

Information content in $F(R)$ brane models with nonconstant curvatureR. A. C. Correia,^{1,*} P. H. R. S. Moraes,^{2,†} A. de Souza Dutra,^{3,‡} and Roldão da Rocha^{4,§}¹*CCNH, Universidade Federal do ABC, Santo André, São Paulo 09210-580, Brazil*²*Departamento de Física, ITA (Instituto Tecnológico de Aeronáutica), São José dos Campos, São Paulo 12228-900, Brazil*³*UNESP, Universidade Estadual Paulista, Guaratinguetá, São Paulo 12516-410, Brazil*⁴*CMCC, Universidade Federal do ABC, Santo André, São Paulo 09210-580, Brazil*

(Received 18 August 2015; published 11 December 2015)

In this work we investigate the entropic information measure in the context of braneworlds with nonconstant curvature. The braneworld entropic information is studied for gravity modified by the square of the Ricci scalar, besides the usual Einstein-Hilbert term. We showed that the minimum value of the brane configurational entropy provides a stricter bound on the parameter that is responsible for the $F(R)$ model differing from the Einstein-Hilbert standard one. Our results are moreover consistent to a negative bulk cosmological constant.

DOI: [10.1103/PhysRevD.92.126005](https://doi.org/10.1103/PhysRevD.92.126005)

PACS numbers: 11.25.-w, 11.27.+d, 89.70.Cf

I. INTRODUCTION

The standard model of cosmology, named the Λ cold dark matter model, derived from Einstein's general relativity, although yielding great accordance between theory and observational data [1], has some shortcomings that question its validity as the truthful model for the origin, structure, and evolution of the Universe. Among those shortcomings, one could quote the cosmological constant (CC), the coincidence and dark matter problems, missing satellites, the hierarchy problem, etc. (see [2] and references therein).

The CC problem is the most critical among those issues, since it lacks a convincing explanation for the physical meaning of dark energy, which composes $\sim 70\%$ of the Universe and, in principle, is responsible for the cosmic acceleration predicted by type Ia supernovae observational data [3,4].

In order to evade some of those shortcomings, it is common to consider generalized theories of gravity, such as the $F(R)$ theories (check [5,6] for instance), as the starting point for alternative cosmological models. Such a formalism successfully describes both the inflationary era [7,8] and the current phase of accelerated expansion our Universe is undergoing [9,10], the latter with no need of a CC.

On the other hand, the hierarchy problem, for instance, may be solved from the approach of braneworld models [11,12]. This occurs since in such a universe setup gravity is allowed to propagate through the bulk (a five-dimensional anti-de Sitter space-time) differently from

the other fundamental forces of nature. This explains the "weakness" of gravity in the observable Universe.

Note that important outcomes are also raised from the approach of generalized $F(R)$ gravity in braneworld models. In [13], for instance, the authors obtained exact solutions for the scalar field, warp factor, and energy density in a scenario with nonconstant curvature. Analytical solutions for the equations of motion in the case of constant curvature were presented in [14]. The modified Einstein equations were solved for a flat brane in [15]. Furthermore, cosmological solutions for a fourth-order $F(R)$ brane gravity are presented in [16]. For other works on $F(R)$ branes, check [17–21].

Despite the amount of applications to which $F(R)$ brane models have been applied recently, no efforts have been accomplished yet in the framework of the so-called configurational entropy (CE) in these scenarios. Gleiser and Stamatopoulos (GS) first proposed in [22] such a new physical quantity, which brings additional information about some parameters of a given model for which the energy density is localized. It has been shown that the higher the energy that approximates the actual solution, the higher its relative CE, which is defined as the absolute difference between the actual function CE and the trial function CE. As pointed out in [22], the CE can resolve situations where the energies of the configurations are degenerate. In this case, the CE can be used to select the best configuration.

