Photon self-interaction on deformed spacetime

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A novel nonlocal four-photon interaction on the deformed spacetime is derived and studied in the three selected models (I, II, III). The first two models (I, II) are obtained via two distinct second-order θ-exact Seiberg-Witten maps of the noncommutative U(1) gauge field strength on Moyal space. The third one (III), inspired by the manifestly gauge invariant structures emerging in the first two, due to the model generality has been constructed with a different set of freedom parameters. The physical relevancy of all models is analyzed by evaluating the four-photon-tadpole diagram, which, when combined with the bubble graph, enables us to fully consider all contributions to the one-loop photon polarization tensor. For an arbitrary noncommutative matrix $\theta^{\mu\nu}$, the full quadratic IR divergence cancellation in the one-loop photon two-point function is obtained with particular combinations of Seiberg-Witten map/gauge-symmetry freedom parameters in models I and II. Finally, our model III enables complete elimination of all pathological terms in the photon polarization tensor at one-loop order if a special value for the θ matrix is chosen.

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I. INTRODUCTION

The application of the θ-exact Seiberg-Witten map (SW) [1] expansions [2–14] represents the current state-of-the-art in the field of the noncommutative (NC) gauge field theories [15–21] and the associated particle-physics phenomenology [22–26]. A summation over all orders in the antisymmetric tensor $\theta^{\mu\nu}$ at tree level is automatically achieved via this approach, leading to various interesting results. More importantly, the θ-exact approach allows an access to nonlocal effects within the perturbative approach, most pronouncedly the quadratic UV/IR mixing [27–30] in the two-point functions at one loop [15,21].

A particular topic related to the SW map approach to the noncommutative gauge theories is the implementation of the gauge freedoms into the (inter)action. This step is generally regarded as favorable as far as the control over various divergences in the perturbative quantum corrections is concerned. The original employment was via a θ-iterative construction [31]. Later on a θ-exact substitute was suggested [32] and generalized to a second-order expansion afterwards [33].

While after the first order, it is natural to consider the second-order SW map (either in the perturbative or in the θ-exact approach) in the perturbative quantum field theory, in the past the second order has been much less investigated [15,34]. The reason was obviously technical: The second-order SW map solution is inherently much more complicated than the first-order one and requires consequently more effort to obtain explicitly the gauge invariant action and to implement the gauge freedom. Recently the model building works based on the θ^2 order SW map of non-Abelian gauge fields have received more attention [18,19,35,36] when the order-θ correction vanishes, yet studies on quantum corrections are still absent. Going from the θ expansion to θ exact, the second-order SW map adds its unique additional difficulties. Two expansion solutions sharing the identical first order do exist [5,13]. Each solution involves its own type of 3-products ($\star_3$ and $\star_{3'}$ as in [33]), and while the leading order of these two solutions with respect to θ can be shown to be connected by gauge transformation, the full solutions are not [33].

The gauge freedom structure at second order in the θ-exact approach is considerably more complicated than the first order, and also more complicated than its θ^2-order counterpart. Besides the existence of two solutions, the field strength expansion from each solution contains distinct gauge freedom structures [33]. Analyzing the θ^2 order indicates that more gauge freedom structures will show up after performing the integration-by-part in the action. Performing the same procedure θ-exactly meets the difficulty from the noncommutativity and/or nonassociativity of the generalized star products. This issue is much less pressing at the e^2 order since the $\star_2$ product satisfies the so-called 3-cyclicity [37]

$$\int f(x)(g(x) \star h(x)) = \int h(x)(g(x) \star f(x)),$$

$$\int f(x)(g(x) \star_2 h(x)) = \int h(x)(g(x) \star_2 f(x)). \quad (1)$$

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However, there is no 4-cyclicity in general ($\star_2$ and $\star_{3'}$ products have some cyclicity left, while $\star_3$ has practically no cyclicity left.) In this paper we will show that the proper substitute to integration-by-part in the θ-exact computation on Moyal space is to Fourier-transforming the whole computation to the momentum space and achieve the explicit gauge-invariant action simultaneously with the verification of the Ward identity of the vertex.

Following the preliminary work in [33], we present in this paper, for the first time, the full SW-map based/inspired nonlocally deformed four-photon couplings on Moyal space, with all reasonable gauge freedom parameters included, and their quantum correction induced by such coupling via the one-loop four-photon-tadpole diagram. We use two distinct second-order θ-exact SW map expansions for the gauge field strength (which we call model I and II, as dubbed recently in [33]) with all the freedom/ambiguity/deformation parameters included. From these field strengths we then derive the corresponding actions and show that they can be expressed explicitly in terms of the commutative U(1) field strength $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ by working out the integration-by-part procedure required. Our final form for both SW map based actions further indicates that the θ-exact freedom parameters we suggested before [32] would not deplete all the θ-iterative possibilities in [31]. The two additional gauge freedom/ambiguity/deformation parameters are thus introduced by hand in turn.

We then determine the four-photon coupling vertices with all possible gauge freedom/ambiguity/deformation parameters included from models I and II and write down the one-loop four-photon-tadpole integrals for these models. We find that various momentum factors in the second-order SW map expansion reduce either to unity or to the common nonlocal factor $\sin^2\frac{p\theta k}{2}/(\frac{p\theta k}{2})^2$ in the tadpole, the same as found in the three-photon-bubble diagram studied previously [21].

It is long known that the massless tadpole integrals all vanish at the integration dimensions $D \geq 4$ under dimensional regularization [38]. However, this result is modified by the nonlocal factors [20,29]. In order to precisely verify this effect, we evaluate the tadpole integral using two different methods: The first method simply introduces a pair of identical numerators and denominators to turn the tadpole into the bubble integral then evaluate the tadpole using the protocol established for the bubble diagram, as we did in [21]. The second method generalizes the n-nested zero regulator [38] for the commutative tadpole diagram. We compute the four-photon-tadpole contributions to the photon two-point function as a function of unspecified number of the integration dimensions D. Then we specify gauge field theory dimension d by taking the limits $D \to d$. Next we especially discuss the $d = 4$ case. In the end, we find that both approaches reveal the same *purely* quadratic IR divergent result in the $D \to 4 - \epsilon$ limit, verifying the soundness of our computation.

The UV/IR mixing phenomenon, reflecting the inherent nonlocality of the full theory and arising from the high-momentum region of integration in the Feynman integrals, shows up as a IR divergence when the spatial extension of the NC string of size $|\theta p|$ gets reduced to a point. A related anomaly is a nonanalytical behavior in the NC parameter θ, when the limit $\theta \to 0$ is undertaken. In the NC gauge theory, however, a quadratic IR divergence coexists with the logarithmic divergence which matches the UV behavior. Our study indicates that the quadratic IR divergence is clearly connected with tadpole integrals. Therefore the gauge invariant four photon interaction we found may serve as counterterms to cancel quadratic IR divergence. For this purpose the tadpole induced quadratic IR divergence is summed together with the corresponding contribution from the photon bubble diagram [21]. We find that while it is indeed possible to do so in both models I and II, the procedure requires fixing the first-order gauge freedom parameter [32] $\kappa = 1$. Subsequently, a third possible action (III), inspired by the structures of the first two, is introduced. It involves a SW-map inspired gauge invariance deformation in a more general way.

In this action each manifestly gauge invariant four-photon coupling term starting at θ^2 order is assigned an independent freedom parameter, which is shown to be sufficient to cancel any quadratic IR divergences from the bubble diagram for the two typical θ values. (The UV divergences still require separated fine-tuning.)

This paper is structured as follows: In the first two sections we describe two θ-exact Seiberg-Witten map models up to the e^3 order and construct the four-photon self-interaction, as a model definition. Section III is devoted to the computation, presentation, and discussion of the θ-exact tadpole in D and four dimensions, while in Sec. IV we analyze a sum of the bubble and the tadpole diagram to show elimination of IR divergences for arbitrary θ-matrix. In Sec. V we introduce a generalized model of the deformed four-photon interaction and show the elimination of all divergences from the bubble plus the tadpole diagram for the special choice of the θ-matrix and freedom parameters. Sections VI and VII present the discussion and conclusion, respectively. In this article the capital letters denote noncommutative objects, while the small letters denote the commutative ones.

II. A MODEL DEFINITION OF THE THREE- AND FOUR-PHOTON SELF-INTERACTIONS

A. Definitions and construction of the actions

As usual we consider the formal $U_\star(1)$ NC gauge theory action

$$S = -\frac{1}{4e^2} \int F_{\mu\nu}(ea_\mu, \theta^{\mu\nu}; \kappa, \eta, \ldots) \star F^{\mu\nu}(ea_\mu, \theta^{\mu\nu}; \kappa, \eta, \ldots),$$

(2)

where the formal NC gauge field strength $F_{\mu\nu}(ea_\mu, \theta^{\mu\nu}, \kappa, \eta, \ldots)$ is regarded as a composite operator built-up using the commutative gauge field operator a_μ and the NC parameter $\theta^{\mu\nu}$ via the SW map procedure. The set of parameters $(\kappa, \eta, \ldots)$ represents in principle the SW map/gauge-invariance freedoms. The commutative coupling constant e is attached to the commutative gauge field operator a_μ due to the charge quantization issue [39]. As a bonus feature it also serves as the ordering parameter for the θ-exact SW map expansion, i.e.,

$$F_{\mu\nu} = ef_{\mu\nu} + F_{\mu\nu}^{e^2} + F_{\mu\nu}^{e^3} + \mathcal{O}(e^4). \tag{3}$$

To e^2 order, the gauge field strength SW map $F_{\mu\nu}^{e^2}$ expansion is fairly universal [21,32,33],

$$F_{\mu\nu}^{e^2} = e^2\theta^{ij}(\kappa f_{\mu i} \star_2 f_{\nu j} - a_i \star_2 \partial_j f_{\mu\nu}). \tag{4}$$

Note that κ deformation in the settings of this paper (4) and in a recent works [32,33], corresponds to the κ_g^{-1} for the previous κ_g-deformation of [21].[1] In this paper the κ

deformation is dubbed the κ settings, where we have the following triple-photon action:

$$S^e = -\frac{e}{2}\int \theta^{ij}f^{\mu\nu}\left(\kappa f_{\mu i} \star_2 f_{\nu j} - \frac{1}{4}f_{ij} \star_2 f_{\mu\nu}\right), \tag{5}$$

responsible to the contribution to the photon polarization tensor arising from the photon bubble diagram [21].

The (profound) structure of the θ-exact SW map of a U(1) gauge theory is summarized in [33], where two distinct gauge field SW maps were found and analyzed up to the $e^3 \sim a_\mu^4$ order. Expanding (2) up to order a_μ^4 with (3) gives the following general form for the four-photon interaction

$$S^{e^2} = -\frac{1}{4e^2}\int F_{\mu\nu}^{e^2}F^{e^2\mu\nu} + 2ef^{\mu\nu}F_{\mu\nu}^{e^3}, \tag{6}$$

where the following two distinct solutions for the e^3 order gauge field strength have been found and given explicitly in [33]. The first solution is resolved from SW differential equation [33],

$$F_{\mu\nu_{(I)}}^{e^3}(x)_{\kappa,\kappa_1,\kappa_2} = \frac{e^3}{2}\theta^{ij}\theta^{kl}\bigg[\kappa_1([f_{\mu k}f_{\nu i}f_{lj}]_{\star_{3'}} + [f_{\nu l}f_{\mu i}f_{kj}]_{\star_{3'}}) - \kappa a_i \star_2 \partial_j(f_{\mu k} \star_2 f_{\nu l})$$
$$- \kappa_2([f_{\nu l}a_i\partial_j f_{\mu k}]_{\star_{3'}} + [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}} + [a_k\partial_l(f_{\mu i}f_{\nu j})]_{\star_{3'}} - 2a_i \star_2 \partial_j(f_{\mu k} \star_2 f_{\nu l}))$$
$$+ [a_i\partial_j a_k\partial_l f_{\mu\nu}]_{\star_{3'}} + [\partial_l f_{\mu\nu}a_i\partial_j a_k]_{\star_{3'}} + [a_k a_i\partial_l\partial_j f_{\mu\nu}]_{\star_{3'}} - \frac{1}{2}([a_i\partial_k a_j\partial_l f_{\mu\nu}]_{\star_{3'}} + [\partial_l f_{\mu\nu}a_i\partial_k a_j]_{\star_{3'}})\bigg], \tag{7}$$

while the second one is obtained by inverting the known solution for the inverted SW map [5] in [33],

$$F_{\mu\nu_{(II)}}^{e^3}(x)_{\kappa,\kappa_1',\kappa_2'} = e^3\theta^{ij}\theta^{kl}\bigg[\kappa_1'(f_{\mu i} \star_2 (f_{jk} \star_2 f_{l\nu}) + f_{l\nu} \star_2 (f_{jk} \star_2 f_{\mu i}) - [f_{\mu i}f_{jk}f_{l\nu}]_{\star_3})$$
$$- \kappa_2'((a_i \star_2 \partial_j f_{\mu k}) \star_2 f_{\nu l} + (a_i \star_2 \partial_j f_{\nu l}) \star_2 f_{\mu k} - [a_i\partial_j(f_{\mu k}f_{\nu l})]_{\star_3}) - \kappa a_i \star_2 \partial_j(f_{\mu k} \star_2 f_{\nu l})$$
$$+ (a_i \star_2 \partial_j a_k) \star_2 \partial_l f_{\mu\nu} + a_i \star_2 (\partial_j a_k \star_2 \partial_l f_{\mu\nu}) + a_i \star_2 (a_k \star_2 \partial_j\partial_l f_{\mu\nu}) - [a_i\partial_j a_k\partial_l f_{\mu\nu}]_{\star_3}$$
$$- \frac{1}{2}(a_i \star_2 (\partial_k a_j \star_2 \partial_l f_{\mu\nu}) + (a_i \star_2 \partial_k a_j) \star_2 \partial_l f_{\mu\nu} - [a_i\partial_k a_j\partial_l f_{\mu\nu}]_{\star_3} + [a_i a_k\partial_j\partial_l f_{\mu\nu}]_{\star_3})\bigg]. \tag{8}$$

Definitions of generalized star products and the momentum dependent functions $f_{\star_2}(p,q)$, $f_{\star_3}(p,q,k)$ and $f_{\star_{3'}}(p,q,k)$ are given in the Appendix A.

