Lepton mixing patterns from combinations of elementary correlations

Shu-jun Rong[*]

Department of Physics, Shaanxi University of Technology, Hanzhong, Shaanxi 723000, China
(Received 23 October 2015; published 12 January 2016)

Recent data of reactor neutrino experiments set more stringent constraints on leptonic mixing patterns. We examine all possible patterns on the basis of combinations of elementary correlations of elements of leptonic mixing matrix. We obtain 62 viable mixing patters at 3σ level of mixing parameters. Most of these patterns can be paired via the $\mu - \tau$ interchange which changes the octant of θ_{23} and the sign of $\cos\delta$. All viable patterns can be classified into two groups: the perturbative patterns and nonperturbative patterns. The former can be obtained from perturbing tri-bimaximal mixing. The latter cannot be obtained from perturbing any mixing pattern whose θ_{13} is zero. Different predictions of Dirac CP phase δ of these two types of mixing patterns are discussed. Evolutions of mass matrices of neutrinos with small mixing parameters are discussed via special mixing patterns on the basis of flavor groups. In general cases, a small variation of $\sin\theta_{13}$ may bring about large modifications to alignment of vacuum expectation values in a mixing model. Therefore, small but nonzero $\sin\theta_{13}$ brings a severer challenge to leptonic mixing models on the basis of flavor groups than usual views.

I. INTRODUCTION

Recent reactor neutrino experiments have confirmed a nonzero θ_{13} [1–5], which opens the window to determine the Dirac CP violation phase and the octant of θ_{23}. These experiment results also set more stringent constraints on the theoretical models of neutrinos. As a challenge brought by the progress of experiments, few models based on symmetries [6] can fit the experiment results completely. Perturbations or other phenomenological considerations should be introduced in the typical mixing patterns such as tri-bi-maximal (TBM) [7,8], golden ratio mixing (GRM) [9,10], Toorop-Feruglio-Hagedorn mixing (TFHM) [11,12] etc. [13–15]. However, even so, the candidate mixing pattern is not unique, we need to determine all viable leptonic mixing patterns and find differences between their physical origins and predictions. In this article, we extract all viable mixing patterns at 3σ level of mixing parameters on the basis of various combinations of correlation of elements of mixing matrix. Special correlations have been widely discussed in the literature, see Refs. [16–18] for example. We search for more general correlations to construct the leptonic mixing pattern. The method to generate general correlations is employing the combinations of elementary correlations. We propose two types of elementary correlations. By examining all possible combinations of these elementary correlations, we obtain 62 viable mixing patters, including some familiar types such as tri-maximal TM_1, TM_2 [19–22], $\mu - \tau$ symmetric mixing [23–27] and many new types. Most of these patterns could be paired via the $\mu - \tau$ interchange which changes the octant of θ_{23} and

the sign of $\cos\delta$. Furthermore, according to the dependence of $\cos\delta$ on small $\sin^2\theta_{13}$ all these patterns can be classified into two types: the perturbative and nonperturbative patterns. The former could be obtained from perturbing TBM. The latter cannot be obtained from perturbing any mixing pattern whose θ_{13} is zero.

In order to study the robustness of leptonic mixing models on the basis of flavor groups, the dependence of mass matrix of neutrinos on small mixing parameter $\sin\theta_{13}$ is discussed via special perturbative and nonperturbative mixing patterns. In general cases, a small variation of $\sin\theta_{13}$ may correspond to large modifications to alignment of vacuum expectation values in a mixing model on the basis of flavor groups.

The article is organized as follows: In Sec. II we introduce the general framework of our program, including the unitary parametrization of mixing matrix, constraints on the parameters at 3σ level, two types of elementary correlations and the $\mu - \tau$ interchange. In Sec. III we show all viable combinations of elementary correlations, including the combination of two correlations and the combination of three correlations, mixing angles, and Dirac CP phases of viable patterns. In Sec. IV we discuss evolutions of mass matrices of neutrinos via special leptonic mixing patterns. Finally, we present a summary.

II. GENERAL FRAMEWORK

A. Unitary parametrization of leptonic mixing matrix

In order to simplify calculations of combinations of correlations, we adopt the following parametrization of leptonic mixing matrix:

[*]rongshj@snut.edu.cn

$$|U| = \begin{pmatrix} \sqrt{\frac{-1-k+N}{N}} & \sqrt{\frac{k}{N}} & \sqrt{\frac{1}{N}} \\ \sqrt{\frac{-l-j+N}{N}} & \sqrt{\frac{l}{N}} & \sqrt{\frac{j}{N}} \\ \sqrt{\frac{1+l+j+k-N}{N}} & \sqrt{\frac{-l-k+N}{N}} & \sqrt{\frac{-1-j+N}{N}} \end{pmatrix}, \tag{1}$$

where j, k, l, N are non-negative real numbers. By comparison with the standard parametrization of Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matirx [28]:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$, δ is the Dirac CP-violating phase, α_1 and α_2 are Majorana phases. In our parametrization, s_{ij} could be written as:

$$\sin\theta_{13} = \sqrt{\frac{1}{N}},$$

$$\sin\theta_{23} = \sqrt{\frac{j}{N-1}},$$

$$\sin\theta_{12} = \sqrt{\frac{k}{N-1}}. \tag{3}$$

And the Dirac CP phase is expressed as:

$$\cos\delta = \frac{-l(-1+N)^2 - (1+k-N)(-1+N)N + j(k+N+kN-N^2)}{2\sqrt{N}\sqrt{j(-1-j+N)}\sqrt{k(-1-k+N)}}. \tag{4}$$

According to the recent fit data of the Ref. [29], absolute values of elements of mixing matrix in 3σ ranges are

$$|U| = \begin{pmatrix} 0.801 \to 0.845 & 0.514 \to 0.580 & 0.137 \to 0.158 \\ 0.225 \to 0.517 & 0.441 - 0.699 & 0.614 \to 0.793 \\ 0.246 \to 0.529 & 0.464 \to 0.713 & 0.590 \to 0.776 \end{pmatrix}. \tag{5}$$

So constraints on our parameters at 3σ level are

$$\frac{1}{N} = 0.0188 \to 0.0251,$$

$$\frac{l}{N} = 0.1945 \to 0.4886,$$

$$\frac{j}{N} = 0.3770 \to 0.6288,$$

$$\frac{k}{N} = 0.2642 \to 0.3364. \tag{6}$$

B. Elementary correlations

We propose two types of elementary correlations as follows:

$$A: |U_{\alpha i}| = |U_{\beta j}|,$$

$$B: |U_{\alpha i}| = 2|U_{\beta j}|. \tag{7}$$

These two types of correlations are implied in some familiar mixing patterns such as TBM and GRM. We examine correlations of these two types. The parameter-space of a viable correlation (or viable combination of elementary correlations) should satisfy two conditions: First, the magnitude of every element of mixing matrix is in 3σ range; Second, $|\cos\delta(l,j,k,N)| \leq 1$. There are 7 viable type-A elementary correlations in total. They are listed as follows:

$$A1: |U_{\mu 1}| = |U_{\tau 1}|, \tag{8}$$

$$A2: |U_{\mu 2}| = |U_{\tau 2}|, \tag{9}$$

$$A3: |U_{\mu 3}| = |U_{\tau 3}|, \tag{10}$$

$$A4: |U_{e2}| = |U_{\mu 2}|, \tag{11}$$

$$A5: \quad |U_{e2}| = |U_{\tau 2}|, \tag{12}$$

$$A6: \quad |U_{\mu 2}| = |U_{\mu 3}|, \tag{13}$$

$$A7: \quad |U_{\tau 2}| = |U_{\tau 3}|, \tag{14}$$

$$A8: \quad |U_{\mu 1}| = |U_{\mu 2}|, \tag{15}$$

$$A9: \quad |U_{\tau 1}| = |U_{\tau 2}|. \tag{16}$$

There are 14 viable type-B elementary correlations in total. They are written as follows:

$$B1: \quad |U_{e1}| = 2|U_{\mu 1}|, \tag{17}$$

$$B2: \quad |U_{e1}| = 2|U_{\tau 1}|, \tag{18}$$

$$B3: \quad |U_{e2}| = 2|U_{\mu 1}|, \tag{19}$$

$$B4: \quad |U_{e2}| = 2|U_{\tau 1}|, \tag{20}$$

$$B5: \quad |U_{\mu 2}| = 2|U_{\mu 1}|, \tag{21}$$

$$B6: \quad |U_{\tau 2}| = 2|U_{\tau 1}|, \tag{22}$$

$$B7: \quad |U_{\tau 2}| = 2|U_{\mu 1}|, \tag{23}$$

$$B8: \quad |U_{\mu 2}| = 2|U_{\tau 1}|, \tag{24}$$

$$B9: \quad |U_{\mu 3}| = 2|U_{\mu 1}|, \tag{25}$$

$$B10: \quad |U_{\tau 3}| = 2|U_{\tau 1}|, \tag{26}$$

$$B11: \quad |U_{\mu 3}| = 2|U_{\tau 1}|, \tag{27}$$

$$B12: \quad |U_{\tau 3}| = 2|U_{\mu 1}|, \tag{28}$$

$$B13: \quad |U_{\mu 1}| = 2|U_{\tau 1}|, \tag{29}$$

$$B14: \quad |U_{\tau 1}| = 2|U_{\mu 1}|. \tag{30}$$

In order to get general mixing patterns, all combinations of elementary correlations are examined. The number of possible combinations is so large. However, because the octant of θ_{23} and CP violation phase δ are uncertain at 3σ level, one type of combination can be converted to another type through the $\mu - \tau$ interchange. $\mu - \tau$ interchange is equivalent to the following transformation:

$$\begin{cases} \theta_{23} \to \frac{\pi}{2} - \theta_{23} \\ \delta \to \delta + \pi \end{cases} \quad or \quad \begin{cases} j \to N - 1 - j \\ l \to N - l - k. \end{cases} \tag{31}$$

As for type-A correlations, $\mu - \tau$ interchange brings following correspondences:

$$A1 \leftrightarrow A1, \quad A2 \leftrightarrow A2, \quad A3 \leftrightarrow A3,$$
$$A4 \leftrightarrow A5, \quad A6 \leftrightarrow A7, \quad A8 \leftrightarrow A9. \tag{32}$$

And correspondences in type-B correlations are

$$B1 \leftrightarrow B2, \quad B3 \leftrightarrow B4, \quad B5 \leftrightarrow B6, \quad B7 \leftrightarrow B8,$$
$$B9 \leftrightarrow B10, \quad B11 \leftrightarrow B12, \quad B13 \leftrightarrow B14. \tag{33}$$

So we first examine the parameter-space of a combination of elementary correlations. Then the examination of the corresponding combination is carried out easily.

III. COMBINATIONS OF ELEMENTARY CORRELATIONS

A. Viable combinations of two elementary correlations

There are 83 viable combinations of two elementary correlations in total. They can be classified into 3 types: Ai-Aj, Bi-Bj, and Ai-Bj, where the notation Ai-Aj means elementary correlation Ai and Aj hold at the same time. The common combinations of these three types are not counted repeatedly.

There are 12 viable Ai-Aj combinations. They are listed as follows:

$$\begin{array}{cccccc} A1 - A2, & A4 - A5, & A1 - A4, & A1 - A5, & A1 - A6, & A1 - A7, \\ A3 - A4, & A3 - A5, & A5 - A6, & A4 - A7, & A7 - A8, & A6 - A9. \end{array} \tag{34}$$

There are 23 viable Bi-Bj combinations. They are listed as follows:

$$\begin{array}{cccccc} B1 - B2, & B1 - B6, & B2 - B5, & B1 - B8, & B2 - B7, & B1 - B10, \\ B2 - B9, & B1 - B11, & B2 - B12, & B3 - B5, & B4 - B6, & B3 - B7, \\ B4 - B8, & B5 - B7, & B6 - B8, & B5 - B9, & B6 - B10, & B7 - B12, \\ B8 - B11, & B9 - B11, & B10 - B12, & B9 - B12, & B10 - B11. \end{array} \tag{35}$$

There are 48 viable Ai-Bj combinations. They are listed as follows:

$$
\begin{array}{cccccc}
A4 - B14, & A8 - B2, & A2 - B2, & A2 - B1, & A2 - B3, & A2 - B4, \\
A2 - B9, & A2 - B10, & A2 - B11, & A2 - B12, & A2 - B13, & A2 - B14, \\
A3 - B2, & A3 - B1, & A3 - B5, & A3 - B6, & A4 - B2, & A5 - B1, \\
A4 - B1, & A5 - B2, & A4 - B6, & A5 - B5, & A4 - B7, & A5 - B8, \\
A4 - B9, & A5 - B10, & A4 - B10, & A5 - B9, & A4 - B11, & A5 - B12, \\
A4 - B12, & A5 - B11, & A6 - B2, & A7 - B1, & A6 - B1, & A7 - B2, \\
A6 - B4, & A7 - B3, & A6 - B6, & A7 - B5, & A6 - B10, & A7 - B9, \\
A6 - B12, & A7 - B11, & A6 - B13, & A7 - B14, & A8 - B11, & A9 - B12.
\end{array} \tag{36}
$$

Let us describe these viable combinations in more detail:

(i) Most viable combinations are paired via the $\mu - \tau$ interchange. There are only 5 single combinations, among which 3 combinations are invariant under the $\mu - \tau$ interchange, including A1-A2 ($\mu - \tau$ symmetry), B1-B2 (TM$_1$), A4-A5 (TM$_2$). The reason why A8-B2 and A4-B14 are single is that the constraint matrix (5) is not symmetric under the $\mu - \tau$ interchange. So the viable parameter-space of a combination may become unviable under the $\mu - \tau$ interchange, especially when the parameter-space is small.

