

Derivation of local-in-time fourth post-Newtonian ADM Hamiltonian for spinless compact binaries

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The paper gives full details of the computation within the canonical formalism of Arnowitt, Deser, and Misner of the local-in-time part of the fourth post-Newtonian, i.e. of power eight in one over speed of light, conservative Hamiltonian of spinless compact binary systems. The Hamiltonian depends only on the bodies' positions and momenta. Dirac delta distributions are taken as source functions. Their full control is furnished by dimensional continuation, by means of which the occurring ultraviolet (UV) divergences are uniquely regularized. The applied near-zone expansion of the time-symmetric Green function leads to infrared (IR) divergences. Their analytic regularization results in one single ambiguity parameter. Unique fixation of it was successfully performed in T. Damour, P. Jaranowski, and G. Schäfer, *Phys. Rev. D* **89**, 064058 (2014) through far-zone matching. Technically as well as conceptually (backscatter binding energy), the level of the Lamb shift in quantum electrodynamics is reached. In a first run a computation of all terms is performed in three-dimensional space using analytic Riesz-Hadamard regularization techniques. Then divergences are treated locally (i.e., around particles' positions for UV and in the vicinity of spatial infinity for IR divergences) by means of combined dimensional and analytic regularization. Various evolved analytic expressions are presented for the first time. The breakdown of the Leibniz rule for distributional derivatives is addressed as well as the in general nondistributive law when regularizing value of products of functions evaluated at their singular point.

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I. INTRODUCTION

In the early days of Einstein's theory of gravity, the theory of general relativity as Einstein coined it, the weak-field and slow-motion expansion of the field equations, nowadays known as post-Newtonian (PN) expansion, played a crucial role already. The first post-Newtonian (1PN) approximation revealed a convincing understanding of the perihelion advance of Mercury [1]. Later on, in 1919, the light bending at the limb of the Sun was measured in agreement with the 1PN prediction of Einstein's theory [2,3]. A rich 1PN scenario came up with the discovery of the Hulse-Taylor binary pulsar PSR B1913 + 16 in 1974, where after four years of observation even the dissipative two-and-a-half post-Newtonian (2.5PN) effect could be measured in agreement with general relativity in the form of energy loss through gravitational radiation damping [4]. Also the precession of the proper angular momentum (spin) of the pulsar PSR B1913 + 16 from spin-orbit coupling, a 1PN or 1.5PN effect according to the counting of the spin as of 0PN or 0.5PN order, respectively, could be seen [5]. The discovery of the double pulsar system J0737 – 3039 in

2003 allowed measurements of even more 1PN effects than with the Hulse-Taylor binary pulsar [6]. Also in the solar system and in the gravitational field of the Earth the 1PN approximation is crucial [7,8]. In the observation of the Hulse-Taylor pulsar, even 2PN measurement accuracy has been achieved in the periastron advance [9], deduced for the first time in [10] with full details given in [11]. In the long-term run, the observation of the double pulsar system is likely to allow detailed measurements of conservative 2PN and dissipative 3.5PN effects [6,12–14]. On the other side, in the coming years gravitational-wave astronomy will come to operation through advanced (or second-generation) LIGO and Virgo observatories [15,16] and the cryogenic KAGRA detector (formerly known as LCGT) [17]. Even more sensitive third-generation detectors, like ET [18], are already under consideration. Then still higher-order PN effects will become important. Great efforts have already been undertaken in the precise data analysis of possible wave forms (see, e.g., [19–22]), and as well as in the computation of the orbital and spin dynamics through higher PN orders including the gravitational wave emission (see, e.g., [23–30]).

The complete conservative third post-Newtonian (3PN) binary dynamics has been fully achieved for the first time in 2001 [31], based on earlier work [32–35], rooted in the

