# Effective low-energy potential for slow Dirac fermions in Einstein-Cartan gravity with torsion and chameleon field

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We derive the most general effective low-energy potential to order $O(1/m)$ for slow Dirac fermions with mass $m$, coupled to gravitational, chameleon and torsion fields in the Einstein-Cartan gravity. The obtained results can be applied to the experimental analysis of gravitational, chameleon and torsion interactions in terrestrial laboratories. We discuss the use of rotating coordinate systems, caused by rotations of devices, for measurements of the torsion vector and tensor components, caused by minimal torsion-fermion couplings [A. N. Ivanov and M. Wellenzohn, Phys. Rev. D 92, 065006 (2015)]. Using the most general form of a metric tensor of curved spacetimes in rotating coordinate systems, proposed by Obukhov, Silenko, and Teryaev [Phys. Rev. D 84, 024025 (2011)], we extend this metric by the inclusion of the chameleon field and calculate the set of vierbein fields, in terms of which Dirac fermions couple to the torsion vector and tensor components through minimal torsion-fermion couplings. For such a set of vierbein fields we discuss a part of the effective low-energy potential for slow Dirac fermions, coupled to gravitational, chameleon and torsion fields to order $O(1)$ in the large fermion mass expansion.

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## I. INTRODUCTION

In terrestrial laboratories [1–7] gravitational and chameleon interactions are being investigated in terms of cold and ultracold neutrons through some effective low-energy potentials [8–11]. The low-energy torsion-fermion interactions of the pseudoscalar and axial-vector components of the torsion field have been derived and estimated by Lämmerzahl [12] and Obukhov, Silenko, and Teryaev [13]. The most general torsion-fermion interactions of constant torsion fields have been proposed and estimated by Kostelecky, Russell, and Tasson [14]. The results, obtained by Lämmerzahl [12], Obukhov *et al.* [13] and Kostelecky *et al.* [14], have been discussed in Ref. [15]. An attempt of a direct measurement of torsion-fermion interactions with constant torsion fields, proposed by Kostelecky *et al.* [14], has been undertaken by Lehnert, Snow and Yan [16].

In this paper we derive the most general effective low-energy potential to order $1/m$ for slow Dirac fermions with mass $m$, coupled to gravitational, chameleon and torsion fields in the Einstein-Cartan gravity.

The chameleon part of such an effective low-energy potential contains new chameleon-fermion interactions with respect to those calculated in Refs. [10,15]. These new chameleon-fermion interactions can be used for more detailed experimental analysis of the properties of the chameleon field [17,18]. The chameleon field, changing its mass in dependence of the mass density of the environment, has been invented to avoid the problem of the violation of the equivalence principle [19]. In addition the chameleon field can be also identified with a quintessence (canonical scalar field) [20,21], which has been postulated for an explanation of the late-time acceleration of the Universe's expansion [22–24]. The laboratory probes of the chameleon field, coupled to matter in a conformal way [17,18], may also shed light on dark energy dynamics [25–28].

Torsion (in addition to the metric tensor) is a natural geometrical quantity characterizing spacetime geometry through spin-matter interactions [29,30]. It allows one to probe rotational degrees of freedom of spacetime in terrestrial laboratories. Torsion is described by a third-order tensor $\mathcal{T}_{\sigma\mu\nu}$, antisymmetric with respect to indices $\mu$ and $\nu$, i.e. $\mathcal{T}_{\sigma\mu\nu} = -\mathcal{T}_{\sigma\nu\mu}$ [29,30]. It possesses 24 independent components, which can be decomposed into four axial-vector $\mathcal{B}_\mu$, four vector $\mathcal{E}_\mu$ and 16 tensor $\mathcal{M}_{\sigma\mu\nu}$ components [14] [see also Eqs. (6) and (7)]. The torsion part of the effective low-energy potential, derived in our paper, is caused by torsion-fermion minimal couplings for all torsion components only. An importance of this part of the effective low-energy potential is related to a possible solution of the following problem. As has been shown in Ref. [15], only torsion axial-vector $\mathcal{B}_\mu$ components are present in the torsion-fermion minimal couplings in the curved spacetimes with metrics, providing vanishing time-space (space-time) components of the vierbein fields. Since these are usual metrics of spacetimes in terrestrial laboratories, in such spacetimes torsion vector $\mathcal{E}_\mu$ and tensor $\mathcal{M}_{\sigma\mu\nu}$ components, coupled to Dirac fermions, can be introduced only through nonminimal torsion-fermion

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couplings with phenomenological coupling constants [14] (see also Ref. [15]). The presence of phenomenological coupling constants screens real values of the torsion vector $\mathcal{E}_\mu$ and tensor $\mathcal{M}_{\sigma\mu\nu}$ components. Thus, a search for possible ways of measuring torsion vector $\mathcal{E}_\mu$ and tensor $\mathcal{M}_{\sigma\mu\nu}$ components through torsion-fermion minimal couplings is of great importance for understanding the correct values of the torsion.

We show that these measurements can be in principle possible in curved spacetimes, described by metric tensors with nondiagonal components. These metric tensors define nonvanishing time-space (space-time) components of the vierbein fields, in terms of which slow fermions couple to torsion vector $\mathcal{E}_\mu$ and tensor $\mathcal{M}_{\sigma\mu\nu}$ components through minimal torsion-fermion couplings.

It is well known [31] that in rotating coordinate systems spacetimes are described by nondiagonal metric tensors. In terrestrial laboratories spacetimes with nondiagonal metric tensors can be in principle realized by means of rotating devices (neutron interferometers) [32,33] (see also a book by Rauch and Werner [34] for necessary information on neutron interferometry). Thus, we propose to measure torsion vector $\vec{\mathcal{E}}$ and tensor $\mathcal{M}_{\sigma\mu\nu}$ components through minimal torsion-fermion couplings in rotating coordinate systems, caused by rotating devices. In our analysis of curved spacetimes in rotating coordinate systems we follow the papers by Hehl and Ni [35] and Obukhov, Silenko, and Teryaev [36,37].

The paper is organized as follows. In Sec. II we derive the most general Hamilton operator for the Dirac fermions in the Einstein-Cartan gravity with chameleon and torsion fields. We adduce the Schrödinger-Pauli equation for slow Dirac fermions, coupled to the effective low-energy potential, caused by gravitational, chameleon and torsion fields. In Sec. III we i) calculate the vierbein fields, related to the most general metric tensors of the curved spacetimes in rotating coordinate systems, proposed by Obukhov, Silenko, and Teryaev [37], ii) derive the effective low-energy torsion-fermion potential to order $O(1)$ in the large fermion mass expansion and iii) discuss possible measurements of torsion vector and tensor components in such curved spacetimes. In the Appendix we give a detailed derivation of the effective low-energy potential for slow Dirac fermions, coupled to gravitational, chameleon and torsion fields in the Einstein-Cartan gravity.

## II. SLOW DIRAC FERMIONS IN THE EINSTEIN-CARTAN GRAVITY WITH TORSION AND CHAMELEON

In gravitational theories with a chameleon field fermions couple to the chameleon field $\phi(x)$ through the metric $\tilde{g}_{\mu\nu}(x)$ in the Jordan frame related to the metric $g_{\mu\nu}(x)$ in the Einstein frame by $\tilde{g}_{\mu\nu}(x) = f^2(x) g_{\mu\nu}(x)$, where $f(x) = e^{\beta\phi(x)/M_{\rm Pl}}$ is the conformal factor [17,18] (see also Ref. [11]), $\beta$ is the chameleon-matter coupling constant and $M_{\rm Pl} = 1/\sqrt{8\pi G_N} = 2.435 \times 10^{27}$ eV is the reduced Planck mass and $G_N$ is the gravitational coupling [38].

For the derivation of the required effective low-energy potential we start with the analysis of the Dirac equation for fermions with mass $m$, coupled to the chameleon field in the spacetime with torsion and the metric $d\tilde{s}^2 = \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu$ (the Jordan-frame metric). We take the Dirac-fermion action in the following form [39] (see also Ref. [15]):

$$\mathrm{S}_\psi = \int d^4x \sqrt{-\tilde{g}} \left( i \frac{1}{2} \tilde{e}^\mu_{\hat{\lambda}}(x) \bar{\psi}(x) \gamma^{\hat{\lambda}} \overleftrightarrow{D}_\mu \psi(x) - m \bar{\psi}(x) \psi(x) \right), \tag{1}$$

where $\tilde{e}^\mu_{\hat{\lambda}}(x)$ and $\gamma^{\hat{\lambda}}$ are the vierbein fields, mapping the curved spacetime onto the Minkowski spacetime [39] (see also Refs. [11,15]), and the Dirac matrices in the Minkowski spacetime [40], respectively. The first term in the brackets of Eq. (1) takes the form $\tilde{e}^\mu_{\hat{\lambda}}(x) \bar{\psi}(x) \gamma^{\hat{\lambda}} \overleftrightarrow{D}_\mu \psi(x) = \tilde{e}^\mu_{\hat{\lambda}}(x) (\bar{\psi}(x) \gamma^{\hat{\lambda}} D_\mu \psi(x) - (\bar{\psi}(x) \bar{D}_\mu) \gamma^{\hat{\lambda}} \psi(x)$ [39], where $D_\mu \psi(x)$ and $(\bar{\psi}(x) \bar{D}_\mu)$ are the covariant derivatives defined by [11,15]

$$\begin{aligned} D_\mu \psi(x) &= \partial_\mu \psi(x) - \tilde{\Gamma}_\mu(x) \psi(x), \\ (\bar{\psi}(x) \bar{D}_\mu) &= \partial_\mu \bar{\psi}(x) - \gamma^{\hat{0}} \tilde{\Gamma}^\dagger_\mu(x) \gamma^{\hat{0}}. \end{aligned} \tag{2}$$

The spin affine connection $\tilde{\Gamma}_\mu(x)$ is given by [15]

$$\tilde{\Gamma}_\mu(x) = \frac{i}{4} \tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}}(x) \sigma^{\hat{\alpha}\hat{\beta}}, \tag{3}$$

where $\sigma^{\hat{\alpha}\hat{\beta}} = (i/2)(\gamma^{\hat{\alpha}} \gamma^{\hat{\beta}} - \gamma^{\hat{\beta}} \gamma^{\hat{\alpha}})$ are the Dirac matrices in the Minkowski spacetime [40] and the spin connection $\tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}}(x)$ is related to the vierbein fields $\tilde{e}^\mu_{\hat{\lambda}}(x)$ and the affine connection $\tilde{\Gamma}^\alpha{}_{\mu\nu}(x) = \{\widetilde{{}^\alpha{}_{\mu\nu}}\} + \tilde{\mathcal{K}}^\alpha{}_{\mu\nu}(x)$ as follows [15]:

$$\tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}}(x) = -\eta_{\hat{\alpha}\hat{\varphi}} (\partial_\mu \tilde{e}^{\hat{\varphi}}_\nu(x) - \tilde{\Gamma}^\alpha{}_{\mu\nu}(x) \tilde{e}^{\hat{\varphi}}_\alpha(x)) \tilde{e}^\nu_{\hat{\beta}}(x), \tag{4}$$

where $\{\widetilde{{}^\alpha{}_{\mu\nu}}\}$ are the Christoffel symbols [31]

$$\{\widetilde{{}^\alpha{}_{\mu\nu}}\} = \frac{1}{2} \tilde{g}^{\alpha\lambda} \left( \frac{\partial \tilde{g}_{\lambda\mu}}{\partial x^\nu} + \frac{\partial \tilde{g}_{\lambda\nu}}{\partial x^\mu} - \frac{\partial \tilde{g}_{\mu\nu}}{\partial x^\lambda} \right) \tag{5}$$

and $\tilde{\mathcal{K}}^\alpha{}_{\mu\nu}(x) = -\frac{1}{2}(\tilde{\mathcal{T}}^\alpha{}_{\mu\nu}(x) - \tilde{\mathcal{T}}_\mu{}^\alpha{}_\nu(x) - \tilde{\mathcal{T}}_\nu{}^\alpha{}_\mu(x))$ is the contorsion tensor, expressed in terms of the torsion field $\tilde{\mathcal{T}}^\alpha{}_{\mu\nu}(x) = \tilde{g}^{\alpha\sigma}(x) \tilde{\mathcal{T}}_{\sigma\mu\nu}(x)$ [15]. The torsion field $\tilde{\mathcal{T}}_{\sigma\mu\nu}(x)$ can be represented in the following irreducible form [14] (see also Ref. [15]):

$$\tilde{\mathcal{T}}_{\sigma\mu\nu}(x) = \frac{1}{3}(\tilde{g}_{\sigma\mu}(x)\tilde{\mathcal{E}}_{\nu}(x) - \tilde{g}_{\sigma\nu}(x)\tilde{\mathcal{E}}_{\mu}(x)) + \frac{1}{3}\tilde{\varepsilon}_{\sigma\mu\nu\alpha}(x)\tilde{\mathcal{B}}^{\alpha}(x) + \tilde{\mathcal{M}}_{\sigma\mu\nu}(x), \tag{6}$$

where the 4-vector $\tilde{\mathcal{E}}_{\nu}(x)$ and axial 4-vector $\tilde{\mathcal{B}}^{\alpha}(x)$ fields, possessing four independent components each, are defined by

$$\tilde{\mathcal{E}}_{\nu}(x) = \tilde{g}^{\sigma\mu}(x)\tilde{\mathcal{T}}_{\sigma\mu\nu}(x), \qquad \tilde{\mathcal{B}}^{\alpha}(x) = \frac{1}{2}\tilde{\varepsilon}^{\alpha\sigma\mu\nu}(x)\tilde{\mathcal{T}}_{\sigma\mu\nu}(x). \tag{7}$$

Here $\tilde{\varepsilon}_{\sigma\mu\nu\alpha}(x) = \sqrt{-\tilde{g}(x)}\epsilon_{\sigma\mu\nu\alpha}$ and $\tilde{\varepsilon}^{\alpha\sigma\mu\nu}(x) = \epsilon^{\alpha\sigma\mu\nu}/\sqrt{-\tilde{g}(x)}$ are covariant Levi-Civita tensors in the curved spacetime with the Jordan metric $\tilde{g}_{\mu\nu}(x)$ and the definition $\epsilon^{0123} = -\epsilon_{0123} = +1$ [31]. For the derivation of the axial-vector field $\tilde{\mathcal{B}}^{\alpha}(x)$ in terms of the torsion tensor field $\tilde{\mathcal{T}}_{\sigma\mu\nu}(x)$ we have used the relation $\epsilon^{\alpha\sigma\mu\nu}\epsilon_{\sigma\mu\nu\beta} = -6\delta^{\alpha}_{\beta}$ [40].

