

Synchrotron radiation from a weakly magnetized Schwarzschild black holeAndrey A. Shoom^{*}*Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2E1, Canada
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We consider a synchrotron radiation from a charged particle moving in a bound orbit around a weakly magnetized Schwarzschild black hole (a static black hole immersed into a constant uniform magnetic field) in its equatorial plane, perpendicular to the magnetic field. In particular, we study the case when the Lorentz force acting on the charged particle is directed outward from the black hole. In this case, for sufficiently large values of the particle's energy, the particle moves in a nongeodesic bound orbit with loops. Due to a synchrotron radiation, such an orbit decays to a nongeodesic circular stable orbit. We study this transition and calculate the radiated power and energy loss of the particle.

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I. INTRODUCTION

Recent observations indicate that there is a strong magnetic field of several hundred gauss around a supermassive black hole at the center of the Milky Way [1]. A magnetic field around a black hole can be present due to dynamo mechanism in the plasma of an accretion disk around a black hole (see, e.g., [2]). It can also originate from a pulsar orbiting the black hole. For instance, it can be shown that a black hole located far enough from a magnetar and close to its equatorial plane is immersed into a nearly homogeneous magnetic field [3,4]. We call such a black hole a *weakly magnetized black hole* (for details see Sec. II). The presence of such a magnetic field can significantly modify a test charged particle's motion. In particular, if a Lorentz force acting on a charged particle orbiting a weakly magnetized Schwarzschild black hole is directed outward, then, for a sufficiently large value of the magnetic field, the particle orbit can come arbitrary close to the black hole horizon (see, e.g., [5–9]). Motion of a test charged particle around a static black hole in the presence of a magnetic field was studied in many works. For example, a study of small radial and latitudinal harmonic oscillations about circular orbits around a weakly magnetized Schwarzschild black hole showed that such orbits are stable (see, e.g., [5,6]). An extensive analysis of such oscillations was done in [10]. A study of circular orbits and their stability in the Ernst metric with a uniform magnetic field was done in [5], and in a uniform electric, as well as in a uniform magnetic field, in [11]. A dynamics of an axisymmetric string loop carrying an electric current in the spacetime of a weakly magnetized Schwarzschild black hole was studied in [12]. Critical escape velocity and chaotic motion of a charged particle moving around a weakly magnetized Schwarzschild black hole was studied

in [13]. In these works, the effect of an electromagnetic radiation from a charged particle was not considered. On the other side, a magnetic field can cause a charged particle to move with an ultrarelativistic velocity. An ultrarelativistic particle emits an electromagnetic radiation whose properties are very similar to the properties of the synchrotron radiation from electrons in accelerators. In this paper we shall study a synchrotron radiation from a charged particle moving in the equatorial plane of a weakly magnetized Schwarzschild black hole.

A synchrotron radiation in a strong gravitational field generated by a black hole was studied quite widely. For example, a synchrotron radiation from a charged particle moving in a circular nongeodesic orbit around a weakly magnetized Schwarzschild and Kerr black hole was studied in, e.g., [14] and [15,16], respectively. High-frequency electromagnetic and gravitational radiation from a relativistic particle falling into a Schwarzschild and a Kerr black hole was considered and the spectral and angular distributions of the radiation power were calculated in [17]. The spectrum of electromagnetic radiation power from radially free falling monopole and pointlike dipole into a Schwarzschild black hole was found in [18]. Characteristic features of the electromagnetic spectrum of radiation from a radially free falling dipole were studied subsequently in [19]. An electromagnetic bremsstrahlung spectrum of a dipole falling into a Schwarzschild black hole along a spiral orbit was studied in [20]. An electromagnetic radiation spectrum from charged particles moving slowly in an eccentric bound equatorial orbit around a Schwarzschild black hole immersed in to a weak dipolar magnetic field was derived in [21].

In this paper, we shall study a synchrotron radiation from a charged particle moving in an equatorial plane around a weakly magnetized Schwarzschild black hole. We shall consider the case when the Lorentz force acting on the particle is directed outward. In this case, the particle can move in a bound orbit with loops. Such a bound orbit can

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be considered as a combination of a circular motion and a slow motion of its guiding center. The circular motion is a cyclotron revolution due to the magnetic field and the guiding center motion is a Keplerian motion due to the gravitational field of the black hole (for details see [9]). The particle radiates a synchrotron radiation during its cyclotron revolution. As a result, its energy decreases, that results in decrease of the size of the loops and a subsequent transition of the orbit with curls into a circular orbit. Here we shall study such a transition. We shall calculate the radiation power, the corresponding energy loss, the characteristic frequency of the synchrotron radiation, the proper time of the transition, and the portion of the particle orbit where the transition takes place.

This paper is organized as follows: In Sec. II we describe briefly a weakly magnetized Schwarzschild black hole. In Sec. III we review an equatorial motion of a charged particle around a weakly magnetized Schwarzschild black hole. Section IV contains a description of synchrotron radiation in a strong gravitational field. In Sec. V we calculate the energy loss of a charged particle due to its radiation. In Sec. VI we study the synchrotron radiation from a charged particle moving in a bound orbit in the equatorial plane of a weakly magnetized Schwarzschild black hole. Section VII contains a summary of the derived results.