The approach presented in [22] has been used to study the nonequilibrium dynamics of spontaneous symmetry breaking [23], obtain the stability bound for compact objects [24], investigate the emergence of localized objects during inflationary preheating [25], and, moreover, distinguish configurations with energy-degenerate spatial profiles [26]. Furthermore, in a recent work [27], solitons, Lorentz symmetry breaking, supersymmetry, and entropy were employed using the CE concept. In such a work, the

*rafael.couceiro@ufabc.edu.br

†moraes.phrs@gmail.com

‡dutra@feg.unesp.br

§roldao.rocha@ufabc.edu.br

CE for travelling solitons reveals that the best value of the parameter responsible for breaking the Lorentz symmetry is 1 where the energy density is distributed equally around the origin. In this way, it was argued that the information-theoretical measure of travelling solitons in Lorentz symmetry violation scenarios can be very important to probe situations where the parameters responsible for breaking the symmetries are arbitrary. In this case, the CE was shown to select the best value of the parameter in the model. Another interesting work about CE was presented in Ref. [28], where the CE is responsible for identifying the critical point in the context of continuous phase transitions. Finally, in braneworld scenarios [29] it was shown that CE can be employed to demonstrate a high organizational degree in the structure of the system configuration for large values of a parameter of the sine-Gordon model.

In this work we are interested in answering the following issues. Can the CE be calculated in $F(R)$ brane scenarios? If it does, how is its profile? Furthermore, what might the information content in $F(R)$ brane models with nonconstant curvature reveal?

We show that the CE provides a stricter bound on the parameter that is responsible for the $F(R)$ model differing from the standard gravity one.

This paper is organized as follows. In the next section, we present a brief review of $F(R)$ brane models. In particular, we review the results presented by Bazeia and collaborators [13]. In Sec. III, we present an overview regarding CE measure, and we calculate the entropic information for the $F(R)$ brane models. In Sec. IV, we show a comparison between the results of the information-entropic measure of $F(R)$ brane models and what is obtained via cosmology. In Sec. V, we present our conclusions and final remarks.

II. A BRIEF REVIEW OF $F(R)$ BRANE MODELS

In this section a brief overview regarding $F(R)$ brane-world models is presented. Let us start by writing the action of five-dimensional gravity coupled to a real scalar field ϕ as

$$S = \int d^5x \sqrt{|g|} \left[-\frac{1}{4}F(R) + \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi) \right]. \quad (1)$$

Here, $4\pi G_{(5)} = 1$ and $g = \det(g_{ab})$, with field, space, and time variables being dimensionless. $F(R)$ stands for a generic function of the Ricci scalar R . Furthermore, the signature of the metric is adopted as $(+ - - -)$. It should be stressed that $V(\phi)$ is the potential that describes the theory.

We study the case where the metric is represented by

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \quad (2)$$

where y denotes the extra dimension, $\eta_{\mu\nu}$ is the usual Minkowski metric, and e^{2A} stands for the so-called warp factor, which depends only on the extra dimension. Moreover, let us assume that the field ϕ also depends solely upon y . Hence, from the action (1) the corresponding equation of motion for the scalar field reads

$$\phi'' + 4A'\phi' = V_\phi, \quad (3)$$

wherein the primes stand for derivatives with respect to the extra dimension and $V_\phi = dV/d\phi$.

Here, the energy density ρ is given by

$$\rho = -e^{2A}\mathcal{L}, \quad (4)$$

with

$$\mathcal{L} = \frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi). \quad (5)$$

Now, after straightforward manipulations, the modified Einstein equations acquire the form

$$-\frac{2}{3}\phi'^2 = -\frac{1}{3}A''F'_R + \frac{1}{3}F''_R + A''F_R, \quad (6)$$

$$-\frac{1}{2}\phi'^2 + V(\phi) = 2(A'' + A'^2)F_R - 2A'F'_R - \frac{1}{4}F, \quad (7)$$

with $F_R := dF/dR$. The Ricci scalar is assumed to be an arbitrary function of the extra dimension, i.e., $R = R(y)$, yielding from Eq. (6) that

$$\phi'^2 = -\frac{3}{2}A''F_R + \frac{1}{2}(A'R' - R'')F_{RR} - \frac{1}{2}F_{RRR}R'^2. \quad (8)$$