The residual 16 independent components of the torsion field $\tilde{\mathcal{T}}_{\sigma\mu\nu}(x)$ can be attributed to the tensor field $\tilde{\mathcal{M}}_{\sigma\mu\nu}(x)$, which obeys the constraints $\tilde{g}^{\sigma\mu}(x)\tilde{\mathcal{M}}_{\sigma\mu\nu}(x) = \tilde{\varepsilon}^{\alpha\sigma\mu\nu}(x)\tilde{\mathcal{M}}_{\sigma\mu\nu}(x) = 0$ [14].

We have carried out the derivation of the Dirac equation in the curved spacetime with the metric tensor $\tilde{g}_{\mu\nu}(x)$ and torsion in the Appendix of Ref. [15]. The result is

$$\Big(i\tilde{e}^{\mu}_{\hat{\lambda}}(x)\gamma^{\hat{\lambda}}D_{\mu} - \frac{1}{2}i\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\gamma^{\hat{\lambda}} - \frac{1}{2}i\tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x) \times \Big(\eta^{\hat{\lambda}\hat{\beta}}\gamma^{\hat{\alpha}} + \frac{1}{4}i[\sigma^{\hat{\alpha}\hat{\beta}}, \gamma^{\hat{\lambda}}]\Big) - m\Big)\psi(x) = 0, \tag{8}$$

where $[\sigma^{\hat{\alpha}\hat{\beta}}, \gamma^{\hat{\lambda}}] = \sigma^{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\lambda}} - \gamma^{\hat{\lambda}}\sigma^{\hat{\alpha}\hat{\beta}} = 2i(\eta^{\hat{\beta}\hat{\lambda}}\gamma^{\hat{\alpha}} - \eta^{\hat{\lambda}\hat{\alpha}}\gamma^{\hat{\beta}})$ [15]. The Dirac equation (8) agrees well with that derived by Kostelecky [see Eq. (18) of Ref. [39]]. The vierbein fields $\tilde{e}^{\hat{\alpha}}_{\mu}(x)$ and $\tilde{e}^{\mu}_{\hat{\alpha}}(x)$ in the Jordan frame are related to the vierbein fields $e^{\hat{\alpha}}_{\mu}(x)$ and $e^{\mu}_{\hat{\alpha}}(x)$ in the Einstein frame by [11]

$$\tilde{e}^{\hat{\alpha}}_{\mu}(x) = f(x)e^{\hat{\alpha}}_{\mu}(x), \qquad \tilde{e}^{\mu}_{\hat{\alpha}}(x) = e^{\mu}_{\hat{\alpha}}(x)/f(x). \tag{9}$$

The Dirac equation (8) in its standard form reads

$$i\frac{\partial\psi(t,\vec{r})}{\partial t} = \mathrm{H}\psi(t,\vec{r}), \tag{10}$$

where $\psi(t,\vec{r})$ is the Dirac wave function of the fermion with mass $m$. The Hamilton operator H in its nonperturbative form is given by [15]

$$\mathrm{H} = \tilde{E}^{\hat{0}}_{0}(x)\gamma^{\hat{0}}m - \tilde{E}^{\hat{0}}_{0}(x)\tilde{e}^{0}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial t} - \tilde{E}^{\hat{0}}_{0}(x)\tilde{e}^{j}_{\hat{\lambda}}(x)i\gamma^{\hat{0}}\gamma^{\hat{\lambda}}\frac{\partial}{\partial x^{j}} + \frac{1}{2}i\tilde{E}^{\hat{0}}_{0}(x)\gamma^{\hat{0}}(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\gamma^{\hat{\lambda}} + \tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}}\gamma^{\hat{\alpha}}) + \frac{1}{4}\tilde{E}^{\hat{0}}_{0}(x)\tilde{\omega}_{\mu\hat{\alpha}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\epsilon^{\hat{\lambda}\hat{\alpha}\hat{\beta}\hat{\rho}}\gamma^{\hat{0}}\gamma_{\hat{\rho}}\gamma^{5}, \tag{11}$$

where the vierbein field $\tilde{E}^{\hat{0}}_{0}(x)$ is defined by $\tilde{E}^{\hat{0}}_{0}(x) = \tilde{e}^{\hat{0}}_{0}(x)/(1-\tilde{e}^{\hat{0}}_{\hat{j}}(x)\tilde{e}^{j}_{\hat{0}}(x)) = \tilde{e}^{\hat{0}}_{0}(x)/(1-\tilde{e}^{\hat{0}}_{\hat{j}}(x)\tilde{e}^{\hat{j}}_{\hat{0}}(x)) = 1/\tilde{e}^{0}_{\hat{0}}(x)$. The definition for the vierbein field $\tilde{E}^{\hat{0}}_{0}(x)$ follows from the relations $\tilde{e}^{\mu}_{\hat{\alpha}}(x)\tilde{e}^{\hat{\beta}}_{\mu}(x) = \delta^{\hat{\beta}}_{\hat{\alpha}}$ and $\tilde{e}^{\mu}_{\hat{\alpha}}(x)\tilde{e}^{\hat{\alpha}}_{\nu}(x) = \delta^{\mu}_{\nu}$. In addition we have used that $\{\sigma^{\hat{\alpha}\hat{\beta}}, \gamma^{\hat{\lambda}}\} = \sigma^{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\lambda}} + \gamma^{\hat{\lambda}}\sigma^{\hat{\alpha}\hat{\beta}} = -2\epsilon^{\hat{\lambda}\hat{\alpha}\hat{\beta}\hat{\rho}}\gamma_{\hat{\rho}}\gamma^{5}$ [15] and $\gamma^{5} = i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}}$ [40]. For the derivation of the effective low-energy potential it is convenient to transcribe the Hamilton operator (11) into the form

$$\mathrm{H} = \tilde{E}^{\hat{0}}_{0}(x)\gamma^{\hat{0}}m + \frac{1}{2}i\tilde{E}^{\hat{0}}_{0}(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{0}}(x) + \tilde{\omega}_{\mu\hat{0}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}}) + \frac{1}{4}\tilde{E}^{\hat{0}}_{0}(x)(\tilde{\omega}_{\mu\hat{j}\hat{k}}(x)\tilde{e}^{\mu}_{\hat{0}}(x) + \tilde{\omega}_{\mu[\hat{0}\hat{j}]}(x)\tilde{e}^{\mu}_{\hat{k}}(x))\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} - \tilde{E}^{\hat{0}}_{0}(x)\tilde{e}^{j}_{\hat{0}}(x)i\frac{\partial}{\partial x^{j}} - \tilde{E}^{\hat{0}}_{0}(x)\tilde{e}^{j}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial x^{j}} - \tilde{E}^{\hat{0}}_{0}(x)\tilde{e}^{0}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial t} + \frac{1}{2}\tilde{E}^{\hat{0}}_{0}(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{j}}(x) + \tilde{\omega}_{\mu\hat{j}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}})i\gamma^{\hat{0}}\gamma^{\hat{j}} - \frac{1}{4}\tilde{E}^{\hat{0}}_{0}(x)\tilde{\omega}_{\mu\hat{j}\hat{k}}(x)\tilde{e}^{\mu}_{\hat{\ell}}(x)\epsilon^{\hat{j}\hat{k}\hat{\ell}}\gamma^{5}, \tag{12}$$

where we have denoted $\tilde{\omega}_{\mu[\hat{0}\hat{j}]}(x) = \tilde{\omega}_{\mu\hat{0}\hat{j}}(x) - \tilde{\omega}_{\mu\hat{j}\hat{0}}(x)$ and used $\epsilon^{\hat{\ell}\hat{j}\hat{k}\hat{0}} = -\epsilon^{\hat{j}\hat{k}\hat{\ell}}$ [40]. It is well known [13] that the Hamilton operator (12) is not Hermitian. This is because of the factor $\sqrt{-\tilde{g}}$ in the definition of the four-dimensional covariant volume element $d^4x\sqrt{-\tilde{g}}$ in the curved spacetime. In order to deal with the Hermitian Hamilton operator we have to make the following transformation of the fermion wave function and the Hamilton operator [13]:

$$\psi(x) = \Big(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x)\Big)^{-1/2}\psi'(x),$$
$$\mathrm{H}' = \Big(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x)\Big)^{+1/2}\mathrm{H}\Big(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x)\Big)^{-1/2} - i\Big(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x)\Big)^{+1/2}\frac{\partial}{\partial t}\Big(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x)\Big)^{-1/2}. \tag{13}$$

We would like to note that the last term in the Hamilton operator $\mathrm{H}'$ acts on the fermion wave function as a multiplication operator and does not differentiate it with respect to time. This means that we do not need to add in the definition of the Hamilton operator $\mathrm{H}'$ a time derivative operator $i\partial/\partial t$ [41–43]. The Dirac equation for the wave function $\psi'(t,\vec{r})$ retains its standard form

$$i\frac{\partial\psi'(t,\vec{r})}{\partial t} = \mathrm{H}'\psi'(t,\vec{r}), \tag{14}$$

where the Hermitian Hamilton operator $\mathrm{H}'$ is given by

$$\begin{aligned}
\mathrm{H}' &= \tilde{E}^{\hat{0}}_0(x)\gamma^{\hat{0}}m + \frac{1}{2}i\tilde{E}^{\hat{0}}_0(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{0}}(x) + \tilde{\omega}_{\mu\hat{0}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}}) + \frac{1}{2}i\tilde{E}^{\hat{0}}_0(x)\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial t}(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x))\\
&+ \frac{1}{4}\tilde{E}^{\hat{0}}_0(x)(\tilde{\omega}_{\mu\hat{j}\hat{k}}(x)\tilde{e}^{\mu}_{\hat{0}}(x) + \tilde{\omega}_{\mu[\hat{0}\hat{j}]}(x)\tilde{e}^{\mu}_{\hat{k}}(x))\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} + \frac{1}{2}i(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^{j}_{\hat{0}}(x)\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x))\\
&- \tilde{E}^{\hat{0}}_0(x)\tilde{e}^{j}_{\hat{0}}(x)i\frac{\partial}{\partial x^j} - \tilde{E}^{\hat{0}}_0(x)\tilde{e}^{j}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial x^j} - \tilde{E}^{\hat{0}}_0(x)\tilde{e}^{0}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial t}\\
&+ \frac{1}{2}(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^{j}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x)) + \frac{1}{2}(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^{0}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial t}(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x))\\
&+ \frac{1}{2}\tilde{E}^{\hat{0}}_0(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{j}}(x) + \tilde{\omega}_{\mu\hat{j}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}})i\gamma^{\hat{0}}\gamma^{\hat{j}} - \frac{1}{4}\tilde{\omega}_{\mu\hat{j}\hat{k}}(x)\tilde{E}^{\hat{0}}_0(x)\tilde{e}^{\mu}_{\hat{\ell}}(x)\epsilon^{\hat{j}\hat{k}\hat{\ell}}\gamma^{\hat{5}}.
\end{aligned} \tag{15}$$