In both solutions we have included the following freedom parameters: κ from $F_{\mu\nu}^{e^2}$, while in $F_{\mu\nu}^{e^3}$ we have $(\kappa, \kappa_{1,2})$ for model I and $(\kappa, \kappa_{1,2}')$ for model II, respectively. From those field strengths we have found the following two actions at the a_μ^4 order,

$$S_{(I)}^{e^2} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int \kappa^2(f_{\mu i} \star_2 f_{\nu j})(f^\mu{}_k \star_2 f^\nu{}_l) - \kappa(f_{ij} \star_2 f_{\mu\nu})(f^\mu{}_k \star_2 f^\nu{}_l) + 2\kappa_1 f^{\mu\nu}[f_{\mu i}f_{\nu k}f_{jl}]_{\star_{3'}}$$
$$+ 2\kappa_2 f^{\mu\nu}(a_i \star_2 \partial_j(f_{\mu k} \star_2 f_{\nu l}) - [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}} - [a_i f_{\mu k}\partial_j f_{\nu l}]_{\star_{3'}}) + (a_i \star_2 \partial_j f_{\mu\nu})(a_k \star_2 \partial_l f^{\mu\nu})$$
$$+ \frac{1}{2}f^{\mu\nu}(2[a_i\partial_j a_k\partial_l f_{\mu\nu}]_{\star_{3'}} + 2[\partial_l f_{\mu\nu}a_i\partial_j a_k]_{\star_{3'}} + 2[a_i a_k\partial_j\partial_l f_{\mu\nu}]_{\star_{3'}} - [a_i\partial_k a_j\partial_l f_{\mu\nu}]_{\star_{3'}} - [\partial_l f_{\mu\nu}a_i\partial_k a_j]_{\star_{3'}}) \tag{9}$$

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[1]See also the discussion after Eq. (26) in [32]. We first substitute $\kappa_g = \kappa^{-1}$, then extract out from all terms in the action a factor κ^{-2}. That factor is than absorbed as an overall rescaling of the field redefinition [32]. Further on we name this paper settings the κ-settings.

and

$$
S^{e^2}_{(II)} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int \kappa^2(f_{\mu i} \star_2 f_{\nu j})(f^\mu{}_k \star_2 f^\nu{}_l) - \kappa(f_{ij} \star_2 f_{\mu\nu})(f^\mu{}_k \star_2 f^\nu{}_l) + 2\kappa'_1 f^{\mu\nu}(2f_{\mu i} \star_2 (f_{jk} \star_2 f_{l\nu}) - [f_{\mu i} f_{jk} f_{l\nu}]_{\star_3})
$$
$$
- 4\kappa'_2 f^{\mu\nu}((a_i \star_2 \partial_j f_{\mu k}) \star_2 f_{\nu l} - [a_i \partial_j f_{\mu k} f_{\nu l}]_{\star_3}) + (a_i \star_2 \partial_j f_{\mu\nu})(a_k \star_2 \partial_l f^{\mu\nu})
$$
$$
+ f^{\mu\nu}(2(a_i \star_2 (\partial_j a_k \star_2 \partial_l f_{\mu\nu}) + (a_i \star_2 \partial_j a_k) \star_2 \partial_l f_{\mu\nu} - [a_i \partial_j a_k \partial_l f_{\mu\nu}]_{\star_3})
$$
$$
- (a_i \star_2 (\partial_k a_j \star_2 \partial_l f_{\mu\nu}) + (a_i \star_2 \partial_k a_j) \star_2 \partial_l f_{\mu\nu} - [a_i \partial_k a_j \partial_l f_{\mu\nu}]_{\star_3}) + 2a_i \star_2 (a_k \star_2 \partial_j \partial_l f_{\mu\nu}) - [a_i a_k \partial_j \partial_l f_{\mu\nu}]_{\star_2}). \tag{10}
$$

One can reduce (9) and (10) to the leading/θ^2 order and perform integration-by-part to obtain

$$
S^{\theta^2}_{(I)} = S^{\theta^2}_{(II)}
$$
$$
= -\frac{e^2}{4}\int \kappa^2 f_{\mu i} f_{\nu j} f^\mu{}_k f^\nu{}_l - \kappa f^{\mu\nu} f_{\mu i} f_{\nu j} f_{kl}
$$
$$
+ 2\kappa_1 f^{\mu\nu} f_{\mu i} f_{\nu k} f_{jl} - \frac{1}{4} f^{\mu\nu} f_{\mu\nu} f_{ik} f_{jl}
$$
$$
+ \frac{1}{8} f^{\mu\nu} f_{ij} f_{kl} f_{\mu\nu}. \tag{11}
$$

We observe two crucial facts from this formula: First, two more gauge invariant structures $f^{\mu\nu} f_{\mu\nu} f_{ik} f_{jl}$ and $f^{\mu\nu} f_{ij} f_{kl} f_{\mu\nu}$ emerge after the integration-by-part. Second,

these five terms deplete all possible combination of four U(1) field strength $f_{\mu\nu}$ contracted with two θ and two metric tensors. One would naturally wonder what are the θ-exact completions of these two terms and if there are more structures emerging after a θ-exact integration-by-part procedure performed. This is, however, not an easy task, since the universal integration-by-part/cyclicity identities on Moyal space only exist for the two-products/integral over the product of three functions. Yet it is still possible to convert both θ-exact interactions (9) and (10) fully in terms of the commutative field strength $f_{\mu\nu}$ by applying a series of integration-by-part identities resulting from verification of the Ward identity on the four-photon coupling vertex. A detailed procedure for model I is given in Appendix A, while here we only list the final result for both models I and II:

$$
S^{e^2}_{(I)} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int \kappa^2(f_{\mu i} \star_2 f_{\nu j})(f^\mu{}_k \star_2 f^\nu{}_l) - \kappa(f_{ij} \star_2 f_{\mu\nu})(f^\mu{}_k \star_2 f^\nu{}_l) + 2\kappa_1 f^{\mu\nu}[f_{\mu i} f_{\nu k} f_{jl}]_{\star_{3'}}
$$
$$
+ 2\kappa_2 f^{\mu\nu}(a_i \star_2 \partial_j(f_{\mu k} \star_2 f_{\nu l}) - [f_{\mu k} a_i \partial_j f_{\nu l}]_{\star_{3'}} - [a_i f_{\mu k} \partial_j f_{\nu l}]_{\star_{3'}}) - \frac{1}{4} f^{\mu\nu}[f_{\mu\nu} f_{ik} f_{jl}]_{\star_{3'}}
$$
$$
+ \frac{1}{8}(f^{\mu\nu} \star_2 f_{ij})(f_{kl} \star_2 f_{\mu\nu}) + \frac{1}{2}\theta^{pq} f^{\mu\nu}[\partial_i f_{jk} f_{lp} \partial_q f_{\mu\nu}]_{\mathcal{M}_{(I)}}, \tag{12}
$$

and

$$
S^{e^2}_{(II)} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int \kappa^2(f_{\mu i} \star_2 f_{\nu j})(f^\mu{}_k \star_2 f^\nu{}_l) - \kappa(f_{ij} \star_2 f_{\mu\nu})(f^\mu{}_k \star_2 f^\nu{}_l)
$$
$$
+ 2\kappa'_1 f^{\mu\nu}(2f_{\mu i} \star_2 (f_{jk} \star_2 f_{l\nu}) - [f_{\mu i} f_{jk} f_{l\nu}]_{\star_3}) - 4\kappa'_2 f^{\mu\nu}((a_i \star_2 \partial_j f_{\mu k}) \star_2 f_{\nu l} - [a_i \partial_j f_{\mu k} f_{\nu l}]_{\star_3})
$$
$$
- \frac{1}{4} f^{\mu\nu}(3f_{ik} \star_2 (f_{jl} \star_2 f_{\mu\nu}) - 2[f_{ik} f_{jl} f_{\mu\nu}]_{\star_3}) + \frac{1}{8} f^{\mu\nu}(2f_{ij} \star_2 (f_{kl} \star_2 f_{\mu\nu}) - [f_{ij} f_{kl} f_{\mu\nu}]_{\star_3})
$$
$$
- \frac{1}{4}\theta^{pq}\theta^{rs} f^{\mu\nu}[\partial_k f_{ri} \partial_j f_{lp} \partial_q \partial_s f_{\mu\nu} + \partial_i \partial_r f_{jk} \partial_s(f_{lp} \partial_q f_{\mu\nu})]_{\mathcal{M}_{(II)}}. \tag{13}
$$

The products $\mathcal{M}_{(I,II)}$ are defined via the momentum structures $f_{(I,II)}$ in Appendix A.

As stated above, the five terms of order θ^2 deplete all possible indices arrangements. Also $\theta^{ij}\theta^{kl} f_{ik} f_{jl} f_{\mu\nu}$ and $\theta^{ij}\theta^{kl} f_{ij} f_{kl} f_{\mu\nu}$ terms can be easily generated via the

θ-iterative procedure [31]. Therefore it is reasonable to introduce two additional freedom parameters (κ_3, κ_4) and (κ'_3, κ'_4), in (I) and (II) respectively, as the θ-exact completion of these two freedoms. In this way we produce the final forms for the a^4_μ-order actions (I,II):

$$S^{e^2}_{(\mathrm{I})_{\kappa,\kappa_1,\kappa_2,\kappa_3,\kappa_4}} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int \kappa^2(f_{\mu i} \star_2 f_{\nu j})(f^\mu{}_k \star_2 f^\nu{}_l) - \kappa(f_{ij} \star_2 f_{\mu\nu})(f^\mu{}_k \star_2 f^\nu{}_l)$$

$$+ 2\kappa_1 f^{\mu\nu}[f_{\mu i}f_{\nu k}f_{jl}]_{\star_{3'}} + 2\kappa_2 f^{\mu\nu}(a_i \star_2 \partial_j(f_{\mu k} \star_2 f_{\nu l}) - [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}} - [a_i f_{\mu k}\partial_j f_{\nu l}]_{\star_{3'}})$$

$$- \frac{\kappa_3}{4}f^{\mu\nu}[f_{\mu\nu}f_{ik}f_{jl}]_{\star_{3'}} + \frac{\kappa_4}{8}(f^{\mu\nu} \star_2 f_{ij})(f_{kl} \star_2 f_{\mu\nu}) + \frac{1}{2}\theta^{pq}f^{\mu\nu}[\partial_i f_{jk}f_{lp}\partial_q f_{\mu\nu}]_{\mathcal{M}_{(\mathrm{I})}}, \quad (14)$$

and

$$S^{e^2}_{(\mathrm{II})_{\kappa,\kappa'_1,\kappa'_2,\kappa'_3,\kappa'_4}} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int \kappa^2(f_{\mu i} \star_2 f_{\nu j})(f^\mu{}_k \star_2 f^\nu{}_l) - \kappa(f_{ij} \star_2 f_{\mu\nu})(f^\mu{}_k \star_2 f^\nu{}_l)$$

$$+ 2\kappa'_1 f^{\mu\nu}(2f_{\mu i} \star_2 (f_{jk} \star_2 f_{l\nu}) - [f_{\mu i}f_{jk}f_{l\nu}]_{\star_3}) - 4\kappa'_2 f^{\mu\nu}((a_i \star_2 \partial_j f_{\mu k}) \star_2 f_{\nu l} - [a_i\partial_j f_{\mu k}f_{\nu l}]_{\star_3})$$

$$- \frac{\kappa'_3}{4}f^{\mu\nu}(3f_{ik} \star_2 (f_{jl} \star_2 f_{\mu\nu}) - 2[f_{ik}f_{jl}f_{\mu\nu}]_{\star_3}) + \frac{\kappa'_4}{8}f^{\mu\nu}(2f_{ij} \star_2 (f_{kl} \star_2 f_{\mu\nu}) - [f_{ij}f_{kl}f_{\mu\nu}]_{\star_3})$$

$$- \frac{1}{4}\theta^{pq}\theta^{rs}f^{\mu\nu}[\partial_k f_{ri}\partial_j f_{lp}\partial_q\partial_s f_{\mu\nu} + \partial_i\partial_r f_{jk}\partial_s(f_{lp}\partial_q f_{\mu\nu})]_{\mathcal{M}_{(\mathrm{II})}}. \quad (15)$$

Note that κ-terms are identical for both above actions, as they should be in the κ-settings.

B. Feynman rule for the four-photon interaction

Since the triple-photon Feynman rules from action (5) is given previously in [32] it is not necessary to be repeated here. From the actions (14) and (15) we read out the corresponding four-photon interactions in the momentum space, with all four momenta p_i in Fig. 1 being the incoming ones

$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_{(\mathrm{I})}(p_1, p_2, p_3, p_4) = -i\frac{e^2}{4}[(\kappa^2\Gamma^{\mu_1\mu_2\mu_3\mu_4}_{\mathrm{A}}(p_1, p_2, p_3, p_4) + \kappa\Gamma^{\mu_1\mu_2\mu_3\mu_4}_{\mathrm{B}}(p_1, p_2, p_3, p_4)$$

$$+ \kappa_1\Gamma^{\mu_1\mu_2\mu_3\mu_4}_1(p_1, p_2, p_3, p_4) + \kappa_2\Gamma^{\mu_1\mu_2\mu_3\mu_4}_2(p_1, p_2, p_3, p_4) + \kappa_3\Gamma^{\mu_1\mu_2\mu_3\mu_4}_3(p_1, p_2, p_3, p_4)$$

$$+ \kappa_4\Gamma^{\mu_1\mu_2\mu_3\mu_4}_4(p_1, p_2, p_3, p_4) + \Gamma^{\mu_1\mu_2\mu_3\mu_4}_5(p_1, p_2, p_3, p_4))$$

$$+ \text{all } S_4 \text{ permutations over}\{p_i\} \text{ and } \{\mu_i\} \text{ simultaneously}]\delta(p_1 + p_2 + p_3 + p_4). \quad (16)$$

and

$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_{(\mathrm{II})}(p_1, p_2, p_3, p_4) = -i\frac{e^2}{4}[(\kappa^2\Gamma^{\mu_1\mu_2\mu_3\mu_4}_{\mathrm{A}}(p_1, p_2, p_3, p_4) + \kappa\Gamma^{\mu_1\mu_2\mu_3\mu_4}_{\mathrm{B}}(p_1, p_2, p_3, p_4)$$

$$+ \kappa'_1\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_1(p_1, p_2, p_3, p_4) + \kappa'_2\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_2(p_1, p_2, p_3, p_4) + \kappa'_3\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_3(p_1, p_2, p_3, p_4)$$

$$+ \kappa'_4\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_4(p_1, p_2, p_3, p_4) + \Gamma'^{\mu_1\mu_2\mu_3\mu_4}_5(p_1, p_2, p_3, p_4))$$

$$+ \text{all} S_4 \text{ permutations over}\{p_i\} \text{ and } \{\mu_i\} \text{ simultaneously}]\delta(p_1 + p_2 + p_3 + p_4). \quad (17)$$
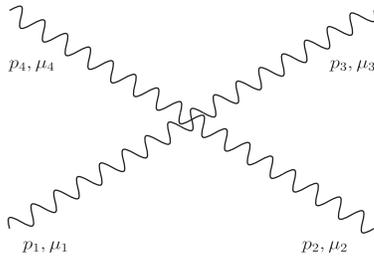

FIG. 1. Four-photon field vertex $\Gamma^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4)$ with all incoming momenta.

with Γ_A, Γ_B, Γ_i and Γ'_i's being given in the Appendix B.