(ii) For more than half of the viable combinations, their 2-dimensional parameter-spaces are not compressed obviously. There are 21 combinations whose parameter-spaces are reduced more than half compared with the noncorrelation case. These combinations are A4-A5, A7-A8, A6-A9, B1-B8, B2-B7, B1-B7, B2-B5, B3-B7, B4-B8, B4-B6, B9-B11, B10-B12, A2-B13, A2-B14, A4-B8, A4-B14, A5-B11, A6-B13, A8-B2, A8-B11, A9-B12.

(iii) There are 17 combinations, whose octants of θ_{23} are uncertain. These combinations are A4-A5, B1-B2, B1-B10, B1-B11, B2-B5, B2-B9, B2-B12, A2-B2, A2-B1, A4-B2, A4-B6, A4-B10, A4-B11, A5-B1, A5-B5, A5-B9, A5-B12.

B. Viable combinations of three elementary correlations

There are 62 viable combinations of three elementary correlations in total. They can be classified into 4 types: Ai-Aj-Ak, Bi-Bj-Bk, Ai-Bj-Bk, and Ai-Aj-Bk. The common combinations of these four types are not counted repeatedly. In detail, there are 3 viable Ai-Aj-Ak combinations. Their mixing patterns are listed as follows:

$$
A1 - A4 - A7: \ |U| = \begin{pmatrix} \sqrt{\frac{5-8s}{7}} & \sqrt{\frac{2+s}{7}} & \sqrt{s} \\ \sqrt{\frac{1+4s}{7}} & \sqrt{\frac{2+s}{7}} & \sqrt{\frac{4-5s}{7}} \\ \sqrt{\frac{1+4s}{7}} & \sqrt{\frac{3-2s}{7}} & \sqrt{\frac{3-2s}{7}} \end{pmatrix}, \tag{37}
$$

$$
A1 - A5 - A6: \ |U| = \begin{pmatrix} \sqrt{\frac{5-8s}{7}} & \sqrt{\frac{2+s}{7}} & \sqrt{s} \\ \sqrt{\frac{1+4s}{7}} & \sqrt{\frac{3-2s}{7}} & \sqrt{\frac{3-2s}{7}} \\ \sqrt{\frac{1+4s}{7}} & \sqrt{\frac{2+s}{7}} & \sqrt{\frac{4-5s}{7}} \end{pmatrix}, \tag{38}
$$

$$
A1 - A4 - A5: \ |U| = \begin{pmatrix} \sqrt{\frac{2-3s}{3}} & \sqrt{\frac{1}{3}} & \sqrt{s} \\ \sqrt{\frac{1+3s}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1+3s}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \tag{39}
$$

where $s = \frac{1}{N} = \sin^2 \theta_{13}$.

There are 5 viable Bi-Bj-Bk combinations. Their mixing patterns are listed as follows:

$$
B1 - B10 - B11: \ |U| = \begin{pmatrix} \sqrt{\frac{7+s}{10}} & \sqrt{\frac{3-11s}{10}} & \sqrt{s} \\ \sqrt{\frac{7+s}{40}} & \sqrt{\frac{13+19s}{40}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1-s}{8}} & \sqrt{\frac{3+5s}{8}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \tag{40}
$$

$$
B2 - B9 - B12: \ |U| = \begin{pmatrix} \sqrt{\frac{7+s}{10}} & \sqrt{\frac{3-11s}{10}} & \sqrt{s} \\ \sqrt{\frac{1-s}{8}} & \sqrt{\frac{3+5s}{8}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{7+s}{40}} & \sqrt{\frac{13+19s}{40}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \tag{41}
$$

$$B1 - B6 - B10\colon \ |U| = \begin{pmatrix} \frac{4}{3}\sqrt{\frac{2}{5}} & \sqrt{\frac{13}{45} - s} & \sqrt{s} \\ \frac{2}{3}\sqrt{\frac{2}{5}} & \sqrt{\frac{4}{15} + s} & \frac{1}{3}\sqrt{5 - 9s} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix},$$

(42)

$$B2 - B5 - B9\colon \ |U| = \begin{pmatrix} \frac{4}{3}\sqrt{\frac{2}{5}} & \sqrt{\frac{13}{45} - s} & \sqrt{s} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3}\sqrt{\frac{2}{5}} & \sqrt{\frac{4}{15} + s} & \frac{1}{3}\sqrt{5 - 9s} \end{pmatrix},$$

(43)

$$B3 - B5 - B7\colon \ |U| = \begin{pmatrix} \sqrt{\frac{2}{3} - s} & \sqrt{\frac{1}{3}} & \sqrt{s} \\ \frac{1}{2}\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{\frac{7}{3}} \\ \sqrt{\frac{1}{4} + s} & \sqrt{\frac{1}{3}} & \sqrt{\frac{5}{12} - s} \end{pmatrix}.$$

(44)

There are 35 viable Ai-Bj-Bk combinations. Their mixing patterns are listed as follows:

$$A4 - B1 - B2\colon \ |U| = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3} - s} & \sqrt{s} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3} - s} & \sqrt{\frac{1}{2} + s} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3} + 2s} & \sqrt{\frac{1}{2} - 2s} \end{pmatrix},$$

(45)

$$A5 - B1 - B2\colon \ |U| = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3} - s} & \sqrt{s} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3} + 2s} & \sqrt{\frac{1}{2} - 2s} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3} - s} & \sqrt{\frac{1}{2} + s} \end{pmatrix},$$

(46)

$$A4 - B1 - B6\colon \ |U| = \begin{pmatrix} \sqrt{\frac{5-2s}{7}} & \sqrt{\frac{2-5s}{7}} & \sqrt{s} \\ \sqrt{\frac{5-2s}{28}} & \sqrt{\frac{2-5s}{7}} & \sqrt{\frac{15+22s}{28}} \\ \sqrt{\frac{3+10s}{28}} & \sqrt{\frac{3+10s}{7}} & \sqrt{\frac{13-50s}{28}} \end{pmatrix},$$

(47)

$$A5 - B2 - B5\colon \ |U| = \begin{pmatrix} \sqrt{\frac{5-2s}{7}} & \sqrt{\frac{2-5s}{7}} & \sqrt{s} \\ \sqrt{\frac{3+10s}{28}} & \sqrt{\frac{3+10s}{7}} & \sqrt{\frac{13-50s}{28}} \\ \sqrt{\frac{5-2s}{28}} & \sqrt{\frac{2-5s}{7}} & \sqrt{\frac{15+22s}{28}} \end{pmatrix},$$

(48)

$$A4 - B1 - B10\colon \ |U| = \begin{pmatrix} 2\sqrt{\frac{3+2s}{17}} & \sqrt{\frac{5-25s}{17}} & \sqrt{s} \\ \sqrt{\frac{3+2s}{17}} & \sqrt{\frac{5-25s}{17}} & \sqrt{\frac{9+23s}{17}} \\ \sqrt{\frac{2-10s}{17}} & \sqrt{\frac{7+50s}{17}} & 2\sqrt{\frac{2-10s}{17}} \end{pmatrix},$$

(49)

$$A5 - B2 - B9\colon \ |U| = \begin{pmatrix} 2\sqrt{\frac{3+2s}{17}} & \sqrt{\frac{5-25s}{17}} & \sqrt{s} \\ \sqrt{\frac{2-10s}{17}} & \sqrt{\frac{7+50s}{17}} & 2\sqrt{\frac{2-10s}{17}} \\ \sqrt{\frac{3+2s}{17}} & \sqrt{\frac{5-25s}{17}} & \sqrt{\frac{9+23s}{17}} \end{pmatrix},$$

(50)

$$A4 - B7 - B12\colon \ |U| = \begin{pmatrix} \sqrt{\frac{7-14s}{10}} & \sqrt{\frac{3+4s}{10}} & \sqrt{s} \\ \sqrt{\frac{1-2s}{10}} & \sqrt{\frac{3+4s}{10}} & \sqrt{\frac{3-s}{5}} \\ \sqrt{\frac{1+8s}{5}} & \sqrt{\frac{2-4s}{5}} & \sqrt{\frac{2-4s}{5}} \end{pmatrix},$$

(51)

$$A5 - B8 - B11\colon \ |U| = \begin{pmatrix} \sqrt{\frac{7-14s}{10}} & \sqrt{\frac{3+4s}{10}} & \sqrt{s} \\ \sqrt{\frac{1+8s}{5}} & \sqrt{\frac{2-4s}{5}} & \sqrt{\frac{2-4s}{5}} \\ \sqrt{\frac{1-2s}{10}} & \sqrt{\frac{3+4s}{10}} & \sqrt{\frac{3-s}{5}} \end{pmatrix},$$

(52)

$$A4 - B6 - B10\colon \ |U| = \begin{pmatrix} \sqrt{\frac{13}{18} - s} & \frac{1}{3}\sqrt{\frac{5}{2}} & \sqrt{s} \\ \sqrt{\frac{1}{6} + s} & \frac{1}{3}\sqrt{\frac{5}{2}} & \frac{1}{3}\sqrt{5 - 9s} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix},$$

(53)

$$A5 - B5 - B9: |U| = \begin{pmatrix} \sqrt{\frac{13}{18} - s} & \frac{1}{3}\sqrt{\frac{5}{2}} & \sqrt{s} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \sqrt{\frac{1}{6} + s} & \frac{1}{3}\sqrt{\frac{5}{2}} & \frac{1}{3}\sqrt{5 - 9s} \end{pmatrix},$$

(54)

$$A4 - B9 - B11: |U| = \begin{pmatrix} \sqrt{\frac{5-2s}{7}} & \sqrt{\frac{2-5s}{7}} & \sqrt{s} \\ \sqrt{\frac{1+s}{7}} & \sqrt{\frac{2-5s}{7}} & 2\sqrt{\frac{1+s}{7}} \\ \sqrt{\frac{1+s}{7}} & \sqrt{\frac{3+10s}{7}} & \sqrt{\frac{3-11s}{7}} \end{pmatrix},$$

(55)

$$A5 - B10 - B12: |U| = \begin{pmatrix} \sqrt{\frac{5-2s}{7}} & \sqrt{\frac{2-5s}{7}} & \sqrt{s} \\ \sqrt{\frac{1+s}{7}} & \sqrt{\frac{3+10s}{7}} & \sqrt{\frac{3-11s}{7}} \\ \sqrt{\frac{1+s}{7}} & \sqrt{\frac{2-5s}{7}} & 2\sqrt{\frac{1+s}{7}} \end{pmatrix},$$

(56)

$$A4 - B1 - B11: |U| = \begin{pmatrix} 2\sqrt{\frac{4-s}{23}} & \sqrt{\frac{7-19s}{23}} & \sqrt{s} \\ \sqrt{\frac{4-s}{23}} & \sqrt{\frac{7-19s}{23}} & 2\sqrt{\frac{3+5s}{23}} \\ \sqrt{\frac{3+5s}{23}} & \sqrt{\frac{9+38s}{23}} & \sqrt{\frac{11-43s}{23}} \end{pmatrix},$$