Using the relation [see Eq. (A-9) of Ref. [15]]

$$\frac{1}{2}i\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial}{\partial t}(\sqrt{-\tilde{g}}\tilde{e}^{0}_{\hat{0}}(x)) = -\frac{1}{2}i\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{0}}(x) - \frac{1}{2}i\tilde{\omega}_{\mu\hat{0}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}} - \frac{1}{2}i\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}}\tilde{e}^{j}_{\hat{0}}(x)) \tag{16}$$

we transcribe the Hamilton operator (15) into the form

$$\begin{aligned}
\mathrm{H}' &= \tilde{E}^{\hat{0}}_0(x)\gamma^{\hat{0}}m - \frac{1}{2}i\tilde{E}^{\hat{0}}_0(x)\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}}\tilde{e}^{j}_{\hat{0}}(x)) + \frac{1}{2}i(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^{j}_{\hat{0}}(x)\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x))\\
&+ \frac{1}{4}\tilde{E}^{\hat{0}}_0(x)(\tilde{\omega}_{\mu\hat{j}\hat{k}}(x)\tilde{e}^{\mu}_{\hat{0}}(x) + \tilde{\omega}_{\mu[\hat{0}\hat{j}]}(x)\tilde{e}^{\mu}_{\hat{k}}(x))\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} - \tilde{E}^{\hat{0}}_0(x)\tilde{e}^{j}_{\hat{0}}(x)i\frac{\partial}{\partial x^j}\\
&- \tilde{E}^{\hat{0}}_0(x)\tilde{e}^{j}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial x^j} - \tilde{E}^{\hat{0}}_0(x)\tilde{e}^{0}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial t} + \frac{1}{2}(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^{j}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x))\\
&- \frac{1}{2}(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^{0}_{\hat{j}}(x)i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}(x)}\tilde{e}^{j}_{\hat{0}}(x)) + \frac{1}{2}\tilde{E}^{\hat{0}}_0(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{j}}(x) + \tilde{\omega}_{\mu\hat{j}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}})i\gamma^{\hat{0}}\gamma^{\hat{j}}\\
&- \frac{1}{2}(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^{0}_{\hat{j}}(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{0}}(x) + \tilde{\omega}_{\mu\hat{0}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}})i\gamma^{\hat{0}}\gamma^{\hat{j}} - \frac{1}{4}\tilde{\omega}_{\mu\hat{j}\hat{k}}(x)\tilde{E}^{\hat{0}}_0(x)\tilde{e}^{\mu}_{\hat{\ell}}(x)\epsilon^{\hat{j}\hat{k}\hat{\ell}}\gamma^{\hat{5}}.
\end{aligned} \tag{17}$$

According to the Foldy-Wouthuysen classification [44], the operators in the first two lines of Eq. (17) are *even*, whereas all other operators are *odd*. For a derivation of a low-energy effective Hamilton operator of slow fermions all *odd* operators should be removed by some unitary transformations [44]. Skipping standard intermediate Foldy-Wouthuysen calculations, which are given in the Appendix, we arrive at the Schrödinger-Pauli equation

$$i\frac{\partial\Psi(t,\vec{r})}{\partial t} = \left(-\frac{1}{2m}\Delta + \Phi_{\rm eff}(t,\vec{r},\vec{\sigma})\right)\Psi(t,\vec{r}), \tag{18}$$

where $\Psi(t,\vec{r})$ is the large component of the Dirac wave function of slow fermions with mass $m$ and $\Phi_{\rm eff}(t,\vec{r},\vec{\sigma})$ is the effective low-energy potential for slow fermions, coupled to gravitational, chameleon and torsion fields, and $\vec{\sigma}$ are the $2\times 2$ Pauli matrices [40]. The exact expression of the potential $\Phi_{\rm eff}(t,\vec{r},\vec{\sigma})$, calculated to order $O(1/m)$, is given by Eq. (A15) of the Appendix.

## III. CONCLUSIVE DISCUSSION

We have analyzed the low-energy approximation of the Dirac equation for fermions with mass $m$ in the

Einstein-Cartan gravity with torsion and chameleon fields. Using the Foldy-Wouthuysen transformations we have derived the most general low-energy potential to order $1/m$ for slow Dirac fermions, coupled to gravitational, torsion and chameleon fields. The aim of the derivation of such an effective low-energy potential is addressed to the investigation of spacetimes in which torsion vector $\vec{\mathcal{E}}$ and tensor $\mathcal{M}_{jk\ell}$ components, coupled minimally to slow Dirac fermions, can be in principle observable. As has been shown in Ref. [15] for metric tensors, yielding vanishing nondiagonal time-space (space-time) components of the vierbein fields, in the perturbative regime for gravitational, torsion and chameleon fields only torsion axial-vector components survive in the low-energy approximation of the minimal torsion-fermion couplings.

In the Appendix for the derivation of the effective low-energy potential $\Phi_{\rm eff}(t,\vec{r},\vec{\sigma})$ [see Eq. (A15)] we have introduced the operators $A$, $B$, $C^{\hat{\ell}}$, $D^{j}_{\hat{j}}$, $F_{\hat{j}}$, $G_{\hat{j}}$, $K$ and $L^{j}$, which are defined in Eq. (A2). In the approximation [15] applied to our approach these operators behave as follows:

$$
\begin{aligned}
&A = 1 + O(g,\phi), \qquad B = 0,\\
&C^{\hat{\ell}} = \frac{1}{4}\mathcal{B}^{\hat{\ell}} + O(g,\phi), \qquad D^{j}_{\hat{j}} = \delta^{j}_{\hat{j}} + O(g,\phi),\\
&F_{\hat{j}} = 0, \qquad G_{\hat{j}} = O(g,\phi),\\
&K = -\frac{1}{4}\mathcal{K} + O(g,\phi), \qquad L^{j} = 0,
\end{aligned}
\tag{19}
$$

where $\mathcal{B}^{\hat{\ell}} = \frac{1}{2}\epsilon^{\hat{\ell}\hat{j}\hat{k}}(\mathcal{T}_{\hat{j}\hat{k}\hat{0}} + \mathcal{T}_{\hat{k}\hat{0}\hat{j}} + \mathcal{T}_{\hat{0}\hat{j}\hat{k}})$ and $\mathcal{K} = \frac{1}{2}\epsilon^{\hat{j}\hat{k}\hat{\ell}}\mathcal{T}_{\hat{\ell}\hat{j}\hat{k}}$ are the torsion axial-vector and pseudoscalar components [15], and $O(g,\phi)$ are the linear-order contributions of gravitational and chameleon fields. The nontrivial linear-order contributions of the torsion vector and tensor components can appear only in spacetimes with nondiagonal metric tensors, yielding nonvanishing nondiagonal time-space (space-time) components of the vierbein fields. It is well known that in the rotating coordinate system spacetime is described by a nondiagonal metric tensor [31] with the nonvanishing time-space (space-time) components $g_{0j}(x)$ proportional to the angular velocity (see also Refs. [35–37]).

The phase shift induced by a rotational motion of an optical interferometer was first proposed by Sagnac [45] and observed by Michelson, Gale, and Pearson [46]. In spite of the fact that the inertial properties of photons and neutrons are different, the analogous effect for the phase shift of slow neutrons was predicted by Page [47] and measured by Werner *et al.* [48], Atwood *et al.* [32] and Mashhoon [33]. For the measurement of such a phase shift Atwood *et al.* [32] and Mashhoon [33] used the rotating two-crystal neutron interferometer and the neutron interferometer in the rotating reference frame, respectively. According to an equivalence between a rotating coordinate system and a gravitational field or a curved spacetime with a corresponding metric tensor [31], the experimental setup of the experiments by Atwood *et al.* [32] and Mashhoon [33] should determine metric tensors of curved spacetimes, created by rotating neutron interferometers in the gravitational field of the Earth.

Following such an equivalence, for the experimental analysis of fermion-torsion interactions, described by the effective low-energy potential $\Phi_{\rm eff}(t,\vec{r},\vec{\sigma})$ given by Eq. (A15), we have to determine the metric tensor of the curved spacetime and calculate the vierbein fields, caused by the experimental setup of possible experiments. We take the line element in spacetimes, created by rotating devices in an arbitrary gravitational field, in the most general form, proposed by Obukhov, Silenko, and Teryaev [37]:

$$
\begin{aligned}
d\tilde{s}^2 = {}& \tilde{V}^2(x)dt^2 + \eta_{\hat{j}\hat{\ell}}\tilde{W}^{\hat{j}}_{j}(x)\tilde{W}^{\hat{\ell}}_{\ell}(x)(dx^{j} - K^{j}(x)dt)\\
&\times (dx^{\ell} - K^{\ell}(x)dt),
\end{aligned}
\tag{20}
$$

where the functions $\tilde{V}^2(x)$ and $\tilde{W}^{\hat{j}}_{j}(x)$ are defined by an arbitrary gravitational field. In comparison with Obukhov *et al.* [37] they are modified by the chameleon field. In turn, the functions $K^{j}(x)$ are caused by rotations. The components of the metric tensor $\tilde{g}_{\mu\nu}(x)$ are equal to

$$
\begin{aligned}
&\tilde{g}_{00}(x) = \tilde{V}^2(x) + \eta_{\hat{j}\hat{\ell}}\tilde{W}^{\hat{j}}_{j}(x)\tilde{W}^{\hat{\ell}}_{\ell}(x)K^{j}(x)K^{\ell}(x),\\
&\tilde{g}_{0j}(x) = -\eta_{\hat{j}\hat{\ell}}\tilde{W}^{\hat{j}}_{j}(x)\tilde{W}^{\hat{\ell}}_{\ell}(x)K^{\ell}(x),\\
&\tilde{g}_{j\ell}(x) = \eta_{\hat{j}\hat{\ell}}\tilde{W}^{\hat{j}}_{j}(x)\tilde{W}^{\hat{\ell}}_{\ell}(x).
\end{aligned}
\tag{21}
$$

The vierbein fields $\tilde{e}^{\hat{\alpha}}_{\mu}(x)$ are defined by the relation [11,15]

$$
\tilde{g}_{\mu\nu}(x) = \eta_{\hat{\alpha}\hat{\beta}}\tilde{e}^{\hat{\alpha}}_{\mu}(x)\tilde{e}^{\hat{\beta}}_{\mu}(x). \tag{22}
$$

Solving Eq. (22) for $\tilde{g}_{00}(x)$ and $\tilde{g}_{j\ell}(x)$ we get

$$
\begin{aligned}
&\tilde{e}^{\hat{0}}_{0}(x) = \sqrt{\tilde{V}^2(x) + (1-\xi)\eta_{\hat{j}\hat{\ell}}\tilde{W}^{\hat{j}}_{j}(x)\tilde{W}^{\hat{\ell}}_{\ell}(x)K^{j}(x)K^{\ell}(x)},\\
&\tilde{e}^{\hat{j}}_{0}(x) = -\sqrt{\xi}\tilde{W}^{\hat{j}}_{j}(x)K^{j}(x),\\
&\tilde{e}^{\hat{0}}_{j}(x) = 0, \qquad \tilde{e}^{\hat{j}}_{j}(x) = \tilde{W}^{\hat{j}}_{j}(x),
\end{aligned}
\tag{23}
$$

where $\xi$ is a parameter, which can be fixed from Eq. (21) for $\tilde{g}_{0j}(x)$. Indeed, using the vierbein fields (23) we obtain

$$
\tilde{g}_{0j}(x) = -\sqrt{\xi}\eta_{\hat{j}\hat{\ell}}\tilde{W}^{\hat{j}}_{j}(x)\tilde{W}^{\hat{\ell}}_{\ell}(x)K^{\ell}(x). \tag{24}
$$

From the comparison of Eq. (24) with Eq. (21) we obtain $\xi = 1$. This gives the vierbein fields, given by Eq. (23), equal to

$$\tilde{e}^{\hat{0}}_{0}(x) = \tilde{V}(x), \qquad \tilde{e}^{\hat{j}}_{0}(x) = -\tilde{W}^{\hat{j}}_{j}(x)K^{j}(x),$$
$$\tilde{e}^{\hat{0}}_{j}(x) = 0, \qquad \tilde{e}^{\hat{j}}_{j}(x) = \tilde{W}^{\hat{j}}_{j}(x). \tag{25}$$

For the calculation of the vierbein fields $\tilde{e}^{\mu}_{\hat{\alpha}}(x)$ we use the relations [11,15]

$$\delta^{\hat{\alpha}}_{\hat{\beta}} = \tilde{e}^{\hat{\alpha}}_{\mu}(x)\tilde{e}^{\mu}_{\hat{\beta}}(x), \qquad \delta^{\mu}_{\nu} = \tilde{e}^{\mu}_{\hat{\alpha}}(x)\tilde{e}^{\hat{\alpha}}_{\nu}(x). \tag{26}$$

Skipping intermediate calculations we obtain

$$\tilde{e}^{0}_{\hat{0}}(x) = \frac{1}{\tilde{V}(x)}, \qquad \tilde{e}^{0}_{\hat{j}}(x) = 0,$$
$$\tilde{e}^{j}_{\hat{0}}(x) = \frac{K^{j}(x)}{\tilde{V}(x)}, \qquad \tilde{e}^{j}_{\hat{j}}(x) = W^{j}_{\hat{j}}(x). \tag{27}$$