III. FOUR-PHOTON INTERACTION AT ONE LOOP

Following the successful derivation of four-photon self-coupling vertices we move on to study the simplest perturbative quantum loop correction induced by this coupling: the one-loop, four-photon-tadpole diagram contribution to the photon polarization tensor. We first read off from Fig. 2 the tadpole integral. After some arithmetics we
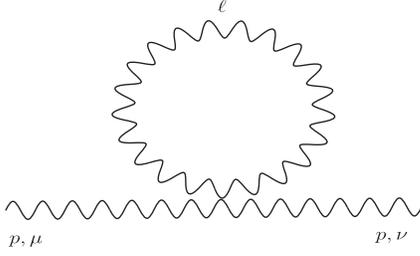
FIG. 2. Four-photon-tadpole contribution to the photon two-point function $T^{\mu\nu}(p)$.

find that various nonlocal factors listed in the Appendix A simplify into two cases: either one or $f_{\star_2}^2(p, \ell)$, i.e.,

$$
\begin{aligned}
T_{(\mathrm{I,II})}^{\mu\nu}(p) &= \frac{1}{2}\mu^{d-D}\int\frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho\sigma}}{\ell^2}\Gamma_{(\mathrm{I,II})}^{\mu\nu\rho\sigma}(p,-p,\ell,-\ell)\\
&= e^2\tau_{(\mathrm{I,II})}^{\mu\nu}\mu^{d-D}\int\frac{d^D\ell}{(2\pi)^D}\frac{\ell^2}{\ell^2}\\
&\quad + e^2\mathcal{T}_{(\mathrm{I,II})}^{\mu\nu\rho\sigma}\mu^{d-D}\int\frac{d^D\ell}{(2\pi)^D}\frac{\ell_\rho\ell_\sigma}{\ell^2}f_{\star_2}^2(p,\ell).
\end{aligned}
\tag{18}
$$

Since the first integral in the above Eq. (18) vanishes according to the dimensional regularization prescription [38], the only remaining integral is the second one. The tensors $\tau_{(\mathrm{I,II})}^{\mu\nu}$ and $\mathcal{T}_{(\mathrm{I,II})}^{\mu\nu\rho\sigma}$ are given in the Appendix C.

A. Tadpole integral

Now we compute our D-dimensional tadpole integral:

$$
I^{\mu\nu} = \int\frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu}{\ell^2}f_{\star_2}^2(p,\ell).
\tag{19}
$$

We have encounter similar computation in our prior works [21,39]. Here to ensure the consistency of our methodology we choose to evaluate the integral in two different ways, both are extensions of sound methods for commutative field theories. As shown below these two methods agrees with each other at the $D \to 4$ limit, as expected.

(1) The first is the conventional method which multiplies the tadpole integral with an identity $1 = (\ell + p)^2/(\ell + p)^2$ to turn it into a bubble integral. We then parameterizing the integral as in our prior works [21], which ultimately yields the following expression in D dimensions

$$
\begin{aligned}
I^{\mu\nu} &= g^{\mu\nu}\frac{1}{D-1}\left(4(\theta p)^{-\frac{D}{2}}\mathcal{K}\left[\frac{D}{2}-1;0,0\right] - \frac{4p^2}{(\theta p)^2}\left((1-D)\mathcal{K}\left[\frac{D}{2}-2;0,1\right] + 2D\mathcal{K}\left[\frac{D}{2}-2;1,1\right]\right)\right.\\
&\quad\left. - p^2\left((1-D)\mathcal{W}\left[\frac{D}{2}-1;0,1\right] - 2D\mathcal{W}\left[\frac{D}{2}-1;1,1\right]\right)\right)\\
&\quad + p^\mu p^\nu\left(\frac{4}{(\theta p)^2}\left((D-2)\mathcal{K}\left[\frac{D}{2}-2;1,0\right] + 2(1-D)\mathcal{K}\left[\frac{D}{2}-2;1,1\right]\right)\right.\\
&\quad\left. - \left((1-D)\mathcal{W}\left[\frac{D}{2}-1;1,0\right] + 2D\mathcal{W}\left[\frac{D}{2}-1;1,1\right]\right)\right)\\
&\quad + (\theta p)^\mu(\theta p)^\nu\frac{1}{D-1}\left(-4D(\theta p)^{-1-\frac{D}{2}}\mathcal{K}\left[\frac{D}{2}-1;0,0\right] + \frac{4p^2}{(\theta p)^4}\left((1-D)\mathcal{K}\left[\frac{D}{2}-2;0,1\right] + (2D-1)\mathcal{K}\left[\frac{D}{2}-2;1,1\right]\right)\right.\\
&\quad\left. + \frac{p^2}{(\theta p)^2}\left((1-D)\mathcal{W}\left[\frac{D}{2}-1;0,1\right] + 2D\mathcal{W}\left[\frac{D}{2}-1;1,1\right]\right)\right),
\end{aligned}
\tag{20}
$$

where the special function integrals are defined as follows

$$
\mathcal{K}[\nu;a,b] = 2^\nu(\theta p)^{-\nu}\int_0^1 dx\, x^a(1-x)^b X^\nu K_\nu[X],
\tag{21}
$$

$$
\mathcal{W}[\nu;a,b] = \int_0^1 dx\, x^a(1-x)^b W_\nu[X].
\tag{22}
$$

Here the $K_\nu[X]$ is the modified Bessel function of second kind, while

$$
\begin{aligned}
W_\nu[X] = (\theta p)^{-2\nu}&\left[X^{2\nu}\Gamma[-\nu]\,_1F_2\left(\frac{1}{2};\frac{3}{2},\nu+1;\frac{X^2}{4}\right)\right.\\
&\left. - \frac{2^{2\nu}}{1-2\nu}\Gamma[\nu]\,_1F_2\left(\frac{1-2\nu}{2};1-\nu,\frac{3-2\nu}{2};\frac{X^2}{4}\right)\right],
\end{aligned}
\tag{23}
$$

and

$$
X = (x(1-x)p^2(\theta p)^2)^{\frac{1}{2}}.
\tag{24}
$$

In the limit $D \to 4 - \epsilon$ after a lengthy manipulations the tadpole integral $I^{\mu\nu}$ reduces to the following two terms

$$I^{\mu\nu} = \frac{1}{6\pi^2} \frac{1}{(\theta p)^4} \left(g^{\mu\nu} - 4\frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^2} \right). \qquad (25)$$

The integrals $\mathcal{K}[\frac{D}{2} - 2; 0, 1]$ and $\mathcal{K}[\frac{D}{2} - 2; 1, 0]$ become UV divergent when $D \to 2 - \epsilon$. This hard $1/\epsilon$ divergence occurs at the same place as the IR divergence when $D \to 4 - \epsilon$; i.e., the UV and IR divergence get "mixed" in the same term now.

(2) As a test a second computation using the n-nested zero regulator [38] is also performed. In this approach we first parameterize the integral as follows

$$\begin{aligned}
I^{\mu\nu} &= \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{\ell^2} f_{\star_2}^2(p, \ell) \\
&= -2 \int_0^\infty d\lambda \lambda^2 \int_0^{1/\lambda} dy \left(y - \frac{1}{\lambda} \right) \\
&\quad \times \int \frac{d^D\ell}{(2\pi)^D} \left(\frac{\ell^2}{D} g^{\mu\nu} - \frac{y^2}{4} (\theta p)^\mu (\theta p)^\nu \right) \\
&\quad \times \exp\left[-\lambda \ell^2 - \frac{\lambda y^2}{4} (\theta p)^2 \right].
\end{aligned} \qquad (26)$$

Next we introduce the loop momenta integral,

$$\int \frac{d^D\ell}{(2\pi)^D} \exp[-\lambda \ell^2] = (4\pi\lambda)^{-\frac{D}{2}} \exp\left[-\lambda f\left(\frac{D}{2} \right) \right], \qquad (27)$$

where $f(\frac{D}{2})$ is the n-nested zero regulator [38] which satisfies the following properties:

(i) $f(\frac{D}{2})$ is a nonzero analytic function
(ii) $f(\frac{D}{2}) = 0$ when $D \in Z^+$
(iii) $f^{(\ell)}(\frac{D}{2}) = 0$ when $D \in Z^+$ and $\ell \le \ell_0 \in N$
(iv) $\forall \operatorname{Re} D \notin Z^+, \exists \operatorname{Im} D, \operatorname{Re}[f(\frac{D}{2})] > 0$.

The inclusion of the $f(\frac{D}{2})$ regulator leads to a planar-modified Bessel-hypergeometric function combinations similar to the first approach with but a variable $(f(\frac{D}{2})(\theta p)^2)^{\frac{1}{2}}$, instead of $(x(1-x)p^2(\theta p)^2)^{\frac{1}{2}}$, and no x integration. In the end, we have found

$$I^{\mu\nu} = I \cdot \left(-\frac{g^{\mu\nu}}{D} + \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^2} \right), \qquad (28)$$

with

$$\begin{aligned}
I &= (4\pi)^{-\frac{D}{2}} \left(\frac{(\theta p)^2}{4} \right)^{-1} \left\{ \left(f\left(\frac{D}{2} \right) \right)^{\frac{D}{2}-1} \Gamma\left(1 - \frac{D}{2} \right) \right. \\
&\quad - W_{\frac{D}{2}} \left[\left(f\left(\frac{D}{2} \right) (\theta p)^2 \right)^{\frac{1}{2}} \right] \\
&\quad - 2 \left(f\left(\frac{D}{2} \right) \right)^{\frac{D}{4}-\frac{1}{2}} \left(\frac{(\theta p)^2}{4} \right)^{-\frac{1}{2}-\frac{D}{4}} \\
&\quad \left. \times K_{\frac{D}{2}-1} \left[\left(f\left(\frac{D}{2} \right) (\theta p)^2 \right)^{\frac{1}{2}} \right] \right\}.
\end{aligned} \qquad (29)$$

The above scalar integral I reduces to a single IR divergent term $-\frac{2}{3\pi^2} (\theta p)^{-4}$ when $D \to 4 - \epsilon$, with all others being suppressed by the $f(\frac{D}{2})$ regulator. This matches the result (25) obtained from the other method.

B. Four-photon-tadpole contributions in the limit $D \to 4 - \epsilon$

By combining the partial tensor reduction from above and the master integral in $D = 4$, we obtain the following results in the κ-settings,

$$\begin{aligned}
T^{\mu\nu}_{(\mathrm{I,II})}(p) &= \frac{e^2}{(4\pi)^2} \{ [g^{\mu\nu} p^2 - p^\mu p^\nu] T_{1_{(\mathrm{I,II})}}(p) \\
&\quad + (\theta p)^\mu (\theta p)^\nu T_{2_{(\mathrm{I,II})}}(p) + [g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu} p^2 \\
&\quad + p^{\{\mu}(\theta\theta p)^{\nu\}}] T_{3_{(\mathrm{I,II})}}(p) \\
&\quad + [(\theta\theta)^{\mu\nu}(\theta p)^2 + (\theta\theta p)^\mu (\theta\theta p)^\nu] T_{4_{(\mathrm{I,II})}}(p) \\
&\quad + (\theta p)^{\{\mu}(\theta\theta\theta p)^{\nu\}} T_{5_{(\mathrm{I,II})}}(p) \}, \qquad (30)
\end{aligned}$$

with model I coefficients

$$T_{1_{(\mathrm{I})}}(p) = \frac{4}{3} \left(\frac{tr\theta\theta}{(\theta p)^4} + 4\frac{(\theta\theta p)^2}{(\theta p)^6} \right) (\kappa_4 - 1),$$

$$\begin{aligned}
T_{2_{(\mathrm{I})}}(p) &= \frac{16}{3} \frac{1}{(\theta p)^4} (2\kappa^2 - 4\kappa + 6\kappa_1 + 2\kappa_2 \\
&\quad - 2\kappa_3 + \kappa_4 - 1),
\end{aligned}$$

$$T_{3_{(\mathrm{I})}}(p) = \frac{16}{3} \frac{1}{(\theta p)^4} (2\kappa^2 - 2\kappa + \kappa_1 + \kappa_2),$$

$$T_{4_{(\mathrm{I})}}(p) = \frac{32}{3} \frac{p^2}{(\theta p)^6} (\kappa^2 - 2\kappa + \kappa_1 + \kappa_2),$$

$$T_{5_{(\mathrm{I})}}(p) = \frac{16}{3} \frac{p^2}{(\theta p)^6} (2\kappa - \kappa_1 - \kappa_2), \qquad (31)$$

and model II

$$T_{1_{(\mathrm{II})}}(p) = \frac{4}{3} \left(\frac{tr\theta\theta}{(\theta p)^4} + 4\frac{(\theta\theta p)^2}{(\theta p)^6} \right) (2\kappa_2' + \kappa_4' - 3),$$

$$T_{2_{(\mathrm{II})}}(p) = \frac{32}{3} \frac{1}{(\theta p)^4} (\kappa^2 - 2\kappa + 2\kappa_1' + 2\kappa_2' - \kappa_3'),$$

$$T_{3_{(\mathrm{II})}}(p) = \frac{16}{3} \frac{1}{(\theta p)^4} (2\kappa^2 - 2\kappa + 2\kappa_2'),$$

$$T_{4_{(\mathrm{II})}}(p) = \frac{32}{3} \frac{p^2}{(\theta p)^6} (\kappa^2 - 2\kappa + 2\kappa_2'),$$

$$T_{5_{(\mathrm{II})}}(p) = \frac{32}{3} \frac{p^2}{(\theta p)^6} (\kappa - \kappa_2'). \qquad (32)$$

The tensor structure remains exactly the same as for the photon bubble diagram (Fig. 3), as one would expect. However, we notice immediately the absence of UV and logarithmic divergent terms contrary to the photon bubble
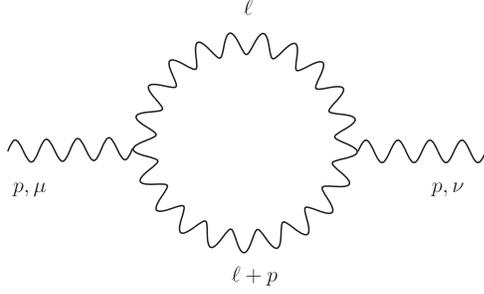
FIG. 3. Three-photon-bubble contribution to the photon two-point function $\mathcal{B}^{\mu\nu}(p)$.

diagram results [21]. In addition, the tadpole diagram produces no finite terms either. The tadpole contribution for model I can be made equal to that of model II when setting

$$\kappa_1 + \kappa_2 = 2\kappa_2',$$
$$4\kappa_1 - 2\kappa_3 = 4\kappa_1' - 4\kappa_3' - \kappa_4' + 3,$$
$$\kappa_4 = 2\kappa_3' + \kappa_4' - 2, \tag{33}$$

and in particular $T^{\mu\nu}_{(\mathrm{I})}(p) = T^{\mu\nu}_{(\mathrm{II})}(p)$, for $\kappa_i = \kappa_i' = 1, \forall i$. We also notice that T_3, T_4, and T_5 can only be set to zero

simultaneously if $\kappa = 0$. Since the value κ directly affects the divergences in the bubble diagram [21], we therefore conclude that the tadpole divergences should be analyzed only after being summed with the bubble diagram contribution.