It is worth mentioning that $\lim_{y \rightarrow 0} R(y) = 20B^2k^2$, where $\lim_{y \rightarrow \infty} R(y) = 8Bk^2$ [14]. The potential can be obtained also from (7):

$$V(\phi) = -\frac{1}{4}F + \frac{1}{4}F_R(8A'^2 + 5A'') - \frac{1}{4}F_{RR}(A'' + 7A'R') - \frac{1}{4}F_{RRR}R'^2. \quad (9)$$

Hence, substituting the equation $R = 8A'' + 20A'^2$ for the Ricci scalar into Eqs. (8)–(9) yields

$$\begin{aligned} \phi'^2 = & -\frac{3}{2}F_RA'' + 4F_{RR}(5A'^2A'' - 5A''^2 - 4A'A''' - A''''') \\ & - 32F_{RRR}(5A'A'' + A''')^2, \end{aligned} \quad (10)$$

$$\begin{aligned}
 V(\phi) = & -\frac{1}{4}F + \frac{1}{4}F_R(5A'' + 8A'^2) \\
 & -F_{RR}(70A'^2A'' + 10A''^2 + 24A'''A' + A''''') \\
 & -16F_{RRR}(5A'A'' + A''''')^2. \quad (11)
 \end{aligned}$$

In order to explicitly find solutions for the above equations, we can apply the following function to them [14,21],

$$F(R) = R + \alpha R^2, \quad (12)$$

where $\alpha \in \mathbb{R}$. Hence Eqs. (10)–(11) can be recast as

$$\begin{aligned}
 \phi'^2 = & -\frac{3}{2}A'' - 4\alpha(16A''^2 + 5A'^2A'' + 2A'''' + 8A'A'''''), \\
 V(\phi) = & -\frac{3}{4}A'' - 3A'^2 - 2\alpha(10A'^4 + 24A'A'''' \\
 & + 8A''^2 + 2A'''' + 69A'^2A''). \quad (13)
 \end{aligned}$$

Moreover, Eq. (4) can be expressed as

$$\begin{aligned}
 \rho(y) = e^{2A} \left[\frac{3}{2}A'' + 3A'^2 + 4\alpha(5A'^4 + 16A'A'''' \right. \\
 \left. 37A'^2A'' + 12A''^2 + 2A''''') \right], \quad (14)
 \end{aligned}$$

which can be rewritten as

$$\begin{aligned}
 \rho(y) = -\frac{d}{dy} \left\{ e^{2A} \left[\frac{3}{2}A' + 4\alpha \left(\frac{13}{3}A'^3 + 2A'''' \right. \right. \right. \\
 \left. \left. \left. + 12A'A''') \right] \right\} + \frac{44\alpha}{3}A'^4e^{2A}. \quad (15)
 \end{aligned}$$

Therefore, once the first term in the above equation does not contribute to the energy, we can immediately write the energy of the configurations as

$$E = \frac{44\alpha}{3} \int dy A'^4 e^{2A}. \quad (16)$$

Now, in order to work with analytical solutions, the warp function is adopted to be [30]

$$A(y) = B \ln [\operatorname{sech}(ky)], \quad (17)$$

where $B > 0$ and $k > 0$. Hence, the energy density reads [13]

$$\rho(y) = g_1 \operatorname{sh}^{2B}(ky) + g_2 \operatorname{sh}^{2B+2}(ky) + g_3 \operatorname{sh}^{2B+4}(ky), \quad (18)$$

with $\operatorname{sh} \equiv \operatorname{sech}$ and

$$\begin{aligned}
 g_1 = & -3B^2k^2(20\alpha B^2k^2 + 1), \\
 g_2 = & 3Bk^2 \left[4\alpha k^2(10B^3 + 37B^2 + 32B + 8) + B + \frac{1}{2} \right], \\
 g_3 = & -12\alpha k^4 B(5B^3 + 37B^2 + 44B + 12). \quad (19)
 \end{aligned}$$