We shall use the conventions adopted in [22] and units where $G = c = 1$. Occasionally, we shall use the Gaussian system of units.

II. WEAKLY MAGNETIZED SCHWARZSCHILD BLACK HOLE

In this section we briefly review the construction of a weakly magnetized Schwarzschild black hole. An extended review of magnetized black holes is presented in [6]. To construct a weakly magnetized Schwarzschild black hole of the mass M we assume that there is a weak magnetic field of the characteristic strength B , such that the spacetime curvature due to this magnetic field is much less than the spacetime curvature near the black hole alone, i.e., $GB^2/c^4 \ll c^4/G^2M^2$. This gives

$$B \ll \frac{c^4}{G^{3/2}M_\odot} \left(\frac{M_\odot}{M} \right) \sim 10^{19} \left(\frac{M_\odot}{M} \right) \text{ G}, \quad (1)$$

where M_\odot is the solar mass. In this case, the back reaction of the magnetic field on the spacetime geometry can be neglected. As a result, the spacetime metric is that of the Schwarzschild one,

$$ds^2 = - \left(1 - \frac{r_g}{r} \right) dt^2 + \left(1 - \frac{r_g}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \quad (2)$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2, \quad r_g = 2M.$$

This spacetime has isometries defined by the Killing vectors $\xi_{(t)}^\mu = \delta_t^\mu$, the generator of time translations, and $\xi_{(\phi)}^\mu = \delta_\phi^\mu$, the generator of azimuthal rotations. Let the magnetic field be static and axisymmetric. Such a magnetic field can be constructed as follows (see, e.g., [6,23]): The Maxwell equation for an electromagnetic 4-potential A^μ in the Lorenz gauge $A^\mu{}_{;\mu} = 0$ reads $A^{\mu;\nu}{}_{;\nu} = R^\mu{}_\nu A^\nu$. On the other side, a Killing vector obeys the equation $\xi^{\mu;\nu}{}_{;\nu} = -R^\mu{}_\nu \xi^\nu$. Thus, because the metric (2) is Ricci flat, one can take

$$A^\mu = \frac{B}{2} \xi_{(\phi)}^\mu, \quad (3)$$

where $B = \text{const}$. This 4-vector potential defines a magnetic field which is static, axisymmetric, and homogeneous at the spatial infinity, where it has the constant strength $B > 0$ and directed upward, orthogonal to the equatorial plane $\theta = \pi/2$.

The 4-potential (3) is invariant with respect to the isometries of the Killing vectors, i.e., $\mathcal{L}_{\xi_{(i)}} A = 0$, $i = (t, \phi)$. The corresponding electromagnetic field tensor $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$ has the following form:

$$F_{\mu\nu} = 2Br \sin\theta (\sin\theta \delta_{[\mu}^t \delta_{\nu]}^\phi + r \cos\theta \delta_{[\mu}^\theta \delta_{\nu]}^\phi). \quad (4)$$

The Schwarzschild metric (2) together with the vector potential (3) define a weakly magnetized Schwarzschild black hole—a Schwarzschild black hole immersed into a constant uniform magnetic field.

III. EQUATORIAL MOTION OF A CHARGED PARTICLE

In this section we review the main results derived in [9]. A dynamical equation for a charged particle of the mass m and the electric charge q moving in the vicinity of a weakly magnetized Schwarzschild black hole reads

$$m \frac{D^2 x^\mu}{d\tau^2} = q F^\mu{}_\nu u^\nu. \quad (5)$$

Here $D/d\tau$ stands for the covariant derivative defined in the metric (2), τ is the particle's proper time, $F_{\mu\nu}$ is given by Eq. (4), and $u^\mu = dx^\mu/d\tau$ is the particle 4-velocity, $u^\mu u_\mu = -1$. For this motion there are two conserved quantities associated with the spacetime Killing vectors: the energy $E > 0$ and the generalized azimuthal angular momentum $L_\phi \in (-\infty, +\infty)$,

$$E \equiv -\xi_{(t)}^\mu P_\mu = m \frac{dt}{d\tau} \left(1 - \frac{r_g}{r} \right), \quad (6)$$

$$L_\phi \equiv \xi_{(\phi)}^\mu P_\mu = \left(m \frac{d\phi}{d\tau} + \frac{qB}{2} \right) r^2 \sin^2\theta. \quad (7)$$

Here $P_\mu = mu_\mu + qA_\mu$ is the generalized 4-momentum of the particle, where the 4-potential A^μ is given by Eq. (3). An invariance of the 4-potential with respect to the spacetime isometries [$\mathcal{L}_{\xi^{(i)}} A = 0$, $i = (t, \phi)$] ensures the conservation of E and L_ϕ along dynamical orbits defined by Eq. (5). Note that L_ϕ is defined with respect to the axis $r = 0$ where fixed points of the Killing vector $\xi^\mu_{(\phi)}$ are located.

One can check that the θ -component of the dynamical equation (5) allows for a solution $\theta = \pi/2$. This is a motion in the equatorial plane of the black hole, orthogonal to the magnetic field. For an equatorial motion the conserved quantities (6) and (7) are sufficient for the complete integrability of the dynamical equation.

