Quantum discord of cosmic inflation: Can we show that CMB anisotropies are of quantum-mechanical origin?

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We investigate the quantumness of primordial cosmological fluctuations and its detectability. The quantum discord of inflationary perturbations is calculated for an arbitrary splitting of the system, and shown to be very large on super-Hubble scales. This entails the presence of large quantum correlations, due to the entangled production of particles with opposite momenta during inflation. To determine how this is reflected at the observational level, we study whether quantum correlators can be reproduced by a nondiscordant state, i.e. a state with vanishing discord that contains classical correlations only. We demonstrate that this can be done for the power spectrum, the price to pay being twofold: first, large errors in other two-point correlation functions that cannot however be detected since they are hidden in the decaying mode; second, the presence of intrinsic non-Gaussianity, the detectability of which remains to be determined but which could possibly rule out a nondiscordant description of the cosmic microwave background. If one abandons the idea that perturbations should be modeled by quantum mechanics and wants to use a classical stochastic formalism instead, we show that any two-point correlators on super-Hubble scales can be exactly reproduced regardless of the squeezing of the system. The latter becomes important only for higher order correlation functions that can be accurately reproduced only in the strong squeezing regime.

I. INTRODUCTION

According to the most recent cosmological observations [1–3], inflation [4–9] provides a mechanism for generating large scale fluctuations [10–15] that fits the data very well [16–20]. The scope of this mechanism is not limited to broad phenomenological predictions but is also deeply connected to the realm of quantum gravity, which makes it particularly interesting from a theoretical point of view. Indeed, according to inflation, the large scale structure in our Universe and the cosmic microwave background (CMB) anisotropies are nothing but quantum fluctuations of the gravitational and inflaton fields, stretched over cosmological distances.

However, the above claim is often taken with a grain of salt probably because, in practice, astronomers analyze the data with purely classical techniques and apparently never need to rely on the quantum formalism to understand them. This situation raises the question of the observational signature of the quantum nature of inflationary perturbations [21–51]. Is quantum mechanics really necessary in order to explain the properties of the CMB anisotropies or can everything be described in a classical context (in which case doubts could be cast on the quantum origin of the perturbations)? To put it differently, what would be wrong and which properties would be missed if the analysis were performed in a fully classical framework? The goal of this article is to address this question both at the theoretical and observational levels. The question is particularly timely since, in quantum information theory, new tools that are directly relevant to the problems discussed in this article have recently been introduced, such as the quantum discord [52,53].

This paper is organized as follows. In the next section, Sec. II, we briefly review the theory of inflationary cosmological perturbations of quantum mechanical origin. We pay special attention to the quantum state in which the fluctuations are placed, namely a two-mode squeezed state. In Sec. III, we introduce the quantum discord [52,53], which is a quantity that has recently received a lot of attention in the quantum information community and which is designed to measure the quantumness of correlations present in a quantum state. It was first applied to cosmological fluctuations in Ref. [54], where, in the case where decoherence of perturbations [55–63] occurs due to environment degrees of freedom, the quantum discord of the joint primordial perturbations-environment system was

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calculated. In this paper, we show that, even in the absence of an environment, particles with wave numbers $\boldsymbol{k}$ and $-\boldsymbol{k}$ are created by the same quantum event, and, as such, are highly entangled. This gives rise to a large quantum discord. In practice, we find that it is proportional to the squeezing parameter which is itself proportional to the exponential of the number of e-folds spent between the Hubble radius exit during inflation and the end of inflation (typically ~ 50 e-folds for the scales observed in the CMB). We conclude that the CMB is a highly nonclassical system that is placed in an entangled quantum state. We also consider the discord for an arbitrary splitting of the system, showing that it is always nonvanishing except for the case where the two subsystems correspond to the real and imaginary parts of the Mukhanov-Sasaki variable. This shows that a purely classical description of cosmological perturbations necessarily fails to reproduce all its quantum correlations. To make this statement more explicit, we then calculate these correlators explicitly. In Sec. IV, we first derive the predictions obtained from the quantum formalism when the system is placed in a two-mode squeezed state. We formulate the problem in the language of quantum information theory in Sec. IV A, a new result already interesting in itself, and then, in Sec. IV B, we establish the expression of two-point correlation functions. Then we ask whether these correlations can be reproduced by other means. In Sec. IV, we derive the correlators that would be produced by a quantum state with "classical correlations" only (i.e. a state with vanishing quantum discord). For two-point correlation functions, see Sec. V A, we find that, in practice, all the nonclassical features are in fact hidden in unobservable quantities. However, in Sec. V B, we calculate the four-point correlation functions and show that, in this situation, one expects non-Gaussianities, at a level that remains to be established but which could possibly lead to the indirect detection of the large CMB quantumness. In Sec. VI, we address the same question but for a classical stochastic distribution. We first present some general considerations in Sec. VI A and then, in Sec. VI B, we study the correlation functions in the case of a Gaussian distribution. In this case, we find that all observable quantities are correctly reproduced on large scales, and we highlight the role that squeezing plays in concealing quantum correlations from observable quantities. In particular, we argue that it plays no role in our ability to reproduce the two-point correlation functions. Finally, in Sec. VII, we present a few concluding remarks, and we end the paper with several appendixes containing various technical aspects.

II. INFLATIONARY COSMOLOGICAL PERTURBATIONS

Inflation is an epoch of accelerated expansion, driven by a scalar field (the inflaton field) and taking place before the standard hot big bang phase. Introduced for the first time nearly 35 years ago, it has gradually become the standard paradigm for the early Universe because it offers simple and elegant solutions to the puzzles of the big bang theory.

As already mentioned in the introduction, one of the most interesting aspects of inflation is that it gives an explanation for the origin of the large scale structure and CMB anisotropies. According to inflation, they are vacuum quantum fluctuations amplified by gravitational instability and stretched over cosmological distances. At the technical level, two types of perturbations are produced: gravitational waves and scalar fluctuations. In the rest of this article, we consider scalar fluctuations only but the present analysis could straightforwardly be applied to tensor modes (since, at leading order in slow roll, they have the same dynamics as scalar perturbations). The scalar sector can be described by a single quantity, the so-called Mukhanov-Sasaki variable [11,64], related to the curvature perturbation $\zeta(\eta, \boldsymbol{x})$ through

$$v(\eta, \boldsymbol{x}) = a M_{\mathrm{Pl}} \sqrt{2\epsilon_1} \zeta(\eta, \boldsymbol{x}). \tag{1}$$

Here, $a(\eta)$ is the Friedmann-Lemaître-Roberston-Walker scale factor, η is the conformal time (related to the cosmic time t by $\mathrm{d}t = a\mathrm{d}\eta$), M_{Pl} denotes the Planck mass and $\epsilon_1 \equiv 1 - (a'/a)'/(a'/a)^2$ is the first slow-roll parameter [65,66]. This last quantity controls whether inflation takes place or not. Indeed, since $\ddot{a}/a = (\dot{a}/a)^2(1 - \epsilon_1)$, where a dot means derivative with respect to cosmic time, inflation is equivalent to having $\epsilon_1 < 1$. Moreover, the Planck data, in particular, the measurement [2] of the scalar spectral index $n_{\mathrm{S}} = 0.968 \pm 0.006$ at 68% confidence level, and the tensor-to-scalar ratio 95% constraint $r < 0.11$, indicate that we not only have $\epsilon_1 < 1$ but in fact $\epsilon_1 \ll 1$, a regime known as slow-roll inflation [65–69].

The next step consists in deriving an equation of motion for $v(\eta, \boldsymbol{x})$. Expanding the action of the system (i.e. Einstein-Hilbert action plus the action of a scalar field) up to second order in the perturbations, one obtains the following expression [70]:

$$^{(2)}\delta S = \frac{1}{2} \int \mathrm{d}^4 x \left[(v')^2 - \delta^{ij}\partial_i v \partial_j v + \frac{(a\sqrt{\epsilon_1})''}{a\sqrt{\epsilon_1}} v^2 \right]. \tag{2}$$

Instead of working in real space, it turns out to be convenient to go to Fourier space. We define the Fourier transform of $v(\eta, \boldsymbol{x})$ by

$$v(\eta, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{d}^3 \boldsymbol{k} \, v_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{3}$$

where the vector $\boldsymbol{k}$ denotes the comoving wave vector. Here, because $v(\eta, \boldsymbol{x})$ is real, one has $v_{-\boldsymbol{k}} = v_{\boldsymbol{k}}^*$ (where a star denotes the complex conjugation). As will be discussed in the following, this relation is important since it shows that, in Fourier space, all degrees of freedoms are not independent. This is of course expected since we now

have a description of curvature perturbations in terms of a complex function while we started from a real one. Inserting Eq. (3) into Eq. (2), one obtains

$$^{(2)}\delta S = \frac{1}{2}\int_{\mathbb{R}^3} d\eta d^3\boldsymbol{k}\left[v_{\boldsymbol{k}}' v_{\boldsymbol{k}}^{*\prime} - \left(k^2 - \frac{z''}{z}\right)v_{\boldsymbol{k}} v_{\boldsymbol{k}}^*\right], \quad (4)$$

where the integration is performed in the full Fourier space and we have defined the background quantity $z(\eta)$ by $z(\eta) \equiv a M_{\mathrm{Pl}}\sqrt{2\epsilon_1}$. Then, under the subtraction of the total derivative $(v_{\boldsymbol{k}} v_{\boldsymbol{k}}^* z'/z)'/2$ in the Lagrangian, which leaves the action unchanged, Eq. (4) becomes

$$^{(2)}\delta S = \frac{1}{2}\int_{\mathbb{R}^3} d\eta d^3\boldsymbol{k}\left[v_{\boldsymbol{k}}' v_{\boldsymbol{k}}^{*\prime} - \frac{z'}{z}(v_{\boldsymbol{k}}' v_{\boldsymbol{k}}^* + v_{\boldsymbol{k}} v_{\boldsymbol{k}}^{*\prime})\right.$$
$$\left. + \left(\frac{z'^2}{z^2} - k^2\right)v_{\boldsymbol{k}} v_{\boldsymbol{k}}^*\right]. \quad (5)$$

From this action, the Euler-Lagrange equation reads

$$\frac{d}{d\eta}\left(\frac{\delta\mathcal{L}}{\delta v_{\boldsymbol{k}}^{*\prime}}\right) - \frac{\delta\mathcal{L}}{\delta v_{\boldsymbol{k}}^*} = \frac{1}{2}\left[v_{\boldsymbol{k}}'' + \left(k^2 - \frac{z''}{z}\right)v_{\boldsymbol{k}}\right] = 0, \quad (6)$$

or $v_{\boldsymbol{k}}'' + \omega^2(k,\eta)v_{\boldsymbol{k}} = 0$ with $\omega^2(k,\eta) \equiv k^2 - z''/z$. One recognizes the equation of motion of a parametric oscillator, that is to say an oscillator with time-dependent frequency. Here, the time dependence is due to the interaction with a classical source, the background gravitational field described by the scale factor $a(\eta)$. As discussed in Ref. [71], this is very similar to the Schwinger effect, the only difference being the nature of the source, an Abelian gauge field for the Schwinger effect and, here, as already mentioned, the gravitational field of the expanding Universe. As was noticed in Refs. [72,73], the cosmological perturbations form a system which is also very similar to what one finds in quantum optics. In particular, we can already anticipate that, upon quantization, one will be led to the concept of squeezing [74–76]. Indeed, it is well known that, if quantization of harmonic oscillators naturally leads to coherent states, quantization of parametric oscillators leads to squeezed states.

Let us now apply the Hamiltonian formalism. Here, in order to deal with independent variables only, we split the integral in the action into two parts and change $\boldsymbol{k}$ into $-\boldsymbol{k}$ in the second part. This leads to the following expression,

$$^{(2)}\delta S = \frac{1}{2}\int_{\mathbb{R}^{3+}} d\eta d^3\boldsymbol{k}\left[v_{\boldsymbol{k}}' v_{\boldsymbol{k}}^{*\prime} + v_{\boldsymbol{k}}^{*\prime} v_{\boldsymbol{k}}' - 2\frac{z'}{z}(v_{\boldsymbol{k}}' v_{\boldsymbol{k}}^* + v_{\boldsymbol{k}} v_{\boldsymbol{k}}^{*\prime})\right.$$
$$\left. + \left(\frac{z'^2}{z^2} - k^2\right)(v_{\boldsymbol{k}} v_{\boldsymbol{k}}^* + v_{\boldsymbol{k}}^* v_{\boldsymbol{k}})\right], \quad (7)$$

where the integral over $\boldsymbol{k}$ is now performed in half the Fourier space, $\boldsymbol{k} \in \mathbb{R}^{3+}$. Then our next move is to define the conjugate momentum,

$$p_{\boldsymbol{k}} = \frac{\delta\mathcal{L}}{\delta v_{\boldsymbol{k}}^{*\prime}} = v_{\boldsymbol{k}}' - \frac{z'}{z} v_{\boldsymbol{k}}. \quad (8)$$

It is easy to see that if $v_{\boldsymbol{k}}$ is a measure of the curvature perturbation $\zeta_{\boldsymbol{k}} = v_{\boldsymbol{k}}/z$, the conjugate momentum $p_{\boldsymbol{k}}$ is a measure of its time derivative since

$$\zeta_{\boldsymbol{k}}'(\eta) = \frac{p_{\boldsymbol{k}}(\eta)}{z(\eta)}. \quad (9)$$

On large scales (i.e. scales larger than the Hubble radius during inflation), the solution to Eq. (6) implies that $\zeta_{\boldsymbol{k}}(\eta) \sim A_{\boldsymbol{k}}(1 + \#_1 k^2\eta^2 + \#_2 k^4\eta^4 + \cdots) + B_{\boldsymbol{k}}\int^\eta d\tau/z^2(\tau)(1 + \cdots)$ where $A_{\boldsymbol{k}}$ and $B_{\boldsymbol{k}}$ are two scale dependent constants and $\#_1$ and $\#_2$ two scale independent constants. During inflation, the first branch (the "growing mode") is constant while the second one (the "decaying mode") is decaying (hence the claim that curvature perturbation are conserved). From the above expansion, one has $\zeta_{\boldsymbol{k}}'(\eta) \sim 2A_{\boldsymbol{k}}\#_1 k^2\eta + B_{\boldsymbol{k}}/z^2(\eta)$ and we see that, in the super-Hubble limit where $k\eta \to 0^-$, the second term of the growing mode dominates over the leading term in the decaying mode. This means that $\zeta_{\boldsymbol{k}}'(\eta) \propto 1/z(\eta)$ and, as a consequence, $p_{\boldsymbol{k}}$ tends to a constant.

Let us now calculate the Hamiltonian of the system, defined as

$$H = \int_{\mathbb{R}^{3+}} d^3\boldsymbol{k}(p_{\boldsymbol{k}} v_{\boldsymbol{k}}^{*\prime} + p_{\boldsymbol{k}}^* v_{\boldsymbol{k}}' - \mathcal{L}). \quad (10)$$

Notice that the terms $p_{\boldsymbol{k}} v_{\boldsymbol{k}}^{*\prime}$ and $p_{\boldsymbol{k}}^* v_{\boldsymbol{k}}'$ are summed up only in $\mathbb{R}^{3+}$ [77]. Plugging in the Lagrangian $\mathcal{L}$ obtained in Eq. (7), one has

$$H = \int_{\mathbb{R}^{3+}} d^3\boldsymbol{k}\left[p_{\boldsymbol{k}} p_{\boldsymbol{k}}^* + \frac{z'}{z}(p_{\boldsymbol{k}}^* v_{\boldsymbol{k}} + p_{\boldsymbol{k}} v_{\boldsymbol{k}}^*) + k^2 v_{\boldsymbol{k}} v_{\boldsymbol{k}}^*\right]. \quad (11)$$

Quantization consists in introducing the creation and annihilation operators $\hat{c}_{\boldsymbol{k}}$ and $\hat{c}_{\boldsymbol{p}}^\dagger$ obeying the commutation relations $[\hat{c}_{\boldsymbol{k}}, \hat{c}_{\boldsymbol{p}}^\dagger] = \delta(\boldsymbol{k} - \boldsymbol{p})$. From now on, quantum operators are denoted with a hat. The Mukhanov-Sasaki variable and its conjugate momentum are related to these operators through the following formulas

$$\hat{v}_{\boldsymbol{k}} = \frac{1}{\sqrt{2k}}(\hat{c}_{\boldsymbol{k}} + \hat{c}_{-\boldsymbol{k}}^\dagger), \quad (12)$$

$$\hat{p}_{\boldsymbol{k}} = -i\sqrt{\frac{k}{2}}(\hat{c}_{\boldsymbol{k}} - \hat{c}_{-\boldsymbol{k}}^\dagger). \quad (13)$$

Notice that Eqs. (12) and (13) mix creation and annihilation operators associated to the modes $\boldsymbol{k}$ and $-\boldsymbol{k}$. This is different from what is usually done since the common

practice is to work "mode by mode." These definitions are necessary since they ensure that $\hat{v}_{-\boldsymbol{k}} = \hat{v}_{\boldsymbol{k}}^{\dagger}$.

The final step is to express the Hamiltonian in terms of the creation and annihilation operators. We obtain

$$\hat{H} = \int_{\mathbb{R}^3} \mathrm{d}^3 \boldsymbol{k} \left[\frac{k}{2} (\hat{c}_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}}^{\dagger} + \hat{c}_{-\boldsymbol{k}} \hat{c}_{-\boldsymbol{k}}^{\dagger}) \right.$$
$$\left. - \frac{i}{2} \frac{z'}{z} (\hat{c}_{\boldsymbol{k}} \hat{c}_{-\boldsymbol{k}} - \hat{c}_{-\boldsymbol{k}}^{\dagger} \hat{c}_{\boldsymbol{k}}^{\dagger}) \right]. \tag{14}$$

In this expression, the first term represents a collection of free oscillators with energy $\omega = k$ (compatible with the fact that we deal with massless excitations) while the second one describes the interaction between the quantized perturbations and the classical source. Of course, if the scale factor is constant (Minkowski spacetime), then z is a constant and this term disappears. This term is responsible for the creation of pairs of particles. We observe that the structure $\hat{c}_{-\boldsymbol{k}}^{\dagger} \hat{c}_{\boldsymbol{k}}^{\dagger}$ implies that the particles are created with opposite momenta, thus ensuring momentum conservation.

Let us now discuss the equation of motion. It is given by the Heisenberg equation, namely $\mathrm{d}\hat{c}_{\boldsymbol{k}}/\mathrm{d}\eta = -i[\hat{c}_{\boldsymbol{k}}, \hat{H}]$. Using Eq. (14), this leads to

$$i \frac{\mathrm{d}\hat{c}_{\boldsymbol{k}}}{\mathrm{d}\eta} = k\hat{c}_{\boldsymbol{k}} + i \frac{z'}{z} \hat{c}_{-\boldsymbol{k}}^{\dagger}. \tag{15}$$

This can be solved by mean of the Bogoliubov transformation

$$\hat{c}_{\boldsymbol{k}}(\eta) = u_k(\eta) \hat{c}_{\boldsymbol{k}}(\eta_{\mathrm{ini}}) + v_k(\eta) \hat{c}_{-\boldsymbol{k}}^{\dagger}(\eta_{\mathrm{ini}}), \tag{16}$$

where $u_k(\eta)$ and $v_k(\eta)$ (not to be confused with the Mukhanov-Sasaki variable) are two functions depending on the wave vector modulus only[1] and obeying

$$i \frac{\mathrm{d}u_k(\eta)}{\mathrm{d}\eta} = ku_k(\eta) + i \frac{z'}{z} v_k^*(\eta), \tag{17}$$

$$i \frac{\mathrm{d}v_k(\eta)}{\mathrm{d}\eta} = kv_k(\eta) + i \frac{z'}{z} u_k^*(\eta). \tag{18}$$

They must moreover satisfy $|u_k(\eta)|^2 - |v_k(\eta)|^2 = 1$ in order for the commutation relation between the creation and annihilation operators to be properly normalized. Let us also notice that the combination $u_k + v_k^*$ obeys the same equation of motion as the Mukhanov-Sasaki variable, namely $(u_k + v_k^*)'' + \omega^2 (u_k + v_k^*) = 0$.

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[1]Indeed, one can see that the equations of motion (17) and (18) for $u_k(\eta)$ and $v_k(\eta)$ only depend on the modulus of $\boldsymbol{k}$. So if the initial state is chosen to be the vacuum, rotationally invariant and hence dependent on k only, $u_k(\eta)$ and $v_k(\eta)$ remain independent of the orientation of $\boldsymbol{k}$ at any time.

From the above considerations, we see that solving the time dependence of the system is equivalent to solving the Bogoliubov system (17) and (18). For this purpose, let us now introduce two operators. The first one is the two-mode squeezing operator $\hat{S}(r_k, \varphi_k)$ defined by $\hat{S}(r_k, \varphi_k) = e^{\hat{B}_k}$ with

$$\hat{B}_k \equiv r_k e^{-2i\varphi_k} \hat{c}_{-\boldsymbol{k}}(\eta_{\mathrm{ini}}) \hat{c}_{\boldsymbol{k}}(\eta_{\mathrm{ini}}) - r_k e^{2i\varphi_k} \hat{c}_{-\boldsymbol{k}}^{\dagger}(\eta_{\mathrm{ini}}) \hat{c}_{\boldsymbol{k}}^{\dagger}(\eta_{\mathrm{ini}}). \tag{19}$$

Clearly, from the above definition, one has $\hat{B}_k^{\dagger} = -\hat{B}_k$. It is characterized by two parameters, the squeezing parameter r_k and the squeezing angle φ_k. The second operator is the rotation operator $\hat{R}_k(\theta_{k,1}, \theta_{k,2})$ that can be expressed as $\hat{R}_k(\theta_{k,1}, \theta_{k,2}) = e^{\hat{D}_k}$ with

$$\hat{D}_k \equiv -i\theta_{k,1} \hat{c}_{\boldsymbol{k}}^{\dagger}(\eta_{\mathrm{ini}}) \hat{c}_{\boldsymbol{k}}(\eta_{\mathrm{ini}}) - i\theta_{k,2} \hat{c}_{-\boldsymbol{k}}^{\dagger}(\eta_{\mathrm{ini}}) \hat{c}_{-\boldsymbol{k}}(\eta_{\mathrm{ini}}). \tag{20}$$

We notice that we also have $\hat{D}_k^{\dagger} = -\hat{D}_k$. The rotation operator is *a priori* characterized by two parameters, the rotation angles $\theta_{k,1}$ and $\theta_{k,2}$.

Let us now calculate the quantity $\hat{R}_k^{\dagger} \hat{S}_k^{\dagger} \hat{c}_{\boldsymbol{k}}(\eta_{\mathrm{ini}}) \hat{S}_k \hat{R}_k$. Using the formula $e^{-\hat{O}} \hat{A} e^{\hat{O}} = \hat{A} + [\hat{O}, \hat{A}] + [\hat{O}, [\hat{O}, \hat{A}]]/2 + \cdots$, it is easy to show that

$$\hat{R}_k^{\dagger} \hat{S}_k^{\dagger} \hat{c}_{\boldsymbol{k}}(\eta_{\mathrm{ini}}) \hat{S}_k \hat{R}_k = e^{i\theta_{k,1}} \cosh r_k \hat{c}_{\boldsymbol{k}}(\eta_{\mathrm{ini}})$$
$$+ e^{-i\theta_{k,2}+2i\varphi_k} \sinh r_k \hat{c}_{-\boldsymbol{k}}^{\dagger}(\eta_{\mathrm{ini}}), \tag{21}$$

$$\hat{R}_k^{\dagger} \hat{S}_k^{\dagger} \hat{c}_{-\boldsymbol{k}}(\eta_{\mathrm{ini}}) \hat{S}_k \hat{R}_k = e^{i\theta_{k,2}} \cosh r_k \hat{c}_{-\boldsymbol{k}}(\eta_{\mathrm{ini}})$$
$$+ e^{-i\theta_{1,k}+2i\varphi_k} \sinh r_k \hat{c}_{\boldsymbol{k}}^{\dagger}(\eta_{\mathrm{ini}}). \tag{22}$$

This relation coincides with the transport equation (16) for $\hat{c}_{\boldsymbol{k}}$ provided that

$$\hat{c}_{\boldsymbol{k}}(\eta) = \hat{R}_k^{\dagger} \hat{S}_k^{\dagger} \hat{c}_{\boldsymbol{k}}(\eta_{\mathrm{ini}}) \hat{S}_k \hat{R}_k, \tag{23}$$

with $\theta_{k,1} = \theta_{k,2} \equiv \theta$, and

$$u_k(\eta) = e^{i\theta_k} \cosh r_k, \tag{24}$$

$$v_k(\eta) = e^{-i\theta_k+2i\varphi_k} \sinh r_k. \tag{25}$$

One checks that, indeed, $|u_k(\eta)|^2 - |v_k(\eta)|^2 = 1$.