(57)

$$A5 - B2 - B12: |U| = \begin{pmatrix} 2\sqrt{\frac{4-s}{23}} & \sqrt{\frac{7-19s}{23}} & \sqrt{s} \\ \sqrt{\frac{3+5s}{23}} & \sqrt{\frac{9+38s}{23}} & \sqrt{\frac{11-43s}{23}} \\ \sqrt{\frac{4-s}{23}} & \sqrt{\frac{7-19s}{23}} & 2\sqrt{\frac{3+5s}{23}} \end{pmatrix},$$

(58)

$$A4 - B2 - B7: |U| = \begin{pmatrix} \sqrt{\frac{5-2s}{7}} & \sqrt{\frac{2-5s}{7}} & \sqrt{s} \\ \sqrt{\frac{3+10s}{28}} & \sqrt{\frac{2-5s}{7}} & \sqrt{\frac{17+10s}{28}} \\ \sqrt{\frac{5-2s}{28}} & \sqrt{\frac{3+10s}{7}} & \sqrt{\frac{11-38s}{28}} \end{pmatrix},$$

(59)

$$A4 - B2 - B9: |U| = \begin{pmatrix} 2\sqrt{\frac{5-s}{29}} & \sqrt{\frac{9-25s}{29}} & \sqrt{s} \\ \sqrt{\frac{4+5s}{29}} & \sqrt{\frac{9-25s}{29}} & 2\sqrt{\frac{4+5s}{29}} \\ \sqrt{\frac{5-s}{29}} & \sqrt{\frac{11+50s}{29}} & \sqrt{\frac{13-49s}{29}} \end{pmatrix},$$

(60)

$$A5 - B1 - B10: |U| = \begin{pmatrix} 2\sqrt{\frac{5-s}{29}} & \sqrt{\frac{9-25s}{29}} & \sqrt{s} \\ \sqrt{\frac{5-s}{29}} & \sqrt{\frac{11+50s}{29}} & \sqrt{\frac{13-49s}{29}} \\ \sqrt{\frac{4+5s}{29}} & \sqrt{\frac{9-25s}{29}} & 2\sqrt{\frac{4+5s}{29}} \end{pmatrix},$$

(61)

$$A4 - B10 - B11: |U|$$
$$= \begin{pmatrix} \frac{1}{4}\sqrt{11 - 11s} & \frac{1}{4}\sqrt{5 - 5s} & \sqrt{s} \\ \frac{1}{4}\sqrt{3 + 13s} & \frac{1}{4}\sqrt{5 - 5s} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1-s}{8}} & \sqrt{\frac{3+5s}{8}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \quad (62)$$

$$A5 - B9 - B12: |U|$$
$$= \begin{pmatrix} \frac{1}{4}\sqrt{11 - 11s} & \frac{1}{4}\sqrt{5 - 5s} & \sqrt{s} \\ \sqrt{\frac{1-s}{8}} & \sqrt{\frac{3+5s}{8}} & \sqrt{\frac{1-s}{2}} \\ \frac{1}{4}\sqrt{3 + 13s} & \frac{1}{4}\sqrt{5 - 5s} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \quad (63)$$

$$A2 - B1 - B2: |U| = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3} - s} & \sqrt{s} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{2+3s}{6}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{2+3s}{6}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \quad (64)$$

$$A2 - B1 - B10: |U| = \begin{pmatrix} 2\sqrt{\frac{8+s}{46}} & \sqrt{\frac{7-25s}{23}} & \sqrt{s} \\ \sqrt{\frac{8+s}{46}} & \sqrt{\frac{16+25s}{46}} & \sqrt{\frac{11-13s}{23}} \\ \sqrt{\frac{6-5s}{46}} & \sqrt{\frac{16+25s}{46}} & 2\sqrt{\frac{6-5s}{46}} \end{pmatrix},$$

(65)

$$A2 - B2 - B9: \ |U| = \begin{pmatrix} 2\sqrt{\frac{8+s}{46}} & \sqrt{\frac{7-25s}{23}} & \sqrt{s} \\ \sqrt{\frac{6-5s}{46}} & \sqrt{\frac{16+25s}{46}} & 2\sqrt{\frac{6-5s}{46}} \\ \sqrt{\frac{8+s}{46}} & \sqrt{\frac{16+25s}{46}} & \sqrt{\frac{11-13s}{23}} \end{pmatrix}, \tag{66}$$

$$A2 - B1 - B11: \ |U| = \begin{pmatrix} 2\sqrt{\frac{6+s}{34}} & \sqrt{\frac{5-19s}{17}} & \sqrt{s} \\ \sqrt{\frac{6+s}{34}} & \sqrt{\frac{12+19s}{17}} & 2\sqrt{\frac{4-5s}{34}} \\ \sqrt{\frac{4-5s}{34}} & \sqrt{\frac{12+19s}{17}} & \sqrt{\frac{9-7s}{17}} \end{pmatrix}, \tag{67}$$

$$A2 - B2 - B12: \ |U| = \begin{pmatrix} 2\sqrt{\frac{6+s}{34}} & \sqrt{\frac{5-19s}{17}} & \sqrt{s} \\ \sqrt{\frac{4-5s}{34}} & \sqrt{\frac{12+19s}{17}} & \sqrt{\frac{9-7s}{17}} \\ \sqrt{\frac{6+s}{34}} & \sqrt{\frac{12+19s}{17}} & 2\sqrt{\frac{4-5s}{34}} \end{pmatrix}, \tag{68}$$

$$A3 - B1 - B10: \ |U| = \begin{pmatrix} 2\sqrt{\frac{7+s}{40}} & \sqrt{\frac{3-11s}{10}} & \sqrt{s} \\ \sqrt{\frac{7+s}{40}} & \sqrt{\frac{13+19s}{40}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1-s}{8}} & \sqrt{\frac{3+5s}{8}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \tag{69}$$

$$A3 - B2 - B9: \ |U| = \begin{pmatrix} 2\sqrt{\frac{7+s}{40}} & \sqrt{\frac{3-11s}{10}} & \sqrt{s} \\ \sqrt{\frac{1-s}{8}} & \sqrt{\frac{3+5s}{8}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{7+s}{40}} & \sqrt{\frac{13+19s}{40}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \tag{70}$$

$$A7 - B1 - B2: \ |U| = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}-s} & \sqrt{s} \\ \sqrt{\frac{1}{6}} & \frac{1}{2}\sqrt{1+4s} & \sqrt{\frac{7}{12}-s} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}} \end{pmatrix}, \tag{71}$$

$$A6 - B1 - B2: \ |U| = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}-s} & \sqrt{s} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{5}{12}} & \sqrt{\frac{5}{12}} \\ \sqrt{\frac{1}{6}} & \frac{1}{2}\sqrt{1+4s} & \sqrt{\frac{7}{12}-s} \end{pmatrix}, \tag{72}$$

$$A7 - B1 - B6: \ |U| = \begin{pmatrix} \frac{4}{3}\sqrt{\frac{2}{5}} & \sqrt{\frac{13}{45}-s} & \sqrt{s} \\ \frac{2}{3}\sqrt{\frac{2}{5}} & \sqrt{\frac{4}{15}+s} & \frac{1}{3}\sqrt{5-9s} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}, \tag{73}$$

$$A6 - B2 - B5: \ |U| = \begin{pmatrix} \frac{4}{3}\sqrt{\frac{2}{5}} & \sqrt{\frac{13}{45}-s} & \sqrt{s} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3}\sqrt{\frac{2}{5}} & \sqrt{\frac{4}{15}+s} & \frac{1}{3}\sqrt{5-9s} \end{pmatrix}, \tag{74}$$

$$A7 - B1 - B11: \ |U| = \begin{pmatrix} 2\sqrt{\frac{6+2s}{35}} & \sqrt{\frac{11-43s}{35}} & \sqrt{s} \\ \sqrt{\frac{6+2s}{35}} & \sqrt{\frac{9+38s}{35}} & \sqrt{\frac{4-8s}{7}} \\ \sqrt{\frac{1-2s}{7}} & \sqrt{\frac{3+s}{7}} & \sqrt{\frac{3+s}{7}} \end{pmatrix}, \tag{75}$$

$$A6 - B2 - B12: \ |U| = \begin{pmatrix} 2\sqrt{\frac{6+2s}{35}} & \sqrt{\frac{11-43s}{35}} & \sqrt{s} \\ \sqrt{\frac{1-2s}{7}} & \sqrt{\frac{3+s}{7}} & \sqrt{\frac{3+s}{7}} \\ \sqrt{\frac{6+2s}{35}} & \sqrt{\frac{9+38s}{35}} & \sqrt{\frac{4-8s}{7}} \end{pmatrix}, \tag{76}$$

$$A7 - B2 - B9: \ |U| = \begin{pmatrix} 2\sqrt{\frac{7+2s}{41}} & \sqrt{\frac{13-49s}{41}} & \sqrt{s} \\ \sqrt{\frac{6-10s}{41}} & \sqrt{\frac{11+50s}{41}} & 2\sqrt{\frac{6-10s}{41}} \\ \sqrt{\frac{7+2s}{41}} & \sqrt{\frac{17-s}{41}} & \sqrt{\frac{17-s}{41}} \end{pmatrix}, \tag{77}$$

$$A6 - B1 - B10: |U| = \begin{pmatrix} 2\sqrt{\frac{7+2s}{41}} & \sqrt{\frac{13-49s}{41}} & \sqrt{s} \\ \sqrt{\frac{7+2s}{41}} & \sqrt{\frac{17-s}{41}} & \sqrt{\frac{17-s}{41}} \\ \sqrt{\frac{6-10s}{41}} & \sqrt{\frac{11+50s}{41}} & 2\sqrt{\frac{6-10s}{41}} \end{pmatrix},$$

(78)

$$A7 - B3 - B5: |U| = \begin{pmatrix} \sqrt{\frac{9-17s}{13}} & 2\sqrt{\frac{1+s}{13}} & \sqrt{s} \\ \sqrt{\frac{1+s}{13}} & 2\sqrt{\frac{1+s}{13}} & \sqrt{\frac{8-5s}{13}} \\ \sqrt{\frac{3+16s}{13}} & \sqrt{\frac{5-8s}{13}} & \sqrt{\frac{5-8s}{13}} \end{pmatrix}.$$

(79)

There are 19 viable Ai-Aj-Bk combinations. Their mixing patterns are listed as follows:

$$A7 - A4 - B1: |U| = \begin{pmatrix} 2\sqrt{\frac{2-4s}{11}} & \sqrt{\frac{3+5s}{11}} & \sqrt{s} \\ \sqrt{\frac{2-4s}{11}} & \sqrt{\frac{3+5s}{11}} & \sqrt{\frac{6-s}{11}} \\ \sqrt{\frac{1+20s}{11}} & \sqrt{\frac{5-10s}{11}} & \sqrt{\frac{5-10s}{11}} \end{pmatrix},$$

(80)

$$A6 - A5 - B2: |U| = \begin{pmatrix} 2\sqrt{\frac{2-4s}{11}} & \sqrt{\frac{3+5s}{11}} & \sqrt{s} \\ \sqrt{\frac{1+20s}{11}} & \sqrt{\frac{5-10s}{11}} & \sqrt{\frac{5-10s}{11}} \\ \sqrt{\frac{2-4s}{11}} & \sqrt{\frac{3+5s}{11}} & \sqrt{\frac{6-s}{11}} \end{pmatrix},$$

(81)

$$A7 - A4 - B2: |U| = \begin{pmatrix} 2\sqrt{\frac{3-4s}{17}} & \sqrt{\frac{5-s}{17}} & \sqrt{s} \\ \sqrt{\frac{2+20s}{17}} & \sqrt{\frac{5-s}{17}} & \sqrt{\frac{10-19s}{17}} \\ \sqrt{\frac{3-4s}{17}} & \sqrt{\frac{7+2s}{17}} & \sqrt{\frac{7+2s}{17}} \end{pmatrix},$$

(82)

$$A6 - A5 - B1: |U| = \begin{pmatrix} 2\sqrt{\frac{3-4s}{17}} & \sqrt{\frac{5-s}{17}} & \sqrt{s} \\ \sqrt{\frac{3-4s}{17}} & \sqrt{\frac{7+2s}{17}} & \sqrt{\frac{7+2s}{17}} \\ \sqrt{\frac{2+20s}{17}} & \sqrt{\frac{5-s}{17}} & \sqrt{\frac{10-19s}{17}} \end{pmatrix},$$

(83)

$$A7 - A4 - B9: |U| = \begin{pmatrix} \sqrt{\frac{10-19s}{14}} & \sqrt{\frac{4+5s}{14}} & \sqrt{s} \\ \sqrt{\frac{2-s}{14}} & \sqrt{\frac{4+5s}{14}} & 2\sqrt{\frac{2-s}{14}} \\ \sqrt{\frac{1+10s}{7}} & \sqrt{\frac{3-5s}{7}} & \sqrt{\frac{3-5s}{7}} \end{pmatrix},$$