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canonical formalism of Arnowitt, Deser, and Misner (ADM) with application of a coordinate system of maximal isotropy in three-dimensional space [36]. Around 2004 and later in 2011 the final results of [31–35] became fully confirmed [37–40], in using completely different techniques but yet all with the employment of harmonic coordinate conditions. Let us stress that the rather tedious (but much easier than reported in the present paper) calculations performed in [31–35] were not just a matter of straightforward computational efforts. The standard regularization Hadamard “partie-finie” technique together with the Hadamard “partie-finie” based Riesz analytic continuation supplemented by the Schwartz distribution theory in $(3 + 1)$ -dimensional space-time resulted in ambiguities [32,33], which yet could be parametrized by two variables only, ω_s and ω_k [34] (for a detailed presentation of the mentioned techniques, see [41] and Appendix A of the present paper). Matching to the Brill-Lindquist initial value solution for a two-black-hole system [42], a solution which was shown derivable from “fictitious” point-mass sources of Dirac δ -function type (one for each black hole) [43], brought out (by the very definition of ω_s) $\omega_s = 0$ [33,34], and implementation of Lorentz invariance in the form of fulfillment of the Poincaré algebra resulted in $\omega_k = 41/24$ (see [35] and Sec. IX of the present paper). Another procedure to obtain $\omega_s = 0$ could have been to insist on the “tweedling of products” structure in the course of computation of regularized value of the product of singular functions (see [33,44] and Appendix A 4 of the current paper) or, connected herewith, by taking just the finite terms (without limiting procedure) in the two-body restriction of the many-body static potential derived in [45]. Only by making use of the technique of dimensional regularization a uniform treatment was achieved in Ref. [31], with confirmation of the numerical values for ω_s and ω_k found earlier (a summary of the applied dimensional-regularization techniques is presented in [46] and in Appendix A below). For extensive application of the dimensional regularization in the generalized ADM formalism for spinning compact binaries the reader may consult [28].

It might be worth comparing the derivation of the 3PN ADM Hamiltonian in [31–35] with the history of the harmonic-coordinate-based calculations, which led to the results achieved in [39], where finally also dimensional regularization played the crucial role. In the papers [47,48] a manifest Lorentz-invariant “extended” Hadamard regularization procedure was developed, which allowed the calculation of the 3PN binary dynamics with only one ambiguity parameter λ . The comparison, for circular motion, of energy in terms of orbital angular frequency (which is a coordinate-invariant or gauge-invariant relation) revealed the relation $\lambda = -3\omega_s/11 - 1987/3080$ [49]. Evidently, the application of the Poincaré algebra in [35] is the equivalent of the manifest Lorentz invariance in [47,48]. However, as shown in [39], the extended

Hadamard regularization procedure, being incompatible with Schwartz distribution theory [see, e.g., Eqs. (7.17) and (8.34) in [47], which should be compared with Eqs. (A90b) and (A92b), respectively, of our paper], could not be made compatible with dimensional regularization [see, e.g., Sec. III D in [39]]. Therefore all terms of the extended Hadamard procedure different from the standard Hadamard (“pure Hadamard-Schwartz” in [39]) ones, had to be traced back to their origins and eliminated [see, e.g., Eq. (3.55) in [39]] before dimensional generalization and regularization could be employed. There exists another derivation of the 3PN binary dynamics in harmonic coordinates made in [40], using an effective field theory approach with space-Fourier transformed fields, which was from the very beginning put on a $(d + 1)$ -dimensional space-time footing with dimensional regularization.

It is interesting to note that the still other derivation of the 3PN binary dynamics in [37,38], based on the Einstein-Infeld-Hoffmann surface integral technique [50], is a purely $(3 + 1)$ -dimensional one. The used surface-integral technique allowed a manifest Lorentz-invariant calculation with surface integrals defined in the smooth vacuum regime. Divergent integrals entered only on technical reasons in simplifying computations. (In the paper by Einstein, Infeld, and Hoffmann [50] the field singularities were already seen in correspondence with Dirac δ -functions, though their treatment as full field sources had had to await for later developments in the form of Infeld’s “good δ -functions,” see, e.g., the book by Infeld and Plebański [44].) Remarkably, the Brill-Lindquist initial value solution in $(3 + 1)$ -dimensional space-time mentioned above has originally been obtained from the pure vacuum Einstein field equations without facing any physical or geometrical divergences [42].