We conclude from the previous analysis that the full Hilbert space of the system $\mathcal{E}$ can be factorized into independent products of Hilbert spaces for modes $\boldsymbol{k}$ and $-\boldsymbol{k}$: $\mathcal{E} = \Pi_{k \in \mathbb{R}^{3+}} \mathcal{E}_{\boldsymbol{k}} \otimes \mathcal{E}_{-\boldsymbol{k}}$. Each of these terms is a composite or a bipartite system, and evolves from the initial vacuum state $|0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle$ into a two-mode squeezed state

$$\hat{S}(r_k, \varphi_k) \hat{R}(\theta_k) |0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle. \tag{26}$$

One can explicitly check that the vacuum state is rotationally invariant, $\hat{R}(\theta_k)|0_k, 0_{-k}\rangle = |0_k, 0_{-k}\rangle$, which implies that if the initial state is the vacuum state, θ_k cancels out in all physical quantities, as will be checked in what follows. More explicitly, using operator ordering theorems [74], one has

$$\hat{S}(r_k, \varphi_k) = \exp\left[e^{-2i\varphi_k} \tanh r_k \hat{c}_{-k}^\dagger(\eta_{\text{ini}})\hat{c}_k^\dagger(\eta_{\text{ini}})\right]$$
$$\times \exp\{-\ln(\cosh r_k)[\hat{c}_k^\dagger(\eta_{\text{ini}})\hat{c}_k(\eta_{\text{ini}})$$
$$+ \hat{c}_{-k}(\eta_{\text{ini}})\hat{c}_{-k}^\dagger(\eta_{\text{ini}})]\}$$
$$\times \exp\left[-e^{2i\varphi_k} \tanh r_k \hat{c}_{-k}(\eta_{\text{ini}})\hat{c}_k(\eta_{\text{ini}})\right], \quad (27)$$

and straightforward calculations lead to

$$\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)|0_k, 0_{-k}\rangle$$
$$= \frac{1}{\cosh r_k}\sum_{n=0}^{\infty} e^{-2in\varphi_k}\tanh^n r_k|n_k, n_{-k}\rangle. \quad (28)$$

This is a two-mode squeezed state, well known in the context of quantum optics as a highly entangled state. Indeed, in a very sketchy way (restricting the sum to $n = 0$ and $n = 1$ and ignoring the coefficients of the expansion, just for the purpose of illustration), it is of the form

$$|\Psi\rangle \sim \frac{1}{\sqrt{2}}(|0_k\rangle|0_{-k}\rangle + |1_k\rangle|1_{-k}\rangle), \quad (29)$$

which, indeed, is in an inseparable form since it cannot be written as $|\Psi_k\rangle \otimes |\Psi_{-k}\rangle$. Entangled states are considered as the most nonclassical states and the above discussion shows that the CMB is placed in such a state. Moreover, it is known that in the strong squeezing limit, the state (28) exactly tends towards an Einstein-Podolsky-Rosen [78] quantum state. We are therefore led to conclude that the CMB is a highly nonclassical system.

Before concluding this section, it is interesting to derive the equations satisfied by the parameters r_k, φ_k and θ_k. From Eq. (28), we see that their time evolution controls the time evolution of the quantum state (28). Using Eqs. (17), (18), (24) and (25), one obtains

$$\frac{dr_k}{d\eta} = \frac{z'}{z}\cos(2\varphi_k), \quad (30)$$

$$\frac{d\varphi_k}{d\eta} = -k - \frac{z'}{z}\coth(2r_k)\sin(2\varphi_k), \quad (31)$$

$$\frac{d\theta_k}{d\eta} = -k - \frac{z'}{z}\tanh r_k \sin(2\varphi_k). \quad (32)$$

The two first equations form a closed system, and θ_k can be derived once r_k and φ_k are known. This is expected since, as already said, θ_k is a spurious parameter that does not play

any physical role. Exact solutions are available only in the exact de Sitter case, for which we have

$$r_k(\eta) = -\text{arg sinh}\left(\frac{1}{2k\eta}\right), \quad (33)$$

$$\phi_k(\eta) = -\frac{\pi}{4} - \frac{1}{2}\arctan\left(\frac{1}{2k\eta}\right), \quad (34)$$

$$\theta_k(\eta) = -k\eta - \arctan\left(\frac{1}{2k\eta}\right). \quad (35)$$

On super-Hubble scales, at the end of inflation, we have $r_k \to +\infty$ (notice that r_k is positive because conformal time is negative during inflation), $\varphi_k \to 0$ and $\theta_k \to \pi/2$, which remains true for quasi–de Sitter expansions.

III. INFLATIONARY DISCORD

The discussion of the last section indicates that the quantumness of the CMB is *a priori* large. Therefore, it seems important to better quantify this property. Recently, in the context of quantum information theory, Henderson and Vedral [52] and Ollivier and Zurek [53] have introduced the quantum discord, which aims to quantitatively measure the quantumness of a system. The aim of this section is to calculate the quantum discord of inflationary perturbations.

Quantum discord is defined as follows. Let us consider a bipartite system $\mathcal{E} = \mathcal{E}_k \otimes \mathcal{E}_{-k}$ (we write the two systems k and $-k$ for obvious reasons but it should be clear that the following considerations are in fact valid for any bipartite system). The idea is to introduce two ways of calculating the mutual information between the two subsystems that coincide for classical correlations but not necessarily in quantum systems. The difference between the two results will then define the quantum discord. The first measure of mutual information is provided by the von Neumann entropy,

$$\mathcal{I}(k, -k) = S[\hat{\rho}(k)] + S[\hat{\rho}(-k)] - S[\hat{\rho}(k, -k)], \quad (36)$$

where S is the entropy defined by $S = -\text{Tr}(\hat{\rho}\log_2\hat{\rho})$, $\hat{\rho}$ being the density matrix of the system under consideration. Let us recall that the density matrix of the subsystem k (respectively $-k$) is obtained from the full density matrix $\hat{\rho}(k, -k)$ by tracing over the degrees of freedom associated with $-k$ (respectively k), that is to say $\hat{\rho}(k) = \text{Tr}_{-k}[\hat{\rho}(k, -k)]$ [respectively $\hat{\rho}(-k) = \text{Tr}_k[\hat{\rho}(k, -k)]$].

Then let us imagine that a measurement of an observable of the subsystem $-k$ is performed through a projector $\hat{\Pi}_j$ (defined on $\mathcal{E}_{-k}$). This transforms the state into $\hat{\rho} \to \hat{\rho}\hat{\Pi}_j/p_j$, with probability $p_j \equiv \text{Tr}[\hat{\rho}(k, -k)\hat{\Pi}_j]$. If we only access $\mathcal{E}_k$, tracing over $-k$, the state becomes

$$\hat{\rho}(\boldsymbol{k}; \hat{\Pi}_j) = \text{Tr}_{-\boldsymbol{k}} \left[\frac{\hat{\rho}(\boldsymbol{k}, -\boldsymbol{k})\hat{\Pi}_j}{p_j} \right]. \tag{37}$$

Now, if one performs a set of all possible measurements through a complete set[2] of projectors $\{\hat{\Pi}_j\}$, an alternative definition of mutual information is given by

$$\mathcal{J}(\boldsymbol{k}, -\boldsymbol{k}) = S[\hat{\rho}(\boldsymbol{k})] - \sum_j p_j S[\hat{\rho}(\boldsymbol{k}; \hat{\Pi}_j)]. \tag{38}$$

Classically, thanks to Bayes theorem, we have $\mathcal{J}(\boldsymbol{k}, -\boldsymbol{k}) = \mathcal{I}(\boldsymbol{k}, -\boldsymbol{k})$. Quantum mechanically however, this need not be true and the deviation from the previous equality therefore represents the quantumness of the correlations contained in a given state. This is why quantum discord is defined by

$$\delta(\boldsymbol{k}, -\boldsymbol{k}) \equiv \min_{\{\hat{\Pi}_j\}} [\mathcal{I}(\boldsymbol{k}, -\boldsymbol{k}) - \mathcal{J}(\boldsymbol{k}, -\boldsymbol{k})], \tag{39}$$

where we minimize over all possible sets of measurements in order to avoid dependence on the projectors. More detailed discussions on the physical interpretation and meaning of quantum discord can be found in Refs. [52,53,79].

Let us now calculate the quantum discord of inflationary perturbations. For the two-mode squeezed state (28), the density matrix is given by

$$\hat{\rho}(\boldsymbol{k}, -\boldsymbol{k}) = \frac{1}{\cosh^2 r_k} \sum_{n,n'=0}^{\infty} e^{-2i(n-n')\varphi_k}$$
$$\times \tanh^{n+n'} r_k |n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle \langle n'_{\boldsymbol{k}}, n'_{-\boldsymbol{k}}|. \tag{40}$$

The reduced density matrix $\hat{\rho}(\boldsymbol{k})$ is obtained from the full density matrix by tracing out degrees of freedom associated to $-\boldsymbol{k}$. One has

$$\hat{\rho}(\boldsymbol{k}) = \sum_{n=0}^{\infty} \langle n_{-\boldsymbol{k}} | \hat{\rho}(\boldsymbol{k}, -\boldsymbol{k}) | n_{-\boldsymbol{k}} \rangle \tag{41}$$

$$= \frac{1}{\cosh^2 r_k} \sum_{n=0}^{\infty} \tanh^{2n} r_k |n_{\boldsymbol{k}}\rangle \langle n_{\boldsymbol{k}}|, \tag{42}$$

which is a thermal state with inverse temperature $\beta_k = -\ln \tanh^2 r_k$. We have of course a similar equation for $\hat{\rho}(-\boldsymbol{k})$ where $|n_{\boldsymbol{k}}\rangle \langle n_{\boldsymbol{k}}|$ is replaced by $|n_{-\boldsymbol{k}}\rangle \langle n_{-\boldsymbol{k}}|$.

Our next move is to calculate the entropy of the different density matrices appearing in the expression of the discord. As can be shown explicitly, the entropy of $\hat{\rho}(\boldsymbol{k}, -\boldsymbol{k})$ vanishes since we deal with a pure state. Since $\hat{\rho}(\boldsymbol{k})$

represents a thermal state, its entropy is simply given by [80]

$$S[\hat{\rho}(\boldsymbol{k})] = (1 + \langle \hat{n}_k \rangle)\log_2(1 + \langle \hat{n}_k \rangle) - \langle \hat{n}_k \rangle \log_2 \langle \hat{n}_k \rangle, \tag{43}$$

where $\langle \hat{n}_k \rangle = \sinh^2 r_k$ is the mean occupation number. Obviously, this formula is also valid for $\hat{\rho}(-\boldsymbol{k})$. Finally, the quantity $S[\hat{\rho}(\boldsymbol{k}; \hat{\Pi}_j)]$ remains to be calculated. In Appendix A, we show that $\hat{\rho}(\boldsymbol{k}; \hat{\Pi}_j)$ is in fact a pure state and, consequently, its entropy is zero. Therefore, it follows that the discord is given by

$$\delta(\boldsymbol{k}, -\boldsymbol{k}) = S[\hat{\rho}(-\boldsymbol{k})] = \cosh^2 r_k \log_2(\cosh^2 r_k)$$
$$- \sinh^2 r_k \log_2(\sinh^2 r_k). \tag{44}$$

The corresponding function is displayed in Fig. 1. One can see that except for $r_k = 0$, the discord is not zero, and, therefore, the quantum state of the perturbations is not classical. In fact, $r_k = 0$ corresponds to a coherent state [81,82]. Such states are often called "quasiclassical" and are known to be the "most classical" states since they follow the classical trajectory in phase space, with minimal spread. In the strong squeezing limit however, $r_k \to \infty$, we have

$$\delta(\boldsymbol{k}, -\boldsymbol{k}) = \frac{2}{\ln 2} r_k - 2 + \frac{1}{\ln 2} + \mathcal{O}(e^{-2r_k}). \tag{45}$$

Let us recall that for the modes within the CMB window, at the end of inflation, $r_k \sim 50$. We conclude that the CMB is
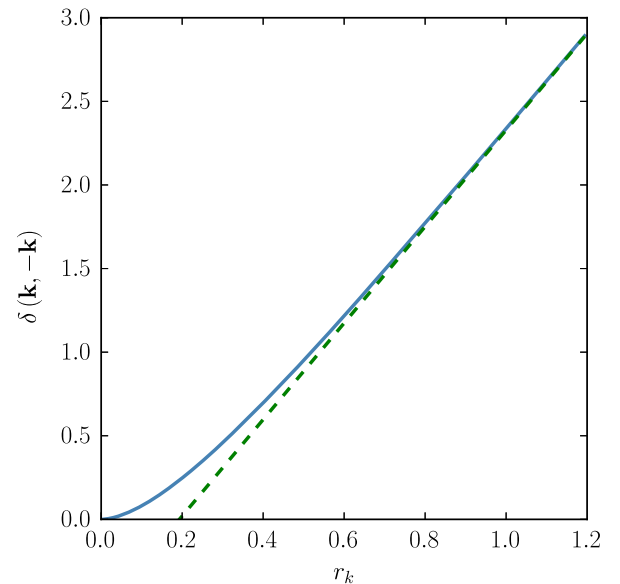

FIG. 1. Quantum discord $\delta(\boldsymbol{k}, -\boldsymbol{k})$ of cosmological scalar perturbations during inflation, as a function of the squeezing parameter r_k. The solid blue line is the result (44) while the dotted green line is the large squeezing expansion (45).

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[2]More precisely, $\{\hat{\Pi}_j\}$ forms a positive operator valued measure, obeying the partition of unity $\Sigma_j \hat{\Pi}_j = 1$. They generalize complete sets in the sense that they do not need to be orthogonal.

placed in a state which is "very quantum." This means that it is certainly impossible to reproduce all the correlation functions in a classical picture as we are going to demonstrate explicitly in the following section. In fact, the results of this section show that, strictly speaking, there is no transition to a classical behavior since the discord only grows. But we will also see that the situation is subtle and that, nevertheless, a classical treatment of the perturbations can partially be employed, in a sense that will be carefully discussed in the following.

Let us also mention that quantum discord is always defined relatively to a given division into two subsystems. For example, in what precedes, the quantum discord was shown to be large for the bipartite system $\mathcal{E} = \mathcal{E}_k \otimes \mathcal{E}_{-k}$. A priori, with a different division, one would have obtained a different result. For example, let us write the Hamiltonian (11) in terms of the real and imaginary (or Hermitian and anti-Hermitian) parts of v_k and p_k, denoted v_k^R, v_k^I, p_k^R, p_k^I, and defined by $v_k = (v_k^R + iv_k^I)/\sqrt{2}$ and $p_k = (p_k^R + ip_k^I)/\sqrt{2}$. Notice that the relation $v_k^\dagger = v_{-k}$ implies that $v_k^R = v_{-k}^R$ and $v_k^I = -v_{-k}^I$. One has

$$H = \int_{\mathbb{R}^{3+}} d^3k \left[\frac{1}{2}(p_k^R)^2 + \frac{k}{2}(v_k^R)^2 + \frac{z}{z'}v_k^R p_k^R \right]$$
$$+ \int_{\mathbb{R}^{3+}} d^3k \left[\frac{1}{2}(p_k^I)^2 + \frac{k}{2}(v_k^I)^2 + \frac{z}{z'}v_k^I p_k^I \right]. \quad (46)$$

One can see that the real and imaginary sectors are in fact independent. This suggests that the quantum discord calculated with respect to $\mathcal{E} = \mathcal{E}_R \otimes \mathcal{E}_I$ should vanish. It is therefore interesting to calculate the quantum discord calculated with respect to an arbitrary subdivision of our system $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$. Starting from $\hat{c}_k$ and $\hat{c}_{-k}$, the most generic pair of independent ladder operators can be written as

$$\hat{a}_1 = \cos\alpha\, \hat{c}_k + \sin\alpha\, \hat{c}_{-k} \quad (47)$$

$$\hat{a}_2 = e^{2i\alpha}(\cos\alpha\, \hat{c}_{-k} - \sin\alpha\, \hat{c}_k), \quad (48)$$

where α is an angle varying from 0 to $\pi/4$. This defines a one-parameter family of subdivision such that, if $\alpha = 0$, then $\mathcal{E} = \mathcal{E}_k \otimes \mathcal{E}_{-k}$ and, if $\alpha = \pi/4$, then $\mathcal{E} = \mathcal{E}_R \otimes \mathcal{E}_I$. One can then proceed and calculate the corresponding discord along the lines already explained before. This is what is done in detail in Appendix B. The result is displayed in Fig. 2 for a few values of $k/(aH)$. As expected, one can check that the discord always vanishes for $\alpha = \pi/4$ ($\mathcal{E} = \mathcal{E}_R \otimes \mathcal{E}_I$) while it is maximal when $\alpha = 0$ ($\mathcal{E} = \mathcal{E}_k \otimes \mathcal{E}_{-k}$). It also increases with the squeezing level, since more pairs of particles with wave numbers $\boldsymbol{k}$ and $-\boldsymbol{k}$ are produced as time proceeds, in agreement with Fig. 1.
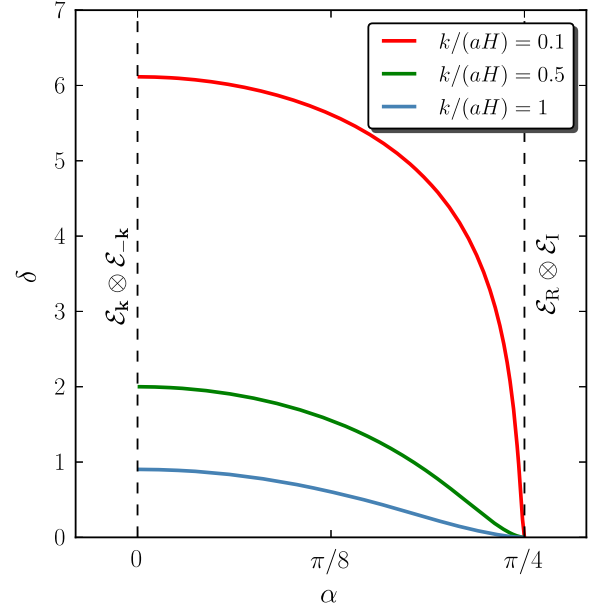

FIG. 2. Quantum discord for different system subdivisions (labeled by α) and different times during inflation.

The above analysis undoubtedly shows that the CMB is a strongly discordant quantum system. In particular, it contains genuine quantum correlations between the $\boldsymbol{k}$ and $-\boldsymbol{k}$ modes (and in fact, as the previous calculation demonstrates, between any "1" and "2" modes except if $\alpha = \pi/4$). A priori, this makes a classical description of the CMB impossible. Therefore, the fact that, in practice, the CMB data are analyzed without any reference to the quantum formalism appears to be rather paradoxical. The rest of this paper is devoted to this question.

IV. QUANTUM PREDICTIONS FOR THE PERTURBATIONS

In practice, the quantities of observational interest are correlation functions. In this section, as a first step, we derive these correlation functions for the full quantum state (28). We make use of the quantum information formalism, which sheds some new light on the standard calculation of inflationary cosmological perturbations.

A. Quantum information theory and cosmological perturbations

A two-mode squeezed state being Gaussian, its characteristic function is Gaussian as well and all correlation functions can be calculated from the covariance matrix. To compute this covariance matrix, we can use the techniques of quantum information theory [80,83]. In order to do so, we must formulate the problem such that this formalism can be applied. This technical question actually plays an important role in the following. The point is that the quantum information formalism is designed mode by mode

while, as already noted, Eqs. (12) and (13) mix creation and annihilation operator associated with two different modes, $\boldsymbol{k}$ and $-\boldsymbol{k}$. We must therefore reformulate the problem such that "coordinates" ("position" and "momentum") are Hermitian operators defined for a single mode. This leads us to introduce the quantities $\hat{q}_{\boldsymbol{k}}$ and $\hat{\pi}_{\boldsymbol{k}}$ such that

$$\hat{q}_{\boldsymbol{k}} = \frac{1}{\sqrt{2k}}(\hat{c}_{\boldsymbol{k}} + \hat{c}_{\boldsymbol{k}}^{\dagger}), \tag{49}$$

$$\hat{\pi}_{\boldsymbol{k}} = -i\sqrt{\frac{k}{2}}(\hat{c}_{\boldsymbol{k}} - \hat{c}_{\boldsymbol{k}}^{\dagger}). \tag{50}$$

Here, and in the rest of this article, we work in the Schrödinger picture where $\hat{q}_{\boldsymbol{k}}$ and $\hat{\pi}_{\boldsymbol{k}}$ are viewed as constant and the state is evolving. Contrary to Eqs. (12) and (13) defining $v_{\boldsymbol{k}}$ and $p_{\boldsymbol{k}}$, Eqs. (49) and (50) do not mix different modes. Making use of Eqs. (12) and (13), one can establish the relation between the two sets of variables, which read

$$\hat{v}_{\boldsymbol{k}} = \frac{1}{2}\left[\hat{q}_{\boldsymbol{k}} + \hat{q}_{-\boldsymbol{k}} + \frac{i}{k}(\hat{\pi}_{\boldsymbol{k}} - \hat{\pi}_{-\boldsymbol{k}})\right], \tag{51}$$

$$\hat{p}_{\boldsymbol{k}} = \frac{1}{2i}[k(\hat{q}_{\boldsymbol{k}} - \hat{q}_{-\boldsymbol{k}}) + i(\hat{\pi}_{\boldsymbol{k}} + \hat{\pi}_{-\boldsymbol{k}})]. \tag{52}$$

In particular, since $\hat{q}_{\boldsymbol{k}} = \hat{q}_{\boldsymbol{k}}^{\dagger}$ and $\hat{\pi}_{\boldsymbol{k}} = \hat{\pi}_{\boldsymbol{k}}^{\dagger}$, one can check that we indeed have $\hat{v}_{\boldsymbol{k}}^{\dagger} = \hat{v}_{-\boldsymbol{k}}$. The quantum information formalism provides us with all correlation functions for combinations of $\hat{q}_{\boldsymbol{k}}$ and $\hat{\pi}_{\boldsymbol{k}}$, and using Eqs. (51) and (52), correlation functions for combinations of $\hat{v}_{\boldsymbol{k}}$ and $\hat{p}_{\boldsymbol{k}}$ can be inferred.

Let us now introduce the characteristic function. It is a real function defined on a four-dimensional real space, given by

$$\chi(\xi) = \text{Tr}[\hat{\rho}\,\hat{\mathcal{W}}(\xi)], \tag{53}$$

where $\hat{\mathcal{W}}(\xi)$ is the Weyl operator, namely

$$\hat{\mathcal{W}}(\xi) = e^{i\xi^{\text{T}}\hat{R}}, \tag{54}$$

with $\hat{R} = (k^{1/2}\hat{q}_{\boldsymbol{k}}, k^{-1/2}\hat{\pi}_{\boldsymbol{k}}, k^{1/2}\hat{q}_{-\boldsymbol{k}}, k^{-1/2}\hat{\pi}_{-\boldsymbol{k}})^{\text{T}}$. In the above expressions, $\hat{\rho}$ is obviously the density operator.