IV. SUMMING OVER BUBBLE AND TADPOLE DIAGRAMS FOR $D \to 4 - \epsilon$

As stated above to complete our analysis for the photon two-point function we have to sum up the tadpole (Fig. 2) (18) and the bubble (Fig. 3) contributions, which has the same structure as (30), but with the coefficients $T_i(p)$ replaced with $B_i(p)$ given below. To do that we express the bubble contribution $\mathcal{B}^{\mu\nu}(p)$, in terms of the κ-settings for the gauge field strength at the e^2 order (4), by converting each B_i-coefficient from the κ_g-setting [21,32] to the κ-setting. We first substitute $\kappa_g = \kappa^{-1}$, then extract out from all coefficients a factor κ^{-2} since it is absorbed as an overall re-scaling factor of the corresponding field redefinition [32]. Remaining divergent parts of the B_i-coefficients are then given next in (34) in terms of the κ-settings, so that they match their tadpole counterparts and could be summed up with.

$$B_1(p) \sim + \left(\frac{1}{3}(1 - 3\kappa)^2 + \frac{1}{3}(1 + 2\kappa)^2 \frac{p^2(\mathrm{tr}\theta\theta)}{(\theta p)^2} + \frac{2}{3}(1 + 4\kappa + \kappa^2) \frac{p^2(\theta\theta p)^2}{(\theta p)^4} \right) \left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) \right]$$
$$- \frac{8}{3}(1 - \kappa)^2 \frac{1}{(\theta p)^6} ((\mathrm{tr}\theta\theta)(\theta p)^2 + 4(\theta\theta p)^2),$$

$$B_2(p) \sim + \left(\frac{4}{3}(1 - \kappa)^2 \frac{p^4(\theta\theta p)^2}{(\theta p)^6} + \frac{1}{3}(1 - 2\kappa - 5\kappa^2) \frac{p^4(\mathrm{tr}\theta\theta)}{(\theta p)^4} + \frac{1}{3}(25 - 86\kappa + 73\kappa^2) \frac{p^2}{(\theta p)^2} \right) \left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) \right]$$
$$- \frac{8}{3}(1 - 3\kappa)(3 - \kappa) \frac{1}{(\theta p)^4} + \frac{16}{3}(1 - \kappa)^2 \frac{p^2}{(\theta p)^8} ((\mathrm{tr}\theta\theta)(\theta p)^2 + 6(\theta\theta p)^2),$$

$$B_3(p) \sim - \frac{1}{6}(1 - 2\kappa - 11\kappa^2) \frac{p^2}{(\theta p)^2} \left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) \right] - \frac{4}{3(\theta p)^4}(1 - 10\kappa + 17\kappa^2),$$

$$B_4(p) \sim -(1 + \kappa)^2 \frac{p^4}{(\theta p)^4} \left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) \right] - \frac{16p^2}{3(\theta p)^6}(1 - 6\kappa + 7\kappa^2),$$

$$B_5(p) \sim + \frac{2}{3}(1 + \kappa + 4\kappa^2) \frac{p^4}{(\theta p)^4} \left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) \right] + \frac{32p^2}{3(\theta p)^6}(1 - \kappa)(1 - 2\kappa). \tag{34}$$

All $B_i(p)$ coefficients are computed for arbitrary κ and $\sim$ means that we have neglected all finite terms in the above equations. We observe the presence of the UV divergences as well as quadratic UV/IR mixing in all B_i's. The logarithmic IR divergences from both planar and nonplanar sources in the bubble diagram appear to have identical coefficient and combine into a single $\ln \mu^2(\theta p)^2$ term. Finally, no single κ value is capable of removing all novel divergences.

A. θ independent elimination of the bubble plus tadpole IR divergences

Since tadpole integrals give only a quadratic IR divergence, we perform a sum over the tadpole and quadratically IR divergent parts of the bubble diagram [21] for an arbitrary choice of the antisymmetric tensor $\theta^{\mu\nu}$ and deformation freedom parameters. Working out the arithmetics, we get for both models I and II in the IR regime the sum $\Pi^{\mu\nu}_{(\mathrm{I,II})}(p)_{\mathrm{IR}} = \mathcal{B}^{\mu\nu}(p)_{\mathrm{IR}} + T^{\mu\nu}_{(\mathrm{I,II})}(p)_{\mathrm{IR}}$ of the bubble and

the tadpole photon contribution to the photon polarization tensor. It has again the structure as in (30), but with coefficients $T_i(p)$ getting replaced with the following sums:

$$\Pi_{i_{(I,II)}}(p)_{IR} = B_i(p)_{IR} + T_{i_{(I,II)}}(p)_{IR}, \quad \forall\, i = 1, \ldots, 5. \quad (35)$$

Since tadpole produces no UV, log and also has no finite contributions, we have

$$\Pi_i(p)_{UV} = B_i(p)_{UV}, \quad \forall\, i = 1, \ldots, 5. \quad (36)$$

According to (35), in the rest of this article label $\Pi^{\mu\nu}$ always represents the sum of bubble and tadpole contributions to the photon polarization tensor. A summation over the leading IR terms in the bubble and tadpole diagrams provides IR results (for overall UV/IR mixings) in model I:

$$\Pi_{1_{(I)}}(p)_{IR} \sim -\frac{4}{3}\frac{1}{(\theta p)^4}\left(\mathrm{tr}\theta\theta + 4\frac{(\theta\theta p)^2}{(\theta p)^2}\right)$$
$$\times (2(\kappa-1)^2 - \kappa_4 + 1),$$

$$\Pi_{2_{(I)}}(p)_{IR} \sim +\frac{8}{3}\frac{1}{(\theta p)^4}(\kappa^2 + 2\kappa + 12\kappa_1 + 4\kappa_2$$
$$- 4\kappa_3 + 2\kappa_4 - 5)$$
$$+ \frac{16}{3}\frac{p^2}{(\theta p)^4}\left(\frac{\mathrm{tr}\theta\theta}{(\theta p)^2} + 6\frac{(\theta\theta p)^2}{(\theta p)^4}\right)(\kappa-1)^2,$$

$$\Pi_{3_{(I)}}(p)_{IR} \sim -\frac{4}{3}\frac{1}{(\theta p)^4}(9\kappa^2 - 2\kappa - 4\kappa_1 - 4\kappa_2 + 1),$$

$$\Pi_{4_{(I)}}(p)_{IR} \sim -\frac{16}{3}\frac{p^2}{(\theta p)^6}(5\kappa^2 - 2\kappa - 2\kappa_1 - 2\kappa_2 + 1),$$

$$\Pi_{5_{(I)}}(p)_{IR} \sim +\frac{16}{3}\frac{p^2}{(\theta p)^6}(4\kappa^2 - 4\kappa - \kappa_1 - \kappa_2 + 2), \quad (37)$$

and in model II:

$$\Pi_{1_{(II)}}(p)_{IR} \sim -\frac{4}{3}\frac{1}{(\theta p)^4}\left(\mathrm{tr}\theta\theta + 4\frac{(\theta\theta p)^2}{(\theta p)^2}\right)$$
$$\times (2(\kappa-1)^2 - 2\kappa_3' - \kappa_4' + 3),$$

$$\Pi_{2_{(II)}}(p)_{IR} \sim +\frac{8}{3}\frac{1}{(\theta p)^4}(\kappa^2 + 2\kappa + 8\kappa_1' + 8\kappa_2' - 4\kappa_3' - 3)$$
$$+ \frac{16}{3}\frac{p^2}{(\theta p)^4}\left(\frac{\mathrm{tr}\theta\theta}{(\theta p)^2} + 6\frac{(\theta\theta p)^2}{(\theta p)^4}\right)(\kappa-1)^2,$$

$$\Pi_{3_{(II)}}(p)_{IR} \sim -\frac{4}{3}\frac{1}{(\theta p)^4}(9\kappa^2 - 2\kappa - 8\kappa_2' + 1),$$

$$\Pi_{4_{(II)}}(p)_{IR} \sim -\frac{16}{3}\frac{p^2}{(\theta p)^6}(5\kappa^2 - 2\kappa - 4\kappa_2' + 1),$$

$$\Pi_{5_{(II)}}(p)_{IR} \sim +\frac{16}{3}\frac{p^2}{(\theta p)^6}(4\kappa^2 - 4\kappa - 2\kappa_2' + 2). \quad (38)$$

Here we see that IR divergence in the coefficients $\Pi_{3,4,5}$ depends only on $\kappa_1 + \kappa_2$ in model I or κ_2' in model II. A bit more calculation shows that $\kappa = \kappa_1 = \kappa_2 = \kappa_2' = 1$ can annihilate $\Pi_{3,4,5_{(I,II)}}(p)_{IR}$, respectively. Furthermore, $\kappa = 1$ also removes the pathological second term in $\Pi_{2_{(I,II)}}(p)_{IR}$. The rest of divergences in (37) and (38) can be readily removed by an appropriate choice of the rest of freedom parameters, proving that the elimination of all IR divergences is independent on the choices of θ-matrix elements.

B. UV divergences

Up to now we see that the quadratic IR divergence can be canceled by selecting the first order gauge freedom parameter $\kappa = 1$ and $\forall\theta$. On the other hand, we also know that there are various UV ($1/\epsilon$) divergences in the bubble diagram which are also precisely connected to the logarithmic divergences. Characterizing this part of the behavior requires some simplifications by special $\theta^{\mu\nu}$ choices which satisfy the condition that $(\theta\theta)^{\mu\nu}$ becomes minus identity within a suitable subspace of dimension $n \le 4$. Here we consider two important examples.

1. The subdimension $n = 2$ and $\theta_1^{\mu\nu}$ matrix

First choice is to set $n = 2$, as used already in [17]. This choice has the potential of preserving unitarity when the NC directions are spatial, and it is manifested in the form of $\theta_1^{\mu\nu}$ matrix[2]:

$$\theta^{\mu\nu} \equiv \theta_1^{\mu\nu} = -\theta^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (39)$$

So, for $d = 4$-dimensional NC field theory and the subspace of dimension $n = 2$, and by noticing that $(\theta\theta)^{\mu\nu}$ is now a projector into a two-dimensional subspace, we find

$$-2\theta^2(\theta p)^\mu(\theta p)^\nu = 2[(\theta\theta)^{\mu\nu}(\theta p)^2 + (\theta\theta p)^\mu(\theta\theta p)^\nu]$$
$$= (\theta p)^{\{\mu}(\theta\theta\theta p)^{\nu\}},$$
$$(\theta\theta\theta p)^\mu = -\theta^2(\theta p)^\mu,$$
$$(\theta p)^2\mathrm{tr}\theta\theta = -2(\theta\theta p)^2 = -2(\theta p)^2\theta^2,$$
$$\forall p; \qquad \theta^2 = 1/\Lambda_{NC}^4, \quad (40)$$

with Λ_{NC} being the scale of noncommutativity. In order to perform computations in this case, we need to use all above relations in general decomposition of the photon polarization tensor, which is the same for both, the bubble and the

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[2]Actually for $d = 4$ any spacelike θ^{ij} can be simplified to this form by a rotation that sends the pseudovector $v_i = \epsilon_{ijk}\theta^{jk}$ to the third axis. Here ϵ_{ijk} is the totally antisymmetric tensor of the three spatial dimensions.

tadpole contribution. A general decomposition, like (30), in $n = 2$ with θ_1 matrix, simplifies to

$$
\begin{aligned}
\Pi^{\mu\nu}(p)|_{n=2}^{\theta_1} = \frac{e^2}{(4\pi)^2}[&(g^{\mu\nu}p^2 - p^\mu p^\nu)\Pi_1 \\
&+ (g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu}p^2 + p^{\{\mu}(\theta\theta p)^{\nu\}})\Pi_3 \\
&+ (\theta p)^\mu(\theta p)^\nu(\Pi_2 - \theta^2\Pi_4 - 2\theta^2\Pi_5)],
\end{aligned} \tag{41}
$$

where (for general θ matrix) the coefficients are given in (37) and/or (38), respectively.

The UV part of (34) obtained by using θ_1 and/or (41) is then given below:

$$
\begin{aligned}
&\Pi^{\mu\nu}(p)|_{UV}^{\theta_1} \\
&= \frac{e^2}{48\pi^2}\left(\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2)\right) \\
&\times \left\{ (g^{\mu\nu}p^2 - p^\mu p^\nu)\left[(3\kappa - 1)^2 - 6\kappa^2 \frac{p^2\theta^2}{(\theta p)^2}\right] \right. \\
&\quad + \frac{1}{2}(g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu}p^2 + p^{\{\mu}(\theta\theta p)^{\nu\}}) \\
&\quad \times \frac{p^2}{(\theta p)^2}[11\kappa^2 + 2\kappa - 1] \\
&\quad \left. + \frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}p^2\left[73\kappa^2 - 86\kappa + 25 + (\kappa - 1)^2\frac{p^2\theta^2}{(\theta p)^2}\right]\right\}.
\end{aligned} \tag{42}
$$

A simple inspection of the above result shows that there is not any single κ value which would simultaneously eliminate IR and UV plus log divergences, thus elimination of UV pathologies has to be treated more carefully. What is more, since in the UV regime we have $\Pi_i = B_i$ because of the absence of the tadpole contributions, the elimination of pathologies goes beyond choices of freedom parameter. However, a fine property is that UV part of polarization tensor (42) in all limits $p \to 0$, $\theta \to 0$, separately and/or simultaneously, behave well enough. This is encouraging enough to consider that the UV plus log divergences in (42) may be removed for any point κ via certain proper subtraction of the linear combinations of dimensionless nonlocal counterterms, as noted in [17],

$$
\mathcal{B}^{\mu\nu} = \xi\partial^2\frac{\text{tr}\theta\theta}{(\theta\partial)^2} + \zeta\partial^2\frac{(\theta\theta\partial)^2}{(\theta\partial)^4}, \tag{43}
$$

with ξ, ζ being free coefficients measuring strengths of the two terms. Those coefficients could be determined during the renormalization procedure, which we leave for the future as a part of the NCGFT renormalization project.

2. The subdimension $n = 4$ and $\theta_2^{\mu\nu}$ matrix

Second choice is to set $n = 4$ and selects the following $\theta_2^{\mu\nu}$ matrix[3] [21],

$$
\begin{aligned}
\theta^{\mu\nu} &\equiv \theta_2^{\mu\nu} \\
&= \theta^2\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \theta^2\begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \\
&\equiv (\sigma_2 \otimes I_2)\theta^2,
\end{aligned} \tag{44}
$$

with σ_2 being famous Pauli matrix. This choice, which we are calling nonunitary, induces following useful relations in four-dimensional Euclidean space-time:

$$
\begin{aligned}
(\theta\theta)^{\mu\nu} &= -g^{\mu\nu}\theta^2, \quad (\theta\theta p)^\mu = -p^\mu\theta^2, \\
(\theta\theta\theta p)^\mu &= -(\theta p)^\mu\theta^2, \\
(\theta p)^2\text{tr}\theta\theta &= -4(\theta\theta p)^2 = -4\theta^4 p^2, \quad \forall p; \quad \theta^2 = 1/\Lambda_{NC}^4.
\end{aligned} \tag{45}
$$

The general tensor structure (30) then simplifies into two parts:

$$
\begin{aligned}
&\Pi^{\mu\nu}(p)|_{n=4}^{\theta_2} \\
&= \frac{e^2}{(4\pi)^2}\left[(g^{\mu\nu}p^2 - p^\mu p^\nu)\left(\Pi_1 + 2\frac{(\theta p)^2}{p^2}\Pi_3 - \frac{(\theta p)^4}{p^4}\Pi_4\right) \right. \\
&\quad \left. + (\theta p)^\mu(\theta p)^\nu\left(\Pi_2 - 2\frac{(\theta p)^2}{p^2}\Pi_5\right)\right].
\end{aligned} \tag{46}
$$

The coefficients Π_i are given in (34), (37), and (38), respectively.

The UV ($1/\epsilon$) divergent and the UV/IR mixing logarithmic terms in the total photon two-point function, come entirely from the bubble diagram, as one can easily be convinced by combining Eqs. (34), (36), and (46),

---

[3]This condition was used in the renormalizability studies of four-dimensional NCGFT without the SW map [40,41]. Note also that this $\theta_2^{\mu\nu}$ is full rank and, thus, breaks in general the unitarity if one performs Wick rotation to the Minkowski space-time [42]. It contains also time-space noncommutativity which breaks causality.