The parameter α is bounded by

$$32k^2(1 + 4B) \leq 3\alpha^{-1} \leq -8k^2(8 + 16B + 5B^2). \quad (20)$$

Now, substituting the warp function (17) into (16), we have the energy E of the brane:

$$E = 11\sqrt{\pi}\alpha k^3 B^4 \Gamma(B) / \Gamma(B + 5/2). \quad (21)$$

Thus, in the next section, we use the approach presented here to obtain the CE in this context. As we will see, the information-entropic measure shows a higher organizational degree in the structure of the system configuration, and consequently we are able to obtain additional information content regarding the system.

III. INFORMATION CONTENT IN $F(R)$ BRANE MODELS

As argued in the introduction, GS have recently proposed a detailed picture of the so-called configurational entropy for the structure of localized solutions in classical field theories [22]. Analogously to that development, we present a CE measure in functional space, from the field configurations where braneworld models can be studied. The framework is revisited here and subsequently applied.

There is an intimate link between information and dynamics, where the entropic measure plays a prominent role. The entropic measure is well known to quantify the informational content of physical solutions to the equations of motion and their approximations, namely, the CE in functional space [22]. GS proposed that nature optimizes not solely by extremizing energy through the plethora of *a priori* available paths, but also from an informational perspective.

To start, let us write the following Fourier transform:

$$\mathcal{F}[\omega] = \frac{1}{\sqrt{2\pi}} \int dy \rho(y) e^{i\omega y}. \quad (22)$$

Now the modal fraction measures the relative weight of each mode ω and is defined by expression [22,24,26,27]:

$$f(\omega) = \frac{|\mathcal{F}[\omega]|^2}{\int d\omega |\mathcal{F}[\omega]|^2}. \quad (23)$$

The CE is inspired by Shannon's information framework, being defined by $\mathfrak{S}_C[f] = -\sum f_n \ln(f_n)$. It represents an absolute limit on the best lossless compression of communication [31]. Hence, the CE at first provided the

informational lining regarding configurations which are compatible to constraints of arbitrary physical systems. When N modes labeled by k carry the same weight, it follows that $f_n = 1/N$ and the discrete CE presents a maximum value at $\mathfrak{S}_C = \ln N$, accordingly. Alternatively, if the system is embodied by merely one mode, consequently $\mathfrak{S}_C = 0$ [22].

Similarly, for arbitrary nonperiodic functions in an open interval, the continuous CE reads

$$\mathfrak{S}_C[f] = - \int d\omega \dot{f}(\omega) \ln[\dot{f}(\omega)], \quad (24)$$

where $\dot{f}(\omega) := f(\omega)/f_{\max}(\omega)$ is defined as the normalized modal fraction, whereas the term $f_{\max}(\omega)$ stands for the maximum fraction. Hence, Eq. (22) engenders the modal fraction to achieve the entropic profile of thick brane solutions. It is worth remarking that Eq. (22) differs from that provided by GS. In this framework we include the warp factor in $\mathcal{F}[\omega]$. Hence, the framework brings further information concerning a warped geometric scenario.