In Appendix C, we show that the two-mode squeezed state has a Gaussian characteristic function given by

$$\chi(\xi) = e^{-\xi^{\text{T}}\gamma\xi/4}, \tag{55}$$

where γ is the covariance matrix, related to the two-point correlation functions by $\langle\hat{R}_j\hat{R}_k\rangle = \gamma_{jk}/2 + iJ_{jk}/2$. Here, J is the commutator matrix, $iJ_{j,k} = [\hat{R}_j, \hat{R}_k]$, given by $J = J_1 \oplus J_1$ with

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{56}$$

The covariance matrix of the two-mode squeezed state (28) has been derived in Appendix C and this leads to

$$\langle\hat{v}_{\boldsymbol{k}}\hat{v}_{\boldsymbol{p}}\rangle = \frac{1}{2k}[\cosh(2r_k) + \sinh(2r_k)\cos(2\varphi_k)]\delta(\boldsymbol{k} + \boldsymbol{p}), \tag{57}$$

$$\langle\hat{p}_{\boldsymbol{k}}\hat{p}_{\boldsymbol{p}}\rangle = \frac{k}{2}[\cosh(2r_k) - \sinh(2r_k)\cos(2\varphi_k)]\delta(\boldsymbol{k} + \boldsymbol{p}), \tag{58}$$

$$\langle\hat{v}_{\boldsymbol{k}}\hat{p}_{\boldsymbol{p}}\rangle = \left[\frac{i}{2} + \frac{1}{2}\sinh(2r_k)\sin(2\varphi_k)\right]\delta(\boldsymbol{k} + \boldsymbol{p}). \tag{59}$$

B. Two-point correlation functions

Using the above results, one can proceed and calculate the power spectrum for the states (28). Using Eq. (3), the correlation function of the Mukhanov-Sasaki variable is given by

$$\langle\hat{v}(\eta, \boldsymbol{x})\hat{v}(\eta, \boldsymbol{y})\rangle = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}d\boldsymbol{p}\langle\hat{v}_{\boldsymbol{k}}(\eta)\hat{v}_{\boldsymbol{p}}(\eta)\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x} + i\boldsymbol{p}\cdot\boldsymbol{y}}. \tag{60}$$

For the two-mode squeezed state, the only correlation function $\langle\hat{v}_{\boldsymbol{k}}(\eta)\hat{v}_{\boldsymbol{p}}(\eta)\rangle$ that does not vanish is the one for which $\boldsymbol{p} = -\boldsymbol{k}$. From Eq. (57), it follows that

$$\langle\hat{v}(\eta, \boldsymbol{x})\hat{v}(\eta, \boldsymbol{y})\rangle = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\frac{1}{2k}[\cosh(2r_k) + \sinh(2r_k)\cos(2\varphi_k)]e^{i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{y})}. \tag{61}$$

We notice that the result only depends on $|\boldsymbol{x} - \boldsymbol{y}|$ as expected in a homogeneous and isotropic universe. As a consequence, one can perform the angular integration and one obtains

$$\langle\hat{v}(\eta, \boldsymbol{x})\hat{v}(\eta, \boldsymbol{y})\rangle = \frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{dk}{k}\frac{\sin(k|\boldsymbol{x} - \boldsymbol{y}|)}{k|\boldsymbol{x} - \boldsymbol{y}|}\frac{k^3}{2k} \\ \times [\cosh(2r_k) + \sinh(2r_k)\cos(2\varphi_k)], \tag{62}$$

or, using Eq. (1),

$$\langle\hat{\zeta}(\eta, \boldsymbol{x})\hat{\zeta}(\eta, \boldsymbol{y})\rangle = \frac{4\pi}{(2\pi)^3}\frac{1}{a^2 M_{\text{Pl}}^2 2\epsilon_1}\int_0^\infty \frac{dk}{k}\frac{\sin(k|\boldsymbol{x} - \boldsymbol{y}|)}{k|\boldsymbol{x} - \boldsymbol{y}|} \\ \times \frac{k^3}{2k}[\cosh(2r_k) + \sinh(2r_k)\cos(2\varphi_k)]. \tag{63}$$

To go further, one needs to specify r_k and φ_k. In the de Sitter limit, they are given by Eqs. (33) and (34), and on super-Hubble scales ($-k\eta \ll 1$), this gives rise to $\cosh(2r_k) \simeq \sinh(2r_k) \simeq 1 + 1/(2k^2\eta^2)$ and $\cos(2\varphi_k) \simeq 1 - 2k^2\eta^2$. As a consequence, the power spectrum P_v of $v(\eta, \boldsymbol{x})$, defined through $\langle \hat{v}_{\boldsymbol{k}} \hat{v}_{\boldsymbol{p}} \rangle = 2\pi^3 \mathcal{P}_v / k^3 \delta(\boldsymbol{k} - \boldsymbol{p})$, becomes $\mathcal{P}_v = 1/(2\pi^2 2\eta^2)$ and the power spectrum of curvature perturbations can be expressed as

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2 M_{\mathrm{Pl}}^2 \epsilon_1}, \tag{64}$$

where we have used that in de Sitter spacetimes, $a(\eta) = -1/(H\eta)$. This formula is the standard result [20,84]. Of course, strictly speaking, we have $\epsilon_1 = 0$ for exact de Sitter spacetime (there are no density perturbations in de Sitter spacetime) so, in fact, the above result is valid at leading order in slow roll only.

Let us now try to evaluate other correlators, in particular, the one involving the curvature perturbation and its time derivative. For a two-mode squeezed state, using Eq. (9), we have

$$\langle \hat{\zeta}(\eta, \boldsymbol{x}) \hat{\zeta}'(\eta, \boldsymbol{y}) + \hat{\zeta}'(\eta, \boldsymbol{x}) \hat{\zeta}(\eta, \boldsymbol{y}) \rangle$$
$$= \frac{1}{(2\pi)^3} \frac{1}{z^2(\eta)} \int d\boldsymbol{k} (\langle \hat{v}_{\boldsymbol{k}} \hat{p}_{-\boldsymbol{k}} \rangle + \langle \hat{p}_{\boldsymbol{k}} \hat{v}_{-\boldsymbol{k}} \rangle) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}. \tag{65}$$

The next step is to use Eq. (59) together with the fact that $[\hat{v}_{\boldsymbol{k}}, \hat{p}_{-\boldsymbol{k}}] = i$. This commutation relation ensures that the term $i/2$ in Eq. (59) cancels out and that the above correlation function is real. Then one arrives at

$$\langle \hat{\zeta}(\eta, \boldsymbol{x}) \hat{\zeta}'(\eta, \boldsymbol{y}) + \hat{\zeta}'(\eta, \boldsymbol{x}) \hat{\zeta}(\eta, \boldsymbol{y}) \rangle$$
$$= \frac{4\pi}{(2\pi)^3} \frac{1}{z^2(\eta)} \int_0^\infty \frac{dk}{k} \frac{\sin(k|\boldsymbol{x}-\boldsymbol{y}|)}{k|\boldsymbol{x}-\boldsymbol{y}|}$$
$$\times k^3 \sinh(2r_k) \sin(2\varphi_k). \tag{66}$$

In the de Sitter super-Hubble limit, this gives rise to $\langle \hat{\zeta}\hat{\zeta}' + \hat{\zeta}'\hat{\zeta} \rangle \sim 1/z \sim \eta$ which is consistent with the fact that $\zeta_{\boldsymbol{k}}$ and $p_{\boldsymbol{k}} = z\zeta_{\boldsymbol{k}}'$ are constant on large scales.

Finally, it is also interesting to evaluate the two-point correlation function $\langle \hat{\zeta}'(\eta, \boldsymbol{x}) \hat{\zeta}'(\eta, \boldsymbol{y}) \rangle$. Straightforward manipulations lead to

$$\langle \hat{\zeta}'(\eta, \boldsymbol{x}) \hat{\zeta}'(\eta, \boldsymbol{y}) \rangle = \frac{1}{(2\pi)^3} \frac{1}{z^2(\eta)} \int d\boldsymbol{k} \langle \hat{p}_{\boldsymbol{k}} \hat{p}_{-\boldsymbol{k}} \rangle e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$
$$= \frac{4\pi}{(2\pi)^3} \frac{1}{z^2(\eta)} \int_0^\infty \frac{dk}{k} \frac{\sin(k|\boldsymbol{x}-\boldsymbol{y}|)}{k|\boldsymbol{x}-\boldsymbol{y}|}$$
$$\times k^3 \frac{k}{2} [\cosh(2r_k) - \sinh(2r_k)\cos(2\varphi_k)]. \tag{67}$$

For the exact de Sitter case, we have $\cosh(2r_k) - \sinh(2r_k)\cos(2\varphi_k) = 1$ and this correlator is proportional to η^2 as expected.

V. CAN THE QUANTUM CORRELATION FUNCTIONS BE OBTAINED WITH A NONDISCORDANT STATE?

In the previous section, we have established the correlation functions of the full quantum state. Given that, as already mentioned, the CMB is usually analyzed in a classical framework, we now study to which extent a "classical state," i.e. a nondiscordant quantum state, can reproduce these results. Notice that, the discord being nonvanishing, we already know for sure that it cannot be the case exactly. As a consequence, here, we instead want to study whether the difference between the exact and the classical approaches is large enough to be seen observationally, which, if so, would be considered as a proof that the perturbations are of quantum-mechanical origin.

A. Two-point correlation functions

The most generic expression for a bipartite system containing classical correlations only is given by [53,85,86]

$$\hat{\rho}_{\mathrm{cl}} = \bigotimes_{\boldsymbol{k} \in \mathbb{R}^{3+}} \sum_{n,m} p_{nm}(\boldsymbol{k}) \hat{\rho}_n(\boldsymbol{k}) \otimes \hat{\rho}_m(-\boldsymbol{k}) \tag{68}$$

$$= \bigotimes_{\boldsymbol{k} \in \mathbb{R}^{3+}} \sum_{n,m} p_{nm}(\boldsymbol{k}) |n_{\boldsymbol{k}}\rangle\langle n_{\boldsymbol{k}}| \otimes |m_{-\boldsymbol{k}}\rangle\langle m_{-\boldsymbol{k}}|, \tag{69}$$

where the p_{nm}'s satisfy $\sum_{n,m} p_{nm}(\boldsymbol{k}) = 1$. The p_{nm}'s represent a joint probability distribution function that we do not need to specify for the moment. Several comments are in order at this point. First, the above density matrix represents a quantum state that is *a priori* anisotropic. Indeed, here, the momentum space is split into two parts, $\mathbb{R}^{3+}$ and $\mathbb{R}^{3-}$, and the number of quantas present in each corresponding mode *a priori* differs. In the following, we denote $\boldsymbol{k}$ a vector belonging to $\mathbb{R}^{3+}$ and $-\boldsymbol{p}$ a vector belonging to $\mathbb{R}^{3-}$. Therefore, in Eq. (69), $|m_{-\boldsymbol{k}}\rangle$ denotes a state with m quanta for $-\boldsymbol{k} \in \mathbb{R}^{3-}$, the vector $-\boldsymbol{k}$ being of course minus the vector $\boldsymbol{k}$. Secondly, in Appendix D, we explicitly show that, as stated above, the quantum discord vanishes for the state (69). Thirdly, it is also important to remark, as demonstrated in Appendix E, that this state is not a Gaussian state. As a consequence, one cannot define a covariance matrix. One has to proceed directly from the above density matrix in order to calculate the different two-point (and higher) correlation functions. Straightforward calculations give

$$\langle \hat{v}_{\boldsymbol{k}} \hat{v}_{\boldsymbol{p}} \rangle_{\mathrm{cl}} = \langle \hat{v}_{-\boldsymbol{k}} \hat{v}_{-\boldsymbol{p}} \rangle_{\mathrm{cl}} = 0, \tag{70}$$

$$\langle \hat{v}_{\boldsymbol{k}} \hat{v}_{-\boldsymbol{p}} \rangle_{\mathrm{cl}} = \langle \hat{v}_{-\boldsymbol{k}} \hat{v}_{\boldsymbol{p}} \rangle_{\mathrm{cl}}$$

$$= \frac{1}{2k} \left[1 + \sum_{n,m} n(p_{nm} + p_{mn}) \right] \delta(\boldsymbol{k} - \boldsymbol{p}). \quad (71)$$

The last expression contains a Dirac function which indicates that the result is nonvanishing only if $\boldsymbol{k} = \boldsymbol{p}$, $\boldsymbol{p} \in \mathbb{R}^{3+}$ being the vector obtained from $-\boldsymbol{p} \in \mathbb{R}^{3-}$ by multiplying by minus one. It is also interesting to notice that it can be written as the correlation function calculated in the Bunch-Davies vacuum, $1/(2k)$, multiplied by $1 + \langle \hat{N} \rangle$, where $\langle \hat{N} \rangle$ is the mean of the particle number operator $\hat{N} = c_{\boldsymbol{k}}^{\dagger} c_{\boldsymbol{k}} + c_{-\boldsymbol{k}}^{\dagger} c_{-\boldsymbol{k}}$, in agreement with Refs. [87,88]. The other autocorrelation functions read

$$\langle \hat{p}_{\boldsymbol{k}} \hat{p}_{\boldsymbol{p}} \rangle_{\mathrm{cl}} = \langle \hat{p}_{-\boldsymbol{k}} \hat{p}_{-\boldsymbol{p}} \rangle_{\mathrm{cl}} = 0, \quad (72)$$

$$\langle \hat{p}_{\boldsymbol{k}} \hat{p}_{-\boldsymbol{p}} \rangle_{\mathrm{cl}} = \langle \hat{p}_{-\boldsymbol{k}} \hat{p}_{\boldsymbol{p}} \rangle_{\mathrm{cl}}$$

$$= \frac{k}{2} \left[1 + \sum_{n,m} n(p_{nm} + p_{mn}) \right] \delta(\boldsymbol{k} - \boldsymbol{p}). \quad (73)$$

In particular, for any distribution p_{nm}, one has the following property,

$$k^2 \langle \hat{v}_{\boldsymbol{k}} \hat{v}_{-\boldsymbol{k}} \rangle_{\mathrm{cl}} = \langle \hat{p}_{\boldsymbol{k}} \hat{p}_{-\boldsymbol{k}} \rangle_{\mathrm{cl}}, \quad (74)$$

while one can check, see Eqs. (57) and (58), that this is not the case for the two-mode squeezed state. We conclude that a classical state (68) cannot reproduce all the correlation functions of the two-mode squeezed state in accordance with the fact that they have a different quantum discord.

Finally, the cross correlation functions can be written as

$$\langle \hat{v}_{\boldsymbol{k}} \hat{p}_{\boldsymbol{p}} \rangle_{\mathrm{cl}} = \langle \hat{v}_{-\boldsymbol{k}} \hat{p}_{-\boldsymbol{p}} \rangle_{\mathrm{cl}} = 0, \quad (75)$$

$$\langle \hat{v}_{\boldsymbol{k}} \hat{p}_{-\boldsymbol{p}} \rangle_{\mathrm{cl}} = \frac{i}{2} \left[1 + \sum_{n,m} n(p_{nm} - p_{mn}) \right] \delta(\boldsymbol{k} - \boldsymbol{p}), \quad (76)$$

$$\langle \hat{v}_{-\boldsymbol{k}} \hat{p}_{\boldsymbol{p}} \rangle_{\mathrm{cl}} = \frac{i}{2} \left[1 + \sum_{n,m} n(p_{mn} - p_{nm}) \right] \delta(\boldsymbol{k} - \boldsymbol{p}). \quad (77)$$

The fact that $\langle \hat{v}_{\boldsymbol{k}} \hat{p}_{-\boldsymbol{p}} \rangle_{\mathrm{cl}} \neq \langle \hat{v}_{-\boldsymbol{k}} \hat{p}_{\boldsymbol{p}} \rangle_{\mathrm{cl}}$ is a consequence of the anisotropy of the quantum state (69). If one does not want to violate homogeneity and isotropy, one simply has to require the joint probability p_{nm} to be symmetric, and the cosmological principle is satisfied when

$$p_{nm} = p_{mn}. \quad (78)$$

Let us now be more accurate and repeat the calculation of the power spectrum for the classical state. Using Eqs. (70) and (71), it is obvious that the standard power spectrum can always be recovered provided the coefficients p_{nm} are chosen such that $\langle \hat{v}_{\boldsymbol{k}}(\eta) \hat{v}_{-\boldsymbol{k}}(\eta) \rangle_{\mathrm{cl}} = \langle \hat{v}_{\boldsymbol{k}}(\eta) \hat{v}_{-\boldsymbol{k}}(\eta) \rangle$, that is to say

$$1 + \sum_{n,m} 2n p_{nm} = \cosh(2r_k) + \sinh(2r_k) \cos(2\varphi_k). \quad (79)$$

A possible solution is to take $p_{nm} = p_n \delta(n - m)$ with p_n being given by a thermal distribution, $p_n = (1 - e^{-\beta_k}) e^{-\beta_k n}$, with

$$\beta_k = - \ln \left[\frac{\cosh(2r_k) + \sinh(2r_k) \cos(2\varphi_k) - 1}{\cosh(2r_k) + \sinh(2r_k) \cos(2\varphi_k) + 1} \right]. \quad (80)$$

In the super-Hubble limit, the corresponding temperature goes to infinity. The classical state (68) can then be written

$$\hat{\rho}_{\mathrm{cl}} = (1 - e^{-\beta_k}) \sum_{n=0}^{+\infty} e^{-\beta_k n} |n_{\boldsymbol{k}}, n_{-\boldsymbol{k}} \rangle \langle n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}|. \quad (81)$$

In other words, we have a thermal "gas" of pairs of particles with momentum $\boldsymbol{k}$ and $-\boldsymbol{k}$. At the level of the power spectrum, this classical description leads to the same result as the quantum one.

As was done before for the two-mode squeezed state, let us now try to evaluate other correlators, involving the curvature perturbation and its time derivative. Using Eqs. (75)–(77) together with Eq. (78), we immediately see that

$$\langle \hat{\zeta}(\eta, \boldsymbol{x}) \hat{\zeta}'(\eta, \boldsymbol{y}) + \hat{\zeta}'(\eta, \boldsymbol{x}) \hat{\zeta}(\eta, \boldsymbol{y}) \rangle_{\mathrm{cl}} = 0. \quad (82)$$

Several comments are in order. First, we see that the classical correlator is not equal to the quantum one; see Eq. (66). It is possible to ensure that the quantum and classical power spectra coincide, but then the price to pay is that the other correlators differ. This is of course conceptually very important. It means that by treating the problem classically, rather than quantum mechanically, we are really doing something wrong: some observables are not correctly calculated. This is consistent with the fact that a two-mode squeezed state has a large discord. Secondly, in practice, the difference between the quantum and the classical results is tiny and unobservable probably forever. Indeed, the classical correlator is exactly zero and the quantum one is proportional to $1/z \sim e^{-N_*}$, where N_* is the number of e-folds between the Hubble radius exit during inflation and the end of inflation. Modes probed in the CMB typically have $N_* \sim 50$ and, hence, the quantum correlator is very small for the CMB scales. We conclude that, at the level of the two-point function, a classical description is acceptable. Thirdly, in any case, it should be clear that the classical description is an effective one only. The state (68) is introduced by hand and is not a solution of the quantum Einstein equations.

Finally, let us repeat the calculation of the correlator involving two derivatives of the curvature perturbations for the classical situation. We now have

$$\langle \hat{\zeta}'(\eta,\boldsymbol{x})\hat{\zeta}'(\eta,\boldsymbol{y})\rangle_{\text{cl}} = \frac{1}{(2\pi)^3}\frac{1}{z^2(\eta)}\int d\boldsymbol{k}\langle \hat{p}_{\boldsymbol{k}}\hat{p}_{-\boldsymbol{k}}\rangle_{\text{cl}}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

$$(83)$$

$$= \frac{4\pi}{(2\pi)^3}\frac{1}{z^2(\eta)}\int_0^\infty \frac{dk}{k}\frac{\sin(k|\boldsymbol{x}-\boldsymbol{y}|)}{k|\boldsymbol{x}-\boldsymbol{y}|}$$
$$\times k^3 k \sum_{n,m} p_{nm}\left(n + \frac{1}{2}\right). \qquad (84)$$

But the coefficients p_{nm} were specified when we required the quantum and classical power spectra to be the same. This means that, as already discussed, the above correlator is in fact explicitly known. From Eq. (79), indeed, one has

$$\langle \hat{\zeta}'(\eta,\boldsymbol{x})\hat{\zeta}'(\eta,\boldsymbol{y})\rangle_{\text{cl}} = \frac{4\pi}{(2\pi)^3}\frac{1}{z^2(\eta)}\int_0^\infty \frac{dk}{k}\frac{\sin(k|\boldsymbol{x}-\boldsymbol{y}|)}{k|\boldsymbol{x}-\boldsymbol{y}|}$$
$$\times k^3 \frac{k}{2}[\cosh(2r_k)+\sinh(2r_k)\cos(2\varphi_k)],$$

$$(85)$$

which is different from Eq. (68). In the de Sitter super-Hubble limit, this gives $\langle \hat{\zeta}'\hat{\zeta}'\rangle_{\text{cl}} \sim \eta^4$. Therefore, we conclude that $\langle \hat{\zeta}'\,\hat{\zeta}'\rangle \sim e^{-2N_*}$ and $\langle \hat{\zeta}'\,\hat{\zeta}'\rangle_{\text{cl}} \sim e^{-4N_*}$ strongly differ ($\langle \hat{\zeta}'\hat{\zeta}'\rangle/\langle \hat{\zeta}'\,\hat{\zeta}'\rangle_{\text{cl}} \sim e^{2N_*} \gg 1$) and that, as before, the classical description cannot reproduce the quantum results. Similarly as before, however, since both results are exponentially small, this piece of information is hidden from all practical experiment.