The fourth post-Newtonian (4PN) conservative dynamics of two-point-mass systems has been completed only quite recently [51], based on previous calculations [52,53]. The results of Refs. [51–53] were in part confirmed by [54–57] (see also [58,59]). The present paper, announced in [51], delivers the details of the involved computations of the 4PN conservative two-point-mass ADM Hamiltonian, where ultraviolet (UV) and infrared (IR) divergences have to be tackled simultaneously and where a mixed dimensional-analytic regularization treatment is needed to be applied (like in [60]). Pure dimensional regularization allows one to control UV divergences only and Refs. [52,53] have uniquely regularized all UV divergences. Regularization of IR divergences turned out to be ambiguous (what is discussed in the present paper) and this ambiguity was resolved in Ref. [51] by taking into account the breakdown of a usual PN scheme (based on a formal near-zone expansion) due to infinite-range tail-transported temporal correlations found in Ref. [61]. Reference [51] showed that the total 4PN conservative Hamiltonian is the sum of instantaneous (local-in-time) near-zone Hamiltonian and time-symmetric but nonlocal-in-time tail Hamiltonian. The present paper is devoted to

details of computation of the 4PN near-zone local-in-time Hamiltonian. A rather nontrivial application of Ref. [51] to 4PN-accurate generalization of effective-one-body approach to compact binary dynamics has been worked out in Ref. [62] already.

Since the seminal work by 't Hooft and Veltman [63], dimensional regularization (for history see [64]) has become very popular in quantum field theory (see, e.g., [65]). Even the famous Lamb shift with simultaneously occurring UV and IR divergences found its elegant computation by applying dimensional regularization [66]. However, whereas UV divergences are nicely controlled by dimensional regularization, IR divergences can pose problems (see, e.g., Refs. [67,68]). This also happens in our approach with the final solution indicated above. It is worth pointing out that also the Lamb shift calculation of Ref. [66] shows up an undefined constant in the IR sector, which gets fixed by some dimensional matching.

A short overview of the various sections will give the reader some help in orientation. Section II presents the ADM formalism for two-point-mass systems developed in $(d + 1)$ -dimensional space-time. It also gives the transition to the Routhian functional which is crucial for the obtention of Hamiltonian which depends on matter variables only. Section III expands the nonpropagating part of the ADM structure for two-point-mass systems through 4PN order. It ends with the formulas for the 4PN-accurate reduced Hamiltonian. Section IV is concerned with the derivation of field equations for the propagating degrees of freedom valid to the next-to-leading order (which is enough for our purposes). It also includes the formal near-zone (and thus PN) expansion of the time-symmetric solution of field equations. Section V develops the 4PN-accurate Routhian functional, without expanding propagating degrees of freedom into the PN series. Section VI presents the 4PN-accurate conservative Hamiltonian dependent on matter variables only, with still non-PN-expanded propagating degrees of freedom. Section VII gives the 4PN-exact near-zone conservative matter Hamiltonian with fully PN-expanded propagating degrees of freedom. Section VIII delivers some details of the computation and regularization of integrals, taking into account the various appendices, and presents the fully explicit result for the total 4PN-accurate matter conservative Hamiltonian. Section IX is devoted to checks with the aid of the Poincaré algebra. All technical calculational details are shifted to three appendices. Appendix A is devoted to all regularization procedures used. Appendix B gives a variety of needed inverse Laplacians. Finally, Appendix C presents field functions in fully explicit forms.

We employ the following notation: $\mathbf{x} = (x^i)$ ($i = 1, \dots, d$) denotes a point in the d -dimensional Euclidean space \mathbb{R}^d endowed with a standard Euclidean metric and a scalar product (denoted by a dot). Spatial latin indices run through $1, \dots, d$, and space-time greek indices vary from 0 to d .

Letters a and b ($a, b = 1, 2$) are particle labels, so $\mathbf{x}_a = (x_a^i) \in \mathbb{R}^d$ denotes the position of the a th point mass. We also define $\mathbf{r}_a := \mathbf{x} - \mathbf{x}_a$, $r_a := |\mathbf{r}_a|$, $\mathbf{n}_a := \mathbf{r}_a/r_a$; and for $a \neq b$, $\mathbf{r}_{ab} := \mathbf{x}_a - \mathbf{x}_b$, $r_{ab} := |\mathbf{r}_{ab}|$, $\mathbf{n}_{ab} := \mathbf{r}_{ab}/r_{ab}$; $|\cdot|$ stands here for the Euclidean length of a vector. The linear momentum vector of the a th particle is denoted by $\mathbf{p}_a = (p_{ai})$, and m_a denotes its mass parameter. An overdot, as in $\dot{\mathbf{x}}_a$, means differentiation with respect to time coordinate t . The partial differentiation with respect to x^i is denoted by ∂_i or by a comma, i.e., $\partial_i \phi \equiv \phi_{,i}$, and the partial differentiation with respect to x_a^i is denoted by ∂_{ai} . We abbreviate Dirac delta distribution $\delta(\mathbf{x} - \mathbf{x}_a)$ by δ_a (both in d and in 3 dimensions); it fulfills the condition $\int d^d x \delta_a = 1$. Flat d -dimensional Laplacian is denoted by Δ_d , whereas Δ (without any subscript) is reserved for Laplacian in $d = 3$ dimensions. Extensive use has been made of the computer algebra system *Mathematica*.