Here, as an interesting example, we calculate the entropic information for the $F(R)$ model. First, the modal fraction can be computed. Thus, substituting Eq. (14) into Eq. (22), we obtain

$$\mathcal{F}[\omega] = \sum_{m=1}^2 \sum_{j=1}^3 A_{j,m} \times {}_2\mathcal{G}_1^{(j,m)}[\gamma_j, \mu_{j,m}; \mu_{j,m} + 1; -1], \quad (25)$$

where ${}_2\mathcal{G}_1[\otimes; \odot; \otimes; \oplus]$ stands for the well-known hypergeometric functions with

$$\mu_{j,m} := \frac{1}{2k} (\gamma_j k - i(-1)^{m+1} \omega).$$

Moreover, $A_{j,m}$ and γ_j are defined as

$$A_{j,m} := \frac{1}{\sqrt{2\pi}} \frac{2^{\gamma_j-1} g_j (\gamma_j k + i(-1)^{m+1} \omega)}{(\omega^2 + \gamma_j^2 k^2)}, \quad (26)$$

$$\gamma_j := 2(B + j - 1). \quad (27)$$

Thus, using Eq. (25), the modal fraction (23) becomes

$$f(\omega) = \frac{\sum_{p,m=1}^2 \sum_{q,j=1}^3 A_{j,m} A_{q,p}^* ({}_2\mathcal{G}_1^{(j,m)}) ({}_2\mathcal{G}_1^{(p,q)})^*}{\sum_{p,m=1}^2 \sum_{q,j=1}^3 \int d\omega A_{j,m} A_{q,p}^* ({}_2\mathcal{G}_1^{(j,m)}) ({}_2\mathcal{G}_1^{(p,q)})^*}. \quad (28)$$

In Fig. 1 the modal fraction is depicted for different values of α . The maximum of the distributions is localized around the mode $\omega = 0$. By taking into account the modal fraction in (28) and its maximum contribution, Eq. (24) can now be solved in order to obtain the brane configurational entropy (BCE). In this case, due to the high complexity of

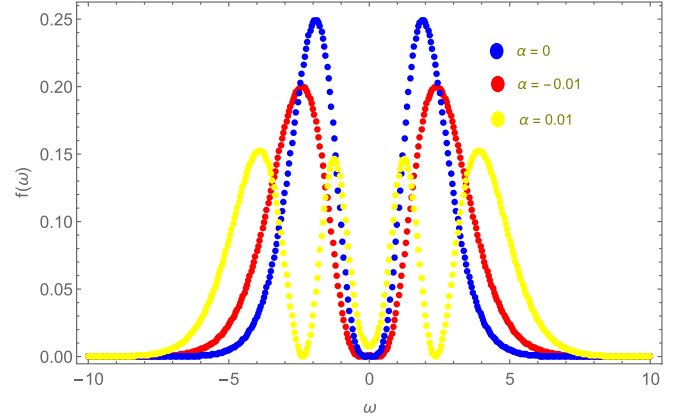


FIG. 1 (color online). Modal fractions for $\alpha = 0$ (blue), $\alpha = -0.01$ (red) and $\alpha = 0.01$ (yellow).

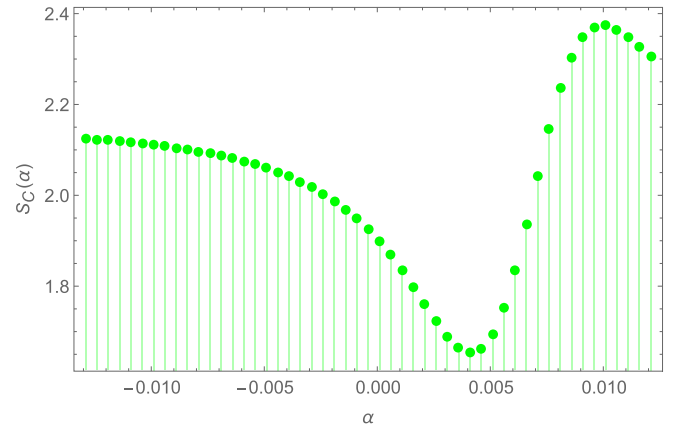


FIG. 2 (color online). Configurational entropy as a function of the parameter α .

integration, Eq. (24) must be integrated numerically. The results are shown in Fig. 2, where the BCE is plotted as a function of α . By using a recent approach presented by GS [22], the BCE is correlated to the energy of the system, in the sense that the lower (higher) the BCE, the lower (higher) the energy of the solutions. Moreover, BCE further provides independent criteria to control the stability of configurations based upon the informational content of their profiles [24]. In fact, the BCE maximum represents the boundary between stability and instability, as the case analysed in [24] for Q -balls.

