eff}}}{g_{rr}} \right) = 0. \quad (55)$$

Using the momenta (51) and (52), solving Eqs. (54) and (55) leads to expressions for the energy

$$E = \frac{-2Mr + r^2 \pm a(Mr - \rho)^{1/2} - a\rho}{r\sqrt{-3Mr + r^2 \pm 2a(Mr - \rho)^{1/2} + 2\rho}}, \quad (56)$$

and axial angular momentum

$$L_z = \pm \frac{[a^2 + r^2 \mp 2a(Mr - \rho)^{1/2}](Mr - \rho)^{1/2} \mp a\rho}{r\sqrt{-3Mr + r^2 \pm 2a(Mr - \rho)^{1/2} + 2\rho}}, \quad (57)$$

where the upper sign is taken for prograde orbits and the lower sign for retrograde orbits. Restoring dimensional factors of G and c and using expressions (56) and (57) we define the Keplerian frequency Ω_ϕ ,

$$\Omega_\phi = -\frac{c^3}{G} \frac{g_{t\phi}E + g_{tt}L_z}{g_{\phi\phi}E + g_{t\phi}L_z}, \quad (58)$$

the radial epicyclic frequency κ_r given by [31]

$$\kappa_r^2 = -\left(\frac{c^3}{G}\right)^2 \frac{\partial^2}{\partial r^2} \left[\frac{V_{\text{eff}}}{g_{rr}(p^r)^2} \right], \quad (59)$$

and the angular epicyclic frequency Ω_θ given by [31]

$$\Omega_\theta^2 = -\left(\frac{c^3}{G}\right)^2 \frac{\partial^2}{\partial \theta^2} \left[\frac{V_{\text{eff}}}{g_{\theta\theta}(p^t)^2} \right], \quad (60)$$

evaluated in the equatorial plane $\theta = \pi/2$. The expressions for these quantities as functions of r , E , and L_z are lengthy but can be evaluated easily using the metric components written down in Sec. IV B.

Following previous authors [26,54], we assume that the fundamental g -mode frequency is given by the maximum value of $\kappa_r/2\pi$ with respect to r , while the fundamental c -mode frequency is given by the Lense-Thirring frequency $\Omega_{\text{LT}} = \Omega_\phi - \Omega_\theta$ (e.g. [28]) evaluated at the innermost stable circular orbit (ISCO) [31]. The radius of the ISCO, r_{ISCO} , for the Kerr-Newman metric is given as the smallest (real) solution to the equation,

$$0 = M(6M - r_{\text{ISCO}})r_{\text{ISCO}}^2 + a^2(3Mr_{\text{ISCO}} - 4\rho) \pm 8a(Mr_{\text{ISCO}} - \rho)^{3/2} - 9Mr_{\text{ISCO}}\rho + 4\rho^2. \quad (61)$$

In Fig. 2 we plot the resulting g - and c -mode frequencies for the Kerr-Newman metric as a function of spin for a range of values of ρ . As ρ increases in the positive direction, the frequency of the relevant g -mode can become very large ($\gtrsim 0.3$ kHz), when $a/M \sim 0.9$ for a $10M_\odot$ object, as can the frequency of the c -mode (\gtrsim kHz). In GR, the g -mode frequency is bounded above by $248.3(M/10M_\odot)$ Hz for the extreme case $a = M$. Allowing for positive values of ρ in $f(R)$ gravity permits the frequency to exceed this bound. Furthermore, the c -mode frequency is undefined in the presence of a naked singularity, which forms for the Kerr-Newman solution when $a > \sqrt{3}M/\sqrt{5}$ for $\rho = 0.4M^2$ as indicated by the spike in the dashed curve. We see that for