In what follows, it is convenient to introduce the dimensionless quantities

$$\begin{aligned} \mathcal{T} &= \frac{t}{r_g}, & \rho &= \frac{r}{r_g}, & \sigma &= \frac{\tau}{r_g}, \\ \mathcal{E} &= \frac{E}{m}, & \ell &= \frac{L_\phi}{mr_g}, & b &= \frac{qBr_g}{2m}. \end{aligned} \quad (8)$$

The first integral of the dynamical equation in the equatorial plane takes the following form:

$$\dot{\mathcal{T}} = \frac{\mathcal{E}\rho}{\rho-1}, \quad \dot{\phi} = \frac{\ell}{\rho^2} - b, \quad \dot{\rho}^2 = \mathcal{E}^2 - U, \quad (9)$$

$$U = \left(1 - \frac{1}{\rho}\right) \left[1 + \frac{(\ell - b\rho^2)^2}{\rho^2}\right], \quad (10)$$

where U is the effective potential. Here and in what follows, the overdot stands for the derivative with respect to σ . The effective potential is positive in the domain of interest $\rho \in (1, +\infty)$. It vanishes at $\rho \rightarrow 1$ and diverges at $\rho \rightarrow +\infty$. Depending on values of ℓ and b the potential is either monotonically increasing or has two extrema, a local maximum and a local minimum.

Equation (9) is invariant with respect to the discrete transformations

$$b \rightarrow -b, \quad \ell \rightarrow -\ell, \quad \phi \rightarrow -\phi. \quad (11)$$

According to the direction of the magnetic field ($B > 0$), for a positive electric charge q we have $b > 0$. Without loss of the generality, we can take $q > 0$. Equations for a particle with a negative charge are derived from those for a particle with a positive charge by the transformation (11).

The parameter ℓ can be positive or negative. For $\ell > 0$ the Lorentz force acting on a charged particle is repulsive, i.e., it is directed outward from the black hole. For $\ell < 0$ the Lorentz force is attractive, i.e., it is directed toward the black hole. In what follows, we shall consider the case when $\ell > 0$. In this case, according to Eq. (9) for the azimuthal coordinate ϕ , there is a critical value

$\rho_{\text{cr}} = \sqrt{\ell/b} > \rho_{\text{min}}$ for which $\dot{\phi} = 0$. For $\rho < \rho_{\text{cr}}$ one has $\dot{\phi} > 0$ and for $\rho > \rho_{\text{cr}}$ one has $\dot{\phi} < 0$. As a result, there are two different types of bound orbits separated by the critical one. The first type corresponds to $\rho_2 < \rho_{\text{cr}}$ and looks similar to a contracted trochoid. The second type corresponds to $\rho_2 > \rho_{\text{cr}}$ and has loops. It looks similar to a prolate trochoid. The critical type is defined by $\rho_2 = \rho_{\text{cr}}$ and has cusps. It looks similar to a cycloid (see Fig. 3 below). Orbits with loops correspond to the larger than $\mathcal{E}_{\text{cr}} = \sqrt{1 - 1/\rho_{\text{cr}}}$ values of the particle's energy. Such orbits can be considered as a result of a superposition of a cyclotron rotation along small circles and a slow drift of the guiding center of a circle (see [9]). For such orbits their local curvature radius is sufficiently smaller than the characteristic scale of the gravitational field. In this case, the wave equation for an electromagnetic radiation in curved spacetime can be approximated by that in a flat spacetime (for details see the next section).

Here we shall consider bound orbits.¹ A study of such orbits may help to analyze motion of particles in a black hole accretion disk. Bound orbits exist if the effective potential has two extrema: $\mathcal{E}_{\text{max}}^2 = U(\rho_{\text{max}})$ and $\mathcal{E}_{\text{min}}^2 = U(\rho_{\text{min}})$. For a bound orbit the radial coordinate ρ oscillates between the minimal ρ_1 and the maximal ρ_2 values, such that $\rho_{\text{max}} \leq \rho_1 \leq \rho_{\text{min}} \leq \rho_2$ and for a particular bound orbit the particle's energy \mathcal{E} obeys the relation $\mathcal{E}_{\text{min}} \leq \mathcal{E} \leq \mathcal{E}_{\text{max}}$.

IV. SYNCHROTRON RADIATION

For an ultrarelativistic particle of a charge q moving in Minkowski spacetime the radiated four-momentum is given by

$$d\mathcal{P}^{\hat{\alpha}} = \frac{2q^2}{3} \frac{d^2 x^{\hat{\mu}}}{d\tau^2} \frac{d^2 x^{\hat{\nu}}}{d\tau^2} u^{\hat{\alpha}} d\tau, \quad (12)$$

where $x^{\hat{\alpha}}$ are natural (Minkowski) coordinates (see, e.g. [24]). Because of nonlocal nature of the process of formation of electromagnetic radiation, this expression generally is not applicable for a charged particle moving in a curved spacetime. This is the case when the particle moves along a spacetime geodesic and the characteristic length δl of the particle's orbit where the dominant part of the radiation is formed is of the order of the characteristic scale of the gravitational field $L \sim r_g$. However, if dynamics of the particle is mostly governed by forces of non-gravitational, e.g. electromagnetic, nature such that the particle's world line is not a geodesic, then the process of formation of electromagnetic radiation can approximately be described by local quantities (see, e.g., [14,25,26]).