(84)

$$A6 - A5 - B10: |U| = \begin{pmatrix} \sqrt{\frac{10-19s}{14}} & \sqrt{\frac{4+5s}{14}} & \sqrt{s} \\ \sqrt{\frac{1+10s}{7}} & \sqrt{\frac{3-5s}{7}} & \sqrt{\frac{3-5s}{7}} \\ \sqrt{\frac{2-s}{14}} & \sqrt{\frac{4+5s}{14}} & 2\sqrt{\frac{2-s}{14}} \end{pmatrix},$$

(85)

$$A7 - A4 - B11: |U| = \begin{pmatrix} \sqrt{\frac{10-13s}{14}} & \sqrt{\frac{4-s}{14}} & \sqrt{s} \\ \sqrt{\frac{2+17s}{14}} & \sqrt{\frac{4-s}{14}} & 2\sqrt{\frac{1-2s}{7}} \\ \sqrt{\frac{1-2s}{7}} & \sqrt{\frac{3+s}{7}} & \sqrt{\frac{3+s}{7}} \end{pmatrix},$$

(86)

$$A6 - A5 - B12: |U| = \begin{pmatrix} \sqrt{\frac{10-13s}{14}} & \sqrt{\frac{4-s}{14}} & \sqrt{s} \\ \sqrt{\frac{1-2s}{7}} & \sqrt{\frac{3+s}{7}} & \sqrt{\frac{3+s}{7}} \\ \sqrt{\frac{2+17s}{14}} & \sqrt{\frac{4-s}{14}} & 2\sqrt{\frac{1-2s}{7}} \end{pmatrix},$$

(87)

$$A7 - A4 - B14: |U|$$
$$= \begin{pmatrix} \frac{1}{4}\sqrt{11-20s} & \frac{1}{4}\sqrt{5+4s} & \sqrt{s} \\ \frac{1}{4}\sqrt{1+4s} & \frac{1}{4}\sqrt{5+4s} & \sqrt{\frac{5}{8}-\frac{s}{2}} \\ \frac{1}{2}\sqrt{1+4s} & \frac{1}{2}\sqrt{\frac{3}{2}-2s} & \sqrt{\frac{3}{8}-\frac{s}{2}} \end{pmatrix},$$

(88)

$$A4 - A5 - B1: |U| = \begin{pmatrix} \sqrt{\frac{2}{3}-s} & \sqrt{\frac{1}{3}} & \sqrt{s} \\ \sqrt{\frac{1}{6}-\frac{s}{4}} & \sqrt{\frac{1}{3}} & \frac{\sqrt{2}+s}{2} \\ \sqrt{\frac{1}{6}+\frac{5s}{4}} & \sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{2-5s} \end{pmatrix},$$

(89)

$$A4-A5-B2: \ |U| = \begin{pmatrix} \sqrt{\frac{2}{3}-s} & \sqrt{\frac{1}{3}} & \sqrt{s} \\ \sqrt{\frac{1}{6}+\frac{5s}{4}} & \sqrt{\frac{1}{3}} & \frac{1}{2}\sqrt{2-5s} \\ \sqrt{\frac{1}{6}-\frac{s}{4}} & \sqrt{\frac{1}{3}} & \frac{\sqrt{2+s}}{2} \end{pmatrix},$$

$$(90)$$

$$A4-A5-B9: \ |U| = \begin{pmatrix} \sqrt{\frac{2}{3}-s} & \sqrt{\frac{1}{3}} & \sqrt{s} \\ \sqrt{\frac{2}{15}} & \sqrt{\frac{1}{3}} & 2\sqrt{\frac{2}{15}} \\ \sqrt{\frac{1}{5}+s} & \sqrt{\frac{1}{3}} & \sqrt{\frac{7}{15}-s} \end{pmatrix},$$

$$(91)$$

$$A4-A5-B10: \ |U| = \begin{pmatrix} \sqrt{\frac{2}{3}-s} & \sqrt{\frac{1}{3}} & \sqrt{s} \\ \sqrt{\frac{1}{5}+s} & \sqrt{\frac{1}{3}} & \sqrt{\frac{7}{15}-s} \\ \sqrt{\frac{2}{15}} & \sqrt{\frac{1}{3}} & 2\sqrt{\frac{2}{15}} \end{pmatrix},$$

$$(92)$$

$$A4-A5-B11: \ |U|$$
$$= \begin{pmatrix} \sqrt{\frac{2}{3}-s} & \sqrt{\frac{1}{3}} & \sqrt{s} \\ \frac{1}{3}\sqrt{2+12s} & \sqrt{\frac{1}{3}} & \frac{2}{3}\sqrt{1-3s} \\ \frac{1}{3}\sqrt{1-3s} & \sqrt{\frac{1}{3}} & \frac{1}{3}\sqrt{5+3s} \end{pmatrix},$$

$$(93)$$

$$A4-A5-B12: \ |U|$$
$$= \begin{pmatrix} \sqrt{\frac{2}{3}-s} & \sqrt{\frac{1}{3}} & \sqrt{s} \\ \frac{1}{3}\sqrt{1-3s} & \sqrt{\frac{1}{3}} & \frac{1}{3}\sqrt{5+3s} \\ \frac{1}{3}\sqrt{2+12s} & \sqrt{\frac{1}{3}} & \frac{2}{3}\sqrt{1-3s} \end{pmatrix},$$

$$(94)$$

$$A3-A4-B2: \ |U| = \begin{pmatrix} \sqrt{\frac{2-2s}{3}} & \sqrt{\frac{1-s}{3}} & \sqrt{s} \\ \sqrt{\frac{1+5s}{6}} & \sqrt{\frac{1-s}{3}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1-s}{6}} & \sqrt{\frac{1+2s}{3}} & \sqrt{\frac{1-s}{2}} \end{pmatrix},$$

$$(95)$$

$$A3-A5-B1: \ |U| = \begin{pmatrix} \sqrt{\frac{2-2s}{3}} & \sqrt{\frac{1-s}{3}} & \sqrt{s} \\ \sqrt{\frac{1-s}{6}} & \sqrt{\frac{1+2s}{3}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1+5s}{6}} & \sqrt{\frac{1-s}{3}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \quad (96)$$

$$A3-A4-B6: \ |U| = \begin{pmatrix} \sqrt{\frac{7-8s}{10}} & \sqrt{\frac{3-2s}{10}} & \sqrt{s} \\ \sqrt{\frac{2+7s}{10}} & \sqrt{\frac{3-2s}{10}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1+s}{10}} & \sqrt{\frac{2+2s}{5}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \quad (97)$$

$$A3-A5-B5: \ |U| = \begin{pmatrix} \sqrt{\frac{7-8s}{10}} & \sqrt{\frac{3-2s}{10}} & \sqrt{s} \\ \sqrt{\frac{1+s}{10}} & \sqrt{\frac{2+2s}{5}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{2+7s}{10}} & \sqrt{\frac{3-2s}{10}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}. \quad (98)$$

Specific mixing angles and Dirac CP phases are listed in Table I–III.

As we can see from the expressions of $|U|$, there are only 6 single combinations, i.e., A1-A4-A5, A2-B1-B2, B3-B5-B7, A4-B2-B7, A7-B3-B5, A7-A4-B14, among which A1-A4-A5 and A2-B1-B2 are $\mu - \tau$ symmetric. The remaining 56 combinations are paired through $\mu - \tau$ interchange. According to the expressions of $\cos\delta$ in tables, all viable patterns can be classified into two groups: perturbative patterns and nonperturbative patterns. There is a factor $\sqrt{s}$ in $\cos\delta$ of the former pattern which could be obtained from perturbing TBM. So we could call it TBM-like pattern. In contrast, there is a factor $\frac{1}{\sqrt{s}}$ in $\cos\delta$ of the latter pattern which cannot be obtained from perturbing the pattern whose θ_{13} is zero. There are 8 perturbative patterns, i.e., A1-A4-A5, A4-B1-B2, A5-B1-B2, A2-B1-B2, A4-A5-B1, A4-A5-B2, A3-A4-B2, A3-A5-B1. The remaining 54 patterns are non-perturbative. In more details, as for $\cos\delta$ of perturbative and nonperturbative patterns, we have following observations:

(i) Because of the factor $\sqrt{s}$, for perturbative patterns, $\cos\delta \to 0$ when $s \to 0$, see A3-A4-B2, A3-A5-B1 in Table III for example. In contrast, for nonperturbative patterns, because of the factor $\frac{1}{\sqrt{s}}$, $\cos\delta \to \infty$ when $s \to 0$, see A3-A4-B6 in Table III for example. So for a nonperturbative pattern, small perturbation to $s = 0$ would bring very large variation of Dirac CP phase. We note that this observation has also been obtained in special mixing patterns in Ref. [30].

(ii) For different perturbative patterns, the difference of $\cos\delta$ is very small. And the variation of δ is focused on small range around $\pm\frac{\pi}{2}$. We show $\cos\delta$

TABLE I. Mixing angles of and Dirac *CP* Phase of viable leptonic mixing patterns.

Combinations	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$	$\cos\delta$
A1-A4-A7	$\frac{2+s}{7-7s}$	$\frac{4-5s}{7-7s}$	s	$\frac{1-5s+10s^2-12s^3}{2\sqrt{s}\sqrt{10-11s-8s^2}\sqrt{12-23s+10s^2}}$
A1-A5-A6	$\frac{2+s}{7-7s}$	$\frac{3-2s}{7-7s}$	s	$-\frac{1-5s+10s^2-12s^3}{2\sqrt{s}\sqrt{10-11s-8s^2}\sqrt{12-23s+10s^2}}$
A1-A4-A5	$\frac{1}{3-3s}$	$\frac{1}{2}$	s	0
B1-B10-B11	$\frac{3-11s}{10-10s}$	$\frac{1}{2}$	s	$\frac{-1-2s+3s^2}{4\sqrt{s}\sqrt{21-74s-11s^2}}$
B2-B9-B12	$\frac{3-11s}{10-10s}$	$\frac{1}{2}$	s	$-\frac{-1-2s+3s^2}{4\sqrt{s}\sqrt{21-74s-11s^2}}$
B1-B6-B10	$\frac{13-45s}{45-45s}$	$\frac{5-9s}{9-9s}$	s	$\frac{5-31s+90s^2}{4\sqrt{s}\sqrt{26-90s}\sqrt{5-9s}}$
B2-B5-B9	$\frac{13-45s}{45-45s}$	$\frac{4}{9-9s}$	s	$-\frac{5-31s+90s^2}{4\sqrt{s}\sqrt{26-90s}\sqrt{5-9s}}$
B3-B5-B7	$\frac{1}{3-3s}$	$\frac{7}{12-12s}$	s	$\frac{-1-4s+12s^2}{\sqrt{7s}\sqrt{5-12s}\sqrt{2-3s}}$
A4-B1-B2	$\frac{1-3s}{3-3s}$	$\frac{1+2s}{2-2s}$	s	$\frac{3(1-5s)\sqrt{s}}{2\sqrt{2}\sqrt{1-3s}\sqrt{1-2s-8s^2}}$
A5-B1-B2	$\frac{1-3s}{3-3s}$	$\frac{1-4s}{2-2s}$	s	$-\frac{3(1-5s)\sqrt{s}}{2\sqrt{2}\sqrt{1-3s}\sqrt{1-2s-8s^2}}$
A4-B1-B6	$\frac{2-5s}{7-7s}$	$\frac{15+22s}{28-28s}$	s	$\frac{9+6s+3s^2(-89+10s)}{2\sqrt{s}\sqrt{(5-2s)(2-5s)}\sqrt{(15+22s)(13-50s)}}$
A5-B2-B5	$\frac{2-5s}{7-7s}$	$\frac{13-50s}{28-28s}$	s	$-\frac{9+6s+3s^2(-89+10s)}{2\sqrt{s}\sqrt{(5-2s)(2-5s)}\sqrt{(15+22s)(13-50s)}}$
A4-B1-B10	$\frac{5-25s}{17-17s}$	$\frac{9+23s}{17-17s}$	s	$\frac{11+224s-1365s^2-150s^3}{8\sqrt{10s}\sqrt{9-22s-115s^2}\sqrt{3-13s-10s^2}}$
A5-B2-B9	$\frac{5-25s}{17-17s}$	$\frac{8-40s}{17-17s}$	s	$-\frac{11+224s-1365s^2-150s^3}{8\sqrt{10s}\sqrt{9-22s-115s^2}\sqrt{3-13s-10s^2}}$
A4-B7-B12	$\frac{3+4s}{10-10s}$	$\frac{3-s}{5-5s}$	s	$-\frac{1+37s-90s^2+24s^3}{2\sqrt{14s}\sqrt{3-2s-8s^2}\sqrt{3-7s+2s^2}}$
A5-B8-B11	$\frac{3+4s}{10-10s}$	$\frac{2-4s}{5-5s}$	s	$\frac{1+37s-90s^2+24s^3}{2\sqrt{14s}\sqrt{3-2s-8s^2}\sqrt{3-7s+2s^2}}$
A4-B6-B10	$\frac{5}{18-18s}$	$\frac{5-9s}{9-9s}$	s	$\frac{7+43s-90s^2}{4\sqrt{s}\sqrt{65-90s}\sqrt{5-9s}}$
A5-B5-B9	$\frac{5}{18-18s}$	$\frac{4}{9-9s}$	s	$-\frac{7+43s-90s^2}{4\sqrt{s}\sqrt{65-90s}\sqrt{5-9s}}$
A4-B9-B11	$\frac{2-5s}{7-7s}$	$\frac{4+4s}{7-7s}$	s	$\frac{1+10s-74s^2+15s^3}{4\sqrt{s}\sqrt{3-8s-11s^2}\sqrt{10-29s+10s^2}}$
A5-B10-B12	$\frac{2-5s}{7-7s}$	$\frac{3-11s}{7-7s}$	s	$-\frac{1+10s-74s^2+15s^3}{4\sqrt{s}\sqrt{3-8s-11s^2}\sqrt{10-29s+10s^2}}$
A4-B1-B11	$\frac{7-19s}{23-23s}$	$\frac{12+20s}{23-23s}$	s	$\frac{3(5+37s-317s^2+19s^3)}{8\sqrt{s}\sqrt{33-74s-215s^2}\sqrt{28-83s+19s^2}}$
A5-B2-B12	$\frac{7-19s}{23-23s}$	$\frac{11-43s}{23-23s}$	s	$-\frac{3(5+37s-317s^2+19s^3)}{8\sqrt{s}\sqrt{33-74s-215s^2}\sqrt{28-83s+19s^2}}$
A4-B2-B7	$\frac{2-5s}{7-7s}$	$\frac{17+10s}{28-28s}$	s	$\frac{-1+74s-325s^2+90s^3}{2\sqrt{s}\sqrt{187-536s-380s^2}\sqrt{10-29s+1s^2}}$
A4-B2-B9	$\frac{9-25s}{29-29s}$	$\frac{16+20s}{29-29s}$	s	$\frac{-1+359s-1735s^2+225s^3}{8\sqrt{s}\sqrt{52-131s-245s^2}\sqrt{45-134s+25s^2}}$
A5-B1-B10	$\frac{9-25s}{29-29s}$	$\frac{13-49s}{29-29s}$	s	$-\frac{-1+359s-1735s^2+225s^3}{8\sqrt{s}\sqrt{52-131s-245s^2}\sqrt{45-134s+25s^2}}$