The vierbein fields in Eq. (25) and Eq. (27) have been calculated under the assumption that the functions $W^{\hat{j}}_{j}(x)$ and $W^{j}_{\hat{j}}(x)$ obey the orthogonality relations

$$W^{\hat{j}}_{j}(x)W^{j}_{\hat{\ell}} = \delta^{\hat{j}}_{\hat{\ell}}, \qquad W^{j}_{\hat{j}}(x)W^{\hat{j}}_{\ell} = \delta^{j}_{\ell}, \tag{28}$$

which are fulfilled for the Schwarzschild metric in the weak gravitational field of the Earth approximation [15]. For the verification of the correctness of the obtained vierbein fields we construct the metric tensor $\tilde{g}^{\mu\nu}(x)$. In terms of the vierbein fields $\tilde{e}^{\mu}_{\hat{\alpha}}(x)$ it is determined by

$$\tilde{g}^{\mu\nu}(x) = \eta^{\hat{\alpha}\hat{\beta}}\tilde{e}^{\mu}_{\hat{\alpha}}(x)\tilde{e}^{\nu}_{\hat{\beta}}(x). \tag{29}$$

Using the vierbein fields (27) for the components of the metric tensor $\tilde{g}^{\mu\nu}(x)$ we obtain the following expressions:

$$\tilde{g}^{00}(x) = \frac{1}{\tilde{V}^{2}(x)}, \qquad \tilde{g}^{0j}(x) = \frac{K^{j}(x)}{\tilde{V}(x)},$$
$$\tilde{g}^{j\ell}(x) = \frac{K^{j}(x)K^{\ell}(x)}{\tilde{V}^{2}(x)} + \eta^{\hat{j}\hat{\ell}}\tilde{W}^{j}_{\hat{j}}(x)\tilde{W}^{\ell}_{\hat{\ell}}(x). \tag{30}$$

One may show that the metric tensors $\tilde{g}_{\mu\nu}(x)$ and $\tilde{g}^{\mu\nu}(x)$, given by Eq. (21) and Eq. (30), respectively, obey the relation $\tilde{g}^{\mu\alpha}(x)\tilde{g}_{\alpha\nu}(x) = \delta^{\mu}_{\nu}$. Then, because of $\tilde{e}^{\hat{0}}_{j}(x) = \tilde{e}^{0}_{\hat{j}}(x) = 0$ for the vierbein fields (25) and (27) we get $\tilde{E}^{\hat{0}}_{0}(x) = \tilde{e}^{\hat{0}}_{0}(x)$.

For the vierbein fields (25) and (27) torsion-fermion interactions are yielded by the operators $C_{\hat{\ell}}$, $G_{\hat{j}}$ and $K$ only. Since the contributions of the torsion-fermion interactions, caused by the operators $G_{\hat{j}}$ and $K$, are suppressed by a factor of $1/m$, below we analyze the contributions of the torsion-fermion interactions, caused by the operator $C^{\hat{\ell}}$, which appear to order $O(1)$ in the large fermion mass expansion. We also take into account the contributions of the operators $A$, $B$ and $L^{j}$ in order to derive a complete set of gravitational, chameleon and torsion interactions with slow fermions to order $O(1)$ in the large fermion mass expansion. The analysis of contributions of the operators $G_{\hat{j}}$ and $K$ goes beyond the scope of this paper. We are planning to perform such an analysis in our forthcoming publication.

For the vierbein fields (25) and (27) the operators (A2) are given by

$$\begin{aligned}
A &= \tilde{e}^{\hat{0}}_{0}(x),\\
B &= -\frac{1}{2}i\tilde{e}^{\hat{0}}_{0}(x)\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial}{\partial x^{j}}\Big(\sqrt{-\tilde{g}}\tilde{e}^{j}_{\hat{0}}(x)\Big)\\
&\quad + \frac{1}{2}i(\tilde{e}^{\hat{0}}_{0}(x))^{2}\tilde{e}^{j}_{\hat{0}}(x)\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^{j}}\Big(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x)\Big),\\
C^{\hat{\ell}} &= \frac{1}{4}\tilde{e}^{\hat{0}}_{0}(x)(\tilde{\omega}_{0\hat{j}\hat{k}}(x)\tilde{e}^{0}_{\hat{0}}(x) + \tilde{\omega}_{\ell\hat{j}\hat{k}}(x)\tilde{e}^{\ell}_{\hat{0}}(x)\\
&\quad + \tilde{\omega}_{\ell\hat{0}\hat{j}}(x)\tilde{e}^{\ell}_{\hat{k}}(x) - \tilde{\omega}_{\ell\hat{j}\hat{0}}(x)\tilde{e}^{\ell}_{\hat{k}}(x))\epsilon^{\hat{j}\hat{k}\hat{\ell}},\\
D^{j}_{\hat{j}} &= -\tilde{e}^{\hat{0}}_{0}(x)\tilde{e}^{j}_{\hat{j}}(x),\\
F_{\hat{j}} &= 0,\\
G_{\hat{j}} &= \frac{1}{2}\tilde{e}^{\hat{0}}_{0}(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\ell}(x)\tilde{e}^{\ell}_{\hat{j}}(x) + \tilde{\omega}_{0\hat{j}\hat{0}}(x)\tilde{e}^{0}_{\hat{0}}(x)\\
&\quad + \tilde{\omega}_{\ell\hat{j}\hat{0}}(x)\tilde{e}^{\ell}_{\hat{0}}(x) + \tilde{\omega}_{\ell\hat{j}\hat{k}}(x)\tilde{e}^{\ell}_{\hat{\ell}}(x)\eta^{\hat{\ell}\hat{k}})\\
&\quad + \frac{1}{2}(\tilde{e}^{\hat{0}}_{0}(x))^{2}\tilde{e}^{j}_{\hat{j}}(x)\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^{j}}\Big(\sqrt{-\tilde{g}(x)}\tilde{e}^{0}_{\hat{0}}(x)\Big),\\
K &= -\frac{1}{4}\tilde{\omega}_{\ell\hat{j}\hat{k}}(x)\tilde{e}^{\hat{0}}_{0}(x)\tilde{e}^{\ell}_{\hat{\ell}}(x)\epsilon^{\hat{j}\hat{k}\hat{\ell}},\\
L^{j} &= -\tilde{e}^{\hat{0}}_{0}(x)\tilde{e}^{j}_{\hat{0}}(x).
\end{aligned} \tag{31}$$

For the analysis of interactions of slow Dirac fermions with gravitational, chameleon and torsion fields, caused by nondiagonal spacetime components $\tilde{e}^{j}_{\hat{0}}(x)$ of the vierbein fields we assume a motion of Dirac fermions with mass $m$ in the curved spacetime with the Schwarzschild metric, taken in the weak gravitational field of the Earth approximation and modified by the contributions of the chameleon field and rotation. The line element of such a spacetime is given by

$$\begin{aligned}
d\tilde{s}^{2} &= (1 + 2U_{+})dt^{2} + 2(1 - 2U_{-})\vec{K}\cdot d\vec{r}dt\\
&\quad - (1 - 2U_{-})d\vec{r}^{2},
\end{aligned} \tag{32}$$

where we have neglected the contribution of the terms of order $\vec{K}^{2}$ that is well justified in the terrestrial laboratories and kept the contributions of the chameleon field to linear order [35]. The potentials $U_{\pm}$ are equal to [15]

$$U_{\pm} = U_{\rm E} \pm \frac{\beta}{M_{\rm Pl}}\phi(x), \tag{33}$$

where $U_{\rm E} = \vec{g}\cdot\vec{r}$ is the Newtonian gravitational potential of the Earth and $\vec{g}$ is the gravitational acceleration [15]. To linear-order contributions of the gravitational and chameleon fields the vierbein fields (25) and (27) read

$$\begin{aligned}
&\tilde{e}^{\hat{0}}_{0}(x) = 1 + U_{+}, \qquad \tilde{e}^{\hat{j}}_{0}(x) = -(1-U_{-})K^{\hat{j}}(x),\\
&\tilde{e}^{\hat{0}}_{j}(x) = 0, \qquad e^{\hat{j}}_{j}(x) = (1-U_{-})\delta^{\hat{j}}_{j},\\
&\tilde{e}^{0}_{\hat{0}}(x) = 1 - U_{+}, \qquad \tilde{e}^{j}_{\hat{0}}(x) = +(1-U_{+})K^{j}(x),\\
&\tilde{e}^{0}_{\hat{j}}(x) = 0, \qquad \tilde{e}^{j}_{\hat{j}}(x) = (1+U_{-})\delta^{j}_{\hat{j}}.
\end{aligned} \tag{34}$$

The diagonal components of the vierbein fields agree well with those, calculated in Ref. [15]. In such a spacetime the operators $A$, $B$, $C^{\hat{\ell}}$ and $L^{j}$ are equal to

$$\begin{aligned}
A &= 1 + U_{+},\\
B &= -\frac{1}{2}i\,{\rm div}\,\vec{K},\\
C^{\ell} &= -\frac{1}{4}({\rm rot}\,\vec{K})^{\hat{\ell}} + \frac{1}{4}\mathcal{B}^{\ell} + \frac{1}{4}\epsilon^{\ell jk}K_{j}\mathcal{T}_{00k} + \frac{1}{4}\epsilon^{\ell jk}\mathcal{T}_{jka}K^{a}\\
&= -\frac{1}{4}({\rm rot}\,\vec{K})^{\hat{\ell}} + \frac{1}{4}\mathcal{B}^{\ell} + \frac{1}{6}\mathcal{K}K^{\ell} + \frac{1}{4}\epsilon^{\ell jk}K_{j}\mathcal{M}_{00k}\\
&\quad + \frac{1}{4}\epsilon^{\ell jk}\mathcal{M}_{jka}K^{a},\\
L^{j} &= -K^{j},
\end{aligned} \tag{35}$$

where in the linear order approximation $\tilde{\mathcal{T}}_{\alpha\mu\nu} = \mathcal{T}_{\alpha\mu\nu}$, $\mathcal{B}^{\ell} = \frac{1}{2}\epsilon^{\ell jk}(\mathcal{T}_{jk0} + \mathcal{T}_{k0j} + \mathcal{T}_{0jk})$ and $\mathcal{K} = \frac{1}{2}\epsilon^{abc}\mathcal{T}_{abc}$. Then, we have used Eq. (6) and $\sqrt{-\tilde{g}} = 1 + U_{+} - 3U_{-}$, calculated to linear order of the gravitational and chameleon fields and at the neglect of the contribution of order $O(\vec{K}^{2})$. For the calculation of the operators in Eq. (35) we have not distinguished indices in the Minkowski and curved spacetime. This is correct, since the operators (35) are defined in the perturbative regime for gravitational, chameleon and torsion fields and describe corresponding interactions of slow Dirac fermions in the Minkowski spacetime.

For curved spacetimes with the metric (32) the contribution of the operators (35) to the effective low-energy potential $\Phi_{\rm eff}(t,\vec{r},\vec{\sigma})$, calculated to order $O(1)$ in the large fermion mass expansion, is given by

$$\begin{aligned}
&\Phi_{\rm eff}(t,\vec{r},\vec{\sigma})\\
&\quad = mU_{+} - \frac{1}{2}i\,{\rm div}\,\vec{K} - \vec{K}\cdot i\vec{\nabla} + \frac{1}{4}\vec{\sigma}\cdot{\rm rot}\,\vec{K} - \frac{1}{4}\vec{\sigma}\cdot\vec{\mathcal{B}}\\
&\quad\quad - \frac{1}{6}\mathcal{K}\vec{\sigma}\cdot\vec{K} + \frac{1}{4}\sigma_{\ell}\epsilon^{\ell jk}K_{j}\mathcal{M}_{00k} + \frac{1}{4}\sigma_{\ell}\epsilon^{\ell jk}\mathcal{M}_{jka}K^{a},
\end{aligned} \tag{36}$$

where $\sigma_{j} = (-\vec{\sigma})_{j}$ and we have used that $\epsilon_{jka0} = \epsilon_{jka}$ with $\epsilon_{123} = +1$ [40]. It is important to note that torsion vector $\vec{\mathcal{E}}$ components do not couple to slow Dirac fermions to order $O(1)$ in the large fermion mass expansion and in the linear order approximation of the torsion field.