$\Pi^{\mu\nu}(p)|_{\text{UV}}^{\theta_2}$

$$= \frac{e^2 p^2}{24\pi^2} \left(\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2) \right) \left[\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) (3\kappa - 1)^2 \right.$$

$$\left. + \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^2} \frac{3}{2} (3\kappa - 1)(9\kappa - 7) \right]. \qquad (47)$$

The above two divergences are certainly removable by selecting the special point $\kappa = 1/3$ [21,32]. However, the UV plus log divergences from bubble diagrams are retained for any other κ point in the $n = 4$, θ_2 case and could be only removed by a proper subtraction of a linear combinations of nonlocal counterterms similar to (43).

3. Photon polarization for the θ_2 matrix and $\kappa = 1/3$ deformation

In this case, because of the elimination of the UV divergences (47), we analyze next the four-dimensional

IR divergences for $\theta^{\mu\nu} = \theta_2^{\mu\nu}$ satisfying $(\theta\theta)^{\mu\nu} = -g^{\mu\nu}\theta^2$ in four-dimensional Euclidean spacetime. So, the three-photon bubble plus four-photon-tadpole one-loop contributions to the photon two-point function, for special choice for θ_2 and $\kappa = 1/3$ is being reduced to two IR (UV/IR mixing) terms from photon tadpole diagram. This is obtained by using decomposition (46) with (31) and (32), respectively. Adding two finite terms from the bubble diagram [21],

$$\Pi^{\mu\nu}_{(\text{I,II})}(p)|_{\kappa=1/3}^{\theta_2}$$

$$= \frac{e^2 p^2}{2\pi^2} \left\{ \frac{7}{27} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) - \frac{1}{2} \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^2} \right\}, \quad (48)$$

we finally obtain the total photon polarization tensor in both models as

$$\Pi^{\mu\nu}_{(\text{I})}(p)|_{\text{IR}}^{\theta_2,\kappa=1/3} = \frac{e^2 p^2}{2\pi^2} \left\{ \frac{7}{27} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left(1 + \frac{4}{7} \frac{1}{p^2(\theta p)^2} \right) \right.$$

$$\left. - \frac{1}{2} \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^2} \left(1 + \frac{4}{27} \frac{1}{p^2(\theta p)^2} (31 - 72\kappa_1 - 36\kappa_2 + 18\kappa_3 - 9\kappa_4) \right) \right\},$$

$$\Pi^{\mu\nu}_{(\text{II})}(p)|_{\text{IR}}^{\theta_2,\kappa=1/3} = \frac{e^2 p^2}{2\pi^2} \left\{ \frac{7}{27} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left(1 + \frac{4}{7} \frac{1}{p^2(\theta p)^2} \right) \right.$$

$$\left. - \frac{1}{2} \frac{(\theta p)^\mu (\theta p)^\nu}{(\theta p)^2} \left(1 + \frac{4}{27} \frac{1}{p^2(\theta p)^2} (22 - 36\kappa'_1 - 72\kappa'_2 + 18\kappa'_3) \right) \right\}. \qquad (49)$$

The "additional" term $(\theta p)^\mu (\theta p)^\nu / (\theta p)^4$ receives multiple corrections from the tadpole diagram and can be easily removed by shifting the second-order gauge freedom parameters. Usual tensor structure $(g^{\mu\nu} p^2 - p^\mu p^\nu)$, however, depends solely on the first order gauge freedom parameter κ in such a way that no real κ value can set it to zero in models I and II.

At the end of this section let us summarize what we have learned about photon polarization tensor pathologies within models I and II:—Summing over the tadpole and bubble diagram not only formally completes the one-loop quantum corrections to the photon polarization tensor, but also appears to be crucial for the elimination of the quadratic IR divergences $\forall\theta$.—The quadratic IR divergences can be removed for arbitrary θ, and for $\kappa = \kappa_1 = \kappa_2 = \kappa'_2 = 1$, plus appropriate choices for the rest of the freedom parameters in both models I and II.—Unfortunately, choice $\kappa = 1$ does not remove UV divergences for arbitrary θ.—Choice $\kappa = 1/3$ and fixing $\theta = \theta_2$ removes UV but not IR for both models I and II.—We conclude that to eliminate simultaneously all divergences from tadpole plus bubble contributions requires a new extended model/procedure of deformation freedom parameter selections.

V. EXTENDING SW MAP-BASED MODELS

A. Generalized SW map inspired four-photon self-interaction model

Our experience in the last two sections suggests that in order to efficiently eliminate pathological divergences in a deformed one-loop photon polarization tensor it is necessary to modify the freedom parameter dependence at least in the four-photon interactions. Here we propose such a modification to the SW mapped models I,[4] which we call model III, to fulfill this requirement. We start by realizing that κ^2 and κ terms in (9) and/or (12) and/or (14) may be varied independently due to the manifestly gauge invariant structure of both terms, so by this means we could assign independent gauge-symmetry (variation) freedom parameters to all five gauge invariant structures starting at θ^2 order as found in (11). Proceeding further with this idea, we get the following action:

---

[4]The same could be done also with the model II action (15), but we choose to discuss only one case for simplicity.

$S_{\eta_1,\eta_2,\eta_3,\eta_4,\eta_5}^{e^2}$

$$
\begin{aligned}
= & -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int \eta_1(f_{\mu i} \star_2 f_{\nu j})(f^\mu{}_k \star_2 f^\nu{}_l) \\
& -\eta_2(f_{ij} \star_2 f_{\mu\nu})(f^\mu{}_k \star_2 f^\nu{}_l) + 2\eta_3 f^{\mu\nu}[f_{\mu i}f_{\nu k}f_{jl}]_{\star_{3'}} \\
& -\frac{\eta_4}{4}f^{\mu\nu}[f_{\mu\nu}f_{ik}f_{jl}]_{\star_{3'}} + \frac{\eta_5}{8}(f^{\mu\nu} \star_2 f_{ij})(f_{kl} \star_2 f_{\mu\nu}) \\
& + 2f^{\mu\nu}(a_i \star_2 \partial_j(f_{\mu k} \star_2 f_{\nu l}) - [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}} \\
& - [a_i f_{\mu k}\partial_j f_{\nu l}]_{\star_{3'}}) + \frac{1}{2}\theta^{pq}f^{\mu\nu}[\partial_i f_{jk}f_{lp}\partial_q f_{\mu\nu}]_{\mathcal{M}_{(1)}}. \quad (50)
\end{aligned}
$$

Comparing (50) with (14) we see the following replacement rules: $\kappa^2 := \eta_1$, $\kappa := \eta_2$, $\kappa_1 := \eta_3$, $\kappa_2 := 1$, $\kappa_3 := \eta_4$, $\kappa_4 := \eta_5$. Here we omitted κ_2 because its corresponding structure starts at θ^4.

Interaction (50) leads to the following result for the tadpole diagram in the η setting,

$$
\begin{aligned}
T^{\mu\nu}(p)_\eta = \frac{e^2}{(4\pi)^2}\{ & [g^{\mu\nu}p^2 - p^\mu p^\nu]T_1(p)_\eta \\
& + (\theta p)^\mu(\theta p)^\nu T_2(p)_\eta \\
& + [g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu}p^2 + p^{\{\mu}(\theta\theta p)^{\nu\}}]T_3(p)_\eta \\
& + [(\theta\theta)^{\mu\nu}(\theta p)^2 + (\theta\theta p)^\mu(\theta\theta p)^\nu]T_4(p)_\eta \\
& + (\theta p)^{\{\mu}(\theta\theta\theta p)^{\nu\}}T_5(p)_\eta\}, \quad (51)
\end{aligned}
$$

with $T_i(p)_\eta$ being

$$
T_1(p)_\eta = \frac{4}{3}\left(\frac{\mathrm{tr}\theta\theta}{(\theta p)^4} + 4\frac{(\theta\theta p)^2}{(\theta p)^6}\right)(\eta_5 - 1),
$$

$$
T_2(p)_\eta = \frac{64}{3}\frac{1}{(\theta p)^4}(2\eta_1 - 4\eta_2 + 6\eta_3 - 2\eta_4 + \eta_5 + 1),
$$

$$
T_3(p)_\eta = \frac{64}{3}\frac{1}{(\theta p)^4}(2\eta_1 - 2\eta_2 + \eta_3 + 1),
$$

$$
T_4(p)_\eta = \frac{128}{3}\frac{p^2}{(\theta p)^6}(\eta_1 - 2\eta_2 + \eta_3 + 1),
$$

$$
T_5(p)_\eta = \frac{64}{3}\frac{p^2}{(\theta p)^6}(2\eta_2 - \eta_3 - 1). \quad (52)
$$

B. Bubble plus tadpole results in model III

We now consider the divergences from bubble plus tadpole diagrams in model III for both θ_1 and θ_2 choices. As we will see below, the quadratic IR divergence cancellation can be achieved for arbitrary κ and both choices of θ, while UV cancellation is only available for θ_2.

1. The IR part for subdimension $n = 2$ and the θ_1 matrix in the η setting

Substituting (52) into (35) then (41), we obtain the following conditions for quadratic IR divergence cancellation:

$$
2(1 - \kappa)^2 - (\eta_5 - 1) = 0,
$$
$$
(1 - 3\kappa)(3 - \kappa) - 2(2\eta_1 - 4\eta_2 + 6\eta_3 - 2\eta_4 + \eta_5 + 1) = 0,
$$
$$
(1 - 10\kappa + 17\kappa^2) - 4(2\eta_1 - 2\eta_2 + \eta_3 + 1) = 0,
$$
$$
(3\kappa^2 - 2\kappa + 1) - 2\eta_1 = 0, \quad (53)
$$

with solutions

$$
\eta_1 = \frac{1}{2}(3\kappa^2 - 2\kappa + 1), \quad \eta_3 = 2\eta_2 + \frac{1}{4}(5\kappa^2 - 2\kappa - 7),
$$

$$
\eta_4 = 4\eta_2 - \frac{1}{2}(11\kappa^2 - 4\kappa - 7), \quad \eta_5 = 1 + 2(\kappa - 1)^2, \quad (54)
$$

which produce the IR free photon polarization tensor for arbitrary κ, as expected.

2. Subdimension $n = 4$ represented by the θ_2 matrix and for the η setting

For NCQFTs in d dimensions, equal to the subspace dimensions n, i.e., in four-dimensional Euclidian spacetime with the $n = 4$ case, and by choosing θ_2 deformation (44), we get the following tadpole

$$
\begin{aligned}
T^{\mu\nu}(p)_\eta|_{\mathrm{IR}}^{n=4,\theta_2} = \frac{e^2}{3\pi^2}\frac{1}{(\theta p)^2}\bigg[& \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)2\eta_1 \\
& + \frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}(2\eta_1 - 8\eta_2 + 8\eta_3 \\
& - 2\eta_4 + \eta_5 + 3)\bigg]. \quad (55)
\end{aligned}
$$

Using the above Eq. (55), together with (30) and (34) for θ_2, we obtain the following bubble plus tadpole contributions to the photon polarization tensor in the IR regime:

$$
\begin{aligned}
\Pi^{\mu\nu}(p)|_{\mathrm{IR}}^{\theta_2,\kappa,\eta_i} & \\
= \frac{e^2}{6\pi^2}\frac{1}{(\theta p)^2}\bigg[& \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)(-3\kappa^2 - 2\kappa + 1 + 4\eta_1) \\
& - \frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}(15\kappa^2 - 26\kappa + 7 - 4\eta_1 + 16\eta_2 \\
& - 16\eta_3 + 4\eta_4 - 2\eta_5 - 6)\bigg]. \quad (56)
\end{aligned}
$$

Here, due to the η settings the quadratic IR tadpole contributions are clearly disentangled from the well-behaving remaining contributions. Since the number of free parameters exceeds the number of tensor structures in a subspace $n = 4$, the quadratic IR divergence cancellation surely exists for arbitrary κ. Furthermore, an interesting choice,

$$\kappa = \frac{1}{3}, \qquad \eta_1 = 0, \qquad \eta_2 = \eta_3, \qquad 2\eta_4 - \eta_5 = 3, \quad (57)$$

eliminates not only IR but, simultaneously due to (47), also the hard UV and logarithmic divergences as well.

VI. DISCUSSION

Following recent progress in the e^3-order θ-exact Seiberg-Witten map expansions [13,14,33], in this paper the resulting four-photon interaction is presented through the construction of three different models (I, II, III). We study their impact on the perturbative quantization of the SW map deformed U(1) gauge field theory via the corresponding one-loop, four-photon-tadpole contributions to the (divergent part of the) photon polarization tensor. Note also that the same term should contribute to the NC phenomenology at extreme energies, for example, tree-level NCQED contributions to the $\gamma\gamma \to \gamma\gamma$ scattering processes [43].

Quadratic IR divergence in NCQFT on Moyal space was first found in the original version of UV/IR mixing in the tadpole integral of the NC ϕ^4 QFT on Moyal space [30], where part of the originally quadratically UV divergent diagram gets UV regularized from the NC phase factor and becomes quadratically IR divergent instead. The UV divergence in the commutative U(1) gauge theory, on the other hand, is logarithmic under the dimensional regularization procedure, yet the IR divergence (steaming from UV/IR mixing) in the NC $U_\star(1)$ gauge theory still starts from the quadratic order [27–29]. It is shown in our previous work on the bubble diagram [21] with the first-order gauge freedom parameter $\kappa_g(= \kappa^{-1})$ that the hard UV ($1/\epsilon$) divergence, the commutative logarithmic divergence $\ln(\mu^2/p^2)$ and the NC logarithmic divergence $\ln p^2(\theta p)^2$, share the same dependence on κ_g, while the quadratic IR divergence bears a completely different one. Thus, the quadratic IR divergence may indeed have a separated origin from "UV/IR mixing" which is logarithmic as it should be.

By evaluating the photon one-loop tadpole diagram Fig. 2 in the first two deformed U(1) gauge theory models based on two distinct gauge field strengths [33] using NC-extended dimensional regularization techniques, we show that the NC massless tadpole integrals produce solely quadratic IR divergence; i.e., there is no UV divergent or finite contributions from tadpole. From this perspective the he result of this paper suggests that the quadratic IR divergence behavior could be simply a tadpole effect, especially because it can be removed by a suitable gauge freedom selection. The two models give rise to different tadpole contributions to the photon polarization tensor for general values of the SW/ gauge freedom deformation parameters κ, κ_i, and κ_i', yet they can be made equal by employing certain simple algebraic relations. In particular, the tadpole integrals from the two models are equal to each other for $\kappa_i = \kappa_i' = 1$, $\forall i$. In fact,

this is not surprising as the limited number of momenta involved in the tadpole evaluation restrict the possible value for the nonlocal factors to either be unity or $\sin^2 \frac{p\theta k}{2}/(\frac{p\theta k}{2})^2$, thus severely constraining the possible combinations in the tensor structure(s).

After summing of bubble and tadpole diagrams in the above three models, we study the possible choices of freedom parameters for divergence eliminations. These choices are expectedly different in the so-called κ-setting models (I, II) and in the third, η-setting model (III). Of importance is that $\kappa = 1$ stands out as the unique quadratic IR divergence cancellation point for arbitrary θ and for both models (I, II), while UV and log/mixing divergence can only be removed at a different $\kappa = \frac{1}{3}$ point with the special choice for θ, $\theta = \theta_2$ matrix. For the third η-setting model, the quadratic IR divergence removal is available for arbitrary κ and both $\theta_{1,2}$ choices, as expected. We, thus, manage to obtain the full divergence cancellation at the $(\kappa, \eta_1) = (\frac{1}{3}, 0)$, point for $\theta = \theta_2$ matrix with the help of extra freedom parameters $\eta_2 = \eta_3$, $2\eta_4 - \eta_5 = 3$ in the four-photon interaction (57).