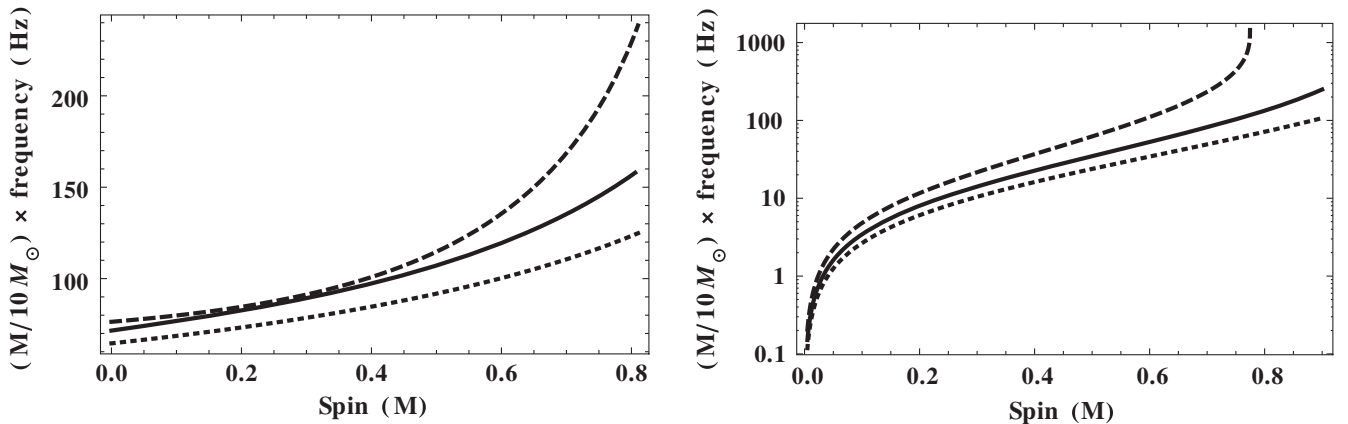


FIG. 2. Fundamental g -mode (left panel) and c -mode (right panel) frequencies as functions of spin with $\rho = 0$ (solid), $\rho = 0.4M^2$ (dashed) and $\rho = -0.4M^2$ (dotted).

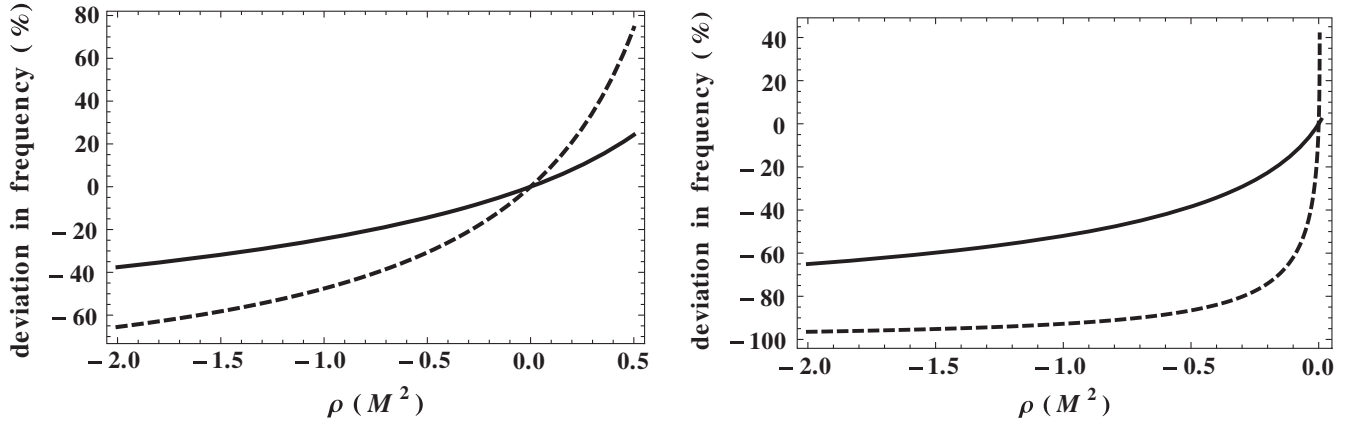


FIG. 3. Deviations in predicted g -mode (solid curve) and c -mode (dashed curve) frequencies from the Kerr frequencies ($\rho = 0$) as a function of ρ for GRO J1655-40 (left panel) with spin parameter $a = 0.29M$ and GRS 1915 + 105 (right panel) with spin parameter $a = 0.998M$.

$0.1 \lesssim a/M \lesssim 0.4$ the g -mode frequencies predicted for $\rho = 0.4M^2$ and $\rho = 0$ are almost indistinguishable.