TABLE II. Mixing angles of and Dirac *CP* Phase of viable leptonic mixing patterns.

| Combinations	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$	$\cos\delta$
A4-B10-B11	$\frac{5}{16}$	$\frac{1}{2}$	s	$\frac{1+15s}{2\sqrt{55}\sqrt{s}}$
A5-B9-B12	$\frac{5}{16}$	$\frac{1}{2}$	s	$\frac{1+15s}{2\sqrt{55}\sqrt{s}}$
A2-B1-B2	$\frac{1-3s}{3-3s}$	$\frac{1}{2}$	s	0
A2-B1-B10	$\frac{7-25s}{23-23s}$	$\frac{11-13s}{23-23s}$	s	$\frac{16+43s+10s^2+75s^3}{8\sqrt{s}\sqrt{56-193s-25s^2}\sqrt{66-133s+65s^2}}$
A2-B2-B9	$\frac{7-25s}{23-23s}$	$\frac{12-10s}{23-23s}$	s	$-\frac{16+43s+10s^2+75s^3}{8\sqrt{s}\sqrt{56-193s-25s^2}\sqrt{66-133s+65s^2}}$
A2-B1-B11	$\frac{5-19s}{17-17s}$	$\frac{8-10s}{17-17s}$	s	$\frac{12+33s+10s^2+57s^3}{8\sqrt{s}\sqrt{30-109s-19s^2}\sqrt{36-73s+35s^2}}$
A2-B2-B12	$\frac{5-19s}{17-17s}$	$\frac{9-7s}{17-17s}$	s	$-\frac{12+33s+10s^2+57s^3}{8\sqrt{s}\sqrt{30-109s-19s^2}\sqrt{36-73s+35s^2}}$

TABLE II. (*Continued*)

Combinations	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$	$\cos\delta$
A3-B1-B10	$\frac{3-11s}{10-10s}$	$\frac{1}{2}$	s	$\frac{(1+3s)(1-s)}{4\sqrt{s}\sqrt{21-74s-11s^2}}$
A3-B2-B9	$\frac{3-11s}{10-10s}$	$\frac{1}{2}$	s	$-\frac{(1+3s)(1-s)}{4\sqrt{s}\sqrt{21-74s-11s^2}}$
A7-B1-B2	$\frac{1-3s}{3-3s}$	$\frac{7-12s}{12-12s}$	s	$\frac{1-11s+30s^2}{2\sqrt{10s}\sqrt{1-3s}\sqrt{7-12s}}$
A6-B1-B2	$\frac{1-3s}{3-3s}$	$\frac{5}{12-12s}$	s	$-\frac{1-11s+30s^2}{2\sqrt{10s}\sqrt{1-3s}\sqrt{7-12s}}$
A7-B1-B6	$\frac{13-45s}{45-45s}$	$\frac{5-9s}{9-9s}$	s	$\frac{5-31s+90s^2}{4\sqrt{s}\sqrt{26-90s}\sqrt{5-9s}}$
A6-B2-B5	$\frac{13-45s}{45-45s}$	$\frac{4}{9-9s}$	s	$-\frac{5-31s+90s^2}{4\sqrt{s}\sqrt{26-90s}\sqrt{5-9s}}$
A7-B1-B11	$\frac{11-43s}{35-35s}$	$\frac{4-8s}{7-7s}$	s	$\frac{9-48s+217s^2+78s^3}{8\sqrt{2s}\sqrt{33-118s-43s^2}\sqrt{3-5s-2s^2}}$
A6-B2-B12	$\frac{11-43s}{35-35s}$	$\frac{3+s}{7-7s}$	s	$-\frac{9-48s+217s^2+78s^3}{8\sqrt{2s}\sqrt{33-118s-43s^2}\sqrt{3-5s-2s^2}}$
A7-B2-B9	$\frac{13-49s}{41-41s}$	$\frac{24-40s}{41-41s}$	s	$\frac{25-728s+1945s^2-90s^3}{8\sqrt{2s}\sqrt{91-317s-98s^2}\sqrt{51-88s+5s^2}}$
A6-B1-B10	$\frac{13-49s}{41-41s}$	$\frac{17-s}{41-41s}$	s	$-\frac{25-728s+1945s^2-90s^3}{8\sqrt{2s}\sqrt{91-317s-98s^2}\sqrt{51-88s+5s^2}}$
A7-B3-B5	$\frac{4+4s}{13-13s}$	$\frac{8-5s}{13-13s}$	s	$\frac{-7-73s+200s^2-72s^3}{4\sqrt{s}\sqrt{9-8s-17s^2}\sqrt{40-89s+40s^2}}$
A7-A4-B1	$\frac{3+5s}{11-11s}$	$\frac{6-s}{11-11s}$	s	$\frac{7-131s+264s^2-60s^3}{4\sqrt{10s}\sqrt{6-13s+2s^2}\sqrt{3-s-10s^2}}$
A6-A5-B2	$\frac{3+5s}{11-11s}$	$\frac{5-10s}{11-11s}$	s	$-\frac{7-131s+264s^2-60s^3}{4\sqrt{10s}\sqrt{6-13s+2s^2}\sqrt{3-s-10s^2}}$
A7-A4-B2	$\frac{5-s}{17-17s}$	$\frac{10-19s}{17-17s}$	s	$\frac{-1+149s-256s^2+36s^3}{4\sqrt{s}\sqrt{70-113s-38s^2}\sqrt{15-23s+4s^2}}$
A6-A5-B1	$\frac{5-s}{17-17s}$	$\frac{7+2s}{17-17s}$	s	$-\frac{-1+149s-256s^2+36s^3}{4\sqrt{s}\sqrt{70-113s-38s^2}\sqrt{15-23s+4s^2}}$
A7-A4-B9	$\frac{4+5s}{14-14s}$	$\frac{4-2s}{7-7s}$	s	$\frac{2-70s+149s^2-45s^3}{2\sqrt{2s}\sqrt{40-26s-95s^2}\sqrt{6-13s+5s^2}}$
A6-A5-B10	$\frac{4+5s}{14-14s}$	$\frac{3-5s}{7-7s}$	s	$-\frac{2-70s+149s^2-45s^3}{2\sqrt{2s}\sqrt{40-26s-95s^2}\sqrt{6-13s+5s^2}}$
A7-A4-B11	$\frac{4-4s}{14-14s}$	$\frac{4-8s}{7-7s}$	s	$\frac{2+50s-91s^2+15s^3}{4\sqrt{s}\sqrt{3-5s-2s^2}\sqrt{40-62s+13s^2}}$
A6-A5-B12	$\frac{4-4s}{14-14s}$	$\frac{3+s}{7-7s}$	s	$-\frac{2+50s-91s^2+15s^3}{4\sqrt{s}\sqrt{3-5s-2s^2}\sqrt{40-62s+13s^2}}$

TABLE III. Mixing angles of and Dirac *CP* phase of viable leptonic mixing patterns.