For rotating coordinate systems with an angular velocity $\vec{\omega}$ the vector functions $K^{j}$ are equal to $K^{j} = -(\vec{\omega}\times\vec{r})^{j}$ [31,35]. This gives ${\rm div}\vec{K} = 0$ and ${\rm rot}\vec{K} = -2\vec{\omega}$. As a result, the effective low-energy potential (36) takes the form

$$\begin{aligned}
&\Phi_{\rm eff}(t,\vec{r},\vec{\sigma})\\
&\quad = mU_{+} - \vec{\omega}\cdot\vec{L} - \vec{\omega}\cdot\vec{S} - \frac{1}{4}\vec{\sigma}\cdot\vec{\mathcal{B}} + \frac{1}{6}\mathcal{K}\vec{\sigma}\cdot(\vec{\omega}\times\vec{r})\\
&\quad\quad - \frac{1}{4}((\vec{\sigma}\cdot\vec{\omega})(\vec{\mathcal{M}}\cdot\vec{r}) - (\vec{\sigma}\cdot\vec{r})(\vec{\mathcal{M}}\cdot\vec{\omega}))\\
&\quad\quad - \frac{1}{4}\sigma_{j}\epsilon^{jk\ell}\mathcal{M}_{k\ell a}\epsilon^{abc}\omega_{b}x_{c},
\end{aligned} \tag{37}$$

where $(\vec{\mathcal{M}})_{k} = -\mathcal{M}_{00k}$. The first term $mU_{+}$ in Eq. (36) corresponds to the Newtonian gravitational potential of the Earth $\vec{g}\cdot\vec{r}$, corrected by the contribution of the chameleon field [8,9]; the second term $-\vec{\omega}\cdot\vec{L}$, where $\vec{L} = -\vec{r}\times i\vec{\nabla}$ is the orbital momentum operator of slow fermions, and the third term $-\vec{\omega}\cdot\vec{S}$, where $\vec{S} = \frac{1}{2}\vec{\sigma}$ is the spin operator of slow fermions, agree well with the results, obtained by Hehl and Ni [35]. The interactions $-\vec{\omega}\cdot\vec{L}$ and $-\vec{\omega}\cdot\vec{S}$ were analyzed in the experiments by Werner, Staudenmann, and Colella [48] and by Mashhoon [33]. The fourth term, describing the torsion-spin-matter interaction of the torsion axial-vector components, was derived by Lämmerzahl [12] and Obukhov, Silenko, and Teryaev [13] (see also Ref. [15]). The other terms in the effective low-energy potential (36) are new. The fifth term $(1/6)\mathcal{K}\vec{\sigma}\cdot(\vec{\omega}\times\vec{r})$ is a new low-energy interaction of the torsion pseudoscalar component $\mathcal{K}$ with slow Dirac fermions. In turn, the last two terms in Eq. (36) describe new low-energy interactions of torsion tensor $\mathcal{M}_{00k}$ and $\mathcal{M}_{jka}$ components with slow Dirac fermions, caused by minimal torsion-fermion couplings without phenomenological coupling constants. According to estimates by Kostelecky *et al.* [14], the constant torsion tensor components $\mathcal{M}_{00k}$ and $\mathcal{M}_{jka}$, multiplied by a phenomenological coupling constant $\xi^{(5)}_{5}$, are restricted by $|\xi^{(5)}_{5}\mathcal{M}_{00k}| < 10^{-27}$ and $|\xi^{(5)}_{5}\mathcal{M}_{jka}| < 10^{-26}$, respectively. Recently the nonminimal torsion-matter couplings have been also discussed by Puetzfeld and Obukhov [49]. As regards torsion vector components $\vec{\mathcal{E}}$, we have found that slow Dirac fermions do not couple to them to order $O(1)$ in the large fermion mass expansion and to linear order of the torsion field approximation.

The upper bound of the linear superposition of the constant torsion vector and axial-vector components $|\zeta| < 9.1\times10^{-23}$ GeV, measured by Lehnert, Snow and

Yan [16] by means of an investigation of a spin rotation of cold neutrons in liquid $^4$He, is larger by a factor of $10^5$ compared with the estimate $|\zeta| < 10^{-27}$ GeV, obtained in Ref. [15].

Thus, we have shown that to linear order of the torsion field approximation in spacetimes of rotating coordinate systems the contributions of only torsion pseudoscalar $\mathcal{K}$ and tensor $\mathcal{M}_{00k}$ and $\mathcal{M}_{k\ell a}$ components, caused by minimal torsion-fermion couplings, appear to order $O(1)$ in the large fermion mass expansion. The certain steps in the realization of curved spacetimes in terrestrial laboratories by using rotating devices (neutron interferometers) were made by Atwood *et al.* [32] and Mashhoon [33].

The measurements of the transition frequencies between quantum gravitational states of ultracold neutrons in the qBounce experiments [1–6] as functions of an angular velocity $\vec{\omega}$ of a rotating mirror should provide a new level of highly precise probes of the properties of the Einstein-Cartan gravity, dark energy and the evolution of the Universe. Of course, the measurements of new gravitational, chameleon and torsion interactions in Eq. (37) as well as other interactions in the effective low-energy potential (A15) can be carried out by using rotating neutron interferometers [7,34].

Now we would like to discuss briefly the Foldy–Wouthuysen method [44], which we use in this paper for the derivation of the effective low-energy potential for slow Dirac fermions, coupled to gravitational, chameleon and torsion fields. Mainly this discussion concerns the uniqueness and accuracy of the Foldy-Wouthuysen representation of the Dirac Hamilton operator, obtained by the Foldy-Wouthuysen transformation. It is well known that the Foldy-Wouthuysen method of a transformation of a Dirac Hamilton operator to a form, containing only *even* (diagonal) operators (as regards the definition of *odd* and *even* operators the reader might consult Ref. [44] or look up in the Appendix to this paper), is not unique and there are some other methods of transformation of a Dirac Hamilton operator to a diagonal form. One can find a very nice survey of possible methods of transformation of a Dirac Hamilton operator for fermions with mass $m$ to a diagonal form, containing only *even* operators, in the paper by de Vries [50]. The Foldy-Wouthuysen method, removing *odd* operators from a Dirac Hamilton operator for fermions with mass $m$ by Foldy-Wouthuysen unitary transformations, allows one to reduce a Dirac Hamilton operator to a nonrelativistic form in the approximation of a large fermion mass expansion by a set of unitary transformations or by the iterative Foldy-Wouthuysen method. The obtained nonrelativistic Hamilton operator is given by an infinite series of *even* operators in powers of $1/m$, which does not seem to give hope for a closed-form operator. A problem of a closed form of a transformed Dirac Hamilton operator, expressed in terms of only *even* operators, was investigated by Eriksen [51]. Eriksen showed that the unitary transformation $e^{iS} = \sqrt{\gamma^0 \mathrm{H}/\sqrt{\mathrm{H}^2}}$, where $\gamma^0$ and H are the Dirac matrix and a Dirac Hamilton operator, allows one to transform a Dirac Hamilton operator H to a square root of an *even* operator. However, Eriksen's unitary operator $e^{iS} = \sqrt{\gamma^0 \mathrm{H}/\sqrt{\mathrm{H}^2}}$, leading to a closed form of a transformed Dirac Hamilton operators, suffers from an ambiguous definition. In order to define the operator $e^{iS} = \sqrt{\gamma^0 \mathrm{H}/\sqrt{\mathrm{H}^2}}$ unambiguously one has to assume that the square root of a unit operator is a unit operator. For a recent discussion of a square root operator definition and analyses of the Dirac Hamilton operators by means of the Eriksen method we direct the reader to the papers by Silenko [52], Neznamov and Silenko [41] and Silenko [42,43,53]. According to Eriksen [51], the unitary operator $e^{iS} = \sqrt{\gamma^0 \mathrm{H}/\sqrt{\mathrm{H}^2}}$, providing an *exact* diagonalization of the Dirac Hamilton operator H, can be defined by $e^{iS} = \sqrt{\gamma^0 \lambda} = (1 + \gamma^0 \lambda)/\sqrt{2 + \gamma^0 \lambda + \lambda \gamma^0}$, where $\lambda = \mathrm{H}/\sqrt{\mathrm{H}^2}$. Another problem of the Foldy-Wouthuysen method concerns the accuracy of the Foldy-Wouthuysen representation of a Dirac Hamilton operator in comparison with a large fermion mass expansion of an *exact* form of a transformed Dirac Hamilton operator. For the first time such a problem was discussed by Eriksen and Kolsrud [54]. Recently this problem has been investigated by Neznamov and Silenko [41] and Silenko [42,43,53]. According to Ref. [54] and Refs. [41–43,53], the Foldy-Wouthuysen representation of a Dirac Hamilton operator, obtained by a set of unitary transformations, cannot coincide with a large fermion mass expansion of an *exact* transformed Dirac Hamilton operator, diagonalized by means of only one unitary transformation (e.g. the Eriksen transformation). As has been pointed out by Neznamov and Silenko [41] and Silenko [42,43,53], such a disagreement can be explained by the noncommutativity of Foldy-Wouthuysen unitary transformations in the iterative Foldy-Wouthuysen method. For example, in our case we have diagonalized the Dirac Hamilton operator (17) by three unitary transformations of the Dirac wave functions $e^{iS_1}$, $e^{iS_2}$ and $e^{iS_3}$ (see the Appendix), respectively. A resulting unitary transformation is equal to $e^{iS} = e^{iS_3} e^{iS_2} e^{iS_1}$. According to Refs. [41–43,53], a coincidence of our result for the effective low-energy potential (A15) with a large fermion mass expansion of an *exact* diagonalized Hamilton operator, obtained by means of only one unitary transformation (e.g. by the Eriksen transformation), can be expected only for the validity of the relation $e^{iS_3} e^{iS_2} e^{iS_1} = e^{i(S_3+S_2+S_1)}$, which demands the commutativity of the operators $[S_i, S_j] = 0$ for $i \neq j = 1, 2, 3$ [41–43,53]. Since the unitary operators do not commute $[S_i, S_j] \neq 0$ for $i \neq j = 1, 2, 3$ (see the Appendix), the relation $e^{iS_3} e^{iS_2} e^{iS_1} = e^{i(S_3+S_2+S_1)}$ is not valid and one may expect some deviations of the effective low-energy potential

(A15) from that derived by a large fermion mass expansion of an *exact* diagonalized Dirac Hamilton operator. However, one may show that any deviations can appear only to order $O(1/m^2)$ [53]. The latter can be justified by the observation that $[S_2, S_1] = O(1/m^2)$ and $[S_3, S_j] = O(1/m^3)$ for $j = 1, 2$. Hence, to order $O(1/m)$, which we have kept for the derivation of the effective low-energy potential $\Phi_{\text{eff}}$ in Eq. (A15), these two fermion mass expansions should coincide. A method of the calculation of the corrections to the Foldy-Wouthuysen representation of a Dirac Hamilton operator has been discussed in detail by Silenko [53]. For example, suppose that two Foldy-Wouthuysen unitary transformations with operators $e^{iS_1}$ and $e^{iS_2}$, performed one after another, diagonalize a Dirac Hamilton operator, i.e. $\mathrm{H} \to \mathrm{H}_{\mathrm{FW}}$, where $\mathrm{H}_{\mathrm{FW}}$ is a Dirac Hamilton operator in the Foldy-Wouthuysen representation. According to Silenko [53], an additional Foldy-Wouthuysen unitary transformation $U_{\text{corr}} = \exp(-\frac{1}{2}[S_1, S_2])$ should allow one to cancel an error of the iterative Foldy-Wouthuysen method in the leading order. Such a correction is valid if the commutators $[S_1, [S_1, S_2]]$ and $[S_2, [S_1, S_2]]$ and commutators of higher orders can be neglected with respect to the commutator $[S_1, S_2]$. Since in our case this constraint is fulfilled, the correction to the effective low-energy potential (A15) can be calculated by means of the unitary transformation $U_{\text{corr}} = \exp(-\frac{1}{2}[S_1, S_2])$. However, in our case $[S_1, S_2] = O(1/m^2)$ and the effective low-energy potential (A15) is calculated to order $O(1/m)$. This might imply that the effective low-energy potential (A15) should in principle coincide with a large fermion mass expansion to order $O(1/m)$ of an *exact* diagonalized Dirac Hamilton operator by, for example, the Eriksen method [41–43,51,53].