B. Non-Gaussianities

We have just seen that two-point correlators for a discordant and for a nondiscordant state differ but that this difference is, in practice, very small. Another way to differentiate these two states is to look at non-Gaussianities. All three-point correlation functions vanish in both the two-mode squeezed and classical states; hence one has to consider four-point correlation functions. For the classical state, making use of the same techniques as above, one finds

$$\langle \hat{\zeta}_{\boldsymbol{k}_1}\hat{\zeta}_{\boldsymbol{k}_2}\hat{\zeta}_{\boldsymbol{k}_3}\hat{\zeta}_{\boldsymbol{k}_4}\rangle_{\text{cl}} = \langle \hat{\zeta}_{\boldsymbol{k}_1}\hat{\zeta}_{\boldsymbol{k}_2}\rangle_{\text{cl}}\langle \hat{\zeta}_{\boldsymbol{k}_3}\hat{\zeta}_{\boldsymbol{k}_4}\rangle_{\text{cl}} + \langle \hat{\zeta}_{\boldsymbol{k}_1}\hat{\zeta}_{\boldsymbol{k}_3}\rangle_{\text{cl}}\langle \hat{\zeta}_{\boldsymbol{k}_2}\hat{\zeta}_{\boldsymbol{k}_4}\rangle_{\text{cl}} + \langle \hat{\zeta}_{\boldsymbol{k}_1}\hat{\zeta}_{\boldsymbol{k}_4}\rangle_{\text{cl}}\langle \hat{\zeta}_{\boldsymbol{k}_2}\hat{\zeta}_{\boldsymbol{k}_3}\rangle_{\text{cl}}$$
$$+ \frac{\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}}{2k_1^2 z^4}\left\{4\sum_n n^2 p_n(k_1) - 4\left[\sum_n n p_n(k_1)\right]^2 - \sum_n n(n+1)p_n(k_1)\right\}$$
$$\times (\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\delta_{\boldsymbol{k}_1,\boldsymbol{k}_3} + \delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_3} + \delta_{\boldsymbol{k}_1,\boldsymbol{k}_2}\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_3}).$$

$$(86)$$

In this expression, we have kept the coefficients $p_n(k)$ unspecified so that the following generic comments can be made. The first line of Eq. (86) corresponds to the standard disconnected part of the four-point correlation function. For the two-mode squeezed state, it is the only term that is obtained, which is in agreement with Wick's theorem given that, as already noted, the two-mode squeezed state is Gaussian. This means that the second and third lines of Eq. (86) correspond to a non-Gaussian contribution. This one is not of the local type and, in fact, corresponds to a scale dependent g_{NL}, notably due to the presence of the δ terms. Therefore, it remains to generate templates corresponding to this structure in order to determine whether such a four-point function is already excluded or not. Let us however notice that for the thermal state introduced above Eq. (80), the term in braces is given by $(\cosh \beta_{k_1} - 1)^{-1}$. On super-Hubble scales, $\beta_{k_1} \to 0$ and this term blows up. Let us also remark that the non-Gaussianities studied here are due to the intrinsic non-Gaussian character [89,90] of the classical state and not to the nonlinearities of the theory. But of course we expect those nonlinearities to produce extra contributions, at the level of both the three- and four-point correlation functions [88,91]. We conclude that measuring the four-point correlation function would be

an important test since it has the potential (depending on the level of classical non-Gaussianity) to rule out non-discordant primordial states, hence indirectly proving the large quantumness of the CMB anisotropies.

VI. CAN THE QUANTUM CORRELATION FUNCTIONS BE OBTAINED IN A CLASSICAL STOCHASTIC DESCRIPTION?

In Sec. V, we studied to what extent a classical state can reproduce the quantum correlation functions. We saw that, in principle, it cannot (even if, in practice, it remains to be seen whether it can be detected). Although it was called classical, the situation was still described in terms of a quantum state. This is why here we go one step further and even ignore the quantum formalism, working only in terms of a stochastic probability distribution.

A. General properties

If an effective classical stochastic description is possible, quantum averages of any operator $\hat{A}$ should be given by an integral of a stochastic distribution over classical phase space, namely

$$\langle \hat{A} \rangle = \langle A \rangle_{\text{stocha}}, \qquad (87)$$

with

$$\langle A \rangle_{\text{stocha}} \equiv \int A(R) W(R) \mathrm{d}^4 R. \qquad (88)$$

As usual, a hat denotes a physical observable $\hat{A} = A(\hat{R})$, while A (no hat) is an ordinary real function. The vector R, previously defined below Eq. (54), represents the classical variables in phase space and, obviously, $\hat{R}$ denotes the corresponding operators $\hat{R} = (k^{1/2} \hat{q}_{\mathbf{k}}, k^{-1/2} \hat{\pi}_{\mathbf{k}}, k^{1/2} \hat{q}_{-\mathbf{k}}, k^{-1/2} \hat{\pi}_{-\mathbf{k}})^{\mathrm{T}}$. The quantity $W(R)$ represents a probability density function. The stochastic description is possible and meaningful if three conditions are met: first, W is positive everywhere and normalized to unity (so that it can be interpreted as a distribution function); second, W obeys the classical equation of motion (see below); and, third, Eq. (87) is valid.

Let us now discuss how W can be concretely found (if it can). For this purpose, we introduce the Weyl transform $\tilde{A}$ of the operator $\hat{A}$ through the following expression,

$$\tilde{A}(q_{\mathbf{k}}, \pi_{\mathbf{k}}, q_{-\mathbf{k}}, \pi_{-\mathbf{k}})$$
$$\equiv \int \mathrm{d}x \mathrm{d}y e^{-i\pi_{\mathbf{k}} x - i\pi_{-\mathbf{k}} y}$$
$$\times \left\langle q_{\mathbf{k}} + \frac{x}{2}, q_{-\mathbf{k}} + \frac{y}{2} \middle| \hat{A} \middle| q_{\mathbf{k}} - \frac{x}{2}, q_{-\mathbf{k}} - \frac{y}{2} \right\rangle, \quad (89)$$

which builds a real function in phase space, $\tilde{A}$, out of the quantum operator $\hat{A}$. It is important to stress that $\tilde{A}$ is a function (i.e. not an operator) which, for instance, means that $\tilde{A} \tilde{B} = \tilde{B} \tilde{A}$. However, $\widetilde{AB} \neq \widetilde{BA}$ in general. It is also important to notice that, a priori, $\tilde{A} \neq A$. In fact, in Appendix F, the following exact result is established,

$$\tilde{R}_i = R_i, \qquad \widetilde{R_j R_k} = R_j R_k + \frac{i}{2} J_{jk}, \qquad (90)$$

where the matrix J is defined in Eq. (56). Moreover, since R and $V \equiv (k^{1/2} v_{\mathbf{k}}, k^{-1/2} p_{\mathbf{k}}, k^{1/2} v_{-\mathbf{k}}, k^{-1/2} p_{-\mathbf{k}})^{\mathrm{T}}$ are linearly related through Eqs. (51) and (52), this relation translates into a similar one for V, namely

$$\tilde{V}_i = V_i, \qquad \widetilde{V_j V_k} = V_j V_k + \frac{i}{2} I_{jk}, \qquad (91)$$

where I is the antidiagonal matrix with coefficients $\{-1, 1, -1, 1\}$ on the antidiagonal and verifies $iI_{j,k} = [\hat{V}_j, \hat{V}_k]$. Therefore, the only nonvanishing terms of the I matrix are the ones corresponding to products of the form $v_{\mathbf{k}} p_{-\mathbf{k}}$. Even more general relations are established in Appendix F.

The fundamental property of the Weyl transform is that

$$\mathrm{Tr}(\hat{A} \hat{B}) = \int \tilde{A}(R) \tilde{B}(R) \mathrm{d}^4 R. \qquad (92)$$

Given that $\langle \tilde{A} \rangle = \mathrm{Tr}(\hat{\rho} \hat{A})$, this immediately suggests taking for W the Weyl transform of the density operator, $W = \tilde{\rho}/(2\pi)^2$ [the factor $(2\pi)^2$ originates from the fact that we consider a system whose phase space is $\mathbb{R}^4$; had we considered a system with phase space $\mathbb{R}^{2n}$, this factor would have been $(2\pi)^n$] namely

$$W(R) \equiv \frac{1}{(2\pi)^2} \int \mathrm{d}x \mathrm{d}y e^{-i\pi_{\mathbf{k}} x - i\pi_{-\mathbf{k}} y}$$
$$\times \left\langle q_{\mathbf{k}} + \frac{x}{2}, q_{-\mathbf{k}} + \frac{y}{2} \middle| \hat{\rho} \middle| q_{\mathbf{k}} - \frac{x}{2}, q_{-\mathbf{k}} - \frac{y}{2} \right\rangle. \quad (93)$$

Then it follows that any expectation value of a physical observable can be obtained through the relation

$$\langle \hat{A} \rangle = \int \tilde{A}(R) W(R) \mathrm{d}^4 R, \qquad (94)$$

or, in other words,

$$\langle \hat{A} \rangle = \langle \tilde{A} \rangle_{\text{stocha}}. \qquad (95)$$

It is worth emphasizing again that this equation is exact. As a consequence, the condition given by Eq. (87) is realized if the Weyl transform equals its classical counterpart, that is to say if

$$\tilde{A} = A. \qquad (96)$$

Let us conclude this subsection by noticing that, as is well known, $W(R)$ given by Eq. (93) is the Wigner function [92,93]. For illustrative purposes, we now give the Wigner function of the two-mode squeezed and classical states. With the covariance matrices calculated in Sec. IV, this gives rise to an explicit expression for the two-mode squeezed state Wigner function, namely (see Appendix G)

$$W = \frac{1}{\pi^2} \exp \left[- \left(k q_{\mathbf{k}}^2 + k q_{-\mathbf{k}}^2 + \frac{\pi_{\mathbf{k}}^2}{k} + \frac{\pi_{-\mathbf{k}}^2}{k} \right) \cosh(2r_k) \right.$$
$$+ 2(q_{\mathbf{k}} \pi_{\mathbf{k}} + q_{-\mathbf{k}} \pi_{-\mathbf{k}}) \sin(2\varphi_k) \sinh(2r_k)$$
$$\left. + 2 \left(k q_{\mathbf{k}} q_{-\mathbf{k}} - \frac{\pi_{\mathbf{k}} \pi_{-\mathbf{k}}}{k} \right) \cos(2\varphi_k) \sinh(2r_k) \right]. \quad (97)$$

It is a positive, Gaussian function. Using Eqs. (51) and (52), the corresponding expression in terms of the $v_{\mathbf{k}}$ and $p_{\mathbf{k}}$ variables can be established as well. One can notice that due to the last term, W can neither be factorized as $W = W_{\mathbf{k}}(q_{\mathbf{k}}, \pi_{\mathbf{k}}) W_{-\mathbf{k}}(q_{-\mathbf{k}}, \pi_{-\mathbf{k}})$ nor as $W = W_{\mathbf{k}}(v_{\mathbf{k}}, p_{\mathbf{k}}) W_{-\mathbf{k}}(v_{-\mathbf{k}}, p_{-\mathbf{k}})$, which reflects the presence of correlations between the modes $\mathbf{k}$ and $-\mathbf{k}$. These correlations vanish
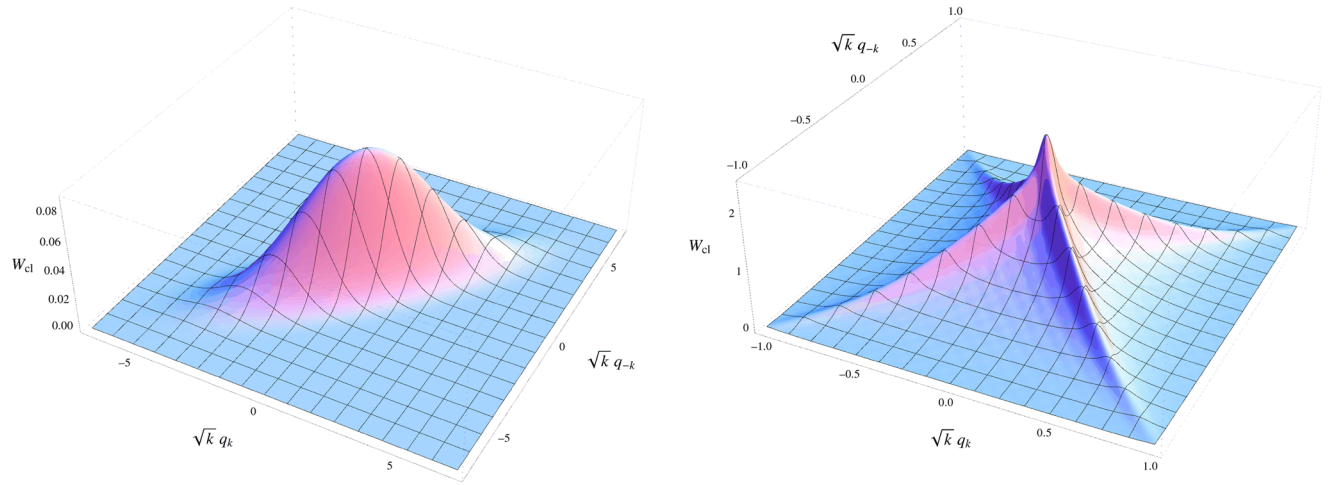
FIG. 3.   Marginalized Wigner function $W(q_k, q_{-k}) = \int W(q_k, \pi_k, q_{-k}, \pi_{-k}) \mathrm{d}\pi_k \mathrm{d}\pi_{-k}$, for the super-Hubble mode $k/(aH) = 0.3$. The left panel corresponds to the two-mode squeezed state while the right panel is for the classical state (81). One can see that as explained in the text and contrary to the two-mode squeezed state, the classical state is both non-Gaussian and unsqueezed.

in the sub-Hubble limit [where $\cos(2\varphi_k)\sinh(2r_k) = 0$], but in the super-Hubble limit, the Wigner function gets squeezed along the directions "$k = -k$,"

$$W \propto \delta(q_k - q_{-k})\delta(\pi_k + \pi_{-k}), \qquad (98)$$

hence the term squeezing.

Let us now calculate the Wigner function of the classical state (69). Combining Eqs. (G19) and (E14), one obtains after a few manipulations

$$W_{\mathrm{cl}} = \frac{1}{(2\pi)^2}\int_0^\infty x\mathrm{d}x \int_0^\infty y\mathrm{d}y$$
$$\times \exp\left[-\frac{\coth(\beta_k/2)}{4}(x^2 + y^2)\right] I_0\left[\mathrm{cosech}\left(\frac{\beta_k}{2}\right)\frac{xy}{4}\right]$$
$$\times J_0\left(x\sqrt{kq_k^2 + \frac{\pi_k^2}{k}}\right) J_0\left(y\sqrt{kq_{-k}^2 + \frac{\pi_{-k}^2}{k}}\right), \qquad (99)$$

where β_k has been defined in Eq. (80) and I_0 and J_0 are Bessel functions. As already pointed out in Sec. V, one can see that the classical state is a non-Gaussian one.[3] It is also interesting to notice that, in the sub-Hubble limit,

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[3]Let us notice that when one of the modes is integrated out, the classical Wigner function becomes Gaussian. For example, the marginalized Wigner function in the plane (q_k, π_k), obtained by marginalizing over q_{-k} and π_{-k}, can be expressed as the Fourier transform of the restrained characteristic function $\chi(\eta_1, \eta_2, 0, 0)$, and one obtains

$$W_{\mathrm{cl}}(q_k, \pi_k) = \frac{1}{\pi}\tanh\left(\frac{\beta_k}{2}\right)e^{-\tanh(\frac{\beta_k}{2})(kq_k^2 + \frac{\pi_k^2}{k})}. \qquad (100)$$

Therefore, the non-Gaussianity of the classical Wigner function directly reflects the (classical) correlations between the two modes k and $-k$.

$\beta_k \to \infty$, the argument of the I_0 function vanishes and the integrals over x and y can be factorized. This means that, in this limit, the Wigner function becomes separable, $W = W_k(q_k, \pi_k)W_{-k}(q_{-k}, \pi_{-k})$. When β_k decreases, this is not the case anymore and the two modes are correlated. This is similar to what was noted for the two-mode squeezed state. However, these correlations are of a different nature. Indeed, since the classical Wigner function depends on the phase-space variables only through the two combinations $kq_k^2 + \pi_k^2/k$ and $kq_{-k}^2 + \pi_{-k}^2/k$, it is rotationally invariant inside each mode. In particular, squeezing does not take place. This is consistent with the results obtained in the previous sections. In Fig. 3, we display the marginalized Wigner functions $W(q_k, q_{-k})$ for the two-mode squeezed and classical states.

B. The case of a Gaussian state

So far, the discussion was fully generic. We now discuss whether the three conditions for the validity of a stochastic classical description mentioned before, namely W being positive everywhere, W obeying the classical equation of motion and the validity of Eq. (87) or, equivalently, of Eq. (96), are satisfied for a Gaussian state.

Let us start with the first condition, i.e. the positivity of the Wigner function. In Appendix G, it is shown that, for Gaussian states, the characteristic function of which is of the form given by Eq. (55), one has

$$W(R) = \frac{1}{\pi^2\sqrt{\det\gamma}}e^{-R^{\mathrm{T}}\gamma^{-1}R}. \qquad (101)$$

In this case, the Wigner function is therefore a (correctly normalized) Gaussian function with covariance matrix γ (hence its name). This also means that it is positive at any time and, thus, indeed satisfies our first condition. Notice that this is a general feature of Gaussian states and is

completely independent from the fact that we have squeezing (large or not). For instance, a coherent state also has a positive Wigner function.

Let us now examine the second condition. In Appendix H, we show that, for any quadratic Hamiltonian, the Wigner function obeys a classical equation of motion. Indeed, if one differentiates Eq. (93) with respect to time and makes use of the Schrödinger equation, one obtains

$$\frac{dW}{d\eta} = \{H_{\boldsymbol{k}}, W\}_{\text{PB}}, \tag{102}$$

where the right-hand side of this equation is the Poisson bracket between the Hamiltonian $H_{\boldsymbol{k}}$ and the Wigner function. The above equation therefore describes a Liouville evolution. This means that if one starts from a collection of pairs of pointlike particles [the first one living in $(q_{\boldsymbol{k}}, \pi_{\boldsymbol{k}})$, the other one in $(q_{-\boldsymbol{k}}, \pi_{-\boldsymbol{k}})$] at time η that mimic the distribution $W(\eta)$, and if one lets them all evolve according to the classical Hamilton equations, the distribution calculated at later time η' matches $W(\eta')$. We therefore conclude that the second condition is also met. Again, this is completely independent of whether we have large squeezing or not and is true for any quadratic Hamiltonian.

Finally, we come to the third condition, namely the equality between quantum and stochastic correlators. Using Eqs. (90) and (95), one has

$$\langle \hat{R}_j \hat{R}_k \rangle = \langle \widetilde{R_j R_k} \rangle_{\text{stocha}} = \left\langle R_j R_k + \frac{i}{2} J_{jk} \right\rangle_{\text{stocha}} \tag{103}$$

$$= \langle R_j R_k \rangle_{\text{stocha}} + \frac{i}{2} J_{jk}. \tag{104}$$

This implies that for combinations of $\hat{R}_j$ and $\hat{R}_k$ such that $J_{jk} = 0$, the stochastic distribution reproduces exactly the two-point quantum correlators. The only cases where $J_{jk} \neq 0$ correspond to mixed terms and we find

$$\langle q_{\boldsymbol{k}} \pi_{\boldsymbol{k}} \rangle_{\text{stocha}} = \langle q_{-\boldsymbol{k}} \pi_{-\boldsymbol{k}} \rangle_{\text{stocha}} = \langle \pi_{\boldsymbol{k}} q_{\boldsymbol{k}} \rangle_{\text{stocha}}$$
$$= \langle \pi_{-\boldsymbol{k}} q_{-\boldsymbol{k}} \rangle_{\text{stocha}} = 0. \tag{105}$$

In terms of the Mukhanov-Sasaki variable and its conjugate momentum, this means that

$$\langle v_{\boldsymbol{k}} v_{\boldsymbol{p}} \rangle_{\text{stocha}} = \frac{1}{2k} [\cosh(2r_k) + \sinh(2r_k)\cos(2\varphi_k)] \delta(\boldsymbol{k} + \boldsymbol{p}), \tag{106}$$

$$\langle p_{\boldsymbol{k}} p_{\boldsymbol{p}} \rangle_{\text{stocha}} = \frac{k}{2} [\cosh(2r_k) - \sinh(2r_k)\cos(2\varphi_k)] \delta(\boldsymbol{k} + \boldsymbol{p}), \tag{107}$$

$$\langle v_{\boldsymbol{k}} p_{\boldsymbol{p}} \rangle_{\text{stocha}} = \frac{1}{2} \sinh(2r_k)\sin(2\varphi_k)\delta(\boldsymbol{k} + \boldsymbol{p}). \tag{108}$$

These equations should be compared to their quantum counterparts, namely Eqs. (57)–(59). We notice that the only correlator which is not identical is the last one: $\langle \hat{v}_{\boldsymbol{k}} \hat{p}_{-\boldsymbol{k}} \rangle \neq \langle v_{\boldsymbol{k}} p_{-\boldsymbol{k}} \rangle_{\text{stocha}}$; see Eqs. (59) and (108). There is an additional $i/2$ in the quantum correlator which makes the difference. However, contrary to the classical state studied in Sec. V, all the correlators of $\hat{\zeta}$ and $\hat{\zeta}'$ are correctly reproduced. This is evident for $\langle \hat{\zeta} \hat{\zeta} \rangle$ and $\langle \hat{\zeta}' \hat{\zeta}' \rangle$ since they do not involve the two-point function $\langle \hat{v}_{\boldsymbol{k}} \hat{p}_{-\boldsymbol{k}} \rangle$ but it is also true that

$$\langle \hat{\zeta}(\eta, \boldsymbol{x}) \hat{\zeta}'(\eta, \boldsymbol{y}) + \hat{\zeta}'(\eta, \boldsymbol{x}) \hat{\zeta}(\eta, \boldsymbol{y}) \rangle$$
$$= \langle \zeta(\eta, \boldsymbol{x}) \zeta'(\eta, \boldsymbol{y}) + \zeta'(\eta, \boldsymbol{x}) \zeta(\eta, \boldsymbol{y}) \rangle_{\text{stocha}}. \tag{109}$$

The reason is that the above operator is Hermitian and, therefore, the complex term $i/2$ cancels out. This means that, as far as the two-point correlators are concerned, the quantum and stochastic descriptions cannot be observationally distinguished. Notice that this conclusion is independent of the value of r_k and is valid for any value of the squeezing parameter.

Actually, squeezing only plays a role when computing higher order observable quantities involving the momentum. For example, in Appendix F, it is shown that $\widetilde{q_{\boldsymbol{k}}^2 \pi_{\boldsymbol{k}}^2} = q_{\boldsymbol{k}}^2 \pi_{\boldsymbol{k}}^2 + 2iq_{\boldsymbol{k}} \pi_{\boldsymbol{k}} - 1/2$ and $\widetilde{\pi_{\boldsymbol{k}}^2 q_{\boldsymbol{k}}^2} = \pi_{\boldsymbol{k}}^2 q_{\boldsymbol{k}}^2 - 2i\pi_{\boldsymbol{k}} q_{\boldsymbol{k}} - 1/2$. Making use of Eq. (95), this means that

$$\langle q_{\boldsymbol{k}}^2 \pi_{\boldsymbol{k}}^2 + \pi_{\boldsymbol{k}}^2 q_{\boldsymbol{k}}^2 \rangle_{\text{stocha}} = \langle q_{\boldsymbol{k}}^2 \pi_{\boldsymbol{k}}^2 + \pi_{\boldsymbol{k}}^2 q_{\boldsymbol{k}}^2 \rangle - 1, \tag{110}$$

so that the difference between the quantum and stochastic correlators is still present for Hermitian operators. However, from the results of Sec. IV A, one has $\langle q_{\boldsymbol{k}}^2 \pi_{\boldsymbol{k}}^2 + \pi_{\boldsymbol{k}}^2 q_{\boldsymbol{k}}^2 \rangle = \cosh^2(2r_k)/2$ which is exponentially large when $r_k \gg 1$. Therefore, in the large squeezing limit, the stochastic description gives accurate results for these correlators too.

As mentioned before, a difference between the quantum and stochastic cases arises only if the corresponding correlators contain $\hat{q}_{\boldsymbol{k}}$ and the momentum. In particular, we show in Appendix F that $\widetilde{q_{\boldsymbol{k}}^n} = q_{\boldsymbol{k}}^n$. However, this does not mean that higher order terms, such as $\langle \hat{\zeta}_{\boldsymbol{k}}^4 \rangle$, will also be correctly and automatically reproduced since $\hat{\zeta}_{\boldsymbol{k}} = \hat{v}_{\boldsymbol{k}}/z$ is expressed in terms of $\hat{q}_{\boldsymbol{k}}$ but also in terms of $\hat{\pi}_{\boldsymbol{k}}$; see Eq. (51). In other words, higher order correlators of $\hat{\zeta}_{\boldsymbol{k}}$ will precisely involve terms with various powers of $\hat{q}_{\boldsymbol{k}}$ and $\hat{\pi}_{\boldsymbol{k}}$ for which the squeezing plays a role.