Combinations	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$	$\cos\delta$
A7-A4-B14	$\frac{5+4s}{16-16s}$	$\frac{5-4s}{8-8s}$	s	$\frac{-7-31s+104s^2-48s^3}{2\sqrt{s}\sqrt{55-56s-80s^2}\sqrt{15-32s+16s^2}}$
A4-A5-B1	$\frac{1}{3-3s}$	$\frac{2+s}{4-4s}$	s	$\frac{3\sqrt{s}(-1+2s)}{\sqrt{2-3s}\sqrt{4-8s-5s^2}}$
A4-A5-B2	$\frac{1}{3-3s}$	$\frac{2-5s}{4-4s}$	s	$-\frac{3\sqrt{s}(-1+2s)}{\sqrt{2-3s}\sqrt{4-8s-5s^2}}$
A4-A5-B9	$\frac{1}{3-3s}$	$\frac{8}{15-15s}$	s	$-\frac{1+13s-30s^2}{4\sqrt{2s}\sqrt{7-15s}\sqrt{2-3s}}$
A4-A5-B10	$\frac{1}{3-3s}$	$\frac{7-15s}{15-15s}$	s	$\frac{1+13s-30s^2}{4\sqrt{2s}\sqrt{7-15s}\sqrt{2-3s}}$
A4-A5-B11	$\frac{1}{3-3s}$	$\frac{4-12s}{9-9s}$	s	$\frac{1+13s-30s^2}{4\sqrt{s}\sqrt{2-3s}\sqrt{5-12s-9s^2}}$
A4-A5-B12	$\frac{1}{3-3s}$	$\frac{5+3s}{9-9s}$	s	$-\frac{1+13s-30s^2}{4\sqrt{s}\sqrt{2-3s}\sqrt{5-12s-9s^2}}$
A3-A4-B2	$\frac{1}{3}$	$\frac{1}{2}$	s	$\frac{3\sqrt{s}}{2\sqrt{2}}$
A3-A5-B1	$\frac{1}{3}$	$\frac{1}{2}$	s	$-\frac{3\sqrt{s}}{2\sqrt{2}}$
A3-A4-B6	$\frac{3-2s}{10-10s}$	$\frac{1}{2}$	s	$\frac{1+5s-6s^2}{2\sqrt{s}\sqrt{21-38s+16s^2}}$
A3-A5-B5	$\frac{3-2s}{10-10s}$	$\frac{1}{2}$	s	$-\frac{1+5s-6s^2}{2\sqrt{s}\sqrt{21-38s+16s^2}}$
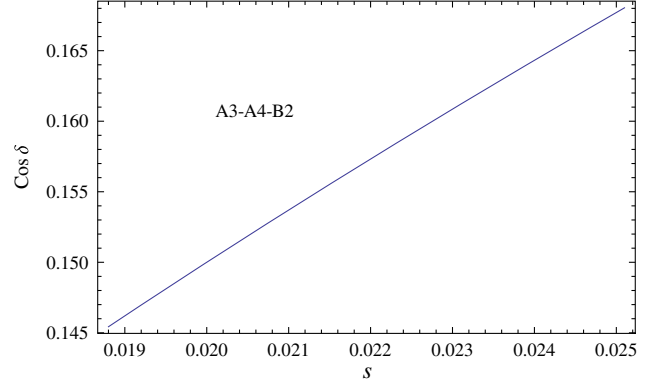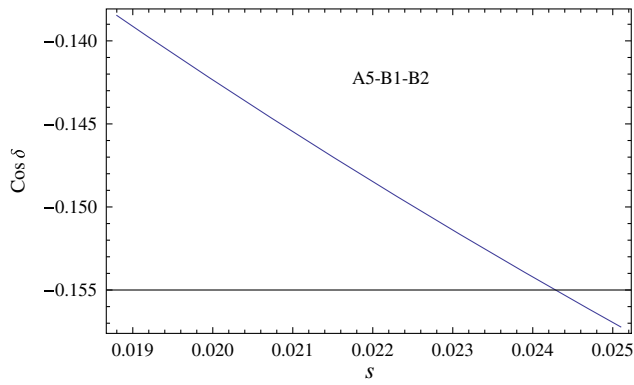
FIG. 1. $\cos\delta$ in viable range of parameter s of perturbative patterns A5-B1-B2 and A3-A4-B2.
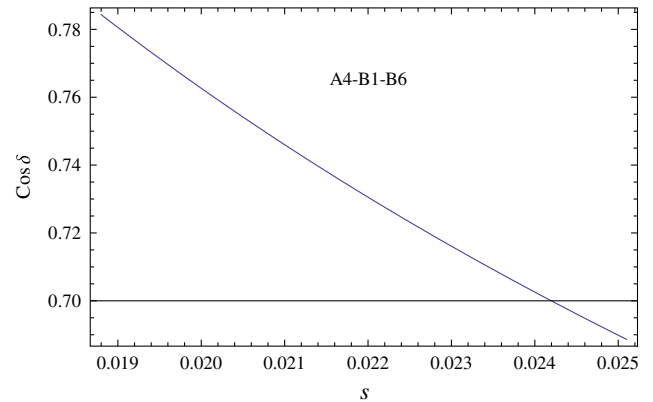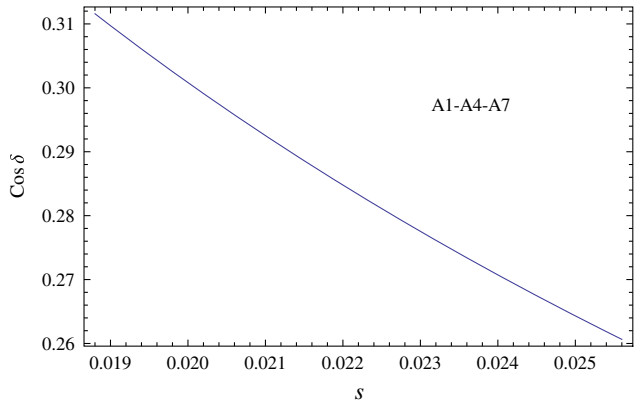
FIG. 2. $\cos\delta$ in viable range of parameter s of nonperturbative patterns A1-A4-A7 and A4-B1-B6.

of A5-B1-B2, A3-A4-B2 as an example in Fig. 1. In contrast, for two similar nonperturbative patterns, the difference of $\cos\delta$ could be notable. Take A1-A4-A7 and A4-B1-B6 for example, the difference between their magnitude of elements of mixing matrix is small, while the difference between their $\cos\delta$ is large, see Fig. 2.
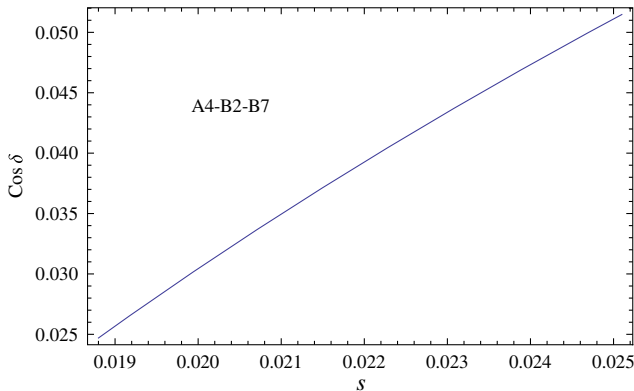
FIG. 3. $\cos\delta$ in viable range of parameter s of nonperturbative pattern A4-B2-B7.

(iii) Except the perturbative patterns A2-B1-B2 (64) and A1-A4-A5 (39) as $\mu - \tau$ symmetric combinations, $\cos\delta = 0$ cannot be obtained at 3σ level in viable patterns. In nonperturbative patterns, we could obtain $\cos\delta \simeq 0$ for A4-B2-B7 (59), see Fig. 3.

(iv) Trivial Dirac CP violation phases $\delta = 0, \pi$ cannot be obtained at 3σ level in viable patterns. In nonperturbative patterns, we could obtain $\cos\delta \simeq -1$ for B34-B5-B7 and A7-A4-B14, see Fig. 4.

(v) The viable range of parameter s or $1/N$ is not reduced obviously for most combinations. The only exceptions are the paired combinations A5-B2-B9 and A4-B1-B10, see Fig. 5.

IV. EVOLUTIONS OF MASS MATRICES OF SPECIAL MIXING PATTERNS

In the previous section, we have extracted all viable leptonic mixing patterns at 3σ level from the general combinations of two types of elementary correlations. These viable combinations could be classified into perturbative and nonperturbative patterns. The dependence of $\cos\delta$ on $\sin^2\theta_{13}$ of perturbative patterns is different to that of nonperturbative patterns. In this section, we study the
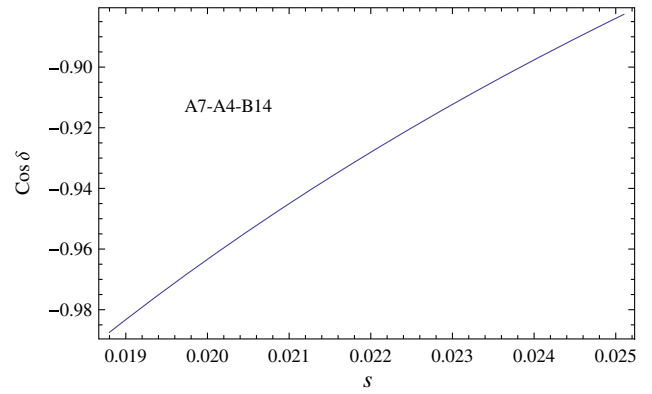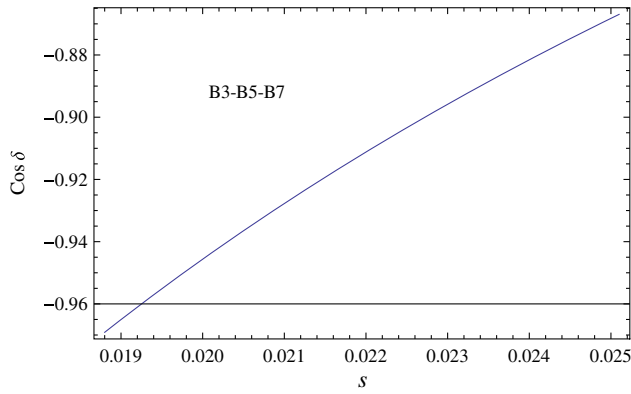
FIG. 4. $\cos \delta$ in viable range of parameter s of nonperturbative patterns B3-B5-B7 and A7-A4-B14.
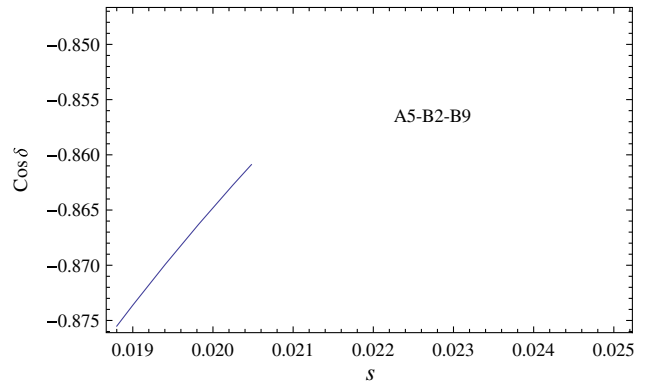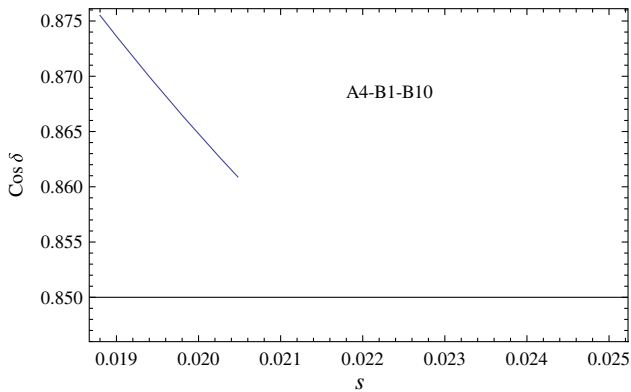
FIG. 5. $\cos \delta$ in viable range of parameter s of nonperturbative patterns A4-B1-B10 and A5-B2-B9.

dependence of neutrinos mass matrix on the small mixing parameter $\sin^2 \theta_{13}$. Our physical motivation to discuss evolutions of mass matrices of special mixing patterns with $\sin^2 \theta_{13}$ is to examine the stability of a mixing model on the basis of flavor groups and their breaking. We want to know whether a small perturbation of $\sin^2 \theta_{13}$ would bring about large modifications to the assumption in a flavor model such as alignment of vacuum expectation values of scalar fields. Our discussion is qualitative and is on the basis of special leptonic mixing patterns. But the conclusions obtained here could be extended to general patterns.

A. Evolutions of mass matrices of perturbative patterns

The general perturbative or TBM-like pattern could be expressed as:

$$|U| = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix}, \quad (99)$$

where ϵ_{ij} is a small correction or perturbation of order $\sin \theta_{13}$. And the Dirac CP phase could be written as:

$$\cos \delta = f(\epsilon_{ij}), \quad (100)$$

where f is a function dependent regularly on small ϵ_{ij}, which means f(0)=0. Take the perturbative pattern A3-A5-B1 for example, we have:

$$|U| = \begin{pmatrix} \sqrt{\frac{2-2s}{3}} & \sqrt{\frac{1-s}{3}} & \sqrt{s} \\ \sqrt{\frac{1-s}{6}} & \sqrt{\frac{1+2s}{3}} & \sqrt{\frac{1-s}{2}} \\ \sqrt{\frac{1+5s}{6}} & \sqrt{\frac{1-s}{3}} & \sqrt{\frac{1-s}{2}} \end{pmatrix}, \quad (101)$$

$$\cos \delta = -\frac{3\sqrt{s}}{2\sqrt{2}}, \quad (102)$$

where $s = \sin^2 \theta_{13}$. As for this type of patterns, one can choose TBM pattern as a zeroth-order approximation first, and then take corrections into consideration. These corrections usually come from leptonic interaction or special vacuum expectation of Higgs field or other scalar fields. A

large number of papers have constructed leptonic mixing models for TBM-like patterns following this way, see Refs. [31–37] for example.