¹Bound orbits of small harmonic radial and latitudinal oscillations around a stable circular orbit were studied in [10].

To see why such a description is possible, let us consider a locally geodesic frame of reference localized somewhere in the vicinity of the midpoint of δl . Such a frame of reference can be defined by the normal Riemann coordinates $y^{\hat{\alpha}}$ (see, e.g., [22]). In this frame we have

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} + \mathcal{O}(y^2), \quad (13)$$

$$\Gamma^{\hat{\alpha}}_{\hat{\mu}\hat{\nu}} = -\frac{1}{3}(R^{\hat{\alpha}}_{\hat{\mu}\hat{\nu}\hat{\gamma}} + R^{\hat{\alpha}}_{\hat{\nu}\hat{\mu}\hat{\gamma}})y^{\hat{\gamma}} + \mathcal{O}(y^2), \quad (14)$$

where $\eta_{\hat{\mu}\hat{\nu}}$ is an orthonormal metric and y is the proper size of the spacetime domain in the vicinity of the particle location covered by the normal Riemann coordinates. Let us now estimate the characteristic wavelength of a synchrotron radiation, λ_* . The angular distribution of the synchrotron radiation is localized in a narrow cone of the opening angle $\delta\varphi \approx \hat{\gamma}^{-1} \ll 1$ around the direction of the particle velocity (see, e.g., [24,27]). This ‘‘projector effect’’ is characteristic for a synchrotron radiation. Let r_c be the local curvature radius of the particle orbit. Then, $\delta l \approx r_c \hat{\gamma}^{-1}$. We shall now find the characteristic wavelength of the radiation. One has $\lambda_* = \hat{T}$, where \hat{T} is the period of the radiated wave measured by a local observer. On the other side, $\delta l = \delta \hat{l}' = (d\hat{l}'/d\hat{t}')\hat{T}$, where \hat{l}' is the time measured in the particle’s frame and \hat{t}' is the time measured by the observer. To calculate $d\hat{l}'/d\hat{t}'$ we shall use the relation $\hat{t} = \hat{t}' + \hat{R}(\hat{t}')$, where $\hat{R}(\hat{t}')$ is the distance measured from the particle to the observer along the light trajectory. Using the obvious expression $d\hat{R}/d\hat{t}' = -\hat{n} \cdot \hat{v} = \hat{v} \cos \delta\varphi$, where \hat{n} is the unit vector in the direction of the radiation, we derive

$$\frac{d\hat{t}}{d\hat{t}'} \approx 1 - \hat{v} + \frac{\hat{v}}{2} \delta\varphi^2 \approx \hat{\gamma}^{-2}. \quad (15)$$

As a result, we find

$$\lambda_* \approx \delta l \hat{\gamma}^{-2} \approx r_c \hat{\gamma}^{-3}. \quad (16)$$

This expression implies that $\lambda_* \ll r_c$. It means that the wave zone of the radiation begins at the distances shorter than r_c .

The wave equation for the radiation 4-vector potential A_{rad}^α reads

$$-A_{\text{rad};\beta}^{\alpha;\beta} + R^\alpha_{\beta} A_{\text{rad}}^\beta = 4\pi J^\alpha, \quad A_{\text{rad};\alpha}^\alpha = 0. \quad (17)$$

In a locally geodesic frame, in a vacuum spacetime, with the use of the expressions (13)–(14) it takes the following linear in $y^{\hat{\alpha}}$ form:

$$-\square A_{\text{rad}}^{\hat{\alpha}} + \frac{2}{3}(R^{\hat{\alpha}\hat{\mu}}_{\hat{\nu}\hat{\gamma}} + R^{\hat{\alpha}}_{\hat{\nu}\hat{\mu}\hat{\gamma}})y^{\hat{\gamma}} A_{\text{rad};\hat{\mu}}^{\hat{\nu}} \approx 4\pi J^{\hat{\alpha}}, \quad A_{\text{rad};\hat{\alpha}}^{\hat{\alpha}} \approx 0, \quad (18)$$

where \square is the wave operator defined in the orthonormal metric $\eta_{\hat{\mu}\hat{\nu}}$. For a synchrotron radiation the terms containing the Riemann tensor components are of the relative order of $\lambda_* y/L^2$. For $y \lesssim L$ the relative order of the terms proportional to the Riemann tensor due to large Lorentz gamma factor is small even for $r_c \sim L$. Thus, such terms can be neglected and the wave equation can be approximated by that in a flat spacetime.

As a result, the synchrotron radiation from an electrically charged particle moving in a nongeodesic world line in the vicinity of a weakly magnetized Schwarzschild black hole can be described in a locally geodesic frame of reference where the spacetime is practically flat and the expression (12) is applicable. Note, however, that this expression correctly covers the entire range of frequencies in flat spacetime only. In a curved spacetime the low frequency part depends on the gravitational field which is not included into consideration. As a result, the low frequency part is neglected when one uses the expression (12) in a curved spacetime.