It is usually said that a stochastic classical distribution successfully reproduces the quantum correlators of cosmological fluctuations in the large squeezing limit. The results of this section shed some light on this statement. The fact that the Wigner function is positive and evolves under the classical equations of motion is independent of the squeezing level and is just a consequence of having a quadratic

TABLE I.   Reproducibility of the two-mode squeezed state correlation functions by a classical state (see Sec. IV) and a classical stochastic distribution (see Sec. VI).

	Classical state		Stochastic distribution	
Correlation function	Generic	Super-Hubble	Generic	Super-Hubble (squeezed)
$\mathcal{P}_\zeta \propto \langle \zeta_k \zeta_{-k} \rangle$	Succeeds		Succeeds	
Other non-Hermitian two-point correlators	Fails	Strongly fails but nonobservable	Fails	Succeeds
Other Hermitian two-point correlators			Succeeds	
Higher order Hermitian correlators	Fails	Strongly fails	Fails	Succeeds

Hamiltonian and a Gaussian state. For observable (i.e. Hermitian) operators, two-point correlators are also correctly reproduced by the Wigner function regardless of the squeezing level, and it is only when considering higher order correlators that squeezing comes into play.[4] Therefore, our ability to calculate exactly the power spectrum in the stochastic approach has nothing to do with the large squeezing limit but is in fact due to the property that the Hamiltonian is quadratic.

VII. DISCUSSION AND CONCLUSION

Let us now summarize our main findings. The presence of quantum correlations in a quantum state can be characterized by the recently proposed quantum discord [52,53]. Cosmological perturbations produced during inflation are placed in a two-mode squeezed state, for which we have calculated the quantum discord and showed that it is very large at the end of inflation. This means that primordial perturbations represent a highly nonclassical system.

We have also computed the discord for a general splitting of the system and shown that, except for the specific case where the two subsystems correspond to the real and imaginary parts of the Mukhanov-Sasaki variable, the quantum discord is always very large on super-Hubble scales. In an ordinary situation, such as a Bell-type experiment where the spin states of two remote systems are measured, the splitting is unambiguous, and the two subsystems are obviously the two particles that have travelled far away from each other. But the situation is less clear when a measurement of a "continuous" system is performed. A priori, a measurement corresponds to a realization which provides us with all the values of the q_k's, i.e. a map of the CMB. How should we split this system into two subsystems? These tricky questions are in fact connected to deeper issues related to the status of the measurement problem in a cosmological context; see Refs. [35,95].

In order to determine how the large quantum discord is reflected at the observational level, we have compared the correlation functions of the two-mode squeezed states with the ones that would be obtained (i) from a classical state, that is to say a state with vanishing quantum discord, that contains classical correlations only; and (ii) from a classical stochastic distribution. The results are summarized in Table I. One can design a classical state that reproduces the observed primordial power spectrum. Large differences with the two-mode squeezed state correlators then appear in other two-point functions but they are typically hidden in the decaying mode, and are hence unobservable. However, such classical states are intrinsically non-Gaussian, at a level that remains to be estimated but which could possibly lead to the indirect detection of the large CMB quantumness. This is a very important conclusion of this work. Then, if one is ready to abandon the idea that perturbations should be described in the framework of quantum mechanics and wants to use a classical stochastic distribution instead, we have seen that because the Hamiltonian is quadratic and the two-mode squeezed state is Gaussian, the Wigner function provides such a distribution, evolving under the classical equation of motion and allowing us to reproduce all observable (i.e. Hermitian) two-point correlators. It is important to stress that the squeezing plays no role at this level. The large squeezing is important only if one wants to correctly reproduce non-Hermitian or higher order correlators.

Let us conclude with a few remarks. First, the fact that a classical stochastic distribution succeeds in reproducing the two-mode squeezed state late time correlators is due to most phase information being contained in the decaying mode, which is not observable in the large squeezing limit in the minimal setup we considered here. More complicated situations, such as multifield inflation [96] where entropic perturbations can source the decaying mode on large scales, or models where features in the potential [97–99] give rise to transient violations of slow roll (hence modified squeezing histories), may therefore lead to different conclusions. Second, in light of the results of this paper, the presence of large quantum correlations in primordial perturbations can be seen as a remarkable feature of inflation. It would be interesting to see whether this property is also present in alternative scenarios [100] to inflation, such as bouncing cosmologies [101] or ekpyrotic setups [102].

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[4]A remaining nontrivial question is whether a stochastic classical description is valid to describe (even approximately) correlators of nonanalytical functions in v and p (or q and π) [94].

The answer to the question raised in the title of this paper is, at the level of two-point correlators, yes in principle, but no in practice. However, other observations, such as the non-Gaussian patterns derived in Sec. V B or Bell-type experiments, may be capable of proving the existence of large quantum correlations in the CMB. We plan to investigate these issues in future publications [103].

ACKNOWLEDGMENTS

APPENDIX A: THE PROJECTED STATE IS A PURE STATE

In this appendix, we show that the projected state defined in Eq. (37) is a pure state, and that, as a consequence, its entropy vanishes. Let us start from a general bipartite state which can always be written as

$$|\psi\rangle = \sum_{n,m} c_{nm} |n_{\boldsymbol{k}}, m_{-\boldsymbol{k}}\rangle. \tag{A1}$$

Then the corresponding density matrix reads

$$\hat{\rho}(\boldsymbol{k}, -\boldsymbol{k}) = \sum_{n,m} \sum_{p,q} c_{nm} c_{pq}^* |n_{\boldsymbol{k}}\rangle\langle p_{\boldsymbol{k}}| \otimes |m_{-\boldsymbol{k}}\rangle\langle q_{-\boldsymbol{k}}|. \tag{A2}$$

The next step, as was discussed around Eq. (37), is to consider the following projector which corresponds to the measurement of an observable of the subsystem $-\boldsymbol{k}$:

$$\hat{\Pi}_j = \hat{\mathrm{Id}}_{\boldsymbol{k}} \otimes |j_{-\boldsymbol{k}}\rangle\langle j_{-\boldsymbol{k}}|. \tag{A3}$$

From this expression, it is easy to show that

$$\hat{\rho}(\boldsymbol{k}, -\boldsymbol{k})\hat{\Pi}_j = \sum_{n,m} \sum_p c_{nm} c_{pj}^* |n_{\boldsymbol{k}}\rangle\langle p_{\boldsymbol{k}}| \otimes |m_{-\boldsymbol{k}}\rangle\langle j_{-\boldsymbol{k}}|, \tag{A4}$$

from which we deduce that

$$\mathrm{Tr}_{-\boldsymbol{k}}[\hat{\rho}(\boldsymbol{k}, -\boldsymbol{k})\hat{\Pi}_j] = \sum_i \langle i_{-\boldsymbol{k}}|\hat{\rho}\hat{\Pi}_j|i_{-\boldsymbol{k}}\rangle$$
$$= \sum_n \sum_p c_{nj} c_{pj}^* |n_{\boldsymbol{k}}\rangle\langle p_{\boldsymbol{k}}|. \tag{A5}$$

We also have $\mathrm{Tr}[\hat{\rho}(\boldsymbol{k}, -\boldsymbol{k})\hat{\Pi}_j] = \sum_n c_{nj} c_{nj}^*$ where, here, the trace is taken over the full state space. Therefore, the state defined in Eq. (37) can be expressed as

$$\hat{\rho}(\boldsymbol{k}; \hat{\Pi}_j) = \frac{1}{\sum_n c_{nj} c_{nj}^*} \sum_n \sum_p c_{nj} c_{pj}^* |n_{\boldsymbol{k}}\rangle\langle p_{\boldsymbol{k}}|. \tag{A6}$$

From this expression, it is straightforward to show that $\hat{\rho}^2(\boldsymbol{k}; \hat{\Pi}_j) = \hat{\rho}(\boldsymbol{k}; \hat{\Pi}_j)$. Indeed, using the above explicit expression, one has

$$\hat{\rho}^2(\boldsymbol{k}; \hat{\Pi}_j) = \frac{1}{\sum_n c_{nj} c_{nj}^*} \frac{1}{\sum_m c_{mj} c_{mj}^*} \sum_{pq} c_{pj} c_{qj}^* |p_{\boldsymbol{k}}\rangle\langle q_{\boldsymbol{k}}|$$
$$\times \sum_{rs} c_{rj} c_{sj}^* |r_{\boldsymbol{k}}\rangle\langle s_{\boldsymbol{k}}| \tag{A7}$$

$$= \frac{1}{\sum_n c_{nj} c_{nj}^*} \frac{1}{\sum_m c_{mj} c_{mj}^*} \sum_{pq} \sum_{rs} c_{pj} c_{qj}^* c_{rj} c_{sj}^* |p_{\boldsymbol{k}}\rangle$$
$$\times \langle q_{\boldsymbol{k}}|r_{\boldsymbol{k}}\rangle\langle s_{\boldsymbol{k}}| \tag{A8}$$

$$= \frac{1}{\sum_n c_{nj} c_{nj}^*} \frac{1}{\sum_m c_{mj} c_{mj}^*} \sum_p \sum_{rs} c_{pj} c_{rj}^* c_{rj} c_{sj}^*$$
$$\times |p_{\boldsymbol{k}}\rangle\langle s_{\boldsymbol{k}}| \tag{A9}$$

$$= \frac{1}{\sum_n c_{nj} c_{nj}^*} \frac{1}{\sum_m c_{mj} c_{mj}^*} \sum_r c_{rj} c_{rj}^*$$
$$\times \sum_p \sum_s c_{pj} c_{sj}^* |p_{\boldsymbol{k}}\rangle\langle s_{\boldsymbol{k}}| \tag{A10}$$

$$= \frac{1}{\sum_n c_{nj} c_{nj}^*} \sum_p \sum_s c_{pj} c_{sj}^* |p_{\boldsymbol{k}}\rangle\langle s_{\boldsymbol{k}}|$$
$$= \hat{\rho}(\boldsymbol{k}; \hat{\Pi}_j). \tag{A11}$$

It is therefore a pure state and, consequently, as noticed in the main text, its entropy vanishes.

APPENDIX B: QUANTUM DISCORD FOR A GENERAL SPLITTING OF THE SYSTEM

In this appendix, we calculate the discord for a general splitting of the quantum system as discussed in Sec. III. The calculations presented here lead to Fig. 2. The state of the system is given by Eq. (28) and can be written as

$$\hat{\rho} = \frac{1}{\cosh^2(r_{\boldsymbol{k}})} \sum_{n,n'=0}^{\infty} e^{-2i(n-n')\varphi_{\boldsymbol{k}}} \tanh^{n+n'}(r_{\boldsymbol{k}}) |n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle\langle n'_{\boldsymbol{k}}, n'_{-\boldsymbol{k}}|$$
$$= \sum_{n,n'=0}^{\infty} a_{n,n'} |n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle\langle n'_{\boldsymbol{k}}, n'_{-\boldsymbol{k}}|, \tag{B1}$$

which defines the coefficients $a_{n,n'}$. Here, the density matrix is expanded over the states $|n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle$, which is especially convenient when the system is split according to $\mathcal{E} = \mathcal{E}_{\boldsymbol{k}} \otimes \mathcal{E}_{-\boldsymbol{k}}$. In this appendix, however, we consider the case where $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$, the corresponding ladder operators being given by Eqs. (47) and (48). It is therefore more convenient to use the states

$$|n_1, m_2\rangle = \frac{(\hat{a}_1^{\dagger})^n}{\sqrt{n!}} \frac{(\hat{a}_2^{\dagger})^m}{\sqrt{m!}} |0_1, 0_2\rangle, \tag{B2}$$

containing n quantas in the mode 1 and m quantas in the mode 2. As a consequence, one must first write

$|n_k, n_{-k}\rangle = \sum_{m,p}\langle m_1, p_2|n_k, n_{-k}\rangle|m_1, p_2\rangle$ and calculate $\langle m_1, p_2|n_k, n_{-k}\rangle$. This quantity can be written as

$$\langle m_1, p_2|n_k, n_{-k}\rangle = \frac{1}{\sqrt{m!p!n!n!}}$$
$$\times \langle 0_1, 0_2|(\hat{a}_1)^m(\hat{a}_2)^p(\hat{c}_k^\dagger)^n(\hat{c}_{-k}^\dagger)^n|0_1, 0_2\rangle, \tag{B3}$$

where we used Eq. (B2) and that $|0_k, 0_{-k}\rangle = |0_1, 0_2\rangle$. Then the idea of the calculation is to permutate the operators such that $(\hat{a}_1)^{n_1}$ and $(\hat{a}_2)^{n_2}$ directly act on $|0_1, 0_2\rangle$. In order to do so, we use the fact [104] that, for two operators $\hat{X}$ and $\hat{Y}$ such that $[\hat{X}, \hat{Y}] = d\hat{I}$, one has

$$[\hat{X}^n, \hat{Y}^m] = \sum_{\ell=1}^{\min(n,m)} d^\ell \ell! \binom{n}{\ell}\binom{m}{\ell}\hat{Y}^{m-\ell}\hat{X}^{n-\ell}, \tag{B4}$$

where the coefficient $\binom{n}{\ell} = n!/[(n-\ell)!\ell!]$. This immediately implies that

$$\hat{X}^n\hat{Y}^m = \sum_{\ell=0}^{\min(n,m)} d^\ell \ell! \binom{n}{\ell}\binom{m}{\ell}\hat{Y}^{m-\ell}\hat{X}^{n-\ell}. \tag{B5}$$

Let us make use of this relation. Since $[\hat{a}_1, \hat{c}_k^\dagger] = \cos\alpha$, see Eq. (47), one has

$$\hat{a}_1^m(\hat{c}_k^\dagger)^n = \sum_{\ell_1=0}^{\min(m,n)} (\cos\alpha)^{\ell_1}\ell_1! \binom{m}{\ell_1}\binom{n}{\ell_1}(\hat{c}_k^\dagger)^{n-\ell_1}\hat{a}_1^{m-\ell_1}. \tag{B6}$$

Moreover, since $\hat{a}_1$ and $\hat{a}_2$ commute, one can first permute them and then use the relation we have just established. In that case Eq. (B3) can be written as

$$\langle m_1, p_2|n_k, n_{-k}\rangle = \frac{1}{\sqrt{m!p!n!n!}} \sum_{\ell_1=0}^{\min(m,n)} (\cos\alpha)^{\ell_1}\ell_1! \binom{m}{\ell_1}\binom{n}{\ell_1}\langle 0_1, 0_2|(\hat{a}_2)^p(\hat{c}_k^\dagger)^{n-\ell_1}\hat{a}_1^{m-\ell_1}(\hat{c}_{-k}^\dagger)^n|0_1, 0_2\rangle. \tag{B7}$$

In order for $\hat{a}_1$ to act on the vacuum, we see that one more permutation is needed. It can be obtained by using again the same trick. Since $[\hat{a}_1, \hat{c}_{-k}^\dagger] = \sin\alpha$, see Eq. (47), one has

$$\hat{a}_1^{m-\ell_1}(\hat{c}_{-k}^\dagger)^n = \sum_{\ell_2=0}^{\min(m-\ell_1,n)} (\sin\alpha)^{\ell_2}\ell_2! \binom{m-\ell_1}{\ell_2}\binom{n}{\ell_2}(\hat{c}_{-k}^\dagger)^{n-\ell_2}\hat{a}_1^{m-\ell_1-\ell_2}, \tag{B8}$$

and, therefore, one can write

$$\langle m_1, p_2|n_k, n_{-k}\rangle = \frac{1}{\sqrt{m!p!n!n!}} \sum_{\ell_1=0}^{\min(m,n)}\sum_{\ell_2=0}^{\min(m-\ell_1,n)} (\cos\alpha)^{\ell_1}\ell_1! \binom{m}{\ell_1}\binom{n}{\ell_1}(\sin\alpha)^{\ell_2}\ell_2! \binom{m-\ell_1}{\ell_2}\binom{n}{\ell_2}$$
$$\times \langle 0_1, 0_2|(\hat{a}_2)^p(\hat{c}_k^\dagger)^{n-\ell_1}(\hat{c}_{-k}^\dagger)^{n-\ell_2}\hat{a}_1^{m-\ell_1-\ell_2}|0_1, 0_2\rangle. \tag{B9}$$

This expression vanishes unless $m - \ell_1 - \ell_2 = 0$ or $\ell_2 = m - \ell_1$. Moreover, clearly, if $n < m - \ell_1$, then $\min(m - \ell_1, n) = n$, the sum over ℓ_2 runs from 0 to n, and ℓ_2 can never reach $m - \ell_1$. In that case, the above expression is zero. On the contrary, if $n > m - \ell_1$, then $\min(m - \ell_1, n) = m - \ell_1$ and the sum over ℓ_2 runs from 0 to $m - \ell_1$. But if $n > m - \ell_1$, it also means that $\ell_1 > m - n$ and, if this quantity is positive, the sum over ℓ_1 must start from this value rather than from zero. In other words, it must start from $\max(0, m - n)$. Therefore, one obtains

$$\langle m_1, p_2|n_k, n_{-k}\rangle = \frac{1}{\sqrt{m!p!n!n!}} \sum_{\ell_1=\max(0,m-n)}^{\min(m,n)} (\cos\alpha)^{\ell_1}\ell_1! \binom{m}{\ell_1}\binom{n}{\ell_1}(\sin\alpha)^{m-\ell_1}(m-\ell_1)! \binom{n}{m-\ell_1}$$
$$\times \langle 0_1, 0_2|(\hat{a}_2)^p(\hat{c}_k^\dagger)^{n-\ell_1}(\hat{c}_{-k}^\dagger)^{n-m+\ell_1}|0_1, 0_2\rangle. \tag{B10}$$

The next step is now to bring the operator $\hat{a}_2$ to the left and the same method we have just described can be applied. After straightforward manipulations, the final expression reads

$$\langle m_1, p_2|n_k, n_{-k}\rangle = \sqrt{\frac{m!p!}{n!n!}} \sum_{\ell_1=\max(0,m-n)}^{\min(m,n)} (\cos\alpha)^{2\ell_1+n-m}(\sin\alpha)^{m+n-2\ell_1}(-1)^{n-\ell_1}e^{2ip\alpha}\binom{n}{m-\ell_1}\binom{n}{\ell_1}\delta(m+p, 2n). \tag{B11}$$

The appearance of the δ function makes sense since it means that the same total number of particles must be contained in both states. In particular, one can check that, in the specific case $\alpha = 0$, the previous expression gives $\delta(m-n)\delta(p-n)$, as it should.

Using the expression previously established, one can then write

$$|n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle = \sum_{m=0}^{+\infty}\sum_{p=0}^{+\infty}\langle m_1, p_2|n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle|m_1, p_2\rangle = \sum_{m=0}^{2n}\langle m_1, (2n-m)_2|n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle|m_1, (2n-m)_2\rangle \tag{B12}$$

$$= [-(\cos\alpha)(\sin\alpha)]^n \sum_{m=0}^{2n} \frac{\sqrt{m!(2n-m)!}}{n!} e^{2i(2n-m)\alpha}(\tan\alpha)^m$$

$$\times \sum_{\ell=\max(0,m-n)}^{\min(m,n)} (\tan\alpha)^{-2\ell}(-1)^{-\ell}\binom{n}{m-\ell}\binom{n}{\ell}|m_1, (2n-m)_2\rangle. \tag{B13}$$

This expression can be further simplified as the second sum can be calculated in terms of a hypergeometric function. Indeed, the sum over m can be split into a sum from $m = 0$ to $m = n$, in which case one has $m \leq n$ and a sum from $m = n+1$ to $m = 2n$, in which case one has $m > n$. But in these two cases, the corresponding sum over ℓ is known since for $m \leq n$, one has

$$\sum_{\ell=0}^{m}\left[-\frac{1}{\tan^2\alpha}\right]^\ell\binom{n}{m-\ell}\binom{n}{\ell} = \binom{n}{m}{}_2F_1\left(-m, -n, 1-m+n, -\frac{1}{\tan^2\alpha}\right), \tag{B14}$$

while, for $m \geq n$, one can write

$$\sum_{\ell=m-n}^{n}\left[-\frac{1}{\tan^2(\alpha)}\right]^\ell\binom{n}{m-\ell}\binom{n}{\ell} = \left[-\frac{1}{\tan^2(\alpha)}\right]^{m-n}\binom{n}{m-n}{}_2F_1\left(m-2n, -n, 1+m-n, -\frac{1}{\tan^2\alpha}\right). \tag{B15}$$

As a consequence, the state $|n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle$ can be expressed as

$$|n_{\boldsymbol{k}}, n_{-\boldsymbol{k}}\rangle = \frac{[-\cos(\alpha)\sin(\alpha)]^n}{n!}e^{4in\alpha}\left[\sum_{m=0}^{n}\sqrt{(m)!(2n-m)!}\,e^{-2im\alpha}(\tan\alpha)^m\right.$$

$$\times \binom{n}{m}{}_2F_1\left(-m, -n, 1-m+n, -\frac{1}{\tan^2\alpha}\right)|m_1, (2n-m)_2\rangle + \sum_{m=n+1}^{2n}\sqrt{(m)!(2n-m)!}\,e^{-2im\alpha}$$

$$\times (-1)^{m-n}(\tan\alpha)^{2n-m}\binom{n}{m-n}{}_2F_1\left(m-2n, -n, 1+m-n, -\frac{1}{\tan^2\alpha}\right)|m_1, (2n-m)_2\rangle\bigg] \tag{B16}$$

$$= \sum_{m=0}^{2n} b_{nm}|m_1, (2n-m)_2\rangle, \tag{B17}$$

which defines the coefficients b_{nm}. Combining Eqs. (B1) and (B17), one can write the expression of the density matrix as

$$\hat{\rho}_{12} = \sum_{n,n'=0}^{\infty}\sum_{m=0}^{2n}\sum_{m'=0}^{2n'} a_{nn'}b_{nm}b_{n'm'}^*|m, 2n-m\rangle\langle m', 2n'-m'|. \tag{B18}$$

We are now in a position to calculate the discord along the lines of Sec. III. The quantity $\mathcal{I}_{12}$ is defined by $\mathcal{I}_{12} = S(\hat{\rho}_1) + S(\hat{\rho}_2) - S(\hat{\rho}_{12})$, where $\hat{\rho}_1$ ($\hat{\rho}_2$) is obtained from $\hat{\rho}_{12}$ by tracing out the degrees of freedom of the subsystem 2 (1). Then let us imagine that we make a measurement on the subsystem 1 corresponding to an operator $\hat{\Pi}_j$. This defines $\hat{\rho}_2(\hat{\Pi}_j) = \text{Tr}_1(\hat{\rho}_{12}\hat{\Pi}_j/p_j)$ where $p_j = \text{Tr}(\hat{\rho}_{12}\hat{\Pi})$ and $\mathcal{J}_{12} = S(\hat{\rho}_2) - \sum_j p_j S[\hat{\rho}_2(\hat{\Pi}_j)]$. However, we have shown in Appendix A that $S[\hat{\rho}_2(\hat{\Pi}_j)] = 0$ since $\hat{\rho}_{12}$ is still a pure state. As a result, $\mathcal{J}_{12} = S(\hat{\rho}_2)$ and, hence, the discord is just given by $\delta_{12} = S(\hat{\rho}_1)$. We see that we only need to calculate the entropy of $\hat{\rho}_1$. In order to

obtain this reduced density matrix, we trace out the second set of degrees of freedom. Using Eq. (B18), this leads to the following expression:

$$\hat{\rho}_1 = \sum_{p=0}^{\infty} \langle p_2 | \hat{\rho} | p_2 \rangle$$

$$= \sum_{p=0}^{\infty} \sum_{n,n'=0}^{\infty} \sum_{m=0}^{2n} \sum_{m'=0}^{2n'} a_{nn'} b_{nm} b_{n'm'}^* \langle p_2 | m, 2n - m \rangle$$

$$\times \langle m', 2n' - m' | p_2 \rangle, \tag{B19}$$

$$= \sum_{n,n'=0}^{\infty} \sum_{m=0}^{2n} a_{n,n'} b_{n,m} b_{n',m+2(n'-n)}^* |m\rangle\langle m + 2(n' - n)|. \tag{B20}$$