As an illustrative case, we consider mixing models of pattern A3-A5-B1. The specific mixing matrix of A3-A5-B1 is expressed as:

$$U = \begin{pmatrix} \sqrt{\frac{2}{3}}c_{13} & \frac{c_{13}}{\sqrt{3}} & e^{-i\delta}s_{13} \\ -\frac{1}{\sqrt{6}} - \frac{e^{i\delta}s_{13}}{\sqrt{3}} & \frac{1}{\sqrt{3}} - \frac{e^{i\delta}s_{13}}{\sqrt{6}} & \frac{c_{13}}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{e^{i\delta}s_{13}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} - \frac{e^{i\delta}s_{13}}{\sqrt{6}} & \frac{c_{13}}{\sqrt{2}} \end{pmatrix}$$
$$\times \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{103}$$

In the basis where the mass matrix of charged lepton is diagonal, the mass matrix of Majorana neutrinos is written as:

$$M_\nu = U\mathrm{diag}(m_1, m_2, m_3)U^T = M_\nu^0 + \delta M_\nu. \tag{104}$$

The zeroth-order mass matrix of neutrinos M_ν^0 is of the form:

$$M_\nu^0 = U_{\mathrm{TBM}}diag(m_1, m_2, m_3)U_{\mathrm{TBM}}^T$$
$$= \begin{pmatrix} \frac{2m_1}{3} + \frac{m_2}{3} & -\frac{m_1}{3} + \frac{m_2}{3} & \frac{m_1}{3} - \frac{m_2}{3} \\ -\frac{m_1}{3} + \frac{m_2}{3} & \frac{m_1}{6} + \frac{m_2}{3} + \frac{m_3}{2} & -\frac{m_1}{6} - \frac{m_2}{3} + \frac{m_3}{2} \\ \frac{m_1}{3} - \frac{m_2}{3} & -\frac{m_1}{6} - \frac{m_2}{3} + \frac{m_3}{2} & \frac{m_1}{6} + \frac{m_2}{3} + \frac{m_3}{2} \end{pmatrix}, \tag{105}$$

where TBM mixing matrix is written as:

$$U_{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \tag{106}$$

Here the Majorana phases have been absorbed into the neutrino mass eigenvalues. And the correction to zeroth-order mass matrix could be written as:

$$(\delta M_\nu)_{11} = -\frac{s_{13}^2}{3}(2m_1 + m_2 - 3m_3 e^{-2i\delta}), \tag{107}$$

$$(\delta M_\nu)_{12} = \frac{1}{3}(1 - c_{13})(m_1 - m_2)$$
$$- \frac{\sqrt{2}}{6}s_{13}c_{13}e^{i\delta}(m_2 + 2m_1) + \frac{\sqrt{2}}{2}s_{13}c_{13}e^{-i\delta}m_3, \tag{108}$$

$$(\delta M_\nu)_{13} = \frac{1}{3}(1 - c_{13})(m_2 - m_1)$$
$$- \frac{\sqrt{2}}{6}s_{13}c_{13}e^{i\delta}(m_2 + 2m_1)$$
$$+ \frac{\sqrt{2}}{2}s_{13}c_{13}e^{-i\delta}m_3, \tag{109}$$

$$(\delta M_\nu)_{22} = \frac{s_{13}^2}{6}e^{2i\delta}(m_2 + 2m_1)$$
$$+ \frac{\sqrt{2}}{3}s_{13}e^{i\delta}(m_1 - m_2) - \frac{s_{13}^2}{2}m_3, \tag{110}$$

$$(\delta M_\nu)_{23} = \frac{s_{13}^2}{6}e^{2i\delta}(m_2 + 2m_1) - \frac{s_{13}^2}{2}m_3, \tag{111}$$

$$(\delta M_\nu)_{33} = \frac{s_{13}^2}{6}e^{2i\delta}(m_2 + 2m_1)$$
$$+ \frac{\sqrt{2}}{3}s_{13}e^{i\delta}(m_2 - m_1) - \frac{s_{13}^2}{2}m_3. \tag{112}$$

In order to construct mixing models, we could choose a special flavor group according to textures of M_ν^0 and δM_ν. For example, we choose A_4 as a flavor group. Following the presentation of A_4 in the Ref. [31,38], we could obtain the general mass matrix of neutrinos [31]:

$$M_\nu = m_0 \begin{pmatrix} a + \frac{2}{3}b_1 & c - \frac{1}{3}b_3 & d - \frac{1}{3}b_2 \\ c - \frac{1}{3}b_3 & d + \frac{1}{3}b_2 & a - \frac{2}{3}b_1 \\ d - \frac{1}{3}b_2 & a - \frac{2}{3}b_1 & c + \frac{1}{3}b_3 \end{pmatrix}, \tag{113}$$

where m_0, a, b_i, c, d come from the vacuum expectation values of scalar fields as singlet or triplet of A_4. Then employing the following equation:

$$m_0 \begin{pmatrix} a + \frac{2}{3}b_1 & c - \frac{1}{3}b_3 & d - \frac{1}{3}b_2 \\ c - \frac{1}{3}b_3 & d + \frac{1}{3}b_2 & a - \frac{2}{3}b_1 \\ d - \frac{1}{3}b_2 & a - \frac{2}{3}b_1 & c + \frac{1}{3}b_3 \end{pmatrix} = M_\nu^0 + \delta M_\nu, \tag{114}$$

we could obtain the special alignment of vacuum expectation values of scalar fields. Now one may consider that we have obtained a neutrinos mixing model that could interpret the special mixing pattern. However, we should answer the important question what the physical origin of special alignment of vacuum expectations is. At the level of zeroth-order mass matrix, the alignment of vacuum expectations could be obtained by residual symmetries such as Z_2, C_3 after spontaneous breaking of A_4, see Ref. [39] for example. However, we should answer the further question whether the assumption of such mixing model is stable under the perturbation of small mixing parameters. Let us examine the evolution of neutrinos mass matrix with s_{13}. At

present, absolute mass scales of neutrinos and ordering of neutrino masses are unknown. The effects of s_{13} on corrections to neutrinos mass matrix include following cases:

(i) Case 1 $|m_1| \sim |m_2| \sim |m_3|$. In this case, corrections of order $s_{13}^2|m_i|$ could be omitted, the correction matrix could be expressed as:

$$\delta M_\nu \simeq \begin{pmatrix} 0 & \epsilon_2 & \epsilon_2 \\ \epsilon_2 & \epsilon_1 & 0 \\ \epsilon_2 & 0 & -\epsilon_1 \end{pmatrix}, \qquad (115)$$

where ϵ_i is of the form:

$$\epsilon_1 = \frac{\sqrt{2}}{3} s_{13} e^{i\delta}(m_1 - m_2),$$

$$\epsilon_2 = \frac{\sqrt{2}}{6} s_{13} c_{13} e^{i\delta}(3e^{-2i\delta}m_3 - m_2 - 2m_1). \qquad (116)$$

(ii) Case 2 $|m_1| < |m_2| < |m_3|$. In this case, corrections of order $s_{13}^2|m_3|$ cannot be omitted. The correction matrix could be expressed as:

$$\delta M_\nu \simeq \begin{pmatrix} \epsilon_1' & \epsilon_2' & \epsilon_2' \\ \epsilon_2' & \epsilon_3' & \epsilon_5' \\ \epsilon_2' & \epsilon_5' & \epsilon_4' \end{pmatrix}, \qquad (117)$$

where ϵ_i' is of the form:

$$\epsilon_1' = m_3 s_{13}^2 e^{-2i\delta}, \qquad \epsilon_5' = -\frac{s_{13}^2}{2}m_3,$$

$$\epsilon_3' = \frac{\sqrt{2}}{3} s_{13} e^{i\delta}(m_1 - m_2) - \frac{s_{13}^2}{2}m_3,$$

$$\epsilon_4' = \frac{\sqrt{2}}{3} s_{13} e^{i\delta}(m_2 - m_1) - \frac{s_{13}^2}{2}m_3, \qquad (118)$$

and the expression of ϵ_2' is the same as ϵ_2 in Case 1.

(iii) Case 3 $|m_3| < |m_1| < |m_2|$. In this case, corrections of order $s_{13}^2|m_i|$ could also be omitted, the correction matrix is the same as that in Case 1.

From corrections of elements of the mass matrix, we can see that small s_{13} corresponds to small modification of the structure of the neutrinos mass matrix in the Case 1 and Case 3. However, small s_{13} would change the neutrinos mass matrix notably in the Case 2. Especially, for example, if $|m_3| \sim 10|m_i|(i = 1, 2)$, the correction ϵ_2' brought by small s_{13} is large compared with the zeroth-order element. Therefore, only when neutrinos mass scales of $|m_i|(i = 1, 2, 3)$ approximate or when neutrinos masses are in inverted ordering, the corrections are small and could be treated as perturbation. So the structure of the neutrinos mass matrix of perturbative pattern is dependent on the

mass scales of neutrinos. A small variation of s_{13} may correspond to large modifications to alignment of vacuum expectation values in mixing models of Case 2. Therefore, in general, the robustness of a mixing model on the basis of flavor group is dependent on neutrinos mass scales. Similar observations have been obtained in Refs. [40,41] with different perturbative methods.

B. Evolutions of mass matrices of nonperturbative patterns

Different to the perturbative patterns, there is no unified formula to depict the nonperturbative mixing patterns. Because the dependence of $\cos\delta$ on $\sin^2\theta_{13}$ is not continuous when $\theta_{13} \to 0$, we cannot obtain a nonperturbative pattern from perturbing TBM pattern. However, as a theoretical program, we could still choose a zeroth-order mixing pattern whose $\theta_{13} \neq 0$, and then take corrections into consideration, see Ref. [42] for example. On the other hand, we could survey various flavor groups and choose special residual symmetries to construct viable mixing models without extra corrections. Following this program, orders of viable discrete flavor groups are very large. And viable mixing patterns on the basis of large finite groups are usually TM$_2$, see Refs. [43–45].

For both of these programs, in order to study the stability of the flavor model of a special mixing pattern, variation of lepton mass matrix should be examined when the perturbation of small mixing parameter is considered. As an illustrative example, we discuss the evolution of neutrinos mass matrix of pattern A5-B9-B12. The mixing matrix of A5-B9-B12 is written as:

$$\begin{pmatrix} \frac{\sqrt{11}}{4}c_{13} & \frac{\sqrt{5}}{4}c_{13} & e^{-i\delta}s_{13} \\ -\sqrt{\frac{5}{32}} - \sqrt{\frac{11}{32}}e^{i\delta}s_{13} & \sqrt{\frac{11}{32}} - \sqrt{\frac{5}{32}}e^{i\delta}s_{13} & \frac{c_{13}}{\sqrt{2}} \\ \sqrt{\frac{5}{32}} - \sqrt{\frac{11}{32}}e^{i\delta}s_{13} & -\sqrt{\frac{11}{32}} - \sqrt{\frac{5}{32}}e^{i\delta}s_{13} & \frac{c_{13}}{\sqrt{2}} \end{pmatrix},$$

$$(119)$$

where $\cos\delta$ is listed in Table II. In the basis where the mass matrix of charged lepton is diagonal, elements of mass matrix of Majorana neutrinos are written as:

$$(M_\nu)_{11} = \frac{11}{16}c_{13}^2 m_1 + \frac{5}{16}c_{13}^2 m_2 + s_{13}^2 e^{-2i\delta}m_3, \qquad (120)$$

$$(M_\nu)_{12} = -\frac{\sqrt{2}}{32}(\sqrt{55}c_{13} + 11s_{13}c_{13}e^{i\delta})m_1$$
$$+ \frac{\sqrt{2}}{32}(\sqrt{55}c_{13} - 11s_{13}c_{13}e^{i\delta})m_2$$
$$+ \frac{\sqrt{2}}{2}s_{13}c_{13}e^{-i\delta}m_3, \qquad (121)$$

$$(M_\nu)_{13} = \frac{\sqrt{2}}{32}(\sqrt{55}c_{13} - 11s_{13}c_{13}e^{i\delta})m_1$$
$$- \frac{\sqrt{2}}{32}(\sqrt{55}c_{13} + 11s_{13}c_{13}e^{i\delta})m_2$$
$$+ \frac{\sqrt{2}}{2}s_{13}c_{13}e^{-i\delta}m_3, \tag{122}$$

$$(M_\nu)_{22} = \frac{1}{32}(\sqrt{5} + \sqrt{11}s_{13}e^{i\delta})^2 m_1$$
$$+ \frac{1}{32}(\sqrt{5} - \sqrt{11}s_{13}e^{i\delta})^2 m_2 + \frac{1}{2}c_{13}^2 m_3, \tag{123}$$

$$(M_\nu)_{23} = \frac{1}{32}(-5 + 11s_{13}^2 e^{2i\delta})m_1$$
$$+ \frac{1}{32}(-11 + 5s_{13}^2 e^{2i\delta})m_2 + \frac{1}{2}c_{13}^2 m_3, \tag{124}$$

$$(M_\nu)_{33} = \frac{1}{32}(\sqrt{5} - \sqrt{11}s_{13}e^{i\delta})^2 m_1$$
$$+ \frac{1}{32}(\sqrt{5} + \sqrt{11}s_{13}e^{i\delta})^2 m_2 + \frac{1}{2}c_{13}^2 m_3. \tag{125}$$