From the θ-dependent photon polarization tensor in the U(1) NCQFT with the special choice for θ_2, we raise one more interesting question: Does our deformation parameter set (κ, η_i) run between IR and UV divergence-eliminating points? To be more precise, let us take the liberty of assuming that there is indeed a possibility that deformation parameters κ and η are energy/momentum-dependent quantities.[5] As a next point after inspecting (47), (56), and (57), we notice the fact that starting with the photon polarization behavior in the deep IR regime, and moving towards the high UV regime, our divergence eliminating freedom parameters κ, η_i decreases: $\kappa = 1 \to \frac{1}{3}$, $\eta_1 = 1 \to 0$, $2\eta_4 - \eta_5 = 7 \to 3$, while $\eta_2 - \eta_3 = 0$. Now taking into account the results of the θ^1 models [44,45], where in the NC SU(N) model [45] was explicitly shown that the θ^1 parameter runs while having a negative β function, we conjecture that β functions associated with the parameters κ, η_1, and $(2\eta_4 - \eta_5)$ could be negative too. The $(\eta_2 - \eta_3)$ difference should have a zero β function. However, a precise computation of the above β functions goes well beyond the scope of this paper and, therefore, such considerations should be left for the future.

VII. CONCLUSION

We introduce three different types of nonlocal four-photon interactions in a full-fledged θ-exact deformed U(1) gauge field theory. The first two actions (I,II) are deformed by the two distinct θ-exact gauge field strength Seiberg-Witten (SW) maps, and the third one (III) involves a

---

[5]That would not be strange at all, since our deformation parameters are actually quantities similar to coupling constants—sitting in the actions—of any QFT.

SW-map inspired gauge invariance deformation in a more general way.

The four-photon interaction induces the tadpole diagram which results in a pure quadratic IR divergence. Based on the computation of that tadpole, we conjecture that the general origin of the quadratic IR divergence, including those from the bubble diagram, stems from the tadpolelike integrals, which although highly UV divergent still vanish under the dimensional regularization in the commutative theory [38].

In conclusion, by performing the sum over one-loop bubble and tadpole diagrams in all three models, we obtain for arbitrary noncommutative tensors $\theta^{\mu\nu}$ and with suitable choices of freedom parameters, the quadratic IR divergence free-photon polarization tensor. A simultaneous UV, quadratic IR, and logarithmic divergences cancellation does occur only in model III by employing the special $\theta = \theta_2$ matrix and the deformation parameter set $\kappa = \frac{1}{3}$, $\eta_1 = 0$, $\eta_2 = \eta_3$, $2\eta_4 - \eta_5 = 3$. The most important fact is definitely a huge freedom due to the SW mapping and gauge invariance, which at the end of the day allows elimination of all pathologies in the one-loop two point function, which is quite aspiring although there is still a very long way before one could speculate a renormalizable U(1) theory on Moyal space. The IR improvement we have achieved till now seems to suggest that there should be some way out within the SW map approach, therefore we believe that further investigation along this line could help reaching an accurate formulation with SW map and provide some answers about the renormalizability of the NCQFT in general.

Besides the splitting of the polarization states driven by the breaking of Lorentz invariance caused by $\theta^{\mu\nu}$, the persistence of a finite additional term/effect (48) of quantum-gravity origin may also affect the low-momentum end of the spectrum in the photon dispersion relation. The serious possibility of eliminating all pathological effects in the photon polarization tensor within NCQFT is our main motivation to continue searching for the noncommutative-geometry/quantum-gravity-inspired UV/IR mixing phenomena and connections between the NCQFT and holography, black hole horizon physics, etc. [16–18,20,22].

APPENDIX A: EXPLICIT GAUGE INVARIANT ACTION

The U(1) gauge invariance of the four-photon interaction (9) and (10) is quite lengthy to verify by directly evaluating the infinitesimal gauge transformation. It is convenient to put them into the same form as (5) but not an easy task because of the loss of cyclicity. In this appendix we show that this step is achievable by analyzing the integration-by-part in the momentum space.

We first notice that the generalized star products based on the constant Moyal deformation parameter θ^{ij} bear a relatively common form in momentum space,

$$[f_1...f_n]_{\mathcal{M}}(x) = \int \prod_{i=1}^n \frac{d^d p_i}{(2\pi)^d} \prod_{i=1}^n \tilde{f}_i(p_i) \exp\left[-i\left(\sum_{i=1}^n p_i\right)x\right]\mathcal{F}(p_1,...,p_n;\theta^{ij}). \tag{A1}$$

A few examples relevant to this paper include

$$(f \star g)(x) = \int e^{-i(p+q)x}\tilde{f}(p)\tilde{g}(q)f_\star(p,q), \qquad (f \star_2 g)(x) = \int e^{-i(p+q)x}\tilde{f}(p)\tilde{g}(q)f_{\star_2}(p,q),$$

$$[fgh]_{\star_3}(x) = \int e^{-i(p+q+k)x}\tilde{f}(p)\tilde{g}(q)\tilde{h}(k)f_{\star_3}(p,q,k), \qquad [fgh]_{\star_{3'}}(x) = \int e^{-i(p+q+k)x}\tilde{f}(p)\tilde{g}(q)\tilde{h}(k)f_{\star_{3'}}(p,q,k), \tag{A2}$$

with

$$f_\star(p,q) = \exp\left(i\frac{p\theta q}{2}\right), \qquad f_{\star_2}(p,q) = \frac{\sin\frac{p\theta q}{2}}{\frac{p\theta q}{2}}, \qquad f_{\star_3}(p,q,k) = \frac{\sin\frac{p\theta k}{2}\sin(\frac{p\theta q}{2}+\frac{p\theta k}{2})}{(\frac{p\theta q}{2}+\frac{p\theta k}{2})(\frac{p\theta k}{2}+\frac{q\theta k}{2})} + \frac{\sin\frac{q\theta k}{2}\sin(\frac{p\theta q}{2}-\frac{q\theta k}{2})}{(\frac{p\theta q}{2}-\frac{q\theta k}{2})(\frac{p\theta k}{2}+\frac{q\theta k}{2})},$$

$$f_{\star_{3'}}(p,q,k) = \frac{\cos(\frac{p\theta q}{2}+\frac{p\theta k}{2}-\frac{q\theta k}{2})-1}{(\frac{p\theta q}{2}+\frac{p\theta k}{2}-\frac{q\theta k}{2})\frac{q\theta k}{2}} - \frac{\cos(\frac{p\theta q}{2}+\frac{p\theta k}{2}+\frac{q\theta k}{2})-1}{(\frac{p\theta q}{2}+\frac{p\theta k}{2}+\frac{q\theta k}{2})\frac{q\theta k}{2}}. \tag{A3}$$

Since $\mathcal{F}(p_1,\ldots,p_n;\theta^{ij})$ is coordinate independent, one can introduce general integration-by-part transformation

$$\int d^d x\, g(x)[f_1\ldots f_n]_{\mathcal{F}'}(x) = \int d^d x\, f_i(x)[f_1\ldots f_{i-1}gf_{i+1}\ldots f_n]_{\mathcal{F}}(x) \tag{A4}$$

by setting

$$\mathcal{F}(p_1,\ldots p_{i-1},q_g,p_{i+1},\ldots,p_n;\theta^{ij})\big|_{q_g=-\sum_{i=1}^n p_i} = \mathcal{F}'(p_1,\ldots,p_n;\theta^{ij}), \tag{A5}$$

because

$$\int d^d x\, f_i(x)[f_1\ldots f_{i-1}gf_{i+1}\ldots f_n]_{\mathcal{F}}(x)$$

$$= \int d^d x \int \frac{d^d q_g}{(2\pi)^d}\prod_{i=1}^n\frac{d^d p_i}{(2\pi)^d}\tilde{g}(q_g)\prod_{i=1}^n\tilde{f}_i(p_i)\exp\left[-i\left(q_g+\sum_{i=1}^n p_i\right)x\right]\mathcal{F}(p_1,\ldots p_{i-1},q_g,p_{i+1},\ldots,p_n;\theta^{ij})$$

$$= \int\frac{d^d q_g}{(2\pi)^d}\prod_{i=1}^n\frac{d^d p_i}{(2\pi)^d}\tilde{g}(q_g)\prod_{i=1}^n\tilde{f}_i(p_i)\mathcal{F}(p_1,\ldots p_{i-1},q_g,p_{i+1},\ldots,p_n;\theta^{ij})(2\pi)^d\delta^d\left(q_g+\sum_{i=1}^n p_i\right)$$

$$= \int\prod_{i=1}^n\frac{d^d p_i}{(2\pi)^d}\prod_{i=1}^n\tilde{f}_i(p_i)(\tilde{g}(q_g)\mathcal{F}(p_1,\ldots p_{i-1},q_g,p_{i+1},\ldots,p_n;\theta^{ij}))\big|_{q_g=-\sum_{i=1}^n p_i}$$

$$= \int\prod_{i=1}^n\frac{d^d p_i}{(2\pi)^d}\prod_{i=1}^n\tilde{f}_i(p_i)\tilde{g}\left(q_g=-\sum_{i=1}^n p_i\right)\mathcal{F}'(p_1,\ldots,p_n;\theta^{ij})$$

$$= \int\frac{d^d q_g}{(2\pi)^d}\prod_{i=1}^n\frac{d^d p_i}{(2\pi)^d}\tilde{g}(q_g)\prod_{i=1}^n\tilde{f}_i(p_i)\mathcal{F}'(p_1,\ldots,p_n;\theta^{ij})(2\pi)^d\delta^d\left(q_g+\sum_{i=1}^n p_i\right)$$

$$= \int d^d x\, g(x)[f_1\ldots f_n]_{\mathcal{F}'}(x). \tag{A6}$$

Once $\mathcal{F}'$ equals $\mathcal{F}$ we obtain certain level of cyclicity in a slightly generalized sense. For example,

$$f_\star(p_3,p_1)\big|_{p_1=-(p_2+p_3)} = f_\star(p_2,p_3), \tag{A7}$$

gives the 3-cyclicity of the Moyal star product:

$$\int f(x)(g(x)\star h(x)) = \int g(x)(h(x)\star f(x)). \tag{A8}$$

Using the same method one can also show the following relations:

$$\int f(g\star_2 h) = \int g(h\star_2 f) = \int h(g\star_2 f), \quad \int f(g\star_2(h\star_2 k)) = \int g(f\star_2(h\star_2 k)), \quad \int f[ghk]_{\star_{3'}} = \int g[fhk]_{\star_{3'}}. \tag{A9}$$

The simple relations above are not yet sufficient to perform the integration-by-part that we want. However, they do indicate that one should look for more momentum space identities first. We achieve this goal by analyzing the Ward identity $p_{i_{\mu_i}}\Gamma^{\cdots\mu_i\cdots}, i = 1,\ldots,4$. Here we consider only the part of the action(s) which is not yet explicitly gauge invariant, for example from (9),

$$S_{(I)_r}^{e^2} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int (a_i \star_2 \partial_j f_{\mu\nu})(a_k \star_2 \partial_l f^{\mu\nu}) + \frac{1}{2}f^{\mu\nu}(2[a_i\partial_j a_k\partial_l f_{\mu\nu}]_{\star_{3'}} + 2[\partial_l f_{\mu\nu}a_i\partial_j a_k]_{\star_{3'}}$$
$$+ 2[a_i a_k\partial_j\partial_l f_{\mu\nu}]_{\star_{3'}} - [\partial_l f_{\mu\nu}a_i\partial_k a_j]_{\star_{3'}} - [a_i\partial_k a_j\partial_l f_{\mu\nu}]_{\star_{3'}}). \tag{A10}$$

From the interaction above we can then read out a vertex

$$\Gamma_{(I)_r}^{\mu_1\mu_2\mu_3\mu_4} \propto (2((p_1 p_4)g^{\mu_1\mu_4} - p_1^{\mu_4}p_4^{\mu_1})((2(\theta p_3)^{\mu_2}(\theta p_4)^{\mu_3} + 2(\theta p_4)^{\mu_2}(\theta p_4)^{\mu_3} - (p_3\theta p_4)\theta^{\mu_2\mu_3})f_{\star_{3'}}[p_4, p_2, p_3]$$
$$+ (2(\theta p_3)^{\mu_2}(\theta p_4)^{\mu_3} - (p_3\theta p_4)\theta^{\mu_2\mu_3})f_{\star_{3'}}[p_4, p_2, p_3] + 2(\theta p_1)^{\mu_2}(\theta p_4)^{\mu_3}f_{\star_2}(p_1, p_2)f_{\star_2}(p_3, p_4))$$
$$+ \text{all } S_4 \text{ permutations over } \{p_i\} \quad \text{and} \quad \{\mu_i\} \text{ simultaneously}) \cdot \delta(p_1 + p_2 + p_3 + p_4). \tag{A11}$$

Next we find that the Ward identity is satisfied once we sum over the following four permutations: identity, $2\leftrightarrow3$, $1\leftrightarrow4$, and the composition of the last two. The rest of the S_4 permutations only change the labeling for the contraction. Two identities emerge in this analysis:

$$f_{\star_{3'}}[p_3, p_3, p_4] + f_{\star_{3'}}[p_3, p_2, -(p_2 + p_3 + p_4)] = 2f_{\star_2}(p_2, p_3 + p_4)f_{\star_2}(p_3, p_4), \tag{A12}$$

$$2(p_2\theta p_3)f_{\star_{3'}}[p_4, p_2, p_3] = (p_3\theta p_4)f_{\star_{3'}}[p_2, p_3, p_4] - (p_3\theta(p_2 + p_4))f_{\star_{3'}}[p_2, p_3, -(p_2 + p_3 + p_4)]$$
$$- (p_2\theta p_4)f_{\star_{3'}}[p_3, p_2, p_4] + (p_2\theta(p_3 + p_4))f_{\star_{3'}}[p_3, p_2, p_1]. \tag{A13}$$

We can construct the following integration-by-part relations out of these two identities,

$$2\int f(g \star_2 (h \star_2 k)) = \int f[ghk]_{\star_{3'}} + k[hgf]_{\star_{3'}}, \tag{A14}$$

and

$$2\theta^{ij}\int f[k\partial_i g\partial_j h]_{\star_{3'}} = \theta^{ij}\int f[g\partial_i h\partial_j k]_{\star_{3'}} + k[g\partial_i h\partial_j f]_{\star_{3'}} - f[h\partial_i g\partial_j k]_{\star_{3'}} - k[h\partial_i g\partial_j f]_{\star_{3'}}. \tag{A15}$$