In the last section we provide the consequences of the model studied above and point to forthcoming perspectives.

IV. COMPARISON WITH COSMOLOGY

Cosmological models have been constantly derived from higher order derivative gravity theories.

For instance, in [21] the authors have used the same functional form used in the present paper for $F(R)$,

i.e., $F(R) = R + \alpha R^2$, to derive analytical solutions for both the warp factor and scalar field as functions of y . As solutions for some cosmological parameters, as the bulk CC, the authors have found $\Lambda_5 = 477/(-6728\alpha\kappa_5^2)$, with κ_5 representing the five-dimensional coupling constant. Moreover, another approach that provides similar results predicts that $\Lambda_5 = -B^2k^2(3 + 20\alpha B^2k^2)$ [13]. Note that in order to obtain a negative bulk CC, α must be positive, which is in accordance with what was developed in the previous section. Note also that the negative bulk CC is responsible for gravity “leaking” from the brane to the extra dimension but still remaining concentrated in our observable Universe. On the other hand, a positive bulk CC would accelerate such a process of leaking (check [32]).

Moreover, in [21], for the brane tension, it was found that $\lambda = 3\kappa_5^2/(784\alpha)$. Such a relation reinforces the positive sign of α , since a negative tension brane is gravitationally unstable by itself (check [19]).

Furthermore, an equation that leads to the singularities of the effective potential on the brane has been constructed [Eq. (48) of [21]]. Such an equation has solutions only when $\alpha \gtrsim 1/(40B^2)$ with $B > 2$. Note that by taking $B = 2.5$ in the latter relation, one obtains exactly the value of α derived via the study of the BCE presented above, i.e., $\alpha = 0.0046$. Indeed, the parameter space of (B, α) was analyzed in [21], with the upper limit of α lying on $\alpha = 0.005$.

In [13], the allowed region of α for distinct values of B was also depicted (check the upper panel of Fig. 2 on such a reference). The result covers $\alpha = 0.0046$ with the energy density having a maximum at $y = 0$.

V. CONCLUDING REMARKS AND OUTLOOK

The entropic information has been studied in braneworld models, with emphasis on the $F(R)$ model, which has been chosen for its very physical content and usefulness. The BCE is moreover exerted to evince a higher organizational degree in the structure of system configuration. The GS technique was employed to achieve a correlation involving the energy of the system and its BCE. Moreover, our analysis is further based upon the CE $\mathfrak{S}(\alpha)$, depicted in Fig. 2. Such configurations for $\alpha \approx 0.0046$ are most probably found by the system. In fact, in such range of α the CE $\mathfrak{S}(\alpha)$ approaches zero. Our results are consistent to the upper limit $\alpha \lesssim 0.005$ in [21], and further impose the value of α corresponding to the best ordering from the BCE point of view. Such a value for α was supported by results obtained purely via $F(R)$ brane cosmological models, as it can be realized in Sec. IV.

Once we have developed the formalism of the BCE and the entropic information as well, we can further apply a procedure similar to what has been studied in the previous sections to other thick braneworld models.

ACKNOWLEDGMENTS

R. A. C. C. thanks CAPES for financial support. P. H. R. S. M. thanks FAPESP for financial support. A. S. D. is thankful to the CNPq for financial support. R. d. R. thanks CNPq Grants No. 303027/2012-6 and No. 473326/2013-2, and FAPESP Grant No. 2015/10270-0, for partial financial support.