In Fig. 3, we plot deviations in the g - and c -mode frequencies from the Kerr value ($\rho = 0$) for two different quasar systems GRS 1915 + 105 and GRO J1655-40. To demonstrate, we take the Middleton *et al.* values [32] for the spin of GRS 1915 + 105 ($a = 0.998M$) and the Motta *et al.* values [58] for the spin of GRO J1655-40 ($a = 0.29M$). These values were obtained by fitting continuum and precession models to the data [59] and are compatible with our estimates computed in Sec. VB. We see that increasing ρ in the positive direction increases the frequencies of the g - and c -modes. For $a = 0.998M$ in GRS 1915 + 105, the model breaks down for $\rho \approx 10^{-2}M^2$ (naked singularity), as indicated by the spike in the c -mode frequency.

B. Kinematic resonance model

In the kinematic resonance model, the measured QPO frequencies are identified with the Kepler frequency Ω_ϕ and the radial epicyclic frequency κ_r , whose ratio is a simple fraction in resonance. Here we determine combinations of a and ρ which are consistent with the measured frequencies.

QPO pairs are typically observed to have frequency ratios of 3/2 or 1/2 [34]. Assuming these ratios are universal, several authors have estimated the spins of black holes assuming a Kerr description (for e.g. GRS 1915 + 105 [52] and others [56,60]). We show here that if the object is actually a Kerr-Newman black hole in $f(R)$ gravity, different spin estimates arise. If, in addition, one has an independent measure of the mass of the black hole, interesting bounds follow on a and ρ . As ρ can be swapped for the mass quadrupole moment (or any higher order moment) of the Kerr-Newman metric, we can also test whether or not the no-hair theorem is violated in GR, if $\rho = 0$ does not allow for any value of a to fit the data. One

needs an independent measurement of the spin to then also test whether or not any $f(R)$ no-hair relations are violated.

Independent measurements of the black hole mass in the systems GRS 1915 + 105 ($M = 14.0 \pm 4.4M_\odot$) and GRO J1665-40 ($M = 5.31 \pm 0.07M_\odot$) have been made using optical ray tracing [61,62] and X-ray timing techniques [58]. Given a QPO pair at a given resonance with frequencies q_1 and q_2 we determine valid combinations of a and ρ as follows. We solve for the radius r_{res} such that $\kappa_r(r_{\text{res}}) = q_1$ and determine values of a and ρ which yield $\Omega_\phi(r_{\text{res}}) = q_2$. We obtain a band of possible values owing to the error bars placed on the measured mass.

Using Doppler spectroscopy, Greiner *et al.* [62] estimated the mass of GRS 1915 + 105 via Kepler's Third Law with an orbital separation of $(108 \pm 4)R_\odot$, (i.e., separation $\gg r_{\text{Horizon}}$) from data taken at the Antu European South Observatory Very Large Telescope. The inferred mass is the same for both the Kerr and Kerr-Newman metrics, which have the same far-field (Newtonian) limit. In contrast, the more accurate mass estimate obtained for GRO J1665-40 through X-ray timing assumes a Kerr metric to analyse epicyclic frequencies [58]. In principle, this approach is incompatible with our analysis, since these frequencies, and hence the mass estimate, are computed from the metric components (5) [see Eqs. (58)–(60)]. Independent optical measurements of the mass of GRO J1665-40, e.g. by Beer and Podsiadlowski [63] who found $M = 5.4 \pm 0.3M_\odot$ from Kepler's Third Law, avoid this problem while remaining consistent with Motta *et al.* [58]. Our aim is to demonstrate that, even with tight error bars on the mass, as achieved by [58], a large range of spins can accommodate QPO observations, a conclusion which is unaltered (but is less evident) if one uses the self-consistent values of the mass from Beer and Podsiadlowski [63]. As better data become available in the future, it would ultimately be worth recalculating the measured mass based on the assumption of a central Kerr-Newman object following the rigorous