Now we return to (A10) and rearrange the first term using (A14) and the last term using (A15), getting

$$\int \theta^{ij}\theta^{kl}(a_i \star_2 \partial_j f_{\mu\nu})(a_k \star_2 \partial_l f^{\mu\nu}) = \frac{1}{2}\int \theta^{ij}\theta^{kl}(\partial_j f_{\mu\nu}[a_i a_k\partial_l f^{\mu\nu}]_{\star_{3'}} + \partial_l f^{\mu\nu}[a_k a_i\partial_j f_{\mu\nu}]_{\star_{3'}})$$
$$= \int \theta^{ij}\theta^{kl}\partial_j f_{\mu\nu}[a_i a_k\partial_l f^{\mu\nu}]_{\star_{3'}}, \tag{A16}$$

$$\int \theta^{ij}\theta^{kl}f_{\mu\nu}[a_i\partial_k a_j\partial_l f^{\mu\nu}]_{\star_{3'}} = \frac{1}{4}\int \theta^{ij}\theta^{kl}(f_{\mu\nu}[a_i\partial_k a_j\partial_l f^{\mu\nu}]_{\star_{3'}} + f_{\mu\nu}[a_i\partial_k a_j\partial_l f^{\mu\nu}]_{\star_{3'}}$$
$$- f_{\mu\nu}[a_j\partial_k a_i\partial_l f^{\mu\nu}]_{\star_{3'}} - f_{\mu\nu}[a_j\partial_k a_i\partial_l f^{\mu\nu}]_{\star_{3'}})$$
$$= \frac{1}{2}\int \theta^{ij}\theta^{kl}f^{\mu\nu}[f_{\mu\nu}\partial_k a_i\partial_l a_j]_{\star_{3'}}. \tag{A17}$$

One can also use the partial 4-cyclicity of $\star_{3'}$ to obtain

$$\int \theta^{ij}\theta^{kl}f^{\mu\nu}[\partial_l f_{\mu\nu}a_i\partial_k a_j]_{\star_{3'}} = \int \theta^{ij}\theta^{kl}\partial_l f_{\mu\nu}[f^{\mu\nu}a_i\partial_k a_j]_{\star_{3'}} = -\frac{1}{2}\int \theta^{ij}\theta^{kl}f^{\mu\nu}[f_{\mu\nu}\partial_l a_i\partial_k a_j]_{\star_{3'}}, \tag{A18}$$

$$\int \theta^{ij}\theta^{kl}f^{\mu\nu}[\partial_l f_{\mu\nu}a_i\partial_j a_k]_{\star_{3'}} = \int \theta^{ij}\theta^{kl}\partial_l f^{\mu\nu}[f_{\mu\nu}a_i\partial_j a_k]_{\star_{3'}} = -\frac{1}{2}\int \theta^{ij}\theta^{kl}f^{\mu\nu}[f_{\mu\nu}\partial_l(a_i\partial_j a_k)]_{\star_{3'}}. \tag{A19}$$

Now (A10) boils down to

$$S_{(I)_r}^{e^2} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int -f^{\mu\nu}[\partial_j a_i a_k \partial_l f_{\mu\nu}]_{\star_{3'}} -\frac{1}{2}f^{\mu\nu}[f_{\mu\nu}a_k\partial_l\partial_j a_i]_{\star_{3'}} +\frac{1}{2}f^{\mu\nu}([f_{\mu\nu}\partial_k a_i\partial_l a_j]_{\star_{3'}} - [f_{\mu\nu}\partial_l a_i\partial_j a_k]_{\star_{3'}})$$

$$= -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int -f^{\mu\nu}[\partial_j a_i a_k\partial_l f_{\mu\nu}]_{\star_{3'}} -\frac{1}{2}f^{\mu\nu}[f_{\mu\nu}a_k\partial_l\partial_j a_i]_{\star_{3'}} -\frac{1}{4}f^{\mu\nu}[f_{\mu\nu}f_{ik}f_{jl}]_{\star_{3'}}. \tag{A20}$$

We then notice that the above first term is

$$\int \theta^{ij}\theta^{kl}f^{\mu\nu}[\partial_j a_i a_k\partial_l f_{\mu\nu}]_{\star_{3'}} = -\int \theta^{ij}\theta^{kl}(\partial_l f^{\mu\nu}[\partial_j a_i a_k f_{\mu\nu}]_{\star_{3'}} + f^{\mu\nu}[\partial_j a_i\partial_l a_k f_{\mu\nu}]_{\star_{3'}} + f^{\mu\nu}[\partial_l\partial_j a_i a_k f_{\mu\nu}]_{\star_{3'}})$$

$$= -\int \theta^{ij}\theta^{kl}\left(\partial_l f^{\mu\nu}[\partial_j a_i a_k f_{\mu\nu}]_{\star_{3'}} + \frac{1}{4}f^{\mu\nu}[f_{ij}f_{kl}f_{\mu\nu}]_{\star_{3'}} + f^{\mu\nu}[\partial_l\partial_j a_i a_k f_{\mu\nu}]_{\star_{3'}}\right); \tag{A21}$$

therefore,

$$S_{(I)_r}^{e^2} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int \frac{1}{2}(\partial_l f^{\mu\nu}[\partial_j a_i a_k f_{\mu\nu}]_{\star_{3'}} - f^{\mu\nu}[\partial_j a_i a_k\partial_l f_{\mu\nu}]_{\star_{3'}}) + \frac{1}{2}(f^{\mu\nu}[\partial_l\partial_j a_i a_k f_{\mu\nu}]_{\star_{3'}}$$

$$- f^{\mu\nu}[f_{\mu\nu}a_k\partial_l\partial_j a_i]_{\star_{3'}}) - \frac{1}{4}f^{\mu\nu}[f_{\mu\nu}f_{ik}f_{jl}]_{\star_{3'}} + \frac{1}{8}f^{\mu\nu}[f_{ij}f_{kl}f_{\mu\nu}]_{\star_{3'}}. \tag{A22}$$

The first two parentheses can be put into a common formula,

$$\theta^{ij}\theta^{kl}\int \frac{1}{2}(\partial_l f^{\mu\nu}[\partial_j a_i a_k f_{\mu\nu}]_{\star_{3'}} - f^{\mu\nu}[\partial_j a_i a_k\partial_l f_{\mu\nu}]_{\star_{3'}}) + \frac{1}{2}(f^{\mu\nu}[\partial_l\partial_j a_i a_k f_{\mu\nu}]_{\star_{3'}} - f^{\mu\nu}[f_{\mu\nu}a_k\partial_l\partial_j a_i]_{\star_{3'}})$$

$$= \frac{1}{4}\theta^{ij}\theta^{kl}\theta^{pq}\int f^{\mu\nu}[\partial_p\partial_j a_i\partial_q a_k\partial_l f_{\mu\nu}]_{\mathcal{M}_{(I)}} - \partial_l f^{\mu\nu}[\partial_p\partial_j a_i\partial_q a_k f_{\mu\nu}]_{\mathcal{M}_{(I)}} + f^{\mu\nu}[\partial_l\partial_j a_i\partial_p a_k\partial_q f_{\mu\nu}]_{\mathcal{M}_{(I)}}$$

$$- \partial_q f^{\mu\nu}[\partial_l\partial_j a_i\partial_p a_k f_{\mu\nu}]_{\mathcal{M}_{(I)}}$$

$$= \frac{1}{4}\theta^{ij}\theta^{kl}\theta^{pq}\int -f^{\mu\nu}[\partial_l\partial_j a_i\partial_k a_p\partial_q f_{\mu\nu}]_{\mathcal{M}_{(I)}} + \partial_q f^{\mu\nu}[\partial_l\partial_j a_i\partial_k a_p f_{\mu\nu}]_{\mathcal{M}_{(I)}} + f^{\mu\nu}[\partial_l\partial_j a_i\partial_p a_k\partial_q f_{\mu\nu}]_{\mathcal{M}_{(I)}}$$

$$- \partial_q f^{\mu\nu}[\partial_l\partial_j a_i\partial_p a_k f_{\mu\nu}]_{\mathcal{M}_{(I)}}$$

$$= \frac{1}{4}\theta^{ij}\theta^{kl}\theta^{pq}\int f^{\mu\nu}[\partial_i f_{jk}f_{lp}\partial_q f_{\mu\nu}]_{\mathcal{M}_{(I)}} - \partial_q f_{\mu\nu}[\partial_i f_{jk}f_{lp}f^{\mu\nu}]_{\mathcal{M}_{(I)}}, \tag{A23}$$

by using a new 3-product

$$[fgh]_{\mathcal{M}_{(I)}}(x) = \int e^{-i(p+q+k)x}\tilde{f}(p)\tilde{g}(q)\tilde{h}(k)f_{(I)}(p,q,k), \tag{A24}$$

where

$$f_{(I)}(p,q,k) = (p\theta q)^{-1}(f_{\star_{3'}}[p,q,-(p+q+k)] - f_{\star_{3'}}[p,q,k])$$

$$= \frac{2}{(\frac{p\theta q}{2} - \frac{p\theta k}{2} - \frac{q\theta k}{2})(\frac{p\theta q}{2} + \frac{p\theta k}{2} - \frac{q\theta k}{2})(\frac{p\theta q}{2} + \frac{p\theta k}{2} + \frac{q\theta k}{2})} + \frac{\cos(\frac{p\theta q}{2} - \frac{p\theta k}{2} - \frac{q\theta k}{2})}{2\frac{p\theta q}{2}(\frac{p\theta q}{2} - \frac{q\theta k}{2})(\frac{p\theta q}{2} - \frac{p\theta k}{2} - \frac{q\theta k}{2})}$$

$$+ \frac{\cos(\frac{p\theta q}{2} + \frac{p\theta k}{2} - \frac{q\theta k}{2})}{2\frac{q\theta k}{2}(\frac{p\theta q}{2} - \frac{q\theta k}{2})(\frac{p\theta q}{2} + \frac{p\theta k}{2} - \frac{q\theta k}{2})} + \frac{\cos(\frac{p\theta q}{2} + \frac{p\theta k}{2} + \frac{q\theta k}{2})}{2\frac{p\theta q}{2}\frac{q\theta k}{2}(\frac{p\theta q}{2} + \frac{p\theta k}{2} + \frac{q\theta k}{2})}. \tag{A25}$$

It is not hard to show that the $\mathcal{M}_{(I)}$ product satisfies an "anticyclicity,"

$$\int f[ghk]_{\mathcal{M}_{(I)}}(x) = -\int k[ghf]_{\mathcal{M}_{(I)}}(x), \tag{A26}$$

because

$$f_{(I)}(p,q,k) = -f_{(I)}(p,q,-(p+q+k)). \qquad (A27)$$

Consequently,

$$S^{e^2}_{(I)_r} = -\frac{e^2}{4}\theta^{ij}\theta^{kl}\int -\frac{1}{4}f^{\mu\nu}[f_{\mu\nu}f_{ik}f_{jl}]_{\star_{3'}}$$
$$+\frac{1}{8}f^{\mu\nu}[f_{ij}f_{kl}f_{\mu\nu}]_{\star_{3'}}$$
$$+\frac{1}{2}\theta^{pq}f^{\mu\nu}[\partial_i f_{jk}f_{lp}\partial_q f_{\mu\nu}]_{\mathcal{M}_{(I)}}. \qquad (A28)$$

Finally, one can use (A14) once more to show

$$\int \theta^{ij}\theta^{kl}f^{\mu\nu}[f_{ij}f_{kl}f_{\mu\nu}]_{\star_{3'}}$$
$$=\frac{1}{2}\int \theta^{ij}\theta^{kl}f^{\mu\nu}([f_{ij}f_{kl}f_{\mu\nu}]_{\star_{3'}} + [f_{kl}f_{ij}f_{\mu\nu}]_{\star_{3'}})$$
$$=\int \theta^{ij}\theta^{kl}(f_{\mu\nu}\star_2 f_{ij})(f_{kl}\star_2 f^{\mu\nu}). \qquad (A29)$$

Thus, in the end we indeed obtain (12) from (9).

Explicit gauge invariance of action (10) can be verified either by adding the ambiguity terms derived in [33] to (12), or by repeating the procedure above and obtaining (13), where a different 3-product $[fgh]_{\mathcal{M}_{(II)}}(x)$ is defined as below:

$$[fgh]_{\mathcal{M}_{(II)}}(x) = \int e^{-i(p+q+k)x}\tilde{f}(p)\tilde{g}(q)\tilde{h}(k)f_{(II)}(p,q,k),$$
$$\qquad (A30)$$

$$f_{(II)}(p,q,k) = \frac{\sin\left(\frac{p\theta q}{2}-\frac{q\theta k}{2}\right)\sin\frac{p\theta k}{2}}{\frac{p\theta q}{2}\frac{p\theta k}{2}\left(\frac{p\theta q}{2}-\frac{q\theta k}{2}\right)\left(\frac{p\theta k}{2}+\frac{q\theta k}{2}\right)}$$
$$-\frac{\sin\left(\frac{p\theta q}{2}+\frac{p\theta k}{2}\right)\sin\frac{q\theta k}{2}}{\frac{p\theta q}{2}\frac{q\theta k}{2}\left(\frac{p\theta q}{2}+\frac{p\theta k}{2}\right)\left(\frac{p\theta k}{2}+\frac{q\theta k}{2}\right)}. \qquad (A31)$$

(Un)like the $\mathcal{M}_{(I)}$ product, the $\mathcal{M}_{(II)}$ product satisfies a cyclicity condition,

$$\int f[ghk]_{\mathcal{M}_{(II)}}(x) = \int k[ghf]_{\mathcal{M}_{(II)}}(x), \qquad (A32)$$

since

$$f_{(II)}(p,q,k) = f_{(II)}(p,q,-(p+q+k)). \qquad (A33)$$

APPENDIX B: VERTICES

In this section we present the detailed definition of each terms in the four-photon interaction vertex in the momentum space $\Gamma^{\mu_1\mu_2\mu_3\mu_4}_{(I,II)}(p_1,p_2,p_3,p_4)$:

$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_A(p_1,p_2,p_3,p_4) = f_{\star_2}(p_1,p_2)f_{\star_2}(p_3,p_4)V^{\mu_1\mu_2\mu_3\mu_4}_A(p_1,p_2,p_3,p_4),$$
$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_B(p_1,p_2,p_3,p_4) = f_{\star_2}(p_1,p_2)f_{\star_2}(p_3,p_4)V^{\mu_1\mu_2\mu_3\mu_4}_B(p_1,p_2,p_3,p_4), \qquad (B1)$$