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## APPENDIX: DERIVATION OF THE EFFECTIVE LOW-ENERGY POTENTIAL $\Phi_{\text{eff}}(t,\vec{r},\vec{\sigma})$ IN EQ. (18)

For the derivation of the effective low-energy potential $\Phi_{\text{eff}}(t, \vec{r}, \vec{\sigma})$ for slow Dirac fermions with mass $m$ in the Schrödinger–Pauli equation (18) we define the Hamilton operator (17) as follows:

$$\mathrm{H}' = A\gamma^{\hat{0}}m + B + C^{\hat{\ell}}\Sigma_{\hat{\ell}} + D^j_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial x^j} + F_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial t} + G_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}} + K\gamma^5 + L^j i\frac{\partial}{\partial x^j} \tag{A1}$$

where we have denoted

$$\begin{aligned}
A &= \tilde{E}^{\hat{0}}_0(x),\\
B &= -\frac{1}{2} i\tilde{E}^{\hat{0}}_0(x)\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}}\tilde{e}^j_{\hat{0}}(x)) + \frac{1}{2} i(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^j_{\hat{0}}(x)\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}(x)}\tilde{e}^0_{\hat{0}}(x)),\\
C^{\hat{\ell}} &= \frac{1}{4}\tilde{E}^{\hat{0}}_0(x)(\tilde{\omega}_{\mu\hat{j}\hat{k}}(x)\tilde{e}^{\mu}_{\hat{0}}(x) + \tilde{\omega}_{\mu[\hat{0}\hat{j}]}(x)\tilde{e}^{\mu}_{\hat{k}}(x))\epsilon^{\hat{j}\hat{k}\hat{\ell}},\\
D^j_{\hat{j}} &= -\tilde{E}^{\hat{0}}_0(x)\tilde{e}^j_{\hat{j}}(x),\\
F_{\hat{j}} &= -\tilde{E}^{\hat{0}}_0(x)\tilde{e}^0_{\hat{j}}(x),\\
G_{\hat{j}} &= \frac{1}{2}\tilde{E}^{\hat{0}}_0(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{j}}(x) + \tilde{\omega}_{\mu\hat{j}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}}) - \frac{1}{2}(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^0_{\hat{j}}(x)(\tilde{\mathcal{T}}^{\alpha}{}_{\alpha\mu}(x)\tilde{e}^{\mu}_{\hat{0}}(x) + \tilde{\omega}_{\mu\hat{0}\hat{\beta}}(x)\tilde{e}^{\mu}_{\hat{\lambda}}(x)\eta^{\hat{\lambda}\hat{\beta}})\\
&\quad + \frac{1}{2}(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^j_{\hat{j}}(x)\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}(x)}\tilde{e}^0_{\hat{0}}(x)) - \frac{1}{2}(\tilde{E}^{\hat{0}}_0(x))^2\tilde{e}^0_{\hat{j}}(x)\frac{1}{\sqrt{-\tilde{g}(x)}}\frac{\partial}{\partial x^j}(\sqrt{-\tilde{g}(x)}\tilde{e}^j_{\hat{0}}(x)),\\
K &= -\frac{1}{4}\tilde{E}^{\hat{0}}_0(x)\tilde{\omega}_{\mu\hat{j}\hat{k}}(x)\tilde{e}^{\mu}_{\hat{\ell}}(x)\epsilon^{\hat{j}\hat{k}\hat{\ell}},\\
L^j &= -\tilde{E}^{\hat{0}}_0(x)\tilde{e}^j_{\hat{0}}(x).
\end{aligned} \tag{A2}$$

For the elimination of the *odd* operators we perform the Foldy-Wouthuysen unitary transformation of the wave function $\psi'(x) = e^{-iS_1}\psi_1(x)$ and the Hamilton operator [44]

$$\mathrm{H}_1 = e^{+iS_1}\mathrm{H}'e^{-iS_1} - ie^{iS_1}\frac{\partial}{\partial t}e^{-iS_1} = \mathrm{H}' - \frac{\partial S_1}{\partial t} + i\left[S_1, \mathrm{H}' - \frac{1}{2}\frac{\partial S_1}{\partial t}\right] + \frac{i^2}{2}\left[S_1, \left[S_1, \mathrm{H}' - \frac{1}{3}\frac{\partial S_1}{\partial t}\right]\right] + \ldots \tag{A3}$$

The time derivative appears because of a time dependence of the chameleon and torsion fields. Then, following Ref. [44] we take the operator $S_1$ in the form

$$S_1 = -\frac{i}{2mA}\gamma^{\hat{0}}\left(D^j_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial x^j} + F_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial t} + G_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}} + K\gamma^5\right). \tag{A4}$$

The time derivative of $S_1$ and the commutators in Eq. (A3) are equal to

$$\frac{\partial S_1}{\partial t} = \frac{1}{2m}\frac{\partial}{\partial t}\left(\frac{D^j_{\hat{j}}}{A}\right)\gamma^{\hat{j}}\frac{\partial}{\partial x^j} + \frac{1}{2m}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\gamma^{\hat{j}}\frac{\partial}{\partial t} + \frac{1}{2m}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right)\gamma^{\hat{j}} - \frac{1}{2m}\frac{\partial}{\partial t}\left(\frac{K}{A}\right) i\gamma^{\hat{0}}\gamma^5 \tag{A5}$$

and

$$\begin{aligned}
&i\left[S_1, \mathrm{H}' - \frac{1}{2}\frac{\partial S_1}{\partial t}\right] \\
&= -D^j_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial x^j} - F_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{\partial}{\partial t} - G_{\hat{j}} i\gamma^{\hat{0}}\gamma^{\hat{j}} - K\gamma^5 - \frac{1}{2} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{D^j_{\hat{j}}}{A}\frac{\partial A}{\partial x^j} - \frac{1}{2} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{F_{\hat{j}}}{A}\frac{\partial A}{\partial t} \\
&+ \frac{1}{2mA} i\gamma^{\hat{j}} D^j_{\hat{j}}\frac{\partial B}{\partial x^j} + \frac{1}{2mA} i\gamma^{\hat{j}} F_{\hat{j}}\frac{\partial B}{\partial t} - \frac{1}{2mA}\eta^{\hat{j}\hat{\ell}} D^j_{\hat{j}}\frac{\partial C_{\hat{\ell}}}{\partial x^j} i\gamma^{\hat{0}}\gamma^5 + \frac{1}{2mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}} D^j_{\hat{j}}\frac{\partial C_{\hat{\ell}}}{\partial x^j} \\
&+ \frac{1}{mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}} D^j_{\hat{j}} C_{\hat{\ell}}\frac{\partial}{\partial x^j} - \frac{1}{2mA}\eta^{\hat{j}\hat{\ell}} F_{\hat{j}}\frac{\partial C_{\hat{\ell}}}{\partial t} i\gamma^{\hat{0}}\gamma^5 + \frac{1}{2mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}} F_{\hat{j}}\frac{\partial C_{\hat{\ell}}}{\partial t} \\
&+ \frac{1}{mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}} F_{\hat{j}} C_{\hat{\ell}}\frac{\partial}{\partial t} + \frac{1}{mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}} G_{\hat{j}} C_{\hat{\ell}} + \frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}} D^k_{\hat{k}}\frac{\partial^2}{\partial x^j \partial x^k} \\
&+ \frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial x^k} + \frac{1}{2m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j} + \frac{1}{2mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^j_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial x^k} \\
&- \frac{1}{2m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j} + \frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}} D^k_{\hat{k}}\frac{\partial^2}{\partial x^k \partial t} + \frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial t}\frac{\partial}{\partial x^k} \\
&+ \frac{1}{2m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t} + \frac{1}{2mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial t}\frac{\partial}{\partial x^k} \\
&- \frac{1}{2m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t} + \frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{j}} D^k_{\hat{k}}\frac{\partial}{\partial x^k} + \frac{1}{2m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{G_{\hat{j}}}{A}\right) \\
&- \frac{1}{2m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{G_{\hat{j}}}{A}\right) + \frac{1}{mA} i\gamma^{\hat{0}}\Sigma^{\hat{k}} K D^k_{\hat{k}}\frac{\partial}{\partial x^k} + \frac{1}{2m} i\gamma^{\hat{0}}\Sigma^{\hat{k}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{K}{A}\right) \\
&+ \frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}} F_{\hat{k}}\frac{\partial^2}{\partial x^j \partial t} + \frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial t} + \frac{1}{2m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j} \\
&+ \frac{1}{2mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^j_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial t} - \frac{1}{2m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j} + \frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}} F_{\hat{k}}\frac{\partial^2}{\partial t^2} \\
&+ \frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t}\frac{\partial}{\partial t} + \frac{1}{2m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t} + \frac{1}{2mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t}\frac{\partial}{\partial t} \\
&- \frac{1}{2m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t} + \frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{j}} F_{\hat{k}}\frac{\partial}{\partial t} + \frac{1}{2m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right) \\
&- \frac{1}{2m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right) + \frac{1}{mA} i\gamma^{\hat{0}}\Sigma^{\hat{k}} K F_{\hat{k}}\frac{\partial}{\partial t} + \frac{1}{2m} i\gamma^{\hat{0}}\Sigma^{\hat{k}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{K}{A}\right) \\
&+ \frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{k}} D^j_{\hat{j}}\frac{\partial}{\partial x^j} + \frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial x^j} + \frac{1}{2mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^j_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial x^j}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}G_{\hat{k}}\frac{\partial}{\partial t}+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial t}+\frac{1}{2mA}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial t}\\
&+\frac{1}{mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}G_{\hat{j}}G_{\hat{k}}+\frac{1}{mA}i\gamma^{\hat{0}}\Sigma^{\hat{k}}KG_{\hat{k}}+\frac{1}{mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}KD^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}+\frac{1}{2mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}D^{j}_{\hat{j}}\frac{\partial K}{\partial x^{j}}\\
&+\frac{1}{mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}KF_{\hat{j}}\frac{\partial}{\partial t}+\frac{1}{2mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}F_{\hat{j}}\frac{\partial K}{\partial t}+\frac{1}{mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}KG_{\hat{j}}+\frac{1}{mA}\gamma^{\hat{0}}K^{2}\\
&-\frac{1}{2mA}\gamma^{\hat{j}}D^{j}_{\hat{j}}\frac{\partial L^{k}}{\partial x^{j}}\frac{\partial}{\partial x^{k}}+\frac{1}{2m}\gamma^{\hat{j}}L^{k}\frac{\partial}{\partial x^{k}}\left(\frac{D^{j}_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^{j}}-\frac{1}{2mA}\gamma^{\hat{j}}F_{\hat{j}}\frac{\partial L^{k}}{\partial t}\frac{\partial}{\partial x^{k}}\\
&+\frac{1}{2m}\gamma^{\hat{j}}L^{k}\frac{\partial}{\partial x^{k}}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}+\frac{1}{2m}\gamma^{\hat{j}}L^{k}\frac{\partial}{\partial x^{k}}\left(\frac{G_{\hat{j}}}{A}\right)-\frac{1}{2m}i\gamma^{\hat{0}}\gamma^{5}L^{k}\frac{\partial}{\partial x^{k}}\left(\frac{K}{A}\right)
\end{aligned}
\tag{A6}
$$

and

$$
\begin{aligned}
&\frac{i^{2}}{2}\left[S_{1},\left[S_{1},\mathrm{H}'-\frac{1}{3}\frac{\partial S_{1}}{\partial t}\right]\right]\\
&=-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial^{2}}{\partial x^{j}\partial x^{k}}-\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}\frac{\partial D^{k}_{\hat{k}}}{\partial x^{j}}\frac{\partial}{\partial x^{k}}-\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}\left(\frac{D^{j}_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^{j}}\\
&-\frac{1}{4mA}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial D^{k}_{\hat{k}}}{\partial x^{j}}\frac{\partial}{\partial x^{k}}+\frac{1}{4m}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}D^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}\left(\frac{D^{j}_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^{j}}-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial^{2}}{\partial t\partial x^{k}}\\
&-\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial D^{k}_{\hat{k}}}{\partial t}\frac{\partial}{\partial x^{k}}-\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}-\frac{1}{4mA}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial D^{k}_{\hat{k}}}{\partial t}\frac{\partial}{\partial x^{k}}\\
&+\frac{1}{4m}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}D^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}G_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}-\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}\left(\frac{G_{\hat{j}}}{A}\right)\\
&+\frac{1}{4m}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}D^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}\left(\frac{G_{\hat{j}}}{A}\right)-\frac{1}{2mA}i\gamma^{\hat{0}}\Sigma^{\hat{k}}KD^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}-\frac{1}{4m}i\gamma^{\hat{0}}\Sigma^{\hat{k}}D^{k}_{\hat{k}}\frac{\partial}{\partial x^{k}}\left(\frac{K}{A}\right)\\
&-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}F_{\hat{k}}\frac{\partial^{2}}{\partial t\partial x^{j}}-\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^{j}}\frac{\partial}{\partial t}-\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{D^{j}_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^{j}}-\frac{1}{4mA}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^{j}}\frac{\partial}{\partial t}\\
&+\frac{1}{4m}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{D^{j}_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^{j}}-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}}\frac{\partial^{2}}{\partial t^{2}}-\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t}\frac{\partial}{\partial t}\\
&-\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}-\frac{1}{4mA}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t}\frac{\partial}{\partial t}+\frac{1}{4m}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}G_{\hat{j}}F_{\hat{k}}\frac{\partial}{\partial t}\\
&-\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right)+\frac{1}{4m}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right)-\frac{1}{2mA}i\gamma^{\hat{0}}\Sigma^{\hat{k}}KF_{\hat{k}}\frac{\partial}{\partial t}-\frac{1}{4m}i\gamma^{\hat{0}}\Sigma^{\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{K}{A}\right)\\
&-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}G_{\hat{k}}\frac{\partial}{\partial x^{j}}-\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial x^{j}}-\frac{1}{4mA}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial x^{j}}-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}G_{\hat{k}}\frac{\partial}{\partial t}\\
&-\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial t}-\frac{1}{4mA}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial t}-\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}G_{\hat{j}}G_{\hat{k}}-\frac{1}{2mA}i\gamma^{\hat{0}}\Sigma^{\hat{k}}KG_{\hat{k}}\\
&-\frac{1}{2mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}D^{j}_{\hat{j}}K\frac{\partial}{\partial x^{j}}-\frac{1}{4mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}D^{j}_{\hat{j}}\frac{\partial K}{\partial x^{j}}-\frac{1}{2mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}F_{\hat{j}}K\frac{\partial}{\partial t}-\frac{1}{4mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}F_{\hat{j}}\frac{\partial K}{\partial t}\\
&-\frac{1}{2mA}i\gamma^{\hat{0}}\Sigma^{\hat{j}}G_{\hat{j}}K-\frac{1}{2mA}\gamma^{\hat{0}}K^{2}-\frac{1}{4mA^{2}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}\frac{\partial}{\partial x^{j}}-\frac{1}{8mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)\\
&-\frac{1}{8mA}\gamma^{\hat{0}}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)-\frac{1}{4mA^{2}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}\frac{\partial}{\partial t}-\frac{1}{8mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{D^k_{\hat{k}}}{A}\frac{\partial A}{\partial x^k}\right)-\frac{1}{4mA^2}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{j}} D^k_{\hat{k}}\frac{\partial A}{\partial x^k}-\frac{1}{4mA^2} i\gamma^{\hat{0}}\Sigma^{\hat{k}} K D^k_{\hat{k}}\frac{\partial A}{\partial x^k}\\
&-\frac{1}{4mA^2}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}} F_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial}{\partial x^j}-\frac{1}{8mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}}\frac{\partial}{\partial x^j}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)-\frac{1}{8mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^j_{\hat{j}}\frac{\partial}{\partial x^j}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)\\
&-\frac{1}{4mA^2}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}} F_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial}{\partial t}-\frac{1}{8mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)-\frac{1}{8mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)\\
&-\frac{1}{4mA^2}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{j}} F_{\hat{k}}\frac{\partial A}{\partial t}-\frac{1}{4mA^2} i\gamma^{\hat{0}}\Sigma^{\hat{k}} K F_{\hat{k}}\frac{\partial A}{\partial t}. \qquad \text{(A7)}
\end{aligned}
$$