In the global Schwarzschild coordinates the radiated four-momentum of a charged particle moving around a weakly magnetized Schwarzschild black hole, as measured by a static observer located in the asymptotically flat region, reads

$$d\mathcal{P}^\alpha = \frac{2q^2}{3} \frac{D^2 x^\mu}{d\tau^2} \frac{D^2 x_\mu}{d\tau^2} dx^\alpha. \quad (19)$$

For the dynamical motion (5) this expression can be written as

$$d\mathcal{P}^\alpha = \frac{2q^4}{3m^2} F^\mu_{\nu} u^\nu F_{\mu\lambda} u^\lambda dx^\alpha. \quad (20)$$

For an equatorial motion ($\theta = \pi/2$) in the ultrarelativistic approximation $\gamma = dt/d\tau \gg 1$, the radiated four-momentum (20) reads

$$d\mathcal{P}^\alpha = \frac{2q^4 B^2}{3m^2} \gamma^2 \left(1 - \frac{r_g}{r}\right)^2 dx^\alpha. \quad (21)$$

One can also calculate the spectral distribution function for the synchrotron radiation. Introducing the radiation intensity

$$I = -dE/dt, \quad (22)$$

where E is the charged particle’s energy, one can derive the following expression (for details see [25,26]):

$$\frac{dI}{dy} = \frac{3\sqrt{3}}{4\pi} \frac{q^4 B^2}{m^2} \gamma^2 \left(1 - \frac{r_g}{r}\right)^3 F(y), \quad (23)$$

where

$$y = \frac{\Omega}{\Omega_c}, \quad \Omega_c = \frac{3qB}{2m} \gamma^2 \left(1 - \frac{r_g}{r}\right)^2, \quad (24)$$

and

$$F(y) = y \int_y^\infty K_{5/3}(x) dx, \quad \int_0^\infty F(y) dy = \frac{8\pi}{9\sqrt{3}}. \quad (25)$$

Here $K_{5/3}(x)$ is the MacDonald function and Ω is the radiation frequency. The function $F(y)$ is localized near its maximum at $y \approx 0.29$. Thus, the main part of the synchrotron radiation is concentrated around the characteristic frequency

$$\Omega_* \approx 0.29\Omega_c. \quad (26)$$

The total radiation intensity I is derived by integrating the expression (23) over $y \in [0, \infty)$,

$$I = \frac{2}{3} \frac{q^4 B^2}{m^2} \gamma^2 \left(1 - \frac{r_g}{r}\right)^3. \quad (27)$$

V. ENERGY LOSS

The expression (27) allows us to calculate the rate of the energy loss measured with respect to the particle's proper time. Using the expressions (8) and (9) we derive the rate of the energy loss,

$$\dot{\mathcal{E}} = -\kappa \mathcal{E}^3, \quad \kappa = \frac{8q^2 b^2}{3mr_g}. \quad (28)$$

Integrating we derive

$$\mathcal{E}(\sigma) = \frac{\mathcal{E}_i}{\sqrt{1 + 2\kappa \mathcal{E}_i^2 \sigma}}, \quad (29)$$

where \mathcal{E}_i is the initial energy. This expression illustrates how the particle's energy decreases with its proper time due to the synchrotron radiation. The rate of the energy decrease is controlled by the parameter κ .

In order to estimate the value of κ , we shall use the Gaussian system of units. We have

$$\kappa \approx 191 \left(\frac{q}{e}\right)^4 \left(\frac{m}{m_e}\right)^{-3} \left(\frac{B}{10^8 \text{ G}}\right)^2 \left(\frac{M}{M_\odot}\right), \quad (30)$$

where e is the elementary electric charge and m_e is the electron mass. According to Piotrovich, Silant'ev, Gnedin, and Natsvlishvili [28], the characteristic scales of the

magnetic field B are of the order of 10^8 G near the horizon of a stellar mass black hole, $M \sim 10M_\odot$, and of the order of 10^4 G near the horizon of a supermassive black hole, $M \sim 10^9 M_\odot$. As a result, we derive $\kappa \approx 1.910 \times 10^3$ for an electron orbiting a stellar mass or a supermassive black hole, and $\kappa \approx 3.085 \times 10^{-7}$ for a proton orbiting a stellar mass or a supermassive black hole.

Thus, ultrarelativistic electrons orbiting astrophysical black holes lose their energy due to their synchrotron radiation relatively fast. On the other side, for protons or heavy ions orbiting astrophysical black holes $\kappa \ll 1$. This implies that protons and heavy ions lose their energy relatively slow.

VI. SYNCHROTRON RADIATION FROM BOUND ORBITS

The shape of the effective potential (10) is controlled by values of the parameters b and ℓ . For a given weakly magnetized Schwarzschild black hole and a charged particle the parameter b is fixed [see Eq. (8)]. In order to have bound orbits (which are possible if the effective potential has two extrema) the dynamical parameter ℓ should take certain values related to the fixed value of b . Let us estimate these values. In the Gaussian system of units the parameter b has the following value:

$$b \approx 8.663 \times 10^9 \left(\frac{q}{e}\right) \left(\frac{m}{m_e}\right)^{-1} \left(\frac{B}{10^8 \text{ G}}\right) \left(\frac{M}{M_\odot}\right). \quad (31)$$