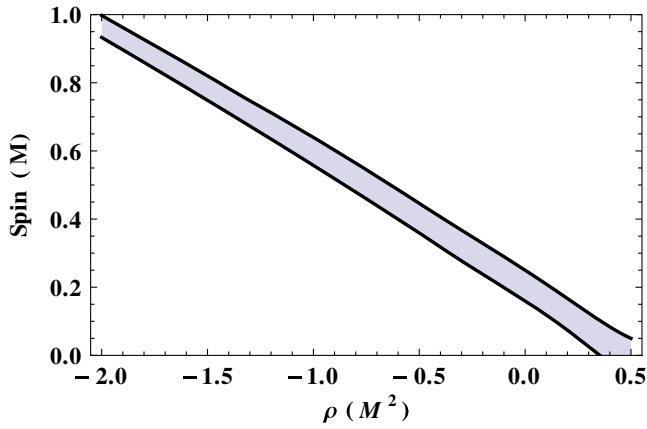


FIG. 4. Values of the spin a and “charge” ρ for the object GRO J1655-40 consistent with a measured mass of $M = 5.31 \pm 0.07M_{\odot}$ and the observed QPO frequencies of 450 Hz and 300 Hz. The shaded region indicates the allowed zone.

statistical approach of Motta *et al.* [58] for an entirely self-consistent analysis, but this lies outside the scope of the present paper.

Méndez *et al.* [64] reported pairs of QPOs from GRS 1915 + 105 with frequencies 35 Hz and 67 Hz, and from GRO J1655-40 with frequencies 300 Hz and 450 Hz. Clearly the QPOs come from 1:2 and 2:3 resonances respectively. For the rough estimates below we assume that these are exact values. The measured error bars are quite tight: 35.1 ± 0.4 Hz and 67.9 ± 0.1 Hz for GRS 1915 + 105, and 289 ± 4 Hz and 446 ± 2 Hz for GRO J1655-40 respectively.

In Fig. 4 we plot the possible parameter space for the object GRO J1655-40 with error bars flowing from M as given by [58]. The $\sim 1\%$ error on M results in a thin band, with $0 \leq a/M \leq 1$ and $-2.0 \lesssim \rho/M^2 \lesssim 0.5$. As ρ increases in the positive direction, the required spin parameter decreases. This is in contrast to the behavior of the quadrupole moment (43), which decreases when either ρ is positive and increasing, or a decreases. Compatibility with the measured QPO frequencies then, for this model, demands that the quadrupole moment *decreases* if ρ is positive and increasing, or *increases* if ρ is negative and decreasing. For $\rho \gtrsim 0.35M^2$ the lower end of the measured mass ($M \lesssim 5.25M_{\odot}$) requires $a < 0$.

In Fig. 5 we repeat the analysis for the object GRS 1915 + 105 with error bars on the mass as given in [61]. We obtain a large range of allowed combinations for a and ρ given the relatively poorly constrained mass measurement of $M = 14.0 \pm 4.4M_{\odot}$. As ρ increases from 0 to $0.25M^2$, the predicted range of a drops from $0.2 \lesssim a/M \lesssim 0.999$ to $0.06 \lesssim a/M \lesssim 0.83$, meaning that GRS 1915 + 105 may not be a near-extreme Kerr black hole as previously suspected [32]. In contrast to the GRO J1655-40 case, we find that for $\rho \gtrsim 0.25M^2$ the top end of the measured mass range ($M \gtrsim 17M_{\odot}$) does not allow for a solution,

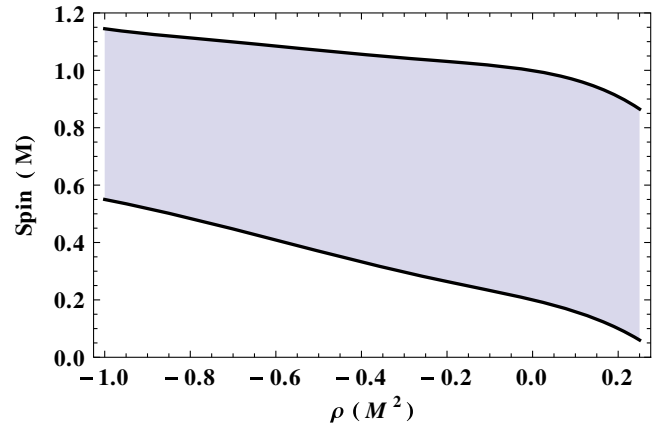


FIG. 5. Values of the spin a and “charge” ρ for the object GRS 1915 + 105 consistent with a measured mass of $M = 14.0 \pm 4.4M_{\odot}$. The large error bars in the mass mean that many possible values of a and ρ allow for the measured QPO frequencies of 67 Hz and 35 Hz. The shaded region indicates the allowed zone.

because the required value of a exceeds the naked singularity threshold $a > \sqrt{3}/2M$.