-
- [1] G. Hinshaw *et al.* (WMAP Collaboration), *Astrophys. J. Suppl. Ser.* **208**, 19 (2013).
- [2] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, *Phys. Rep.* **513**, 1 (2012).
- [3] A. G. Riess *et al.* (Supernova Search Team Collaboration), *Astron. J.* **116**, 1009 (1998).
- [4] S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), *Astrophys. J.* **517**, 565 (1999).
- [5] A. de Felice and S. Tsujikawa, *Living Rev. Relativity* **13**, 3 (2010).
- [6] T. P. Sotiriou and V. Faraoni, *Rev. Mod. Phys.* **82**, 451 (2010).
- [7] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani, and S. Zerbini, *Phys. Rev. D* **77**, 046009 (2008).
- [8] Q.-G. Huang, *J. Cosmol. Astropart. Phys.* **02** (2014) 035.
- [9] V. K. Oikonomou, *Gen. Relativ. Gravit.* **45**, 2467 (2013).
- [10] A. A. Starobinsky, *JETP Lett.* **86**, 157 (2007).
- [11] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 3370 (1999).
- [12] V. Sahni and Y. Shtanov, *J. Cosmol. Astropart. Phys.* **11** (2003) 014.
- [13] D. Bazeia, A. S. Lobão Jr, R. Menezes, A. Y. Petrov, and A. J. da Silva, *Phys. Lett. B* **729**, 127 (2014).
- [14] V. I. Afonso, D. Bazeia, R. Menezes, and A. Y. Petrov, *Phys. Lett. B* **658**, 71 (2007).
- [15] D. Bazeia, A. S. Lobão, Jr., R. Menezes, A. Y. Petrov, and A. J. da Silva, *Phys. Lett. B* **729**, 127 (2014).
- [16] A. Balcerzak and M. P. Dabrowski, *Phys. Rev. D* **81**, 123527 (2010).
- [17] M. Parry, S. Pichler, and D. Deeg, *J. Cosmol. Astropart. Phys.* **04** (2005) 014.
- [18] N. Deruelle, M. Sasaki, and Y. Sendouda, *Prog. Theor. Phys.* **119**, 237 (2008).
- [19] J. Hoff da Silva and M. Dias, *Phys. Rev. D* **84**, 066011 (2011).
- [20] Z. G. Xu, Y. Zhong, H. Yu, and Y. X. Liu, *Eur. Phys. J. C* **75**, 368 (2015).
- [21] Y. X. Liu, Y. Zhong, Z. H. Zhao, and H. T. Li, *J. High Energy Phys.* **06** (2011) 135.

- [22] M. Gleiser and N. Stamatopoulos, *Phys. Lett. B* **713**, 304 (2012).
- [23] M. Gleiser and N. Stamatopoulos, *Phys. Rev. D* **86**, 045004 (2012).
- [24] M. Gleiser and D. Sowinski, *Phys. Lett. B* **727**, 272 (2013).
- [25] M. Gleiser and N. Graham, *Phys. Rev. D* **89**, 083502 (2014).
- [26] R. A. C. Correa, A. de Souza Dutra, and M. Gleiser, *Phys. Lett. B* **737**, 388 (2014).
- [27] R. A. C. Correa, R. da Rocha, and A. de Souza Dutra, *Ann. Phys. (Amsterdam)* **359**, 198 (2015).
- [28] N. Stamatopoulos and M. Gleiser, *Phys. Lett. B* **747**, 125 (2015).
- [29] R. A. C. Correa and R. da Rocha, Configurational Entropy in Braneworld Models: A New Approach to Stability, arXiv:1502.02283.
- [30] M. Gremm, *Phys. Lett. B* **478**, 434 (2000).
- [31] C. E. Shannon, *Bell Syst. Tech. J.* **27**, 379 (1948); **27**, 623 (1948).
- [32] R. Maartens, *Living Rev. Relativity* **13**, 5 (2010).