Since $\hat{\rho}_1$ is *a priori* not a thermal state, one cannot directly apply Eq. (43). In an analogy with Eqs. (49) and (50), let us therefore introduce the quantities $\hat{q}_{1,2}$ and $\hat{\pi}_{1,2}$ defined by $\hat{q}_{1,2} = (\sqrt{2k})^{-1}(a_{1,2} + a_{1,2}^\dagger)$ and $\hat{\pi}_{1,2} = -i\sqrt{k/2}(a_{1,2} - a_{1,2}^\dagger)$. This defines the vector $\hat{P}_j = (k^{1/2}\hat{q}_1, k^{-1/2}\hat{\pi}_1, k^{1/2}\hat{q}_2, k^{1/2}\hat{\pi}_2)$ which is the equivalent of $\hat{R}_j$ defined after Eq. (54). Then the covariance matrix

γ_{jk} is defined by $\langle \hat{P}_j \hat{P}_k \rangle = \gamma_{jk}/2 + iJ_{jk}/2$; see below Eq. (55) in the text and the definition of the matrix J_{jk} given by Eq. (56). In fact, since we want to calculate the entropy of subsystem 1 only, it is sufficient to consider the covariance matrix in this subsystem. Straightforward calculations lead to

$$\gamma_{11} = k\langle \hat{q}_1^2 \rangle = \langle \hat{a}_1^2 + (\hat{a}_1^\dagger)^2 + 2\hat{a}_1\hat{a}_1^\dagger \rangle - 1$$

$$= \mathrm{Tr}\{\hat{\rho}_1[\hat{a}_1^2 + (\hat{a}_1^\dagger)^2 + 2\hat{a}_1\hat{a}_1^\dagger]\} - 1, \tag{B21}$$

$$\gamma_{12} = \gamma_{21} = i\langle (\hat{a}_1^\dagger)^2 - \hat{a}_1^2 \rangle = i\mathrm{Tr}\{\hat{\rho}_1[(\hat{a}_1^\dagger)^2 - \hat{a}_1^2]\}, \tag{B22}$$

$$\gamma_{22} = -\langle \hat{a}_1^2 + (\hat{a}_1^\dagger)^2 - 2\hat{a}_1\hat{a}_1^\dagger \rangle - 1$$

$$= -\mathrm{Tr}\{\hat{\rho}_1[\hat{a}_1^2 + (\hat{a}_1^\dagger)^2 - 2\hat{a}_1\hat{a}_1^\dagger]\} - 1, \tag{B23}$$

which is explicitly known since we have determined the density matrix $\hat{\rho}_1$; see Eq. (B18). As is evident from the above expressions, in order to find γ_{jk}, it is sufficient to determine the three following quantities, $\langle \hat{a}_1^2 \rangle$, $\langle (\hat{a}_1^\dagger)^2 \rangle$ and $\langle \hat{a}_1 \hat{a}_1^\dagger \rangle$. For the sake of illustration, let us calculate the first one. One has

$$\langle \hat{a}_1^2 \rangle = \mathrm{Tr}(\hat{\rho}_1 \hat{a}_1^2) = \sum_{p=0}^{\infty} \sum_{n,n'=0}^{\infty} \sum_{m=0}^{2n} a_{n,n'} b_{n,m} b_{n',m+2(n'-n)}^* \langle p|m\rangle\langle m + 2(n' - n)|\hat{a}_1^2|p\rangle, \tag{B24}$$

$$= \sum_{p=2}^{\infty} \sum_{n,n'=0}^{\infty} \sum_{m=0}^{2n} a_{n,n'} b_{n,m} b_{n',m+2(n'-n)}^* \sqrt{p(p-1)} \langle p|m\rangle\langle m + 2(n' - n)|p - 2\rangle, \tag{B25}$$

$$= \sum_{n=1}^{\infty} \sum_{m=2}^{2n} a_{n,n-1} b_{n,m} b_{n-1,m-2}^* \sqrt{m(m-1)}. \tag{B26}$$

The two other quantities are given by similar calculations. Then it is sufficient to calculate the eigenvalues κ of the matrix $(\gamma/2)J_1$ to directly obtain the symplectic spectrum, thanks to Williamson's theorem [105]. One obtains

$$\kappa = \pm i\sqrt{\left(\langle \hat{a}_1 \hat{a}_1^\dagger \rangle - \frac{1}{2}\right)^2 - \langle \hat{a}_1^2 \rangle\langle (\hat{a}_1^\dagger)^2 \rangle} \equiv \pm i\sigma, \tag{B27}$$

which defines σ. Finally, the von Neumann entropy, and therefore the discord δ_{12}, is given by the following expression:

$$\delta_{12} = S(\hat{\rho}_1) = -\mathrm{Tr}(\hat{\rho}_1 \log \hat{\rho}_1)$$

$$= \left(\sigma + \frac{1}{2}\right)\log_2\left(\sigma + \frac{1}{2}\right) - \left(\sigma - \frac{1}{2}\right)\log_2\left(\sigma - \frac{1}{2}\right). \tag{B28}$$

We have evaluated this expression numerically. It is displayed as a function of the parameter α in Fig. 2. The dependence in $k\eta$ is hidden in the coefficients $a_{nn'}$ which depend on the squeezing parameters, these ones being explicit time-dependent quantities.

APPENDIX C: COVARIANCE MATRIX OF A TWO-MODE SQUEEZED STATE

In this appendix, we calculate the covariance matrix of a two-mode squeezed quantum state. The characteristic function χ has been introduced in Eq. (53) and the Weyl operator W defined in Eq. (54). Using the form (26) for the density matrix, and, as below Eq. (54), defining $\hat{R} = (k^{1/2}\hat{q}_k, k^{-1/2}\hat{\pi}_k, k^{1/2}\hat{q}_{-k}, k^{-1/2}\hat{\pi}_{-k})^\mathrm{T} \equiv (\hat{R}_1, \hat{R}_2, \hat{R}_3, \hat{R}_4)^\mathrm{T}$, the characteristic function $\chi(\xi)$ (where the components of ξ are denoted as ξ_1, ξ_2, ξ_3 and ξ_4) can be written as

$$\chi(\xi) = \mathrm{Tr}[\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)|0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle\langle 0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}|\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)e^{i\xi_1\hat{R}_1 + i\xi_2\hat{R}_2 + i\xi_3\hat{R}_3 + i\xi_4\hat{R}_4}] \tag{C1}$$

$$= \mathrm{Tr}[|0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle\langle 0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}|\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)e^{i\xi_1\hat{R}_1 + i\xi_2\hat{R}_2 + i\xi_3\hat{R}_3 + i\xi_4\hat{R}_4}\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)] \tag{C2}$$

$$= \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\langle n_{\boldsymbol{k}}, n'_{-\boldsymbol{k}}|0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle\langle 0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}|\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)e^{i\xi_1\hat{R}_1 + i\xi_2\hat{R}_2 + i\xi_3\hat{R}_3 + i\xi_4\hat{R}_4}\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)|n_{\boldsymbol{k}}, n'_{-\boldsymbol{k}}\rangle \tag{C3}$$

$$= \langle 0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}|\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)e^{i\xi_1\hat{R}_1 + i\xi_2\hat{R}_2 + i\xi_3\hat{R}_3 + i\xi_4\hat{R}_4}\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)|0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle \tag{C4}$$

$$= \langle 0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}|\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)e^{i\xi_1\hat{R}_1 + i\xi_2\hat{R}_2}e^{+i\xi_3\hat{R}_3 + i\xi_4\hat{R}_4}\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)|0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle \tag{C5}$$

$$= e^{i\xi_1\xi_2/2 + i\xi_3\xi_4/2}\langle 0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}|\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)e^{i\xi_1\hat{R}_1}e^{i\xi_2\hat{R}_2}e^{i\xi_3\hat{R}_3}e^{i\xi_4\hat{R}_4}\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)|0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle, \tag{C6}$$

where we have made use of the Baker-Campbell-Hausdorff formula $e^{\hat{A}+\hat{B}} = e^{-[\hat{A},\hat{B}]/2}e^{\hat{A}}e^{\hat{B}}$, valid if the operators $\hat{A}$ and $\hat{B}$ commute with their commutator $[\hat{A}, \hat{B}]$, which is the case here; see below. The next step consists in introducing the operator $\hat{S}\hat{R}(\hat{S}\hat{R})^\dagger$ (which is unity) between each exponential factor. This means that we must now calculate

$$\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)e^{i\xi_1\hat{R}_1}\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k) = \hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)\sum_{n=0}^{\infty}\frac{1}{n!}i^n\xi_1^n\hat{R}_1^n\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k) \tag{C7}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}i^n\xi_1^n\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)\hat{R}_1^n\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k). \tag{C8}$$

But one has (in order to avoid cumbersome notation, in this equation, we do not write the dependence in the squeezing parameters)

$$\hat{R}^\dagger\hat{S}^\dagger\hat{R}_1^n\hat{S}\hat{R} = \hat{R}^\dagger\hat{S}^\dagger\hat{R}_1\hat{S}\hat{R}\hat{R}^\dagger\hat{S}^\dagger\hat{R}_1^{n-1}\hat{S}\hat{R} = \hat{R}^\dagger\hat{S}^\dagger\hat{R}_1\hat{S}\hat{R}\hat{R}^\dagger\hat{S}^\dagger\hat{R}_1\hat{S}\hat{R}\hat{R}^\dagger\hat{S}^\dagger\hat{R}_1^{n-2}\hat{S}\hat{R} = (\hat{R}^\dagger\hat{S}^\dagger\hat{R}_1\hat{S}\hat{R})^n. \tag{C9}$$

As a consequence, one can rewrite the series as

$$\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)e^{i\xi_1\hat{R}_1}\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k) = \sum_{n=0}^{\infty}\frac{1}{n!}i^n\xi_1^n[\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)\hat{R}_1\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)]^n \tag{C10}$$

$$= e^{i\xi_1\hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k,\varphi_k)\hat{R}_1\hat{S}(r_k,\varphi_k)\hat{R}(\theta_k)}. \tag{C11}$$

Using this result in the expression of the characteristic function, not only for the first exponential term but for the four terms, we arrive at the following expression:

$$\chi(\xi) = e^{i\xi_1\xi_2/2 + i\xi_3\xi_4/2}\langle 0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}|e^{i\xi_1\hat{R}^\dagger\hat{S}^\dagger\hat{R}_1\hat{S}\hat{R}}e^{i\xi_2\hat{R}^\dagger\hat{S}^\dagger\hat{R}_2\hat{S}\hat{R}}e^{i\xi_3\hat{R}^\dagger\hat{S}^\dagger\hat{R}_3\hat{S}\hat{R}}e^{i\xi_4\hat{R}^\dagger\hat{S}^\dagger\hat{R}_4\hat{S}\hat{R}}|0_{\boldsymbol{k}}, 0_{-\boldsymbol{k}}\rangle. \tag{C12}$$

To proceed, one has to evaluate the four terms $\hat{R}^\dagger\hat{S}^\dagger\hat{R}_i\hat{S}\hat{R}$. Using Eqs. (21) and (22), it is easy to show that

$$\hat{\Omega}_1 \equiv \hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)\hat{R}_1\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k) = \hat{R}_1\cosh r_k\cos\theta_k - \hat{R}_2\cosh r_k\sin\theta_k$$
$$+ \hat{R}_3\sinh r_k\cos(\theta_k - 2\varphi_k) + \hat{R}_4\sinh r_k\sin(2\varphi_k - \theta_k), \tag{C13}$$

$$\hat{\Omega}_2 \equiv \hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)\hat{R}_2\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k) = \hat{R}_1\cosh r_k\sin\theta_k + \hat{R}_2\cosh r_k\cos\theta_k$$
$$+ \hat{R}_3\sinh r_k\sin(2\varphi_k - \theta_k) - \hat{R}_4\sinh r_k\cos(2\varphi_k - \theta_k), \tag{C14}$$

$$\hat{\Omega}_3 \equiv \hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)\hat{R}_3\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k)$$
$$= \hat{R}_1\sinh r_k\cos(2\varphi_k - \theta_k) + \hat{R}_2\sinh r_k\sin(2\varphi_k - \theta_k) + \hat{R}_3\cosh r_k\cos\theta_k - \hat{R}_4\cosh r_k\sin\theta_k, \tag{C15}$$

$$\hat{\Omega}_4 \equiv \hat{R}^\dagger(\theta_k)\hat{S}^\dagger(r_k, \varphi_k)\hat{R}_4\hat{S}(r_k, \varphi_k)\hat{R}(\theta_k) = \hat{R}_1\sinh r_k\sin(2\varphi_k - \theta_k) - \hat{R}_2\sinh r_k\cos(2\varphi_k - \theta_k)$$
$$+ \hat{R}_3\cosh r_k\sin\theta_k + \hat{R}_4\cosh r_k\cos\theta_k. \tag{C16}$$

The final step is to express the product of four exponentials in Eq. (C12) in terms of a single exponential. For this purpose, it is interesting to calculate the commutators of the operators $[\hat{\Omega}_i, \hat{\Omega}_j]$ in order to use (again) the Baker-Campbell-Hausdorff formula. One finds

$$[\hat{\Omega}_1, \hat{\Omega}_3] = [\hat{\Omega}_1, \hat{\Omega}_4] = [\hat{\Omega}_2, \hat{\Omega}_3] = [\hat{\Omega}_2, \hat{\Omega}_4] = 0, \qquad [\hat{\Omega}_1, \hat{\Omega}_2] = [\hat{\Omega}_3, \hat{\Omega}_4] = i. \tag{C17}$$

As a consequence, the characteristic function (C12) now takes the form

$$\chi(\xi) = e^{i\xi_1\xi_2/2 + i\xi_3\xi_4/2} \langle 0_k, 0_{-k} | e^{-i\xi_1\xi_2/2} e^{i\xi_1\hat{\Omega}_1 + i\xi_2\hat{\Omega}_2} e^{-i\xi_3\xi_4/2} e^{i\xi_3\hat{\Omega}_3 + i\xi_4\hat{\Omega}_4} | 0_k, 0_{-k} \rangle \tag{C18}$$

$$= \langle 0_k, 0_{-k} | e^{i\xi_1\hat{\Omega}_1 + i\xi_2\hat{\Omega}_2 + i\xi_3\hat{\Omega}_3 + i\xi_4\hat{\Omega}_4} | 0_k, 0_{-k} \rangle \tag{C19}$$

$$= \langle 0_k, 0_{-k} | e^{i\eta_1\hat{R}_1 + i\eta_2\hat{R}_2 + i\eta_3\hat{R}_3 + i\eta_4\hat{R}_4} | 0_k, 0_{-k} \rangle \equiv \chi_{\text{vac}}(\eta_1, \eta_2, \eta_3, \eta_4), \tag{C20}$$

where the coefficients η_i can be expressed as

$$\eta_1 = \xi_1 \cosh r_k \cos\theta_k + \xi_2 \sinh r_k \sin\theta_k + \xi_3 \sinh r_k \cos(2\varphi_k - \theta_k) + \xi_4 \sinh r_k \sin(2\varphi_k - \theta_k), \tag{C21}$$

$$\eta_2 = -\xi_1 \cosh r_k \sin\theta_k + \xi_2 \cosh r_k \cos\theta_k + \xi_3 \sinh r_k \sin(2\varphi_k - \theta_k) - \xi_4 \sinh r_k \cos(2\varphi_k - \theta_k), \tag{C22}$$

$$\eta_3 = \xi_1 \sinh r_k \cos(2\varphi_k - \theta_k) + \xi_2 \sinh r_k \sin(2\varphi_k - \theta_k) + \xi_3 \cosh r_k \cos\theta_k + \xi_4 \cosh r_k \sin\theta_k, \tag{C23}$$

$$\eta_4 = \xi_1 \sinh r_k \sin(2\varphi_k - \theta_k) - \xi_2 \sinh r_k \cos(2\varphi_k - \theta_k) - \xi_3 \cosh r_k \sin\theta_k + \xi_4 \cosh r_k \cos\theta_k. \tag{C24}$$

In Eq. (C20), $\chi_{\text{vac}}(\eta_1, \eta_2, \eta_3, \eta_4)$ represents the characteristic function of the vacuum. But this function is well known and easy to calculate. The vacuum state is a Gaussian state and one has, see for instance Eq. (33) of Ref. [83],

$$\chi_{\text{vac}}(\eta_1, \eta_2, \eta_3, \eta_4) = e^{-\eta^T \text{Id}_4 \eta/4}, \tag{C25}$$

where Id_4 is the identity matrix in four dimensions. Since the definition of the covariance matrix γ is given by the expression

$$\chi(\xi_1, \xi_2, \xi_3, \xi_4) = e^{-\xi^T \gamma \xi/4}, \tag{C26}$$

this implies that

$$\sum_{i=1}^{4} \eta_i^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} \gamma_{ij} \xi_i \xi_j, \tag{C27}$$

which allows us to infer the components of the covariance matrix. Then lengthy but straightforward calculations lead to the following covariance matrix for the two-mode squeezed state:

$$\gamma = \begin{pmatrix} \cosh(2r_k) & 0 & \sinh(2r_k)\cos(2\varphi_k) & \sinh(2r_k)\sin(2\varphi_k) \\ 0 & \cosh(2r_k) & \sinh(2r_k)\sin(2\varphi_k) & -\sinh(2r_k)\cos(2\varphi_k) \\ \sinh(2r_k)\cos(2\varphi_k) & \sinh(2r_k)\sin(2\varphi_k) & \cosh(2r_k) & 0 \\ \sinh(2r_k)\sin(2\varphi_k) & -\sinh(2r_k)\cos(2\varphi_k) & 0 & \cosh(2r_k) \end{pmatrix}. \tag{C28}$$

We notice that γ does not depend on the rotation angle θ_k. Once we have the covariance matrix, we can determine the two-point correlation functions by means of the relation $\langle R_j R_k \rangle = \gamma_{jk}/2 + iJ_{jk}/2$ derived in Appendix F. As mentioned in the main text, the matrix J is defined by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{C29}$$

Using the above expression of the correlation matrix, the explicit form of the two-point correlators is given by

$$\langle \hat{q}_k \hat{q}_k \rangle = \langle \hat{q}_{-k} \hat{q}_{-k} \rangle = \frac{1}{2k} \cosh(2r_k),$$

$$\langle \hat{\pi}_k \hat{\pi}_k \rangle = \langle \hat{\pi}_{-k} \hat{\pi}_{-k} \rangle = \frac{k}{2} \cosh(2r_k), \quad \text{(C30)}$$

$$\langle \hat{q}_k \hat{q}_{-k} \rangle = \frac{1}{2k} \sinh(2r_k) \cos(2\varphi_k),$$

$$\langle \hat{\pi}_k \hat{\pi}_{-k} \rangle = -\frac{k}{2} \sinh(2r_k) \cos(2\varphi_k), \quad \text{(C31)}$$

$$\langle \hat{q}_k \hat{\pi}_{-k} \rangle = \langle \hat{\pi}_k \hat{q}_{-k} \rangle = \frac{1}{2} \sinh(2r_k) \sin(2\varphi_k),$$

$$\langle \hat{q}_k \hat{\pi}_k \rangle = \langle \hat{q}_{-k} \hat{\pi}_{-k} \rangle = -\langle \hat{\pi}_k \hat{q}_k \rangle = -\langle \hat{\pi}_{-k} \hat{q}_{-k} \rangle = \frac{i}{2}. \quad \text{(C32)}$$

These results are used in the main text in order to calculate the two-point correlation functions of $\hat{v}_k$ and $\hat{p}_k$; see Eqs. (57)–(59), related to $\hat{q}_k$ and $\hat{\pi}_k$ through Eqs. (51) and (52).

APPENDIX D: DISCORD OF A CLASSICAL STATE

In this appendix, we show that classical states given by Eq. (69) have vanishing quantum discord. Let us first calculate the reduced density matrix $\hat{\rho}_{\text{class}}(k)$ obtained by tracing out Eq. (69) over degrees of freedom belonging to space $-k$. One obtains

$$\hat{\rho}_{\text{class}}(k) = \sum_m \langle m_{-k}| \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij}|i_k\rangle\langle i_k| \otimes |j_{-k}\rangle\langle j_{-k}|m_{-k}\rangle$$

$$= \sum_{ij} p_{ij}|i_k\rangle\langle i_k|. \quad \text{(D1)}$$

In the same manner, one obtains a similar expression for $\hat{\rho}_{\text{class}}(-k)$ where i_k is replaced by j_{-k} in the above formula. It is easy to show that these matrices are diagonal in the basis $|n_k\rangle$. Indeed,

$$\langle n_k|\hat{\rho}_{\text{class}}(k)|m_k\rangle = \delta_{nm} \sum_j p_{nj}. \quad \text{(D2)}$$

Of course, one has a similar relation (but not identical if the probabilities are not symmetric) in the other half space, namely

$$\langle n_{-k}|\hat{\rho}_{\text{class}}(k)|m_{-k}\rangle = \delta_{nm} \sum_i p_{in}. \quad \text{(D3)}$$

Therefore, it is straightforward to calculate the corresponding entropy. Indeed, the calculation of the entropy requires in general estimating the logarithm of a matrix, which is a nontrivial task. But when the matrix is diagonal, its logarithm is just a diagonal matrix whose entries are given by the logarithm of the entries of the original matrix. As a consequence, one immediately deduces that

$$S[\hat{\rho}_{\text{class}}(k)] = \sum_i \left[\left(\sum_\ell p_{i\ell}\right) \log\left(\sum_j p_{ij}\right)\right],$$

$$S[\hat{\rho}_{\text{class}}(-k)] = \sum_i \left[\left(\sum_\ell p_{\ell i}\right) \log\left(\sum_j p_{ji}\right)\right]. \quad \text{(D4)}$$

On the other hand, since the full density matrix $\hat{\rho}_{\text{class}}(k, -k)$ satisfies

$$\langle n_k m_{-k}|\hat{\rho}_{\text{class}}(k, -k)|n_k' m_{-k}'\rangle = p_{nm}\delta_{nn'}\delta_{mm'}, \quad \text{(D5)}$$

one can write its entropy as

$$S[\hat{\rho}_{\text{class}}(k, -k)] = \sum_{ij} p_{ij} \log p_{ij}. \quad \text{(D6)}$$

It follows that the mutual information $\mathcal{I}(k, -k)$ is given by

$$\mathcal{I}(k, -k) = \sum_i \left[\left(\sum_\ell p_{i\ell}\right) \log\left(\sum_j p_{ij}\right)\right]$$
$$+ \sum_i \left[\left(\sum_\ell p_{\ell i}\right) \log\left(\sum_j p_{ji}\right)\right]$$
$$- \sum_{ij} p_{ij} \log p_{ij}. \quad \text{(D7)}$$

This completes the first step in the calculation of the discord.