We close this section with a suggested experiment to determine all three values M , a , and ρ . In reality, the quasar system may not be in equilibrium, and will be subject to perturbations, such as fluctuations in the accretion flow [65,66]. These perturbations will distort the mass quadrupole moment, resulting in the emission of gravitational waves [35,38], whose luminosities will be sensitive to the governing theory of gravitation [67,68]. Given a combination of gravitational wave detections (from e.g. LIGO [69]) and QPO measurements one could uniquely determine a , M , and ρ (e.g. [70]). Some perturbation theory has been developed for $f(R)$ gravity (e.g. [15,33]) and would be useful in this context. Furthermore, M , a , and ρ may also be determined independently by measuring multipole moments of a central black hole from gravitational wave observations of an extreme-mass ratio inspiral, as pioneered by Ryan [38,71].

VI. DISCUSSION

In this paper we derive a set of mass and angular momentum multipole moments, that generalize earlier works to higher-order curvature theories of gravity, like $f(R)$ theories. The twist 4-vector ω^\dagger used by Hansen [9] is generalized to ensure that there exists a scalar field ω such that $\omega_\alpha = \nabla_\alpha \omega$. The multipole moments then arise by evaluating moment functions, which satisfy Poisson equations.

In Sec. V we calculate the multipole moments associated with a Kerr-Newman spacetime. Interestingly, the latter metric can arise either as a coupled Einstein-Maxwell solution of GR or as a pure vacuum solution to a range

of $f(R)$ theories. Indeed, any GR solution with matter also arises as a vacuum $f(R)$ solution provided the trace of the stress energy tensor, $\text{Tr}(\mathbf{T})$, is constant (as in the case of Maxwell fields in four spacetime dimensions) [17]. This has interesting consequences for the I-Love-Q type results regarding no-hair-like relations (or three-hair relations) among multipole moments for neutron stars and quark stars pioneered by Yunes, Yagi, and others [72–74]. Suppose we make a multipole measurement of a compact object, and it disagrees with the Kerr prediction. In GR, this would indicate the presence of an inherently hairy object such as a neutron star with strong magnetic field [75]. However, it is also possible the object is actually a nonhairy black hole but that the theory of gravity is not GR. Higher-order moment measurements may remove the ambiguity but not if both objects satisfy no-hair-type relations (cf. [76,77]).

Our results demonstrate that accretion disk models of QPOs, specifically the diskoseismology and kinematic resonance models, may be unable to uniquely identify properties of black holes or refute the no-hair theorem when combined with QPO data. We find that a wide range of possible combinations of the spin and “charge” are possible to explain QPO observations of GRS 1915 + 105 and GRO J1665-40. Interestingly, the g -mode frequency for the Kerr case is bounded above due to $a \leq M$, while for the Kerr-Newman case we have a free parameter ρ which, when large, allows for QPOs with kHz frequencies within the diskoseismology paradigm, such as those observed in low mass X-ray binaries like Sco X-1 [78].

Suppose we consider an axisymmetric spacetime that is not stationary. One can separate out the azimuthal terms by constructing a spacelike Killing vector $\xi = \frac{\partial}{\partial \phi}$, allowing the construction of S with respect to the spacelike Killing vector ξ instead of ξ . It would be worth exploring this avenue for generalizing the moment calculations in future work. For example, one could calculate the multipole moments associated with a radiating Vaidya spacetime, which appears as a vacuum solution to some $f(R)$ theories [79]. Additionally, one can apply the results to metrics other than Kerr-Newman, such as those derived using pointlike Lagrangian techniques [80]. In GR, results such as those of Xanthopoulos and Gursel [81,82] demonstrate that each spacetime carries with it a unique set of multipole moments and vice versa. Furthermore, it can be shown that any stationary, asymptotically flat solution to Einstein’s equations approaches the Kerr solution near infinity [10,36]. Since the Geroch-Hansen moments are evaluated at infinity, the two results together show that the Kerr metric is unique in GR. The extension of the moments presented here may help to prove uniqueness results in modified gravity along these lines.