As the Ref [41], we introduce following parameters to describe the structure of mass matrix, i.e.,

$$\varepsilon_a = \frac{(M_\nu)_{12} - (M_\nu)_{13}}{(M_\nu)_{12} + (M_\nu)_{13}}$$
$$= \frac{\sqrt{110}(m_2 - m_1)}{-11\sqrt{2}s_{13}e^{i\delta}(m_1 + m_2) + 16\sqrt{2}s_{13}e^{-i\delta}m_3}, \tag{126}$$

$$\varepsilon_b = \frac{(M_\nu)_{22} - (M_\nu)_{33}}{(M_\nu)_{22} + (M_\nu)_{33}}$$
$$= \frac{2\sqrt{55}s_{13}e^{i\delta}(m_1 - m_2)}{(5 + 11s_{13}^2 e^{2i\delta})(m_1 + m_2) + 16c_{13}^2 m_3}. \tag{127}$$

Through the derivative

$$\frac{d\varepsilon_a}{ds_{13}} \simeq \frac{C}{s_{13}^2} = \frac{-\sqrt{110}(m_2 - m_1)}{[-11\sqrt{2}e^{i\delta}(m_1 + m_2) + 16\sqrt{2}e^{-i\delta}m_3]s_{13}^2}, \tag{128}$$

$$\frac{d\varepsilon_b}{ds_{13}} \simeq \frac{2\sqrt{55}e^{i\delta}(m_1 - m_2)}{(5 + 11s_{13}^2 e^{2i\delta})(m_1 + m_2) + 16c_{13}^2 m_3}, \tag{129}$$

we can see that the structure parameter ε_a depends on small parameter s_{13}^2 singularly when $s_{13}^2 \to 0$. Thus, except the case where the coefficient $|C| \ll 1$, a small perturbation of mixing parameter s_{13} would bring about a notable variation of structure parameter ε_a. So the structure of mass matrix is stable only in special mass ordering of neutrinos such as normal hierarchy, i.e., $|m_1| \ll |m_2| \ll |m_3|$ or quasidegeneracy, i.e., $|m_1| \simeq |m_2| < |m_3|$. Therefore, in general, the robustness of the structure of neutrinos mass matrix of a mixing model of nonperturbative pattern is also dependent on the mass scales of neutrinos. Accordingly, a small variation of s_{13} may correspond to large modifications to alignment of vacuum expectation values in a mixing model on the basis of flavor groups in the special mass ordering of neutrinos.

V. SUMMARY

We have obtained all viable patterns of leptonic mixing matrix on the basis of combinations of elementary correlations of elements of mixing matrix. The elementary correlations include type-A: $|U_{\alpha i}| = |U_{\beta j}|$; type-B: $|U_{\alpha i}| = 2|U_{\beta j}|$. There are 9 viable type-A correlation and 14 viable type-B correlation at 3σ level of mixing parameters. With the help of $\mu - \tau$ interchange of mixing matrix, we examined all viable combinations of elementary correlations. We obtained 83 viable combinations of two elementary correlations and 62 viable combinations of three elementary correlations. A combination of three elementary correlations could determine a neutrinos mixing pattern. All these viable patterns could be classified into two groups: perturbative patterns and nonperturbative patterns. The former can be obtained by perturbing TBM. The latter cannot be obtained from any $\theta_{13} = 0$ pattern. The Dirac CP violating phases of perturbative patterns are compared with those of nonperturbative patterns through function $\cos\delta(s)$ and figures.

In order to study the robustness of leptonic mixing models on the basis of flavor groups, evolutions of mass matrix of neutrinos have been discussed via special perturbative and nonperturbative mixing patterns. We found that the mass matrix of neutrinos is stable under the small variation of mixing parameter $\sin\theta_{13}$ only in special mass ordering of neutrinos. In general cases, a small variation of $\sin\theta_{13}$ may correspond to large modifications to alignment of vacuum expectation values in a mixing model on the basis of flavor groups. Therefore, small but nonzero $\sin\theta_{13}$ brings more stringent constraint on leptonic mixing models on the basis of flavor groups than usual views. A successful mixing model should not only predict viable mixing parameters but also be stable under the small variation of mixing parameter.

Finally, we note that a recent paper [46] has discussed viable patterns on the basis of type-A correlation without discussing evolutions of neutrinos mass matrices. Our work incudes both type-A and type-B correlation. We find that, in 62 viable patterns, only 3 are just from combinations of correlations of type-A. So general methods and results of our work are different from theirs.

ACKNOWLEDGMENTS

[1] K. Abe *et al.* (T2K Collaboration), Indication of Electron Neutrino Appearance from an Accelerator-Produced Off-Axis Muon Neutrino Beam, Phys. Rev. Lett. **107**, 041801 (2011).

[2] P. Adamson *et al.* (MINOS Collaboration), Improved Search for Muon-Neutrino to Electron-Neutrino Oscillations in MINOS, Phys. Rev. Lett. **107**, 181802 (2011).

[3] F. An *et al.* (DAYA-BAY Collaboration), Observation of Electron-Antineutrino Disappearance at Daya Bay, Phys. Rev. Lett. **108**, 171803 (2012).

[4] J. K. Ahn *et al.* (RENO Collaboration), Observation of Reactor Electron Antineutrino Disappearance in the RENO Experiment, Phys. Rev. Lett. **108**, 191802 (2012).

[5] Y. Abe *et al.* (DOUBLE-CHOOZ Collaboration), Indication for the Disappearance of Reactor Electron Antineutrinos in the Double CHOOZ Experiment, Phys. Rev. Lett. **108**, 131801 (2012).

[6] R. M. Fonseca and W. Grimus, Classification of leptonic mixing matrices from finite residual symmetries, J. High Energy Phys. 09 (2014) 033.

[7] P. F. Harrison, D. H. Perkins, and W. G. Scott, Tri-bimaximal mixing and the neutrino oscillation data, Phys. Lett. B **530**, 167 (2002).

[8] P. F. Harrison and W. G. Scott, Symmetries and generalizations of tri-bimaximal neutrino mixing, Phys. Lett. B **535**, 163 (2002).

[9] Y. Kajiyama, M. Raidal, and A. Strumia, The golden ratio prediction for the solar neutrino mixing, Phys. Rev. D **76**, 117301 (2007).

[10] W. Rodejohann, Unified parametrization for quark and leptonic mixing angles, Phys. Lett. B **671**, 267 (2009).

[11] R. de Adelhart Tooorop, F. Feruglio, and C. Hagedorn, Discrete flavour symmetries in light of T2K, Phys. Lett. B **703**, 447 (2011).

[12] G. J. Ding, TFH mixing patterns, large θ_{13} and $\Delta(96)$ flavor symmetry, Nucl. Phys. **B862**, 1 (2012).

[13] C. H. Albright, A. Dueck, and W. Rodejohann, Possible alternatives to tri-bimaximal mixing, Eur. Phys. J. C **70**, 1099 (2010).

[14] H. Fritzsch and Z. Z. Xing, Lepton mass hierarchy and neutrino oscillations, Phys. Lett. B **372**, 265 (1996).

[15] V. D. Barger, S. Pakvasa, T. J. Weiler, and K. Whisnant, Bimaximal mixing of three neutrinos, Phys. Lett. B **437**, 107 (1998).

[16] I. Girardi, S. T. Petcov, and A. V. Titov, Determining the Dirac *CP* violation phase in the neutrino mixing matrix from sum rules, Nucl. Phys. **B894**, 733 (2015).

[17] J. Iizuka, T. Kitabayashi, Y. Minagawa, and M. Yasue, Parameterization of Pontecorvo-Maki-Nakagawa-Sakata mixing matrix based on *CP*-violating bipair neutrino mixing, Mod. Phys. Lett. A **30**, 1550019 (2015).

[18] Z. Z. Xing and S. Zhou, A partial $\mu - \tau$ symmetry and its prediction for leptonic *CP* violation, Phys. Lett. B **737**, 196 (2014).

[19] N. Haba, A. Watanabe, and K. Yoshioka, Twisted Flavors and Tri/bi-Maximal Neutrino Mixing, Phys. Rev. Lett. **97**, 041601 (2006).

[20] X.-G. He and A. Zee, Minimal modification to the tri-bimaximal neutrino mixing, Phys. Lett. B **645**, 427 (2007).

[21] C. S. Lam, Mass independent textures and symmetry, Phys. Rev. D **74**, 113004 (2006).

[22] W. Grimus and L. Lavoura, A model for trimaximal leptonic mixing, J. High Energy Phys. 09 (2008) 106.

[23] T. Fukuyama and H. Nishiura, Mass Matrix of Majorana Neutrinos, arXiv:hep-ph/9702253.

[24] R. N. Mohapatra and S. Nussinov, Bimaximal neutrino mixing and neutrino mass matrix, Phys. Rev. D **60**, 013002 (1999).

[25] C. S. Lam, A 2-3 symmetry in neutrino oscillations, Phys. Lett. B **507**, 214 (2001).

[26] P. F. Harrison and W. G. Scott, mu—tau reflection symmetry in leptonic mixing and neutrino oscillations, Phys. Lett. B **547**, 219 (2002).

[27] W. Grimus and L. Lavoura, A non-standard *CP* transformation leading to maximal atmospheric neutrino mixing, Phys. Lett. B **579**, 113 (2004).

[28] J. Beringer *et al.* (Particle Data Group Collaboration), Review of particle physics, Phys. Rev. D **86**, 010001 (2012).

[29] M. C. Gonzalez-Garcia, M. Maltoni, and T. Schwetz, Updated fit to three neutrino mixing: Status of leptonic *CP* violation, J. High Energy Phys. 11 (2014) 052.

[30] J. Liao, D. Marfatia, and K. Whisnant, Generalized perturbations in neutrino mixing, Phys. Rev. D **92**, 073004 (2015).

[31] B. Brahmachari, S. Choubey, and M. Mitra, A_4 flavor symmetry and neutrino phenomenology, Phys. Rev. D **77**, 073008 (2008).

[32] J. Barry and W. Rodejohann, Deviations from tribimaximal mixing due to the vacuum expectation value misalignment in A4 models, Phys. Rev. D **81**, 093002 (2010).

[33] X.-G. He and A. Zee, Minimal modification to tribimaximal mixing, Phys. Rev. D **84**, 053004 (2011).

[34] S. F. King and C. Luhn, A_4 models of tri-bimaximal-reactor mixing, J. High Energy Phys. 03 (2012) 036.

[35] S. F. King, Tri-bimaximal-Cabibbo mixing, Phys. Lett. B **718**, 136 (2012).

[36] B. Hu, Trimaximal-Cabibbo neutrino mixing: A parametrization in terms of deviations from tri-bimaximal mixing, Phys. Rev. D **87**, 053011 (2013).

[37] L. J. Hall and G. G. Ross, Discrete symmetries and neutrino mass perturbations for θ_{13}, J. High Energy Phys. 11 (2013) 091.

[38] G. Altarelli and F. Feruglio, Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions, Nucl. Phys. **B720**, 64 (2005).

[39] X.-G. He, Y.-Y. Keum, and R. R. Volkas, A_4 flavour symmetry breaking scheme for understanding quark and neutrino mixing angles, J. High Energy Phys. 04 (2006) 039.

[40] W. Rodejohann and X.-J. Xu, Robustness of neutrino mass matrix predictions, Nucl. Phys. **B899**, 463 (2015).

[41] S. Gupta, A. S. Joshipura, and K. M. Patel, How good is $\mu - \tau$ symmetry after results on non-zero θ_{13}?, J. High Energy Phys. 09 (2013) 035.

[42] W. Rodejohann and H. Zhang, Reducing θ_{13} to 9 degrees, Phys. Lett. B **732**, 174 (2014).

[43] R. M. Fonseca and W. Grimus, Classification of leptonic mixing matrices from finite residual symmetries, J. High Energy Phys. 09 (2014) 033.

[44] M. Holthausen, K. S. Lim, and M. Lindner, Leptonic mixing patterns from a scan of finite discrete groups, Phys. Lett. B **721**, 61 (2013).

[45] L. Lavoura and P. O. Ludl, Residual $Z_2 \times Z_2$ symmetries and leptonic mixing, Phys. Lett. B **731**, 331 (2014).

[46] D. Liu and B.-Q. Ma, Examination of pairs in neutrino mixing matrix, Phys. Rev. D **92**, 033011 (2015).