According to [28], we have $b \approx 4.718 \times 10^7$ for a proton orbiting a stellar mass black hole and $b \approx 4.718 \times 10^{11}$ for a proton orbiting a supermassive black hole. For an electron the value of b is $m_p/m_e \approx 1836$ times larger. In order to find the corresponding value of ℓ we use the equation for extrema of the effective potential, $U_{,\rho} = 0$. This equation is equivalent to the equation

$$(3 - 2\rho)\ell^2 - 2b\rho^2\ell + \rho^2(2b^2\rho^3 - b^2\rho^2 + 1) = 0, \quad (32)$$

which is a quadratic polynomial in ℓ . Solving this equation for ℓ we derive

$$\ell_{\pm} = \frac{b\rho^2 \pm \rho\sqrt{4b^2\rho^2(\rho-1)^2 - 3 + 2\rho}}{3 - 2\rho}. \quad (33)$$

The solution ℓ_- is positive in the domain $\rho \in [\rho_c, +\infty)$, where $\rho_c \in (1, 3/2)$. The solution ℓ_+ has two separate branches. The branch corresponding to $\ell_+ > 0$ is defined in the domain $\rho \in [\rho_c, 3/2)$. In what follows, according to our choice ($\ell > 0$), we shall consider this branch of ℓ_+ . This branch and the branch ℓ_- merge at $\rho = \rho_c$. For large values of b , $\ell = \ell_{\pm}$ is large as well. In this case, the branch ℓ_- corresponds to the minimum of U , $\ell = \ell_-(\rho_{\min})$ and the branch ℓ_+ corresponds to the maximum of U ,

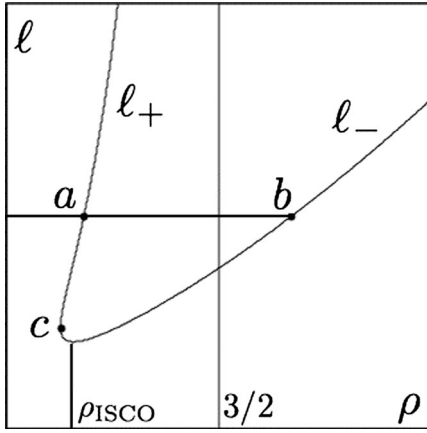


FIG. 1. The branches $\ell_{\pm} > 0$. The branches ℓ_+ and ℓ_- merge at the point c . For sufficiently large values of $\ell = \ell_+(\rho_{\max}) = \ell_-(\rho_{\min})$ the points a and b correspond to the maximum and minimum of the effective potential, respectively. The maximum and minimum of the effective potential merge at $\rho = \rho_{\text{ISCO}}$ where ℓ_- has the minimal value.

$\ell = \ell_+(\rho_{\max})$. The branches ℓ_{\pm} are schematically illustrated in Fig. 1.

Accordingly, for large values of b one can derive the following expansions of the expression (33):

$$\ell_+ = \frac{b\rho_{\max}^2(2\rho_{\max} - 1)}{(3 - 2\rho_{\max})} + \mathcal{O}(b^{-1}), \quad (34)$$

$$\ell_- = b\rho_{\min}^2 + \frac{1}{4b(\rho_{\min} - 1)} + \mathcal{O}(b^{-3}). \quad (35)$$

In the frame of reference where the charged particle is at rest, the dipole radiation is symmetric and, therefore, the radiated momentum is zero. However, in another frame of reference, e.g., in a locally geodesic frame, the dipole radiation is mostly concentrated in a narrow cone in the direction of the particle velocity. Thus, the synchrotron radiation carries away the azimuthal angular momentum of a charged particle. The expression (21) allows us to calculate the rate of the angular momentum loss measured with respect to the particle's proper time, $dL_{\phi}/d\tau = -g_{\phi\phi}d\mathcal{P}^{\phi}/d\tau$. Using the expressions (8) and (9) we derive

$$\dot{\ell} = -\kappa\mathcal{E}^2\rho^2\dot{\phi} = -\kappa\mathcal{E}^2(\ell - b\rho^2). \quad (36)$$

Having this expression we can compare the relative rate of the energy and the angular momentum loss. For a charged particle moving in the vicinity of the black hole we have $\rho \sim 1$ and the expression (35) gives $\ell \sim b$. Using the expressions (28) and (36) we derive

$$\frac{|\dot{\ell}|}{\ell} \sim \frac{|\dot{\mathcal{E}}|}{\mathcal{E}} b^{-2}. \quad (37)$$

Thus, for relatively short time intervals one can neglect the angular momentum loss. In what follows, we shall consider $\ell \approx \text{const}$.

Using Eqs. (10), (34), and (35) one can find the corresponding expansions of the extrema,

$$\mathcal{E}_{\max}^2 = \frac{16b^2\rho_{\max}(\rho_{\max} - 1)^3}{(3 - 2\rho_{\max})^2} + \mathcal{O}(1), \quad (38)$$

$$\mathcal{E}_{\min}^2 = 1 - \frac{1}{\rho_{\min}} + \frac{1}{16b^2\rho_{\min}^3(\rho_{\min} - 1)} + \mathcal{O}(b^{-4}). \quad (39)$$

Note that ℓ_+ and \mathcal{E}_{\max}^2 diverge at $\rho_{\max} = 3/2$. This value of ρ_{\max} corresponds to an unstable circular orbit of the radius $r = 3M$, which is a null geodesic of the Schwarzschild spacetime. However, as it follows from Fig. 1 and Eqs. (9), (10), and (35), unlike in a geodesics motion, the values of the particle's angular momentum ℓ and energy \mathcal{E} remain finite for $\rho_{\min} \rightarrow 3/2$ and for bound orbits lying in the vicinity of $\rho_{\min} = 3/2$. (see Fig. 2). Thus, the expressions in this section are valid for $\rho_{\min} = 3/2$ as well. This result was also presented in [5,6].