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APPENDIX A: TENSOR RELATIONS ON S

In this section we derive some useful relations for tensor fields over S . First, it was shown by Geroch [20] that there is a one-to-one correspondence between tensor fields $\mathcal{T}_{a\dots c}^{b\dots d}$ on S and tensor fields $T_{a\dots c}^{b\dots d}$ on M . Hence the covariant derivative D over S can be written in terms of the covariant derivative ∇ over M as [20,21]

$$D_e \mathcal{T}_{a\dots c}^{b\dots d} = h_e^p h_a^m \dots h_c^n h_r^b \dots h_s^d \nabla_p T_{m\dots n}^{r\dots s}, \quad (\text{A1})$$

where h is the 3-metric on S as in (5). Let k be an arbitrary vector field over S . Then using Eq. (A1) we find

$$D_a D_b k_c = h_a^p h_b^s h_c^t \nabla_p \nabla_s k_t - \lambda^{-1} h_a^p h_b^q h_c^r (\nabla_p \xi_q) \xi^s \nabla_s k_r - \lambda^{-1} h_a^p h_b^q h_c^r (\nabla_p \xi_r) \xi^t \nabla_q k_t. \quad (\text{A2})$$

By antisymmetrizing the above equation to find $(D_a D_b - D_b D_a)k_c$, we obtain an expression for the Riemann tensor \mathcal{R}_{bcd}^a over S :

$$\mathcal{R}_{abcd} = h_{[a}^p h_b^q h_c^r h_d^s [R_{pqrs} + 2\lambda^{-1} (\nabla_p \xi_q) (\nabla_r \xi_s) + 2\lambda^{-1} (\nabla_p \xi_r) (\nabla_q \xi_s)]. \quad (\text{A3})$$

Contracting indices and simplifying the result using the definitions (4) and (13) we find that the Ricci tensor on S $\mathcal{R}_{ab} = \mathcal{R}_{acb}$ satisfies

$$\begin{aligned} \mathcal{R}_{ab} = & \frac{1}{2} \lambda^{-2} [D_a \omega D_b \omega - h_{ab} D^m \omega D_m \omega] \\ & + \frac{1}{2} \lambda^{-1} D_a D_b \lambda - \frac{1}{4} \lambda^{-2} (D_a \lambda) (D_b \lambda) + h_a^m h_b^n R_{mn} \\ & + \frac{1}{2} \lambda^{-2} [Q_a Q_b - D_a \omega Q_b - Q_a D_b \omega - h_{ab} Q^m Q_m \\ & + 2h_{ab} Q^m D_m \omega]. \end{aligned} \quad (\text{A4})$$

From the definitions (3) and (12) for λ and ω , we find that these objects satisfy the Laplace equations [using (A1)]

$$D^a D_a \omega = \frac{3}{2} \lambda^{-1} D_m \omega D^m \lambda - \frac{3}{2} \lambda^{-1} Q_m D^m \lambda + D^a Q_a, \quad (\text{A5})$$

and

$$\begin{aligned}
 D^j D_j \lambda &= \frac{1}{2} \lambda^{-1} (D^m \lambda) (D_m \lambda) \\
 &\quad - \lambda^{-1} [D^m \omega D_m \omega - 2D^m \omega Q_m + Q^m Q_m] \\
 &\quad - 2R_{mn} \xi^m \xi^n. \tag{A6}
 \end{aligned}$$

The fundamental field equations on S are given by (A5) and (A6) in combination with a set of field equations for h_{ij} will depend on the specific theory of gravity. As an

example, the vacuum field equations for $f(R)$ gravity over M read (e.g. [15])