$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_1(p_1,p_2,p_3,p_4) = 2f_{\star_{3'}}(p_2,p_3,p_4)V^{\mu_1\mu_2\mu_3\mu_4}_1(p_1,p_2,p_3,p_4),$$
$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_2(p_1,p_2,p_3,p_4) = (2f_{\star_2}(p_1,p_2)f_{\star_2}(p_3,p_4)-f_{\star_{3'}}(p_2,p_3,p_4)-f_{\star_{3'}}(p_4,p_3,p_2))V^{\mu_1\mu_2\mu_3\mu_4}_2(p_1,p_2,p_3,p_4),$$
$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_3(p_1,p_2,p_3,p_4) = f_{\star_{3'}}(p_4,p_2,p_3)V^{\mu_1\mu_2\mu_3\mu_4}_3(p_1,p_2,p_3,p_4),$$
$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_4(p_1,p_2,p_3,p_4) = f_{\star_2}(p_1,p_2)f_{\star_2}(p_3,p_4)V^{\mu_1\mu_2\mu_3\mu_4}_4(p_1,p_2,p_3,p_4),$$
$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}_5(p_1,p_2,p_3,p_4) = f_{(I)}(p_2,p_3,p_4)V^{\mu_1\mu_2\mu_3\mu_4}_5(p_1,p_2,p_3,p_4), \qquad (B2)$$

$$\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_1(p_1,p_2,p_3,p_4) = (4f_{\star_2}(p_1,p_2)f_{\star_2}(p_3,p_4)-2f_{\star_3}(p_2,p_3,p_4))V^{\mu_1\mu_2\mu_3\mu_4}_1(p_1,p_2,p_3,p_4),$$
$$\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_2(p_1,p_2,p_3,p_4) = 2(f_{\star_3}(p_2,p_3,p_4)-f_{\star_2}(p_2,p_3)f_{\star_2}(p_1,p_4))V^{\mu_1\mu_2\mu_3\mu_4}_2(p_1,p_2,p_3,p_4),$$
$$\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_3(p_1,p_2,p_3,p_4) = (3f_{\star_2}(p_1,p_2)-2f_{\star_3}(p_2,p_3,p_4))V^{\mu_1\mu_2\mu_3\mu_4}_3(p_1,p_2,p_3,p_4),$$
$$\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_4(p_1,p_2,p_3,p_4) = (2f_{\star_2}(p_1,p_2)-f_{\star_3}(p_2,p_3,p_4))V^{\mu_1\mu_2\mu_3\mu_4}_4(p_1,p_2,p_3,p_4),$$
$$\Gamma'^{\mu_1\mu_2\mu_3\mu_4}_5(p_1,p_2,p_3,p_4) = f_{(II)}(p_2,p_3,p_4)V'^{\mu_1\mu_2\mu_3\mu_4}_5(p_1,p_2,p_3,p_4). \qquad (B3)$$

The V_i tensor structures are listed next

$$V_A^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = (p_1p_3)(p_2p_4)\theta^{\mu_1\mu_2}\theta^{\mu_3\mu_4} - (p_1p_3)\theta^{\mu_1\mu_2}(\theta p_4)^{\mu_3}p_2^{\mu_4} + (p_2p_4)\theta^{\mu_1\mu_2}p_1^{\mu_3}(\theta p_3)^{\mu_4}$$
$$+ (p_3\theta p_4)\theta^{\mu_1\mu_2}p_1^{\mu_3}p_2^{\mu_4} - (p_1p_3)(\theta p_2)^{\mu_1}p_4^{\mu_2}\theta^{\mu_3\mu_4} + (p_1p_3)(\theta p_2)^{\mu_1}(\theta p_4)^{\mu_3}g^{\mu_2\mu_4}$$
$$- (\theta p_2)^{\mu_1}p_4^{\mu_2}p_1^{\mu_3}(\theta p_3)^{\mu_4} - (p_3\theta p_4)(\theta p_2)^{\mu_1}p_1^{\mu_3}g^{\mu_2\mu_4} + (p_2p_4)p_3^{\mu_1}(\theta p_1)^{\mu_2}\theta^{\mu_3\mu_4}$$
$$- p_3^{\mu_1}(\theta p_1)^{\mu_2}(\theta p_4)^{\mu_3}g^{\mu_2\mu_4} + (p_2p_4)g^{\mu_1\mu_3}(\theta p_1)^{\mu_2}(\theta p_3)^{\mu_4} + (p_3\theta p_4)g^{\mu_1\mu_3}(\theta p_1)^{\mu_2}p_2^{\mu_4}$$
$$+ (p_1\theta p_2)p_3^{\mu_1}p_4^{\mu_2}\theta^{\mu_3\mu_4} - (p_1\theta p_2)p_3^{\mu_1}(\theta p_4)^{\mu_3}g^{\mu_2\mu_4} + (p_1\theta p_2)g^{\mu_1\mu_3}p_4^{\mu_2}(\theta p_3)^{\mu_4}$$
$$+ (p_1\theta p_2)(p_3\theta p_4)g^{\mu_1\mu_3}g^{\mu_2\mu_4}, \tag{B4}$$

$$V_B^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = 2(\theta p_1)^{\mu_1}((p_2p_3)p_4^{\mu_2}\theta^{\mu_3\mu_4} - (p_2p_3)(\theta p_4)^{\mu_3}g^{\mu_2\mu_4} + p_4^{\mu_2}p_2^{\mu_3}(\theta p_3)^{\mu_4}$$
$$+ (p_3\theta p_4)p_2^{\mu_3}g^{\mu_2\mu_4} - (p_2p_4)p_3^{\mu_2}\theta^{\mu_3\mu_4} + p_3^{\mu_2}(\theta p_4)^{\mu_3}p_2^{\mu_4} - (p_2p_4)g^{\mu_2\mu_3}(\theta p_3)^{\mu_4}$$
$$- (p_3\theta p_4)g^{\mu_2\mu_3}p_2^{\mu_4}), \tag{B5}$$

$$V_1^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = 2(p_1p_2)(p_3^{\mu_1}(\theta p_4)^{\mu_2}\theta^{\mu_3\mu_4} - p_3^{\mu_1}(\theta p_4)^{\mu_3}\theta^{\mu_2\mu_4} + g^{\mu_1\mu_3}(\theta p_4)^{\mu_2}(\theta p_3)^{\mu_4} + (p_3\theta p_4)g^{\mu_1\mu_3}\theta^{\mu_2\mu_4})$$
$$- 2p_1^{\mu_2}((p_2\theta p_4)p_3^{\mu_1}\theta^{\mu_3\mu_4} + p_3^{\mu_1}(\theta p_4)^{\mu_3}(\theta p_2)^{\mu_4} + (p_2\theta p_4)g^{\mu_1\mu_3}(\theta p_3)^{\mu_4} - (p_3\theta p_4)g^{\mu_1\mu_3}(\theta p_2)^{\mu_4})$$
$$+ 2(p_1p_3)(p_2^{\mu_1}(\theta p_4)^{\mu_3}\theta^{\mu_2\mu_4} - p_2^{\mu_1}(\theta p_4)^{\mu_2}\theta^{\mu_3\mu_4} + g^{\mu_1\mu_2}(\theta p_4)^{\mu_3}(\theta p_2)^{\mu_4} + (p_2\theta p_4)g^{\mu_1\mu_2}\theta^{\mu_3\mu_4})$$
$$- 2p_1^{\mu_3}((p_3\theta p_4)p_2^{\mu_1}\theta^{\mu_2\mu_4} + p_2^{\mu_1}(\theta p_4)^{\mu_2}(\theta p_3)^{\mu_4} + (p_3\theta p_4)g^{\mu_1\mu_2}(\theta p_2)^{\mu_4} - (p_2\theta p_4)g^{\mu_1\mu_2}(\theta p_3)^{\mu_4}),$$
$$V_2^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = (\theta p_4)^{\mu_3}((p_1p_2)p_4^{\mu_1}\theta^{\mu_2\mu_4} - (p_1p_2)(\theta p_4)^{\mu_2}g^{\mu_1\mu_4} + p_1^{\mu_2}(\theta p_2)^{\mu_4}p_4^{\mu_1} + p_1^{\mu_2}(p_2\theta p_4)g^{\mu_1\mu_4}$$
$$- (p_1p_4)p_2^{\mu_1}\theta^{\mu_1\mu_4} + p_1^{\mu_4}p_2^{\mu_1}(\theta p_4)^{\mu_2} - (p_1p_4)(\theta p_2)^{\mu_4}g^{\mu_1\mu_2} - p_1^{\mu_4}(p_2\theta p_4)g^{\mu_1\mu_2}),$$
$$V_3^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = -((p_1p_4)g^{\mu_1\mu_4} - p_1^{\mu_4}p_4^{\mu_1})((\theta p_2)^{\mu_3}(\theta p_3)^{\mu_2} + \theta^{\mu_2\mu_3}(p_2\theta p_3)),$$
$$V_4^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = ((p_1p_4)g^{\mu_1\mu_4} - p_1^{\mu_4}p_4^{\mu_1})(\theta p_2)^{\mu_2}(\theta p_3)^{\mu_3},$$
$$V_5^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = -((p_1p_4)g^{\mu_1\mu_4} - p_1^{\mu_4}p_4^{\mu_1})(\theta p_2)^{\mu_2} \cdot ((\theta p_2)^{\mu_3}(p_3\theta p_4) + (\theta p_4)^{\mu_3}(p_2\theta p_3)),$$
$$V_5'^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = ((p_1p_4)g^{\mu_1\mu_4} - p_1^{\mu_4}p_4^{\mu_1}) \cdot ((\theta p_2)^{\mu_2}(p_1\theta p_2)((\theta p_2)^{\mu_3}(p_3\theta p_4) + (\theta p_4)^{\mu_3}(p_2\theta p_3))$$
$$- ((\theta p_2)^{\mu_3}(p_3\theta p_4) + (\theta p_4)^{\mu_3}(p_2\theta p_3)) \cdot ((\theta p_3)^{\mu_2}(p_2\theta p_4) - (\theta p_4)^{\mu_2}(p_2\theta p_3))). \tag{B6}$$

APPENDIX C: PARTIAL TENSOR REDUCTION AND INTEGRATION RESULTS

The partial $\tau(\mathcal{T})$-tensor reduction results are listed below

$$\tau_{(I)}^{\mu\nu} = -\frac{1}{2D}\{[g^{\mu\nu}p^2 - p^\mu p^\nu](tr\theta\theta)(\kappa_3 - 1) + [g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu}p^2 + p^{\{\mu}(\theta\theta p)^{\nu\}}]4(\theta p)^2(\kappa_1 - \kappa_2)$$
$$+ (\theta p)^\mu(\theta p)^\nu(4 + (1 - \kappa_3)D(D-1) - 16\kappa + 8\kappa^2 + 8(D-1)(\kappa_1 - \kappa_2) + 4\kappa_4)\}, \tag{C1}$$

$$\tau_{(II)}^{\mu\nu} = -\frac{1}{2D}\{[g^{\mu\nu}p^2 - p^\mu p^\nu](tr\theta\theta)3(\kappa_3 - 1) + [g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu}p^2 + p^{\{\mu}(\theta\theta p)^{\nu\}}]8(\theta p)^2(\kappa_1' - \kappa_2')$$
$$+ (\theta p)^\mu(\theta p)^\nu(3(1 - \kappa_3')D(D-1) - 16\kappa + 8\kappa^2 + 16(D-1)(\kappa_1' - \kappa_2') + 8\kappa_4')\}, \tag{C2}$$

$$\mathcal{T}_{(\text{I})}^{\mu\nu\rho\sigma} = -\frac{1}{2}\{(g^{\mu\nu}(\theta p)^\rho(\theta p)^\sigma + \theta^{\mu\rho}p^\nu(\theta p)^\sigma + \theta^{\nu\rho}p^\mu(\theta p)^\sigma + \theta^{\mu\rho}\theta^{\nu\sigma}p^2)\cdot 2((D-3)\kappa^2 - 2\kappa + \kappa_1 + \kappa_2)$$

$$+ (2g^{\mu\nu}p^\rho(\theta\theta p)^\sigma - g^{\mu\rho}p^\nu(\theta\theta p)^\sigma - g^{\nu\rho}p^\mu(\theta\theta p)^\sigma - p^\mu(\theta\theta)^{\nu\rho}p^\sigma - p^\nu(\theta\theta)^{\mu\rho}p^\sigma$$

$$+ g^{\mu\rho}(\theta\theta)^{\nu\sigma}p^2 + g^{\nu\rho}(\theta\theta)^{\mu\sigma}p^2)\cdot(2\kappa - \kappa_1 - \kappa_2)$$

$$+ (g^{\mu\rho}(\theta p)^\nu(\theta p)^\sigma + g^{\nu\rho}(\theta p)^\mu(\theta p)^\sigma + \theta^{\mu\rho}(\theta p)^\nu p^\sigma + \theta^{\nu\rho}(\theta p)^\mu p^\sigma)\cdot(-1 - 2\kappa_3 + \kappa_4 + (2+D)\kappa_1$$

$$+ (D-2)(\kappa_2 - 2\kappa) + 4\kappa^2)$$

$$+ (g^{\mu\nu}g^{\rho\sigma}(\theta p)^2 + (\theta\theta)^{\mu\nu}(p^\rho p^\sigma - p^2 g^{\rho\sigma}) + (p^\mu(\theta\theta p)^\nu + p^\nu(\theta\theta p)^\mu)g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}(\theta p)^2 - g^{\mu\rho}(\theta\theta p)^\nu p^\sigma$$

$$- g^{\nu\rho}(\theta\theta p)^\mu p^\sigma)\cdot 2\kappa^2$$

$$+ (\theta p)^\mu(\theta p)^\nu g^{\rho\sigma}\cdot 4(\kappa_1 + \kappa_2 - 2\kappa + (D-1)\kappa_4) - [g^{\mu\nu}p^2 - p^\mu p^\nu](\theta\theta)^{\rho\sigma}(\kappa_4 - 1)\},\quad\text{(C3)}$$

$$\mathcal{T}_{(\text{II})}^{\mu\nu\rho\sigma} = -\frac{1}{2}\{(g^{\mu\nu}(\theta p)^\rho(\theta p)^\sigma + \theta^{\mu\rho}p^\nu(\theta p)^\sigma + \theta^{\nu\rho}p^\mu(\theta p)^\sigma - \theta^{\mu\rho}\theta^{\nu\sigma}p^2)\cdot 2((D-3)\kappa^2 - 2\kappa + 2\kappa_2')$$

$$+ (2g^{\mu\nu}p^\rho(\theta\theta p)^\sigma - g^{\mu\rho}p^\nu(\theta\theta p)^\sigma - g^{\nu\rho}p^\mu(\theta\theta p)^\sigma$$

$$- p^\mu(\theta\theta)^{\nu\rho}p^\sigma - p^\nu(\theta\theta)^{\mu\rho}p^\sigma + g^{\mu\rho}(\theta\theta)^{\nu\sigma}p^2 + g^{\nu\rho}(\theta\theta)^{\mu\sigma}p^2)\cdot 2(\kappa - \kappa_2')$$

$$+ (g^{\mu\rho}(\theta p)^\nu(\theta p)^\sigma + g^{\nu\rho}(\theta p)^\mu(\theta p)^\sigma + \theta^{\mu\rho}(\theta p)^\nu p^\sigma + \theta^{\nu\rho}(\theta p)^\mu p^\sigma)\cdot(-2\kappa_3' - 2(D-2)\kappa + 4\kappa^2 + 2(D-2)\kappa_2')$$

$$+ (g^{\mu\nu}g^{\rho\sigma}(\theta p)^2 + (\theta\theta)^{\mu\nu}(p^\rho p^\sigma - p^2 g^{\rho\sigma}) + (p^\mu(\theta\theta p)^\nu + p^\nu(\theta\theta p)^\mu)g^{\rho\sigma}$$

$$- g^{\mu\rho}g^{\nu\sigma}(\theta p)^2 - g^{\mu\rho}(\theta\theta p)^\nu p^\sigma - g^{\nu\rho}(\theta\theta p)^\mu p^\sigma)\cdot 2\kappa^2$$

$$+ (\theta p)^\mu(\theta p)^\nu g^{\rho\sigma}\cdot(8\kappa_2' - 8\kappa + (D-1)(2\kappa_3' + \kappa_4' - 3)) - [g^{\mu\nu}p^2 - p^\mu p^\nu](\theta\theta)^{\rho\sigma}(2\kappa_3' + \kappa_4' - 3)\}.\quad\text{(C4)}$$

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