We decompose the Hamilton operator $\mathrm{H}_1$ into two parts $\mathrm{H}_1 = \mathrm{H}_{1\,\mathrm{even}} + \mathrm{H}_{1\,\mathrm{odd}}$, where the operators $\mathrm{H}_{1\,\mathrm{even}}$ and $\mathrm{H}_{1\,\mathrm{odd}}$ are equal to

$$
\begin{aligned}
\mathrm{H}_{1\,\mathrm{even}} &= A\gamma^{\hat{0}} m + B + C^{\hat{\ell}}\Sigma_{\hat{\ell}} + L^j i\frac{\partial}{\partial x^j}+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}} D^k_{\hat{k}}\frac{\partial^2}{\partial x^j\partial x^k}\\
&+\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial x^k}+\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j}+\frac{1}{4mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^j_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial x^k}\\
&-\frac{1}{4m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j}+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}} D^k_{\hat{k}}\frac{\partial^2}{\partial x^k\partial t}+\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial t}\frac{\partial}{\partial x^k}\\
&+\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}+\frac{1}{4mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial t}\frac{\partial}{\partial x^k}\\
&-\frac{1}{4m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{j}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}+\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{G_{\hat{j}}}{A}\right)\\
&-\frac{1}{4m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{G_{\hat{j}}}{A}\right)+\frac{1}{2mA} i\gamma^{\hat{0}}\Sigma^{\hat{k}} K D^k_{\hat{k}}\frac{\partial}{\partial x^k}+\frac{1}{4m} i\gamma^{\hat{0}}\Sigma^{\hat{k}} D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{K}{A}\right)\\
&+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}} F_{\hat{k}}\frac{\partial^2}{\partial x^j\partial t}+\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial t}+\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j}\\
&+\frac{1}{4mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^j_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial t}-\frac{1}{4m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j}+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}} F_{\hat{k}}\frac{\partial^2}{\partial t^2}\\
&+\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t}\frac{\partial}{\partial t}+\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}+\frac{1}{4mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t}\frac{\partial}{\partial t}\\
&-\frac{1}{4m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{j}} F_{\hat{k}}\frac{\partial}{\partial t}+\frac{1}{4m}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right)\\
&-\frac{1}{4m}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right)+\frac{1}{2mA} i\gamma^{\hat{0}}\Sigma^{\hat{k}} K F_{\hat{k}}\frac{\partial}{\partial t}+\frac{1}{4m} i\gamma^{\hat{0}}\Sigma^{\hat{k}} F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{K}{A}\right)\\
&+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{k}} D^j_{\hat{j}}\frac{\partial}{\partial x^j}+\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial x^j}+\frac{1}{4mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^j_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial x^j}\\
&+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}} G_{\hat{k}}\frac{\partial}{\partial t}+\frac{1}{4mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} F_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial t}+\frac{1}{4mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} F_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial t}\\
&+\frac{1}{2mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} G_{\hat{j}} G_{\hat{k}}+\frac{1}{2mA} i\gamma^{\hat{0}}\Sigma^{\hat{k}} K G_{\hat{k}}+\frac{1}{2mA} i\gamma^{\hat{0}}\Sigma^{\hat{j}} K D^j_{\hat{j}}\frac{\partial}{\partial x^j}+\frac{1}{4mA} i\gamma^{\hat{0}}\Sigma^{\hat{j}} D^j_{\hat{j}}\frac{\partial K}{\partial x^j}\\
&+\frac{1}{2mA} i\gamma^{\hat{0}}\Sigma^{\hat{j}} K F_{\hat{j}}\frac{\partial}{\partial t}+\frac{1}{4mA} i\gamma^{\hat{0}}\Sigma^{\hat{j}} F_{\hat{j}}\frac{\partial K}{\partial t}+\frac{1}{2mA} i\gamma^{\hat{0}}\Sigma^{\hat{j}} K G_{\hat{j}}+\frac{1}{2mA}\gamma^{\hat{0}} K^2\\
&-\frac{1}{4mA^2}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}} D^k_{\hat{k}}\frac{\partial A}{\partial x^k}\frac{\partial}{\partial x^j}-\frac{1}{8mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}} D^j_{\hat{j}}\frac{\partial}{\partial x^j}\left(\frac{D^k_{\hat{k}}}{A}\frac{\partial A}{\partial x^k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}} D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)-\frac{1}{4mA^{2}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}\frac{\partial}{\partial t}\\
&-\frac{1}{8mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)-\frac{1}{8mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)\\
&-\frac{1}{4mA^{2}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}G_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}-\frac{1}{4mA^{2}} i\gamma^{\hat{0}}\Sigma^{\hat{k}}KD^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}\\
&-\frac{1}{4mA^{2}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial}{\partial x^{j}}-\frac{1}{8mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)\\
&-\frac{1}{8mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)-\frac{1}{4mA^{2}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial}{\partial t}\\
&-\frac{1}{8mA}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)-\frac{1}{8mA}\gamma^{\hat{0}} i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\Sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)\\
&-\frac{1}{4mA^{2}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}G_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t}-\frac{1}{4mA^{2}} i\gamma^{\hat{0}}\Sigma^{\hat{k}}KF_{\hat{k}}\frac{\partial A}{\partial t}
\end{aligned}
\tag{A8}
$$

and

$$
\begin{aligned}
\mathrm{H}_{1\,\mathrm{odd}} = &-\frac{1}{2} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{D^{j}_{\hat{j}}}{A}\frac{\partial A}{\partial x^{j}}-\frac{1}{2} i\gamma^{\hat{0}}\gamma^{\hat{j}}\frac{F_{\hat{j}}}{A}\frac{\partial A}{\partial t}\\
&+\frac{1}{2mA} i\gamma^{\hat{j}}D^{j}_{\hat{j}}\frac{\partial B}{\partial x^{j}}+\frac{1}{2mA} i\gamma^{\hat{j}}F_{\hat{j}}\frac{\partial B}{\partial t}-\frac{1}{2mA}\eta^{\hat{j}\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial C_{\hat{\ell}}}{\partial x^{j}} i\gamma^{\hat{0}}\gamma^{5}+\frac{1}{2mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}}D^{j}_{\hat{j}}\frac{\partial C_{\hat{\ell}}}{\partial x^{j}}\\
&+\frac{1}{mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}}D^{j}_{\hat{j}}C_{\hat{\ell}}\frac{\partial}{\partial x^{j}}-\frac{1}{2mA}\eta^{\hat{j}\hat{\ell}}F_{\hat{j}}\frac{\partial C_{\hat{\ell}}}{\partial t} i\gamma^{\hat{0}}\gamma^{5}+\frac{1}{2mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}}F_{\hat{j}}\frac{\partial C_{\hat{\ell}}}{\partial t}\\
&+\frac{1}{mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}}F_{\hat{j}}C_{\hat{\ell}}\frac{\partial}{\partial t}+\frac{1}{mA}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}}G_{\hat{j}}C_{\hat{\ell}}\\
&-\frac{1}{2mA}\gamma^{\hat{j}}D^{j}_{\hat{j}}\frac{\partial L^{k}}{\partial x^{j}}\frac{\partial}{\partial x^{k}}+\frac{1}{2m}\gamma^{\hat{j}}L^{k}\frac{\partial}{\partial x^{k}}\left(\frac{D^{j}_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^{j}}-\frac{1}{2mA}\gamma^{\hat{j}}F_{\hat{j}}\frac{\partial L^{k}}{\partial t}\frac{\partial}{\partial x^{k}}\\
&+\frac{1}{2m}\gamma^{\hat{j}}L^{k}\frac{\partial}{\partial x^{k}}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}+\frac{1}{2m}\gamma^{\hat{j}}L^{k}\frac{\partial}{\partial x^{k}}\left(\frac{G_{\hat{j}}}{A}\right)-\frac{1}{2m}L^{k}\frac{\partial}{\partial x^{k}}\left(\frac{K}{A}\right) i\gamma^{\hat{0}}\gamma^{5}\\
&-\frac{1}{2m}\frac{\partial}{\partial t}\left(\frac{D^{j}_{\hat{j}}}{A}\right)\gamma^{\hat{j}}\frac{\partial}{\partial x^{j}}-\frac{1}{2m}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\gamma^{\hat{j}}\frac{\partial}{\partial t}-\frac{1}{2m}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right)\gamma^{\hat{j}}+\frac{1}{2m}\frac{\partial}{\partial t}\left(\frac{K}{A}\right) i\gamma^{\hat{0}}\gamma^{5}.
\end{aligned}
\tag{A9}
$$

For the calculation of the Hamilton operator $\mathrm{H}_1 = \mathrm{H}_{1\,\mathrm{even}} + \mathrm{H}_{1\,\mathrm{odd}}$ we have neglected the contributions of order $O(1/m^2)$. In order to remove the *odd* operators we perform the second unitary transformation of the wave function $\psi_1(x) = e^{-iS_2}\psi_2(x)$ and the Hamilton operator $\mathrm{H}_1$:

$$
\mathrm{H}_2 = e^{+iS_2}\mathrm{H}_1 e^{-iS_2} - ie^{iS_2}\frac{\partial}{\partial t}e^{-iS_2} = \mathrm{H}_1 - \frac{\partial S_2}{\partial t} + i\left[S_2, \mathrm{H}_1 - \frac{1}{2}\frac{\partial S_2}{\partial t}\right] + \frac{i^2}{2}\left[S_2, \left[S_2, \mathrm{H}_1 - \frac{1}{3}\frac{\partial S_2}{\partial t}\right]\right] + \ldots, \tag{A10}
$$

where the operator $S_2$ is equal to $S_2 = -(i/2mA)\gamma^{\hat{0}}\mathrm{H}_{1\,\mathrm{odd}}$. Neglecting the contributions of order $O(1/m^2)$ we arrive at the Hamilton operator $\mathrm{H}_2 = \mathrm{H}_{2\,\mathrm{even}} + \mathrm{H}_{2\,\mathrm{odd}}$, where

$$
\begin{aligned}
\mathrm{H}_{2\,\mathrm{even}} = \mathrm{H}_{1\,\mathrm{even}} &+ \frac{1}{4mA^{3}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{j}}\frac{\partial A}{\partial x^{k}}+\frac{1}{4mA^{3}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial x^{j}}\frac{\partial A}{\partial t}\\
&+\frac{1}{4mA^{3}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial A}{\partial x^{k}}+\frac{1}{4mA^{3}}\gamma^{\hat{0}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial A}{\partial t},
\end{aligned}
$$

$$\mathrm{H}_{2\,\mathrm{odd}} = -\frac{1}{2mA^2}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}}D^j_{\hat{j}}C_{\hat{\ell}}\frac{\partial A}{\partial x^j} - \frac{1}{2mA^2}\epsilon^{\hat{j}\hat{\ell}\hat{k}}\gamma_{\hat{k}}F_{\hat{j}}C_{\hat{\ell}}\frac{\partial A}{\partial t} - \frac{1}{4m}\gamma^{\hat{j}}L^k\frac{\partial}{\partial x^k}\left(\frac{D^j_{\hat{j}}}{A^2}\frac{\partial A}{\partial x^j}\right)$$
$$- \frac{1}{4m}\gamma^{\hat{j}}L^k\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A^2}\frac{\partial A}{\partial t}\right) + \frac{1}{4m}\gamma^{\hat{j}}\frac{\partial}{\partial t}\left(\frac{D^j_{\hat{j}}}{A^2}\frac{\partial A}{\partial x^j}\right) + \frac{1}{4m}\gamma^{\hat{j}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A^2}\frac{\partial A}{\partial t}\right). \quad \text{(A11)}$$