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) f'(R) = 0. \tag{A7}$$

Using the identities (A1) and (A4), the $f(R)$ field equations over S read

$$\begin{aligned}
 &\left\{ \frac{1}{2} \lambda^{-2} [D_a \omega D_b \omega - h_{ab} D^m \omega D_m \omega] + \frac{1}{2} \lambda^{-1} D_a D_b \lambda - \frac{1}{4} \lambda^{-2} (D_a \lambda) (D_b \lambda) + h_a^m h_b^n R_{mn} + \frac{1}{2} \lambda^{-2} [Q_a Q_b - D_a \omega Q_b - Q_a D_b \omega \right. \\
 &\quad \left. - h_{ab} Q^m Q_m + 2h_{ab} Q^m D_m \omega] + h_{ab} D^j D_j - D_a D_b \right\} f' \left[-\lambda^{-2} \left(\frac{3}{2} D^m \omega D_m \omega + \frac{3}{2} Q_k Q^k - 3D_p \omega Q^p \right) \right. \\
 &\quad \left. + R_{mn} \left(h^{mn} - \frac{1}{\lambda} \xi^m \xi^n \right) \right] - \frac{1}{2} h_{ab} f \left[-\lambda^{-2} \left(\frac{3}{2} D^m \omega D_m \omega + \frac{3}{2} Q_k Q^k - 3D_p \omega Q^p \right) + R_{mn} \left(h^{mn} - \frac{1}{\lambda} \xi^m \xi^n \right) \right] \\
 &= f'(\mathcal{R}) \mathcal{R}_{ab} - \frac{f(\mathcal{R})}{2} h_{ab} + (h_{ab} D^j D_j - D_a D_b) f'(\mathcal{R}). \tag{A8}
 \end{aligned}$$

The fundamental equations for an $f(R)$ gravity over S with Killing vector ξ are (A5), (A6), and (A8). They reduce to those found by Geroch [8] for $R_{\mu\nu} = 0$ and $f(R) = R$.

APPENDIX B: CONFORMALLY INVARIANT POISSON EQUATIONS

The wave equation for a scalar field ϕ with forcing term f on a curved background,

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = f, \tag{B1}$$

is not conformally invariant; a metric g scaled by some function ψ (i.e. $\tilde{g} = \psi^2 g$) will not be a solution to (B1) unless ψ is trivial. In general, a conformally invariant equation will not exist unless f is of conformal weight $-(1 + n/2)$, i.e. $\tilde{g} \rightarrow \psi^2 g \Rightarrow f \rightarrow \psi^{-1-n/2} f$ [83]. To obtain a conformally invariant equation one must insert a term involving the Ricci scalar of the form (see [35], page 447)

$$\left[g^{\alpha\beta} \nabla_\alpha \nabla_\beta - \frac{n-2}{4(n-1)} R \right] \phi = f, \tag{B2}$$

where we have $n = \dim(M)$. On the space S we have $n = 3$ and as such Eq. (B2) reduces to

$$\left(h^{ab} D_a D_b - \frac{\mathcal{R}}{8} \right) \phi = f, \tag{B3}$$

where h is the 3-metric as in (5). Equation (B3) is precisely the form of the Poisson equations (8) considered in this paper,

where $f = \frac{15}{16} \lambda^{-2} \kappa \phi$. This is the same function f used by Hansen, where it was proved that f is of conformal weight $-5/2$ [9], which is equal to $-1 - n/2$ for $n = 3$. Therefore, Eq. (8) is conformally invariant.

APPENDIX C: THE DIFFERENTIAL EQUATION FOR Q

In order to generalize the twist scalar ω , we require the introduction of a one-form Q obeying

$$dQ = -\star[\xi \wedge R(\xi)], \tag{C1}$$

where the differential equation is solved with Dirichlet boundary conditions to ensure $Q = 0$ for $R_{\nu}^{\mu} \xi^\nu = 0$. In a Boyer-Lindquist coordinate system $\{t, r, \theta, \phi\}$ we may take $\xi^\mu = \delta_t^\mu$ and $Q = (0, Q_r, Q_\theta, Q_\phi)$ without loss of generality. Then (C1) can be written as the coupled system

$$\frac{\partial Q_\phi}{\partial \theta} - \frac{\partial Q_\theta}{\partial \phi} = 2\sqrt{-g} R_t^r, \tag{C2}$$

$$\frac{\partial Q_r}{\partial \phi} - \frac{\partial Q_\phi}{\partial r} = 2\sqrt{-g} R_t^\theta, \tag{C3}$$

$$\frac{\partial Q_r}{\partial \theta} - \frac{\partial Q_\theta}{\partial r} = -2\sqrt{-g} R_t^\phi, \tag{C4}$$

where we make use of the symmetry $\Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha$ of the Christoffel symbols. In practice, the system (C2)–(C4) can be solved through repeated integration (e.g. [84]).

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