Let us now calculate the quantity $\mathcal{J}(k, -k)$. We consider again the projector $\hat{\Pi}_j$ introduced in Eq. (A3). It is easy to show that

$$\hat{\rho}_{\text{class}}(k, -k)\hat{\Pi}_j = \sum_n p_{nj}|n_k\rangle\langle n_k| \otimes |j_{-k}\rangle\langle j_{-k}|. \quad \text{(D8)}$$

Moreover, using this expression, one can also establish that

$$\text{Tr}[\hat{\rho}_{\text{class}}(k, -k)\hat{\Pi}_j] = \sum_n p_{nj}. \quad \text{(D9)}$$

Then, in order to derive a formula for $\hat{\rho}_{\text{class}}(k; \hat{\Pi}_j)$, one has to trace out the degrees of freedom contained in $-k$ in Eq. (D8), see also Eq. (37), and it comes to

$$\hat{\rho}_{\text{class}}(k; \hat{\Pi}_j) = \frac{1}{\sum_\ell p_{\ell j}} \sum_n p_{nj}|n_k\rangle\langle n_k|. \quad \text{(D10)}$$

This density matrix is diagonal with elements $p_{nj}/\sum_\ell p_{\ell j}$ on the diagonal. Consequently, the conditional entropy S_{cond} which, as before, requires the calculation of the logarithm of the above mentioned matrix, can be easily estimated and reads

$$S_{\text{cond}} \equiv \sum_j \left(\sum_\ell p_{\ell j} \right) S[\hat{\rho}_{\text{class}}(\boldsymbol{k}; \hat{\Pi}_j)] = \sum_j \left(\sum_\ell p_{\ell j} \right) \sum_n \frac{p_{nj}}{\sum_m p_{mj}} \log \left(\frac{p_{nj}}{\sum_i p_{ij}} \right) \tag{D11}$$

$$= \sum_{nj} p_{nj} \log \left(\frac{p_{nj}}{\sum_i p_{ij}} \right) = \sum_{nj} p_{nj} \log(p_{nj}) - \sum_{nj} p_{nj} \log \left(\sum_i p_{ij} \right) \tag{D12}$$

$$= \sum_{nj} p_{nj} \log(p_{nj}) - \sum_j \left[\left(\sum_n p_{nj} \right) \log \left(\sum_i p_{ij} \right) \right]. \tag{D13}$$

It follows that the quantity $\mathcal{J}(\boldsymbol{k}, -\boldsymbol{k})$ can be expressed as

$$\mathcal{J}(\boldsymbol{k}, -\boldsymbol{k}) \equiv S[\hat{\rho}_{\text{class}}(\boldsymbol{k})] - S_{\text{cond}} = \sum_i \left[\left(\sum_\ell p_{i\ell} \right) \log \left(\sum_j p_{ij} \right) \right] - \sum_{nj} p_{nj} \log(p_{nj}) + \sum_j \left[\left(\sum_n p_{nj} \right) \log \left(\sum_i p_{ij} \right) \right]$$

$$= \mathcal{I}(\boldsymbol{k}, -\boldsymbol{k}). \tag{D14}$$

Therefore, we see that, for the choice of $\hat{\Pi}_j$ made above, the difference between $\mathcal{J}(\boldsymbol{k}, -\boldsymbol{k})$ and $\mathcal{I}(\boldsymbol{k}, -\boldsymbol{k})$ vanishes. In order to obtain the discord, one must minimize this quantity over the set of projectors; see Eq. (39). However, the discord is a positive definite quantity [86] and, as a consequence, if for one specific choice of $\hat{\Pi}_j$ one proves it is zero, then it obviously means that the minimum is also zero. Therefore, as announced in the main text, we conclude that the quantum discord of the classical state vanishes.

APPENDIX E: CHARACTERISTIC FUNCTION OF A CLASSICAL STATE

In this appendix, we calculate the characteristic function of the classical state (69). Let us recall that the characteristic function χ is defined in Eq. (53) [and the Weyl operator in Eq. (54)]. As a consequence, we have

$$\chi(\xi) = \text{Tr}[\hat{\rho}_{\text{class}}(\boldsymbol{k}, -\boldsymbol{k}) e^{i\xi_1 \hat{R}_1 + i\xi_2 \hat{R}_2 + i\xi_3 \hat{R}_3 + i\xi_4 \hat{R}_4}] \tag{E1}$$

$$= \sum_{n,m} \langle n_{\boldsymbol{k}}, m_{-\boldsymbol{k}} | \sum_{i,j} p_{ij} |i_{\boldsymbol{k}}\rangle \langle i_{\boldsymbol{k}}| \otimes |j_{-\boldsymbol{k}}\rangle \langle j_{-\boldsymbol{k}}| e^{i\xi_1 \hat{R}_1 + i\xi_2 \hat{R}_2 + i\xi_3 \hat{R}_3 + i\xi_4 \hat{R}_4} |n_{\boldsymbol{k}}, m_{-\boldsymbol{k}}\rangle \tag{E2}$$

$$= e^{i\xi_1\xi_2/2 + i\xi_3\xi_4/2} \sum_{n,m} p_{nm} \langle n_{\boldsymbol{k}}, m_{-\boldsymbol{k}} | e^{i\xi_1 \hat{R}_1} e^{i\xi_2 \hat{R}_2} e^{i\xi_3 \hat{R}_3} e^{i\xi_4 \hat{R}_4} |n_{\boldsymbol{k}}, m_{-\boldsymbol{k}}\rangle \tag{E3}$$

$$= e^{i\xi_1\xi_2/2 + i\xi_3\xi_4/2} \sum_{n,m} p_{nm} \int \mathrm{d}(k^{1/2} x_{\boldsymbol{k}}) \mathrm{d}(k^{1/2} x_{-\boldsymbol{k}}) \mathrm{d}(k^{1/2} y_{\boldsymbol{k}}) \mathrm{d}(k^{1/2} y_{-\boldsymbol{k}}) \langle n_{\boldsymbol{k}}, m_{-\boldsymbol{k}} | e^{i\xi_1 \hat{R}_1} e^{i\xi_2 \hat{R}_2} |x_{\boldsymbol{k}}, x_{-\boldsymbol{k}}\rangle$$

$$\times \langle x_{\boldsymbol{k}}, x_{-\boldsymbol{k}} | e^{i\xi_3 \hat{R}_3} e^{i\xi_4 \hat{R}_4} |y_{\boldsymbol{k}}, y_{-\boldsymbol{k}}\rangle \langle y_{\boldsymbol{k}}, y_{-\boldsymbol{k}} | n_{\boldsymbol{k}}, m_{-\boldsymbol{k}}\rangle, \tag{E4}$$

where, in the last equality, we have introduced the closure relation twice. In this expression, $|x_{\boldsymbol{k}}\rangle$ represents the eigenstate of the "position operator" $\hat{q}_{\boldsymbol{k}}$. This is especially convenient because the action of an operator of the form $e^{i\xi_i \hat{R}_i}$ on this state is particular simple. Indeed, one has

$$e^{i\xi_3 \hat{R}_3} e^{i\xi_4 \hat{R}_4} |y_{\boldsymbol{k}}, y_{-\boldsymbol{k}}\rangle = e^{i\xi_3 k^{1/2} \hat{q}_{-\boldsymbol{k}}} e^{i\xi_4 k^{-1/2} \hat{\pi}_{-\boldsymbol{k}}} |y_{\boldsymbol{k}}, y_{-\boldsymbol{k}}\rangle = e^{i\xi_3 k^{1/2} \hat{q}_{-\boldsymbol{k}}} |y_{\boldsymbol{k}}, y_{-\boldsymbol{k}} - \xi_4 k^{-1/2}\rangle \tag{E5}$$

$$= e^{i\xi_3 k^{1/2} (y_{-\boldsymbol{k}} - \xi_4 k^{-1/2})} |y_{\boldsymbol{k}}, y_{-\boldsymbol{k}} - \xi_4 k^{-1/2}\rangle. \tag{E6}$$

In the same way, one can also write

$$e^{i\xi_1 \hat{R}_1} e^{i\xi_2 \hat{R}_2} |x_{\boldsymbol{k}}, x_{-\boldsymbol{k}}\rangle = e^{i\xi_1 k^{1/2} (x_{\boldsymbol{k}} - \xi_2 k^{-1/2})} |x_{\boldsymbol{k}} - \xi_2 k^{-1/2}, x_{-\boldsymbol{k}}\rangle. \tag{E7}$$

As a consequence, using these two last relations, Eq. (E4) now takes the following form:

$$\chi(\xi) = e^{i\xi_1\xi_2/2+i\xi_3\xi_4/2}\sum_{n,m}k^2 p_{nm}\int \mathrm{d}x_k\mathrm{d}x_{-k}\mathrm{d}y_k\mathrm{d}y_{-k}e^{i\xi_1 k^{1/2}x_k}e^{-i\xi_1\xi_2}e^{i\xi_3 k^{1/2}y_{-k}}e^{-i\xi_3\xi_4}\langle n_k, m_{-k}|x_k - \xi_2 k^{-1/2}, x_{-k}\rangle$$

$$\times \langle x_k, x_{-k}|y_k, y_{-k} - \xi_4 k^{-1/2}\rangle\langle y_k, y_{-k}|n_k, m_{-k}\rangle \tag{E8}$$

$$= e^{-i\xi_1\xi_2/2+i\xi_3\xi_4/2}\sum_{n,m}k p_{nm}\int \mathrm{d}x_k\mathrm{d}x_{-k}e^{i\xi_1 k^{1/2}x_k}e^{i\xi_3 k^{1/2}x_{-k}}\langle n_k|x_k - \xi_2 k^{-1/2}\rangle\langle m_{-k}|x_{-k}\rangle\langle x_k|n_k\rangle\langle x_{-k} + \xi_4 k^{-1/2}|m_{-k}\rangle. \tag{E9}$$

Then, in order to proceed, one has to calculate scalar products of the form $\langle x_k|n_k\rangle$. But this is nothing but the wave function of a state containing n particles and, as is well known, it can be expressed in terms of Hermite polynomials H_n. As a consequence, one obtains

$$\chi(\xi) = e^{-i\xi_1\xi_2/2+i\xi_3\xi_4/2}\sum_{n,m}k p_{nm}\int \mathrm{d}x_k\mathrm{d}x_{-k}e^{i\xi_1 k^{1/2}x_k}e^{i\xi_3 k^{1/2}x_{-k}}\frac{1}{\sqrt{\sqrt{\pi}2^n n!}}e^{-(k^{1/2}x_k-\xi_2)^2/2}H_n(k^{1/2}x_k - \xi_2)$$

$$\times \frac{1}{\sqrt{\sqrt{\pi}2^m m!}}e^{-k(x_{-k})^2/2}H_m(k^{1/2}x_{-k})\frac{1}{\sqrt{\sqrt{\pi}2^n n!}}e^{-k(x_k)^2/2}H_n(k^{1/2}x_k)$$

$$\times \frac{1}{\sqrt{\sqrt{\pi}2^m m!}}e^{-(k^{1/2}x_{-k}+\xi_4)^2/2}H_m(k^{1/2}x_{-k} + \xi_4) \tag{E10}$$

$$= \frac{1}{\pi}e^{-i\xi_1\xi_2/2+i\xi_3\xi_4/2}e^{-\xi_1^2/4+i\xi_1\xi_2/2-\xi_2^2/2}e^{-\xi_3^2/4-i\xi_3\xi_4/2-\xi_4^2/4}\sum_{n,m}\frac{p_{nm}}{2^{n+m}n!m!}\int \mathrm{d}u e^{-u^2}H_n\left(u - \frac{\xi_2}{2} + \frac{i\xi_1}{2}\right)H_n\left(u + \frac{\xi_2}{2} + \frac{i\xi_1}{2}\right)$$

$$\times \int \mathrm{d}v e^{-v^2}H_m\left(v - \frac{\xi_4}{2} + \frac{i\xi_3}{2}\right)H_m\left(v + \frac{\xi_4}{2} + \frac{i\xi_3}{2}\right). \tag{E11}$$

Thanks to Eq. (7.377) of Ref. [106], the two integrals involving an exponential function and the product of two Hermite polynomials can be performed exactly. As a result, one obtains

$$\chi(\xi) = e^{-(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)/4}\sum_{n,m}p_{nm}L_n\left(\frac{\xi_1^2}{2} + \frac{\xi_2^2}{2}\right)L_m\left(\frac{\xi_3^2}{2} + \frac{\xi_4^2}{2}\right), \tag{E12}$$

where $L_n^0 \equiv L_n$ is a Laguerre polynomial; see Eq. (8.970.2) of Ref. [106]. Of course, to go further, one needs to specify the distribution p_{nm}. But the last equation has the advantage of showing that, in general, the characteristic function of the classical state has no reason to be Gaussian. If one makes the choice $p_{nm} = (1 - e^{-\beta_k})e^{-\beta_k n}\delta(n - m)$, see the discussion around Eq. (80), then the characteristic function reduces to

$$\chi(\xi) = e^{-(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)/4}(1 - e^{-\beta_k})\sum_{n=0}^{\infty}e^{-\beta_k n}L_n\left(\frac{\xi_1^2}{2} + \frac{\xi_2^2}{2}\right)L_n\left(\frac{\xi_3^2}{2} + \frac{\xi_4^2}{2}\right). \tag{E13}$$

This series can be calculated explicitly by means of Eq. (8.976.1) of Ref. [106]. The result reads

$$\chi(\xi) = e^{-\tanh^{-1}(\frac{\beta_k}{2})\frac{\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2}{4}}I_0\left[\frac{\sqrt{(\xi_1^2+\xi_2^2)(\xi_3^2+\xi_4^2)}}{2\sinh(\beta_k/2)}\right], \tag{E14}$$

where I_0 is a modified Bessel function of order zero [106]. The virtue of this expression is twofold. First, since the argument of the Bessel function is always positive and I_0 is also always positive, the characteristic function is positive everywhere. Second, even with the simple choice of p_{nm} made before, one can see that the characteristic function is not Gaussian (note that in the sub-Hubble limit where $\beta_k \to \infty$, the argument of the Bessel function vanishes and the Bessel function becomes 1; hence the characteristic function is Gaussian in this limit).

APPENDIX F: QUANTUM CORRELATORS AND WEYL TRANSFORMS

In this appendix we show how mean values of quantum operators can be calculated by means of the Wigner function [93]. Using the definition of the Weyl transform given in Eq. (89) and that of the Wigner function, see Eq. (93), let us first calculate the following integral $\int \tilde{A} W$. One has

$$
\int \tilde{A} W \mathrm{d}q_k \mathrm{d}\pi_k \mathrm{d}q_{-k} \mathrm{d}\pi_{-k} = \frac{1}{(2\pi)^2} \int \left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \middle| \hat{A} \middle| q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right\rangle e^{-i\pi_k(x+x')-i\pi_{-k}(y+y')}
$$
$$
\times \left\langle q_k + \frac{x'}{2}, q_{-k} + \frac{y'}{2} \middle| \hat{\rho} \middle| q_k - \frac{x'}{2}, q_{-k} - \frac{y'}{2} \right\rangle \mathrm{d}q_k \mathrm{d}\pi_k \mathrm{d}q_{-k} \mathrm{d}\pi_{-k} \mathrm{d}x \mathrm{d}y \mathrm{d}x' \mathrm{d}y'. \tag{F1}
$$

The integration over π_k and π_{-k} gives rise to Dirac delta functions, which can then be used to perform the integrals over x' and y'. As a consequence, one obtains

$$
\int \tilde{A} W \mathrm{d}q_k \mathrm{d}\pi_k \mathrm{d}q_{-k} \mathrm{d}\pi_{-k} = \int \left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \middle| \hat{A} \middle| q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right\rangle \left\langle q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \middle| \hat{\rho} \middle| q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \right\rangle \mathrm{d}q_k \mathrm{d}q_{-k} \mathrm{d}x \mathrm{d}y. \tag{F2}
$$

The next step consists in performing a change of variables. Instead of working in terms of q_k, x and q_{-k}, y, we define the quantities $u_{\pm k}$ and $w_{\pm k}$ such that $u_k = q_k - x/2$, $w_k = q_k + x/2$, $u_{-k} = q_{-k} - y/2$ and $w_{-k} = q_{-k} + y/2$. The determinant of the Jacobian of these transformations being one, one has $\mathrm{d}q_k \mathrm{d}x = \mathrm{d}u_k \mathrm{d}v_k$ and $\mathrm{d}q_{-k} \mathrm{d}y = \mathrm{d}u_{-k} \mathrm{d}v_{-k}$; hence

$$
\int \tilde{A} W \mathrm{d}q_k \mathrm{d}\pi_k \mathrm{d}q_{-k} \mathrm{d}\pi_{-k} = \int \langle w_k, w_{-k} | \hat{A} | u_k, u_{-k} \rangle \langle u_k, u_{-k} | \hat{\rho} | w_k, w_{-k} \rangle \mathrm{d}u_k \mathrm{d}w_k \mathrm{d}u_{-k} \mathrm{d}w_{-k}
$$
$$
= \int \langle w_k, w_{-k} | \hat{A} \hat{\rho} | w_k, w_{-k} \rangle \mathrm{d}w_k \mathrm{d}w_{-k} \tag{F3}
$$
$$
= \mathrm{Tr}(\hat{A} \hat{\rho}) = \langle \hat{A} \rangle. \tag{F4}
$$

As a consequence, the expectation value of $\hat{A}$ can be obtained by something which is the average of the physical quantity represented by $\tilde{A}$ over phase space with probability density W characterizing the state. Therefore, as explained in the main text, the quantum problem can be replaced by a stochastic approach in $\tilde{A}$. It is thus interesting to study under which conditions A and $\tilde{A}$ can be identified. A general expression for any analytic operator $\hat{A}$ is provided by its Taylor series in terms of the operators $\hat{q}_k$, $\hat{\pi}_k$, $\hat{q}_{-k}$ and $\hat{\pi}_{-k}$. Making use of the canonical commutation relations, this can always be written as

$$
\hat{A} = \sum_{n_1,n_2} a_{n_1 n_2} \hat{q}_k^{n_1} \hat{\pi}_k^{n_2} + \sum_{n_1,n_2} b_{n_1 n_2} \hat{q}_{-k}^{n_1} \hat{\pi}_{-k}^{n_2} + \sum_{n_1,n_2} c_{n_1 n_2} \hat{q}_k^{n_1} \hat{q}_{-k}^{n_2} + \sum_{n_1,n_2} d_{n_1 n_2} \hat{\pi}_k^{n_1} \hat{\pi}_{-k}^{n_2} + \sum_{n_1,n_2} e_{n_1 n_2} \hat{q}_k^{n_1} \hat{\pi}_{-k}^{n_2} + \sum_{n_1,n_2} f_{n_1 n_2} \hat{\pi}_k^{n_1} \hat{q}_{-k}^{n_2}, \tag{F5}
$$

where $a_{n_1 n_2}, \ldots, f_{n_1 n_2}$ are numerical coefficients. Let us calculate the Weyl transform of each of these terms. From the definition (89), one can write that

$$
\widetilde{q_k^{n_1} \pi_k^{n_2}} = \int \mathrm{d}x \mathrm{d}y \, e^{-i\pi_k x - i\pi_{-k} y} \left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \middle| \hat{q}_k^{n_1} \hat{\pi}_k^{n_2} \middle| q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right\rangle \tag{F6}
$$

$$
= \int \mathrm{d}x \mathrm{d}y \, e^{-i\pi_k x - i\pi_{-k} y} \left(q_k + \frac{x}{2} \right)^{n_1} \left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \middle| \hat{\pi}_k^{n_2} \middle| q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right\rangle \tag{F7}
$$

$$
= \int \mathrm{d}x \mathrm{d}y \mathrm{d}\pi_k' \mathrm{d}\pi_k'' \mathrm{d}\pi_{-k}' \mathrm{d}\pi_{-k}'' \, e^{-i\pi_k x - i\pi_{-k} y} \left(q_k + \frac{x}{2} \right)^{n_1} \left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \middle| \pi_k', \pi_{-k}' \right\rangle
$$
$$
\times \langle \pi_k', \pi_{-k}' | \hat{\pi}_k^{n_2} | \pi_k'', \pi_{-k}'' \rangle \left\langle \pi_k'', \pi_{-k}'' \middle| q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right\rangle, \tag{F8}
$$

where we have inserted the identity operator twice. Making use of the relation $\langle q_k|\pi_k\rangle = \exp(iq_k\pi_k)/\sqrt{2\pi}$, the previous expression can be simplified and one obtains

$$\widetilde{q_k^{n_1}\pi_k^{n_2}} = \frac{1}{(2\pi)^2}\int dx dy d\pi_k' d\pi_k'' d\pi_{-k}' d\pi_{-k}'' e^{-i\pi_k x - i\pi_{-k}y}\left(q_k + \frac{x}{2}\right)^{n_1}\langle \pi_k', \pi_{-k}'|\hat{\pi}_k^{n_2}|\pi_k'', \pi_{-k}''\rangle$$

$$\times \exp\left[i\left(q_k + \frac{x}{2}\right)\pi_k' + i\left(q_{-k} + \frac{y}{2}\right)\pi_{-k}' - i\left(q_k - \frac{x}{2}\right)\pi_k'' - i\left(q_{-k} - \frac{y}{2}\right)\pi_{-k}''\right] \tag{F9}$$

$$= \frac{1}{(2\pi)^2}\int dx dy d\pi_k' d\pi_k'' d\pi_{-k}' d\pi_{-k}'' e^{-i\pi_k x - i\pi_{-k}y}\left(q_k + \frac{x}{2}\right)^{n_1}(\pi_k'')^{n_2}\delta(\pi_k' - \pi_k'')$$

$$\times \delta(\pi_{-k}' - \pi_{-k}'')\exp\left[i\left(q_k + \frac{x}{2}\right)\pi_k' + i\left(q_{-k} + \frac{y}{2}\right)\pi_{-k}' - i\left(q_k - \frac{x}{2}\right)\pi_k'' - i\left(q_{-k} - \frac{y}{2}\right)\pi_{-k}''\right] \tag{F10}$$

$$= \frac{1}{(2\pi)^2}\int dx dy d\pi_k' d\pi_{-k}' e^{ix(\pi_k' - \pi_k)+iy(\pi_{-k}' - \pi_{-k})}\left(q_k + \frac{x}{2}\right)^{n_1}(\pi_k')^{n_2} \tag{F11}$$

$$= \frac{1}{2\pi}\int dx d\pi_k' e^{ix(\pi_k' - \pi_k)}\left(q_k + \frac{x}{2}\right)^{n_1}(\pi_k')^{n_2} \tag{F12}$$

$$= \frac{1}{2\pi}\int dx e^{-ix\pi_k}\left(q_k + \frac{x}{2}\right)^{n_1}\int d\pi_k' e^{ix\pi_k'}(\pi_k')^{n_2}. \tag{F13}$$

This expression can be further simplified since the second integral is the Fourier transform of a monomial function. This leads to

$$\widetilde{q_k^{n_1}\pi_k^{n_2}} = (-i)^{n_2}\int dx e^{-ix\pi_k}\left(q_k + \frac{x}{2}\right)^{n_1}\delta^{(n_2)}(x), \tag{F14}$$

where $\delta^{(n_2)}$ stands for the n_2th derivative of the delta function. After integrating by parts n_2 times, one obtains the following formula:

$$\widetilde{q_k^{n_1}\pi_k^{n_2}} = i^{n_2}\int dx \frac{\partial^{n_2}}{\partial x^{n_2}}\left[e^{-ix\pi_k}\left(q_k + \frac{x}{2}\right)^{n_1}\right]\delta(x). \tag{F15}$$

The derivative can be calculated by making use of the binomial formula and this leads to

$$\frac{\partial^{n_2}}{\partial x^{n_2}}\left[e^{-ix\pi_k}\left(q_k + \frac{x}{2}\right)^{n_1}\right] = \sum_{j=0}^{n_2}\binom{n_2}{j}\frac{\partial^j}{\partial x^j}(e^{-ix\pi_k})\frac{\partial^{n_2-j}}{\partial x^{n_2-j}}\left[\left(q_k + \frac{x}{2}\right)^{n_1}\right] \tag{F16}$$

$$= \sum_{j=0}^{n_2}\binom{n_2}{j}(-i\pi_k)^j e^{-ix\pi_k}\frac{n_1!}{2^{n_2-j}(n_1 - n_2 + j)!}\left(q_k + \frac{x}{2}\right)^{n_1 - n_2 + j}\theta(j + n_1 - n_2), \tag{F17}$$

where $\theta(j + n_1 - n_2) = 1$ if $j \geq n_2 - n_1$ and 0 otherwise. Inserting this expression into Eq. (F15), one obtains, after integrating out the $\delta(x)$ function,

$$\widetilde{q_k^{n_1}\pi_k^{n_2}} = \frac{q_k^{n_1 - n_2}}{(-2i)^{n_2}}\sum_{j=0}^{n_2}\binom{n_2}{j}\frac{n_1!(-2i\pi_k q_k)^j}{(n_1 - n_2 + j)!}\theta(j + n_1 - n_2). \tag{F18}$$

Obviously, one finds a similar expression for $\widetilde{q_{-k}^{n_1}\pi_{-k}^{n_2}}$. For $n_1 = 0$, this immediately leads to $\widetilde{\pi_k^n} = \pi_k^n$ while for $n_2 = 0$, one obtains $\widetilde{q_k^n} = q_k^n$. This means that any analytic function of $\hat{q}_k$ only or of $\hat{\pi}_k$ only has a trivial Weyl transform, namely $f(\widetilde{q_k}) = f(q_k)$ and $f(\widetilde{\pi_k}) = f(\pi_k)$. However, this is not true for mixed terms containing both $\hat{q}_k$ and $\hat{\pi}_k$. For example, when $n_1 = n_2 = 1$, one finds $\widetilde{q_k\pi_k} = q_k\pi_k + i/2$.