The maximal and the minimal values of a proton's energy as functions of ρ_{\min} are presented in Fig. 2. One can see that \mathcal{E}_{\max} is of several orders higher than \mathcal{E}_{\min} . For a given value of ρ_{\min} which defines a stable circular orbit about which a charged particle moving in a bound orbit oscillates, one can find the difference $\mathcal{E}_{\max} - \mathcal{E}_{\min}$ which is a measure of the energy carried out by synchrotron radiation. As one can see from Fig. 2, this energy is of the order of \mathcal{E}_{\max} .

A decrease of the energy of a charged particle moving in a bound orbit results in a transition from the prolate trochoid orbit to "cycloid" and to "contracted trochoid" followed by decay of the radial oscillations, so that the

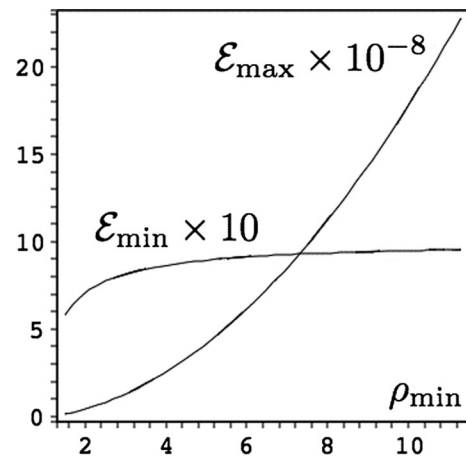


FIG. 2. The minimal and the maximal values of a proton's energy which is moving around a stellar mass black hole in a bound orbit oscillating about the guiding center stable circular orbit of the radius ρ_{\min} . These values vanish for $\rho_{\min} \rightarrow 1$.

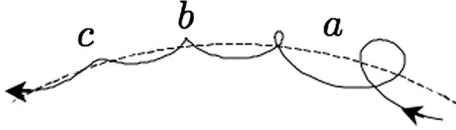


FIG. 3. Schematic illustration of the decay of radial oscillations: a transition from the “prolate trochoid” orbit (section *a*) to “cycloid” (section *b*) and to “contracted trochoid” (section *c*). The dashed line represents the guiding center stable circular orbit of $\rho = \rho_{\min}$.

particle orbit becomes nearly circular. This transition is schematically illustrated in Fig. 3. Note that a real prolate trochoid orbit has a very large number of densely packed loops (see [9]).

Let us now calculate the dimensionless proper time $\Delta\sigma$ required to lower the particle energy from $\mathcal{E}_i = \mathcal{E}_{\max}$ to \mathcal{E}_f . Using the expression (29), we derive

$$\Delta\sigma = \frac{\mathcal{E}_i^2 - \mathcal{E}_f^2}{2\kappa\mathcal{E}_i^2\mathcal{E}_f^2}. \quad (40)$$

One can estimate an upper bound of $\Delta\sigma$. Taking $\mathcal{E}_i = \mathcal{E}_{\max}$ and $\mathcal{E}_f = \mathcal{E}_{\min}$ and using Eq. (39) we derive

$$\Delta\sigma \lesssim \Delta\sigma_{\max} \approx \frac{\rho_{\min}}{2\kappa(\rho_{\min} - 1)}. \quad (41)$$

The corresponding to $\Delta\sigma$ portion of the guiding center stable circular orbit $\Delta\mathcal{C}$ about which the particle makes the radial oscillations can be calculated with the use of its drift angular velocity [cf. Eqs. (9) and (35)],

$$\langle \dot{\phi} \rangle \approx \frac{\ell}{\rho_{\min}^2} - b \approx \frac{1}{4b\rho_{\min}^2(\rho_{\min} - 1)}. \quad (42)$$

We have

$$\Delta\mathcal{C} = \Delta\sigma \frac{\langle \dot{\phi} \rangle}{2\pi} \approx \frac{\Delta\sigma}{8\pi b\rho_{\min}^2(\rho_{\min} - 1)}. \quad (43)$$

The maximal portion of the guiding center circular orbit corresponds to $\Delta\sigma_{\max}$,

$$\Delta\mathcal{C}_{\max} \approx \frac{1}{16\pi\kappa b\rho_{\min}(\rho_{\min} - 1)^2}. \quad (44)$$

The quantities defined above are presented in Fig. 4 as functions of ρ_{\min} . One can see that the dimensionless proper time $\Delta\sigma$ as well as the portion of the guiding center circular orbit $\Delta\mathcal{C}$ rapidly decrease for bound orbits corresponding to larger values of ρ_{\min} . This implies that an intensity of the synchrotron radiation is considerably greater from particles moving in bound orbits of larger radii. Note that because the value of κ for an electron is of many orders larger than