We remove the contribution of the Hamilton operator $\mathrm{H}_{2\,\mathrm{odd}}$ by the third unitary transformation of the wave function $\psi_2(x) = e^{-iS_3}\psi_3(x)$ and the Hamilton operator $\mathrm{H}_2$:

$$\mathrm{H}_3 = e^{+iS_3}\mathrm{H}_2e^{-iS_3} - ie^{iS_3}\frac{\partial}{\partial t}e^{-iS_3} = \mathrm{H}_2 - \frac{\partial S_3}{\partial t} + i\left[S_3, \mathrm{H}_2 - \frac{1}{2}\frac{\partial S_3}{\partial t}\right] + \frac{i^2}{2}\left[S_3, \left[S_3, \mathrm{H}_2 - \frac{1}{3}\frac{\partial S_3}{\partial t}\right]\right] + \ldots, \quad \text{(A12)}$$

where the operator $S_3$ is given by $S_3 = -(i/2mA)\gamma^0\mathrm{H}_{2\,\mathrm{odd}} = O(1/m^2)$. Neglecting the contributions of order $O(1/m^2)$ the low-energy approximation of the Dirac Hamilton operator for slow fermions is equal to $\mathrm{H}_3 = \mathrm{H}_{2\,\mathrm{even}}$, which we denote $\mathrm{H}_3 = \mathrm{H}_{2\,\mathrm{even}} = \mathrm{H}_{\mathrm{FW}}$. Thus, the Dirac equation in the low-energy approximation takes the form

$$i\frac{\partial\psi_3(x)}{\partial t} = \mathrm{H}_{\mathrm{FW}}\psi_3(x), \quad \text{(A13)}$$

where $\psi_3(x) = e^{iS_3}e^{iS_2}e^{iS_1}\psi'(x)$. Following the standard procedure [44] and multiplying both sides of Eq. (A13) by the matrix $(1+\gamma^0)/2$ we arrive at the Schrödinger-Pauli equation

$$i\frac{\partial\Psi(t,\vec{r})}{\partial t} = \left(\frac{1+\gamma^0}{2}\right)\mathrm{H}_{\mathrm{FW}}\psi_3(x) = \left(-\frac{1}{2m}\Delta + \Phi_{\mathrm{eff}}(t,\vec{r},\vec{\sigma})\right)\Psi(t,\vec{r}), \quad \text{(A14)}$$

where $\Psi(t,\vec{r}) = \frac{1+\gamma^0}{2}\psi_3(x)$ is the large component of the slow Dirac fermion wave function and $\Phi_{\mathrm{eff}}(t,\vec{r},\vec{\sigma})$ is the effective low-energy potential

$$\Phi_{\mathrm{eff}}(t,\vec{r},\vec{\sigma}) = (A-1)m + B + C^{\hat{\ell}}\sigma_{\hat{\ell}} + iL^j\frac{\partial}{\partial x^j} - \frac{m}{2A}\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}} - \frac{1}{2A}i\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^k_{\hat{k}}\frac{\partial}{\partial x^k} - \frac{1}{4}i\eta^{\hat{j}\hat{k}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A}\right)$$
$$- \frac{1}{4}\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A}\right) - \frac{1}{2A}i\eta^{\hat{j}\hat{k}}D^j_{\hat{j}}F_{\hat{k}}\frac{\partial}{\partial x^j} - \frac{1}{4A}i\eta^{\hat{j}\hat{k}}D^j_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^j} + \frac{1}{4A}\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^j_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^j} - \frac{1}{A}i\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}}\frac{\partial}{\partial t}$$
$$- \frac{1}{4A}i\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t} - \frac{1}{4}i\eta^{\hat{j}\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right) + \frac{1}{4A}\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t} - \frac{1}{4}\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right) - \frac{1}{2A}i\eta^{\hat{j}\hat{k}}(G_{\hat{j}}F_{\hat{k}} + F_{\hat{j}}G_{\hat{k}})$$
$$+ \frac{1}{A}\sigma^{\hat{j}}F_{\hat{j}}K + \frac{1}{4A^2}i\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^k_{\hat{k}}\frac{\partial A}{\partial x^k} + \frac{1}{4A^2}i\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t} + \frac{1}{2m}\eta^{\hat{j}\hat{k}}\left(\frac{D^j_{\hat{j}}D^k_{\hat{k}}}{A} - \delta^j_{\hat{j}}\delta^k_{\hat{k}}\right)\frac{\partial^2}{\partial x^j\partial x^k}$$
$$+ \frac{1}{4mA}\eta^{\hat{j}\hat{k}}D^j_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial x^k} + \frac{1}{4m}\eta^{\hat{j}\hat{k}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j} + \frac{1}{4mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^j_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial x^k}$$
$$- \frac{1}{4m}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j} + \frac{1}{2mA}\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^k_{\hat{k}}\frac{\partial^2}{\partial x^k\partial t} + \frac{1}{4mA}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial t}\frac{\partial}{\partial x^k} + \frac{1}{4m}\eta^{\hat{j}\hat{k}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}$$
$$+ \frac{1}{4mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial D^k_{\hat{k}}}{\partial t}\frac{\partial}{\partial x^k} - \frac{1}{4m}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t} + \frac{1}{2mA}\eta^{\hat{j}\hat{k}}G_{\hat{j}}D^k_{\hat{k}}\frac{\partial}{\partial x^k} + \frac{1}{4m}\eta^{\hat{j}\hat{k}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{G_{\hat{j}}}{A}\right)$$
$$- \frac{1}{4m}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{G_{\hat{j}}}{A}\right) + \frac{1}{2mA}i\sigma^{\hat{k}}KD^k_{\hat{k}}\frac{\partial}{\partial x^k} + \frac{1}{4m}i\sigma^{\hat{k}}D^k_{\hat{k}}\frac{\partial}{\partial x^k}\left(\frac{K}{A}\right) + \frac{1}{2mA}\eta^{\hat{j}\hat{k}}D^j_{\hat{j}}F_{\hat{k}}\frac{\partial^2}{\partial x^j\partial t}$$
$$+ \frac{1}{4mA}\eta^{\hat{j}\hat{k}}D^j_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial t} + \frac{1}{4m}\eta^{\hat{j}\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j} + \frac{1}{4mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^j_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial x^j}\frac{\partial}{\partial t} - \frac{1}{4m}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{D^j_{\hat{j}}}{A}\right)\frac{\partial}{\partial x^j}$$
$$+ \frac{1}{2mA}\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}}\frac{\partial^2}{\partial t^2} + \frac{1}{4mA}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t}\frac{\partial}{\partial t} + \frac{1}{4m}\eta^{\hat{j}\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t} + \frac{1}{4mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial F_{\hat{k}}}{\partial t}\frac{\partial}{\partial t}$$

$$
\begin{aligned}
&-\frac{1}{4m}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{j}}}{A}\right)\frac{\partial}{\partial t}+\frac{1}{2mA}\eta^{\hat{j}\hat{k}}G_{\hat{j}}F_{\hat{k}}\frac{\partial}{\partial t}+\frac{1}{4m}\eta^{\hat{j}\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right)-\frac{1}{4m}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{G_{\hat{j}}}{A}\right)\\
&+\frac{1}{2mA}i\sigma^{\hat{k}}KF_{\hat{k}}\frac{\partial}{\partial t}+\frac{1}{4m}i\sigma^{\hat{k}}F_{\hat{k}}\frac{\partial}{\partial t}\left(\frac{K}{A}\right)+\frac{1}{2mA}\eta^{\hat{j}\hat{k}}G_{\hat{k}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}+\frac{1}{4mA}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial x^{j}}+\frac{1}{4mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial x^{j}}\\
&+\frac{1}{2mA}\eta^{\hat{j}\hat{k}}F_{\hat{j}}G_{\hat{k}}\frac{\partial}{\partial t}+\frac{1}{4mA}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial t}+\frac{1}{4mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial G_{\hat{k}}}{\partial t}+\frac{1}{2mA}\eta^{\hat{j}\hat{k}}G_{\hat{j}}G_{\hat{k}}+\frac{1}{2mA}i\sigma^{\hat{k}}KG_{\hat{k}}\\
&+\frac{1}{2mA}i\sigma^{\hat{j}}KD^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}+\frac{1}{4mA}i\sigma^{\hat{j}}D^{j}_{\hat{j}}\frac{\partial K}{\partial x^{j}}+\frac{1}{2mA}i\sigma^{\hat{j}}KF_{\hat{j}}\frac{\partial}{\partial t}+\frac{1}{4mA}i\sigma^{\hat{j}}F_{\hat{j}}\frac{\partial K}{\partial t}+\frac{1}{2mA}i\sigma^{\hat{j}}KG_{\hat{j}}+\frac{1}{2mA}K^{2}\\
&-\frac{1}{4mA^{2}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}\frac{\partial}{\partial x^{j}}-\frac{1}{8mA}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)-\frac{1}{8mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)\\
&-\frac{1}{4mA^{2}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}\frac{\partial}{\partial t}-\frac{1}{8mA}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)-\frac{1}{8mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{D^{k}_{\hat{k}}}{A}\frac{\partial A}{\partial x^{k}}\right)\\
&-\frac{1}{4mA^{2}}\eta^{\hat{j}\hat{k}}G_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}-\frac{1}{4mA^{2}}i\sigma^{\hat{k}}KD^{k}_{\hat{k}}\frac{\partial A}{\partial x^{k}}-\frac{1}{4mA^{2}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial}{\partial x^{j}}-\frac{1}{8mA}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)\\
&-\frac{1}{8mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}D^{j}_{\hat{j}}\frac{\partial}{\partial x^{j}}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)-\frac{1}{4mA^{2}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial}{\partial t}-\frac{1}{8mA}\eta^{\hat{j}\hat{k}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)\\
&-\frac{1}{8mA}i\epsilon^{\hat{j}\hat{k}\hat{\ell}}\sigma_{\hat{\ell}}F_{\hat{j}}\frac{\partial}{\partial t}\left(\frac{F_{\hat{k}}}{A}\frac{\partial A}{\partial t}\right)-\frac{1}{4mA^{2}}\eta^{\hat{j}\hat{k}}G_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t}-\frac{1}{4mA^{2}}i\sigma^{\hat{k}}KF_{\hat{k}}\frac{\partial A}{\partial t}+\frac{1}{4mA^{3}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial x^{j}}\frac{\partial A}{\partial x^{k}}\\
&+\frac{1}{4mA^{3}}\eta^{\hat{j}\hat{k}}D^{j}_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial x^{j}}\frac{\partial A}{\partial t}+\frac{1}{4mA^{3}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}D^{k}_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial A}{\partial x^{k}}+\frac{1}{4mA^{3}}\eta^{\hat{j}\hat{k}}F_{\hat{j}}F_{\hat{k}}\frac{\partial A}{\partial t}\frac{\partial A}{\partial t},
\end{aligned}
\tag{A15}
$$

where $\sigma_{\hat{a}}=(-\vec{\sigma})_{\hat{a}}$ and $\sigma_{\hat{a}}=-\sigma^{\hat{a}}$. The effective potential (A15) is the most general effective low-energy potential for slow Dirac fermions in the Einstein-Cartan gravity with torsion and chameleon fields, calculated to order $O(1/m)$.

We would like to note that the effective low-energy potential (A15) is derived by using the Foldy-Wouthuysen method or the Foldy-Wouthuysen unitary transformations of wave functions of Dirac fermions with mass $m$, leading to a nonrelativistic Hamilton operator, expressed in terms of *even* operators only in the form of the large fermion mass expansion in powers of $1/m$. It is known that such a method of the reduction of the Dirac Hamilton operator to a form, containing only *even* operators, is not unique and there are some other methods of unitary transformations such as the Eriksen method [51] and others, which were well discussed by de Vries [50]. In Sec. III we give a short comparison of the Foldy-Wouthuysen with the Eriksen one only, since other methods of unitary transformations of a Dirac Hamilton operator to a form, containing only *even* operators, seem to be cumbersome when compared to the Eriksen method [50]. We discuss also the accuracy of the Foldy-Wouthuysen representation of a Dirac Hamilton operator.


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