This is actually due to the fact that $\hat{q}_k$ and $\hat{\pi}_k$ do not commute. When $n_1 = n_2 = 2$, one finds $\widetilde{q_k^2 \pi_k^2} = q_k^2 \pi_k^2 + 2iq_k\pi_k - 1/2$ and $\widetilde{\pi_k^2 q_k^2} = \pi_k^2 q_k^2 - 2i\pi_k q_k - 1/2$, and so forth. For all other terms in Eq. (F5), the operators commute and the Weyl transform is trivial. For example, let us work out $\widetilde{q_k^{n_1} \pi_{-k}^{n_2}}$ (the other terms proceed in exactly the same way). One has

$$
\widetilde{q_k^{n_1} \pi_{-k}^{n_2}} = \int dx dy \, e^{-i\pi_k x - i\pi_{-k} y}
$$
$$
\times \left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \middle| \hat{q}_k^{n_1} \hat{\pi}_{-k}^{n_2} \middle| q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right\rangle
$$
(F19)

$$
= \int dx dy \, e^{-i\pi_k x - i\pi_{-k} y} \left(q_k + \frac{x}{2} \right)^{n_1}
$$
$$
\times \left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \middle| \hat{\pi}_{-k}^{n_2} \middle| q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right\rangle.
$$
(F20)

At this stage, one can proceed exactly as before (namely, again introduce twice the identity operator and integrate out the delta Dirac functions) and obtain an equation resembling Eq. (F11). It reads

$$
\widetilde{q_k^{n_1} \pi_{-k}^{n_2}} = \frac{1}{(2\pi)^2} \int dx dy d\pi_k' d\pi_{-k}' e^{ix(\pi_k' - \pi_k) + iy(\pi_{-k}' - \pi_{-k})}
$$
$$
\times \left(q_k + \frac{x}{2} \right)^{n_1} (\pi_{-k}')^{n_2}
$$
(F21)

$$
= \frac{1}{2\pi} \int dx d\pi_k' e^{ix(\pi_k' - \pi_k)} \left(q_k + \frac{x}{2} \right)^{n_1} (\pi_{-k})^{n_2}
$$
(F22)

$$
= q_k^{n_1} \pi_{-k}^{n_2},
$$
(F23)

that is to say the announced result. For similar reasons, one also has $\widetilde{q_k^{n_1} q_{-k}^{n_2}} = q_k^{n_1} q_{-k}^{n_2}$, $\widetilde{\pi_k^{n_1} \pi_{-k}^{n_2}} = \pi_k^{n_1} \pi_{-k}^{n_2}$ and $\widetilde{\pi_k^{n_1} q_{-k}^{n_2}} = \pi_k^{n_1} q_{-k}^{n_2}$. In fact, combining these results, the Weyl transform of terms for which $n_1 + n_2 = 1$ can be written in a compact manner, namely

$$
\widetilde{R_j R_k} = R_j R_k + \frac{1}{2} [\hat{R}_j, \hat{R}_k],
$$
(F24)

where the commutator $[\hat{R}_j, \hat{R}_k]$ is given by the matrix J; see Eqs. (56) and (90).

APPENDIX G: WIGNER FUNCTION OF A GAUSSIAN STATE

In this section, we want to write the Wigner function in terms of the characteristic function and the correlation matrix. In order to do so, let us first derive an explicit expression of the characteristic function in terms of the density matrix elements. As already noticed in the text, a first remark is that, if two operators $\hat{A}$ and $\hat{B}$ commute with their commutator ($[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$), the Baker-Campbell-Haussdorf formula reads $e^{\hat{A}+\hat{B}} = e^{-[\hat{A},\hat{B}]/2} e^{\hat{A}} e^{\hat{B}}$. Recalling once more that $\hat{R} \equiv (k^{1/2}\hat{q}_k, k^{-1/2}\hat{\pi}_k, k^{1/2}\hat{q}_{-k}, k^{-1/2}\hat{\pi}_{-k})^{\mathrm{T}} \equiv (\hat{R}_1, \hat{R}_2, \hat{R}_3, \hat{R}_4)^{\mathrm{T}}$, we see that each $\hat{R}_i$ satisfies this property. As a consequence, the Weyl operator defined in Eq. (54) can be written as

$$
\hat{\mathcal{W}}(\xi) \equiv \exp(i\xi_1 k^{1/2}\hat{q}_k + i\xi_2 k^{-1/2}\hat{\pi}_k + i\xi_3 k^{1/2}\hat{q}_{-k} + i\xi_4 k^{-1/2}\hat{q}_{-k})
$$
(G1)

$$
= e^{i(\xi_1\xi_2 + \xi_3\xi_4)/2} e^{i\xi_1 k^{1/2}\hat{q}_k} e^{i\xi_2 k^{-1/2}\hat{\pi}_k} e^{i\xi_3 k^{1/2}\hat{q}_{-k}} e^{i\xi_4 k^{-1/2}\hat{q}_{-k}}.
$$
(G2)

Using the above expression, one can express the characteristic function $\chi(\xi)$ [defined in Eq. (53)] as

$$
\chi(\xi) = \mathrm{Tr}[\hat{\rho} \, \hat{\mathcal{W}}(\xi)]
$$
(G3)

$$
= \int dq_k dq_{-k} \langle q_k, q_{-k} | \hat{\rho} \, \hat{\mathcal{W}}(\xi) | q_k, q_{-k} \rangle
$$
(G4)

$$
= e^{i(\xi_1\xi_2 + \xi_3\xi_4)/2} \int dq_k dq_{-k} \langle q_k, q_{-k} | \hat{\rho} e^{i\xi_1 k^{1/2}\hat{q}_k} e^{i\xi_2 k^{-1/2}\hat{\pi}_k} e^{i\xi_3 k^{1/2}\hat{q}_{-k}} e^{i\xi_4 k^{-1/2}\hat{q}_{-k}} | q_k, q_{-k} \rangle
$$
(G5)

$$
= e^{i(\xi_1\xi_2 + \xi_3\xi_4)/2} \int dq_k dq_{-k} dq_k' dq'_{-k} \langle q_k, q_{-k} | \hat{\rho} | q_k', q'_{-k} \rangle \langle q_k', q'_{-k} | e^{i\xi_1 k^{1/2}\hat{q}_k} e^{i\xi_2 k^{-1/2}\hat{\pi}_k} e^{i\xi_3 k^{1/2}\hat{q}_{-k}} e^{i\xi_4 k^{-1/2}\hat{q}_{-k}} | q_k, q_{-k} \rangle
$$
(G6)

$$
= e^{i(\xi_1\xi_2 + \xi_3\xi_4)/2} \int dq_k dq_{-k} dq_k' dq'_{-k} \langle q_k, q_{-k} | \hat{\rho} | q_k', q'_{-k} \rangle \langle q_k' | e^{i\xi_1 k^{1/2}\hat{q}_k} e^{i\xi_2 k^{-1/2}\hat{\pi}_k} | q_k \rangle \langle q'_{-k} | e^{i\xi_3 k^{1/2}\hat{q}_{-k}} e^{i\xi_4 k^{-1/2}\hat{q}_{-k}} | q_{-k} \rangle,
$$
(G7)

where we have utilized the closure relation in order to simplify the above expression. We see that the characteristic function has indeed been written in terms of the density matrix elements. However, we still need to calculate the two other terms which remain in the integral (G7). Recalling that $\langle q_k | \pi_k \rangle = \exp(i q_k \pi_k)/\sqrt{2\pi}$, one arrives at

$$\langle q'_k | e^{i\xi_1 k^{1/2} \hat{q}_k} e^{i\xi_2 k^{-1/2} \hat{\pi}_k} | q_k \rangle = \int d\bar{q}_k d\bar{\pi}_k \langle q'_k | e^{i\xi_1 k^{1/2} \hat{q}_k} | \bar{q}_k \rangle \langle \bar{q}_k | e^{i\xi_2 k^{-1/2} \hat{\pi}_k} | \bar{\pi}_k \rangle \langle \bar{\pi}_k | q_k \rangle \tag{G8}$$

$$= \frac{1}{2\pi} e^{i\xi_1 k^{1/2} q'_k} \int d\bar{q}_k d\bar{\pi}_k \delta(q'_k - \bar{q}_k) e^{i\xi_2 k^{-1/2} \bar{\pi}_k} e^{i\bar{q}_k \bar{\pi}_k} e^{-i\bar{\pi}_k q_k} \tag{G9}$$

$$= \frac{1}{2\pi} e^{i\xi_1 k^{1/2} q'_k} \int d\bar{\pi}_k e^{i\bar{\pi}_k (k^{-1/2}\xi_2 + q'_k - q_k)} \tag{G10}$$

$$= e^{i\xi_1 k^{1/2} q'_k} \delta(k^{-1/2}\xi_2 + q'_k - q_k). \tag{G11}$$

Of course, one has a similar expression for the last term in Eq. (G7). As a consequence, the expression of the characteristic function takes the following form:

$$\chi(\xi) = e^{i(\xi_1\xi_2 + \xi_3\xi_4)/2} \int dq_k dq_{-k} dq'_k dq'_{-k} \langle q_k, q_{-k} | \hat{\rho} | q'_k, q'_{-k} \rangle e^{i\xi_1 k^{1/2} q'_k} \delta(k^{-1/2}\xi_2 + q'_k - q_k) e^{i\xi_3 k^{1/2} q'_{-k}} \delta(k^{-1/2}\xi_4 + q'_{-k} - q_{-k}) \tag{G12}$$

$$= e^{-i(\xi_1\xi_2 + \xi_3\xi_4)/2} \int dq_k dq_{-k} \langle q_k, q_{-k} | \hat{\rho} | q_k - k^{-1/2}\xi_2, q_{-k} - k^{-1/2}\xi_4 \rangle e^{i(\xi_1 k^{1/2} q_k + \xi_3 k^{1/2} q_{-k})}. \tag{G13}$$

Let us now introduce the quantity $\mathcal{A}(\xi)$ defined by

$$\mathcal{A}(\xi) \equiv \frac{1}{(2\pi)^4} \int d^4\eta\, e^{-i\xi^{\mathrm{T}}\eta} \chi(\eta) d\eta, \tag{G14}$$

and express this phase-space function in terms of the density matrix elements. Using the expression of the characteristic function derived above, one has

$$\mathcal{A}(\xi) = \frac{1}{(2\pi)^4} \int d^4\eta\, dq_k dq_{-k} e^{-i\xi^{\mathrm{T}}\eta} e^{-i(\eta_1\eta_2 + \eta_3\eta_4)/2} e^{i(\eta_1 k^{1/2} q_k + \eta_3 k^{1/2} q_{-k})} \langle q_k, q_{-k} | \hat{\rho} | q_k - k^{-1/2}\eta_2, q_{-k} - k^{-1/2}\eta_4 \rangle. \tag{G15}$$

The integral over η_1 (respectively η_3) can be easily performed and gives rise to a $\delta(k^{1/2}q_k - \eta_2/2 - \xi_1)$ [respectively $\delta(k^{1/2}q_{-k} - \eta_4/2 - \xi_3)$] function. Then this allows us to integrate over η_2 and η_4 and it follows that

$$\mathcal{A}(\xi) = \frac{4}{(2\pi)^2} \int dq_k dq_{-k} \exp\left[2i(\xi_1\xi_2 + \xi_3\xi_4 - k^{1/2}q_k\xi_2 - k^{1/2}q_{-k}\xi_4)\right] \langle q_k, q_{-k} | \hat{\rho} | 2k^{-1/2}\xi_1 - q_k, 2k^{-1/2}\xi_3 - q_{-k} \rangle. \tag{G16}$$

Let us now perform the change of integration variable $q_k = k^{-1/2}\xi_1 + x/2$ and $q_{-k} = k^{-1/2}\xi_3 + y/2$. Then one obtains that $\mathcal{A}(\xi)$ is given by

$$\mathcal{A}(\xi) = \frac{1}{(2\pi)^2} \int dx dy \exp\left[-ik^{1/2}(\xi_2 x + \xi_4 y)\right] \left\langle k^{-1/2}\xi_1 + \frac{x}{2}, k^{-1/2}\xi_3 + \frac{y}{2} \middle| \hat{\rho} \middle| k^{-1/2}\xi_1 - \frac{x}{2}, k^{-1/2}\xi_3 - \frac{y}{2} \right\rangle. \tag{G17}$$

Finally, inserting $\hat{\rho} = |\Psi\rangle\langle\Psi|$ in the above expression yields

$$\mathcal{A}(\xi) = \frac{1}{(2\pi)^2} \int dx dy \exp\left[-ik^{1/2}(\xi_2 x + \xi_4 y)\right] \Psi\left(k^{-1/2}\xi_1 + \frac{x}{2}, k^{-1/2}\xi_3 + \frac{y}{2}\right) \Psi^*\left(k^{-1/2}\xi_1 - \frac{x}{2}, k^{-1/2}\xi_3 - \frac{y}{2}\right), \tag{G18}$$

which exactly coincides with Eq. (93). Therefore, we have shown that

$$W(\xi) = \frac{1}{(2\pi)^4} \int d^4\eta \, e^{-i\xi^T\eta} \chi(\eta), \qquad (G19)$$

which is the standard expression of the Wigner function in terms of the characteristic one.

From here, an expression of W in terms of γ can be obtained. Indeed, as written in Eq. (C26), for a Gaussian state, one has $\chi(\eta) = \exp(-\eta^T\gamma\eta/4)$, where γ is, by definition, the covariance matrix. Inserting this last expression in Eq. (G19), one obtains

$$W(\xi) = \frac{1}{(2\pi)^4} \int d^4\eta \exp\left(-i\xi^T\eta - \frac{1}{4}\eta^T\gamma\eta\right) d\eta \qquad (G20)$$

$$= \frac{1}{(2\pi)^4} \exp(-\xi^T\gamma^{-1}\xi)$$

$$\times \int d^4\eta \exp\left[-\frac{1}{4}(\eta + 2i\gamma^{-1}\xi)^T\gamma(\eta + 2i\gamma^{-1}\xi)\right], \qquad (G21)$$

where we have used the fact that the covariance matrix is symmetric, $\gamma^T = \gamma$, and the fact that ξ and η are real vectors; hence $\eta^T\xi = \xi^T\eta$. After performing the change of integration variable $\eta \to \eta + 2i\gamma^{-1}\xi$, the Gaussian integration finally leads to

$$W(\xi) = \frac{1}{\pi^2\sqrt{\det\gamma}} e^{-\xi^T\gamma^{-1}\xi}, \qquad (G22)$$

which is exactly the formula used in the text, in particular, in Sec. VI.

APPENDIX H: EVOLUTION EQUATION FOR THE WIGNER FUNCTION

In this appendix, we derive the equation controlling the time evolution of the Wigner function. The time derivative of W can be obtained from Eq. (93). Explicitly, one has

$$\frac{d}{d\eta}W = \frac{1}{(2\pi)^2} \int dxdy\left[\frac{d}{d\eta}\Psi^*\left(q_k - \frac{x}{2}, q_{-k} - \frac{y}{2}\right)\right]e^{-i\pi_k x - i\pi_{-k}y}\Psi\left(q_k + \frac{x}{2}, q_{-k} + \frac{y}{2}\right)$$

$$+ \frac{1}{(2\pi)^2} \int dxdy\Psi^*\left(q_k - \frac{x}{2}, q_{-k} - \frac{y}{2}\right)e^{-i\pi_k x - i\pi_{-k}y}\frac{d}{d\eta}\Psi\left(q_k + \frac{x}{2}, q_{-k} + \frac{y}{2}\right) \qquad (H1)$$

$$= \frac{i}{(2\pi)^2} \int dxdy\left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2}\middle|\hat{H}\middle|\Psi\right\rangle^* e^{-i\pi_k x - i\pi_{-k}y}\Psi\left(q_k + \frac{x}{2}, q_{-k} + \frac{y}{2}\right)$$

$$- \frac{i}{(2\pi)^2} \int dxdy\Psi^*\left(q_k - \frac{x}{2}, q_{-k} - \frac{y}{2}\right)e^{-i\pi_k x - i\pi_{-k}y}\left\langle q_k + \frac{x}{2}, q_{-k} + \frac{y}{2}\middle|\hat{H}\middle|\Psi\right\rangle, \qquad (H2)$$

where in the second equality we have used the Schrödinger equation $\partial\Psi/\partial\eta = -i\hat{H}\Psi$ and the fact that the Hamiltonian is Hermitian. In order to proceed, one needs to express the Hamiltonian density in terms of the Gaussian operators $\hat{q}_k$, $\hat{\pi}_k$, $\hat{q}_{-k}$ and $\hat{\pi}_{-k}$. Inserting Eqs. (51) and (52) into Eq. (11), one obtains the following expression:

$$\hat{H}_k = \frac{k^2}{2}(\hat{q}_k^2 + \hat{q}_{-k}^2) + \frac{\hat{\pi}_k^2 + \hat{\pi}_{-k}^2}{2} + \frac{z'}{z}(\hat{q}_k\hat{\pi}_{-k} + \hat{q}_{-k}\hat{\pi}_k). \qquad (H3)$$

The three terms in the above Hamiltonian give rise to three terms for the time evolution of W which, in the following, we denote by $dW_{q^2}/d\eta$, $dW_{\pi^2}/d\eta$ and $dW_{q\pi}/d\eta$. Let us work them out one by one. The first term can be expressed as

$$\frac{dW_{q^2}}{d\eta} = -i\left(\frac{k}{2\pi}\right)^2 \int dxdy\Psi^*\left(q_k - \frac{x}{2}, q_{-k} - \frac{y}{2}\right)$$

$$\times e^{-i\pi_k x - i\pi_{-k}y}(xq_k + yq_{-k})\Psi\left(q_k + \frac{x}{2}, q_{-k} + \frac{y}{2}\right). \qquad (H4)$$

Noticing that differentiating Eq. (93) with respect to π_k and π_{-k} precisely produces factors x and y, it is easy to check that the above expression can also be written as

$$\frac{dW_{q^2}}{d\eta} = k^2\left(q_k\frac{d}{d\pi_k} + q_{-k}\frac{d}{d\pi_{-k}}\right)W. \qquad (H5)$$

Then, recalling that the momentum is represented by $\hat{\pi}_k = -i\partial/\partial q_k$, the second term reads

$$\frac{\mathrm{d}W_{\pi^2}}{\mathrm{d}\eta} = \frac{i/2}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y \left(\frac{\partial^2}{\partial q_k^2} + \frac{\partial^2}{\partial q_{-k}^2} \right) \Psi^* \left(q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right) e^{-i\pi_k x - i\pi_{-k} y} \Psi \left(q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \right)$$
$$- \frac{i/2}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y \Psi^* \left(q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right) e^{-i\pi_k x - i\pi_{-k} y} \left(\frac{\partial^2}{\partial q_k^2} + \frac{\partial^2}{\partial q_{-k}^2} \right) \Psi \left(q_k + \frac{x}{2}, q_{-k} + \frac{y}{2} \right). \tag{H6}$$

In this expression, derivatives with respect to q_k and q_{-k} can be written as derivatives with respect to x and y by noticing that, for example,

$$\frac{\partial^2}{\partial q_k^2} \Psi^* \left(q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right) = -2 \frac{\partial^2}{\partial q_k \partial x} \Psi^* \left(q_k - \frac{x}{2}, q_{-k} - \frac{y}{2} \right), \tag{H7}$$

and similar expressions for the three other double derivatives. Then integrating by part the obtained expression yields the formula

$$\frac{\mathrm{d}W_{\pi^2}}{\mathrm{d}\eta} = -\left(\pi_k \frac{\partial}{\partial q_k} + \pi_{-k} \frac{\partial}{\partial q_{-k}} \right) W. \tag{H8}$$

Finally, the third term remains to be calculated. It is given by

$$\frac{\mathrm{d}W_{q\pi}}{\mathrm{d}\eta} = -\frac{z'/z}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y e^{-i\pi_k x - i\pi_{-k} y} \Psi \left[\left(q_k - \frac{x}{2} \right) \frac{\partial}{\partial q_{-k}} + \left(q_{-k} - \frac{y}{2} \right) \frac{\partial}{\partial q_k} \right] \Psi^*$$
$$- \frac{z'/z}{(2\pi)^2} \int \mathrm{d}x\mathrm{d}y e^{-i\pi_k x - i\pi_{-k} y} \Psi^* \left[\left(q_k + \frac{x}{2} \right) \frac{\partial}{\partial q_{-k}} + \left(q_{-k} + \frac{y}{2} \right) \frac{\partial}{\partial q_k} \right] \Psi, \tag{H9}$$

where, for simplicity, we have omitted the wave function arguments. Using the same manipulations as for the other terms, one obtains the following expression:

$$\frac{\mathrm{d}W_{q\pi}}{\mathrm{d}\eta} = \frac{z'}{z} \left(\pi_{-k} \frac{\partial}{\partial \pi_k} + \pi_k \frac{\partial}{\partial \pi_{-k}} - q_k \frac{\partial}{\partial q_{-k}} - q_{-k} \frac{\partial}{\partial q_k} \right) W. \tag{H10}$$

Combining Eqs. (H5), (H8) and (H10), one finally obtains

$$\frac{\mathrm{d}W}{\mathrm{d}\eta} = \left[k^2 \left(q_k \frac{\partial}{\partial \pi_k} + q_{-k} \frac{\partial}{\partial \pi_{-k}} \right) - \pi_k \frac{\partial}{\partial q_k} - \pi_{-k} \frac{\partial}{\partial q_{-k}} + \frac{z'}{z} \left(\pi_{-k} \frac{\partial}{\partial \pi_k} + \pi_k \frac{\partial}{\partial \pi_{-k}} - q_k \frac{\partial}{\partial q_{-k}} - q_{-k} \frac{\partial}{\partial q_k} \right) \right] W. \tag{H11}$$

It turns out that this last equation is nothing but the classical Liouville equation for the distribution W. Indeed, if one introduces the Poisson bracket $\{\}_{\mathrm{PB}}$ defined by

$$\{f, g\}_{\mathrm{PB}} = \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial \pi_k} - \frac{\partial f}{\partial \pi_k} \frac{\partial g}{\partial q_k} + (\boldsymbol{k} \leftrightarrow -\boldsymbol{k}), \tag{H12}$$

then it is straightforward to write Eq. (H11) as

$$\frac{\mathrm{d}W}{\mathrm{d}\eta} = \{H_k, W\}_{\mathrm{PB}}, \tag{H13}$$

where we have used Eq. (H3). Let us mention that although we have established this result for the Hamiltonian of cosmological perturbations, it can in fact be generalized to any quadratic Hamiltonian.

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