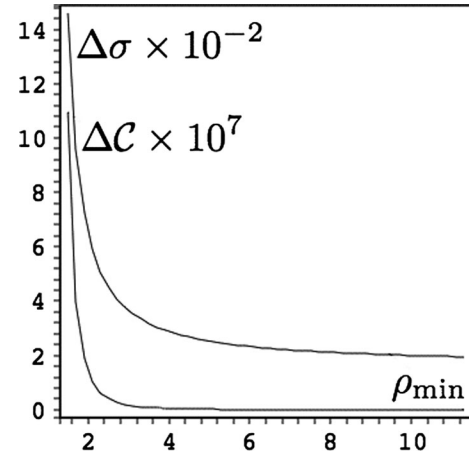


FIG. 4. The dimensionless proper time $\Delta\sigma$ and the portion of the guiding center circular orbit $\Delta\mathcal{C}$ for a proton orbiting a stellar mass black hole. The initial and final energy values of the proton are $\mathcal{E}_i = \mathcal{E}_{\max}$ and $\mathcal{E}_f = \gamma(\rho_{\min} - 1)/\rho_{\min}$ with $\gamma = 100$.

that of a proton or a heavy ion, the corresponding values of $\Delta\sigma$ and $\Delta\mathcal{C}$ are of many orders smaller. Thus, bound orbits of electrons decay much faster than bound orbits of protons and heavy ions.

Let us finally calculate an evolution of the characteristic frequency of the synchrotron radiation. Using the expressions (8), (9), (24), and (26) we derive the dimensionless characteristic frequency,

$$\omega_* = \Omega_* r_g \approx 0.87b\mathcal{E}^2. \quad (45)$$

Then, the expression (29) gives

$$\omega_* \approx \frac{0.87b\mathcal{E}_i^2}{1 + 2\kappa\mathcal{E}_i^2\sigma}. \quad (46)$$

The upper bound on the characteristic frequency is [cf. Eq. (38)]

$$\omega_{*\max} \approx 14b^3 \frac{\rho_{\max}(\rho_{\max} - 1)^3}{(3 - 2\rho_{\max})^2}. \quad (47)$$

According to Fig. 2, the expression (46) implies that the characteristic frequency of the synchrotron radiation is larger for prolate trochoid type bound orbits oscillating about the guiding center stable circular orbit of the radius ρ_{\min} than that for nearly circular orbits of the same ρ_{\min} value. Moreover, the frequency is larger for the orbits of larger value of ρ_{\min} . The rate of the frequency decrease is considerably larger for electrons than that for protons and heavy ions.

VII. SUMMARY

In this paper we studied a synchrotron radiation from a charged particle moving in an equatorial plane of a weakly magnetized Schwarzschild black hole. The Lorentz force

acting on the charged particle is directed outward. The particle originally moves in a bound orbit with loops, which is a combination of a cyclotron motion and a slow Keplerian motion of the guiding center. Due to a synchrotron radiation, the particle loses its energy. The particle's azimuthal angular momentum loss is negligible, comparing to its energy loss. As a result of the energy loss, there is a transition of the orbit with loops to a stable circular orbit of the radius corresponding to the motion of the guiding center. We calculated the radiation power, the corresponding energy loss, the proper time of the transition, the portion of the particle orbit where the transition takes place, and the characteristic frequency of the synchrotron radiation.

Our results show that the radiation power, as measured with respect to a charged particle's proper time, is proportional to the squared magnetic field strength and inversely proportional to the cubed particle's mass. Thus, for example, for electrons the rate of the energy loss of the synchrotron radiation is of many orders larger than that for protons and heavy ions. For a particle of the same energy and charge-to-mass ratio the radiation power is greater for a stellar mass black holes than that for a supermassive ones. Moreover, the power is considerably greater for particles moving in bound orbits whose guiding center orbit has larger radius.

The orbit decay results in a transition from a prolate trochoid bound orbit, which has loops, to cycloid bound orbit, and finally to contracted trochoid bound orbit. The maximal transition time, as measured with respect to a particle's proper time, is inversely proportional to the squared strength of the magnetic field and proportional to the cubed particle's mass. For a given value of the magnetic field strength, it is of many orders shorter for electrons than for protons and heavy ions. The portion of the particle's orbit

where the transition takes place is proportional to the forth power of the particle mass and inversely proportional to the cubed strength of the magnetic field. The portion is smaller for greater radial values of the guiding center orbit. The characteristic frequency of the synchrotron radiation is larger for a charged particle moving in a prolate trochoid type orbit than for the charged particle moving in a cycloid and contracted trochoid type orbit.

The derived results could be useful in the model of a hypothetical thin accretion disk around an astrophysical black hole which is composed of charged test particles. Such particles can exchange energy due to their collision. Suppose that before a collision one of the particles moves in a circular orbit corresponding to the Lorentz force directed outward, and let due to the collision the particle acquires extra energy and begins moving in a bound orbit with loops. Then, because of the synchrotron radiation, the particle's energy is transferred to an electromagnetic wave and the particle's orbit becomes circular again. Thus, in this scenario, according to our results, in such an accretion disk bound orbits with loops have shorter lifetime if compared with circular ones.

In the analysis presented in this paper the radiation effects were considered on "a background of given dynamics" which defines fixed-energy bound orbits. It would be interesting to consider a dynamical analysis where the radiation reaction forces are included into the particle's dynamics.

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