II. THE ADM CANONICAL FORMALISM IN d SPACE DIMENSIONS

We consider here a system of two point masses, i.e. monopolar, pointlike bodies, which interact gravitationally according to general relativity theory. We model point masses by means of Dirac delta distributions δ_a . Let $D := d + 1$ denote the (analytically continued) spacetime dimension. The ADM canonical approach [36] uses a $d + 1$ split of the coupled gravity-plus-matter dynamics and works with the pairs of the canonical variables: positions \mathbf{x}_a and momenta \mathbf{p}_a of point masses, and for the gravitational field the space metric $\gamma_{ij} = g_{ij}$ induced by the full space-time metric $g_{\mu\nu}$ on the hypersurface $t = \text{const}$, and its conjugate momentum $\pi^{ij} = \pi^{ji}$.

The full Einstein field equations in D dimensions in an asymptotically flat space-time and in an asymptotically Minkowskian coordinate system, written in the canonical variables introduced above, are derivable from the Hamiltonian [69] (in units where $16\pi G_D = c = 1$, with G_D denoting the generalized Newtonian gravitational constant and c the speed of light)

$$H = \int d^d x (N\mathcal{H} - N^i \mathcal{H}_i) + \oint_{t^0} d^{d-1} S_i \partial_j (\gamma_{ij} - \delta_{ij} \gamma_{kk}), \quad (2.1)$$

where N and N^i are called *lapse* and *shift* functions and the super-Hamiltonian \mathcal{H} and super-momentum \mathcal{H}_i are defined as follows:

$$\mathcal{H}(\mathbf{x}_a, \mathbf{p}_a, \gamma_{ij}, \pi^{ij}) := \sqrt{\gamma} N^2 (T^{00} - 2G^{00}), \quad (2.2a)$$

$$\mathcal{H}_i(\mathbf{x}_a, \mathbf{p}_a, \gamma_{ij}, \pi^{ij}) := \sqrt{\gamma} N (T_i^0 - 2G_i^0). \quad (2.2b)$$

Here $T^{\mu\nu}$ and $G^{\mu\nu}$ denote the energy-momentum and the Einstein tensor, respectively, and γ is the determinant

of the d -dimensional matrix (γ_{ij}) . In the surface integral in (2.1) i^0 denotes spacelike infinity and $d^{d-1}S_i$ is the $(d-1)$ -dimensional out-pointing surface element there. In terms of lapse and shift functions, the space-time line element reads (we employ signature $d-1, x^0 = t$)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2(dx^0)^2 + \gamma_{ij}(dx^i + N^i dx^0)(dx^j + N^j dx^0). \quad (2.3)$$

The lapse and shift functions are Lagrangian multipliers only and deliver the famous Hamiltonian and momentum constraint equations of the Einstein theory,

$$\mathcal{H} = 0, \quad \mathcal{H}_i = 0. \quad (2.4)$$

The fulfillment of the constraint equations together with $d+1$ additional (independent) coordinate conditions, see Eqs. (2.6) below, reduces the total Hamiltonian of Eq. (2.1) to the reduced one, H_{red} , which obviously is a pure surface expression. This is the ADM Hamiltonian which evolves the independent dynamical degrees of freedom of the total system. Here the lapse and shift functions are not involved at all, only the space-asymptotic numerical value 1 of the lapse function enters. The lapse and shift functions are to be calculated by making use of the coordinate conditions within the remaining set of the Einstein field equations.

The dimensionally continued constraint equations (2.4) written for the two-point-mass system read

$$\sqrt{\gamma}R = \frac{1}{\sqrt{\gamma}} \left(\gamma_{ik} \gamma_{j\ell} \pi^{ij} \pi^{k\ell} - \frac{1}{d-1} (\gamma_{ij} \pi^{ij})^2 \right) + \sum_a (m_a^2 + \gamma_a^{ij} p_{ai} p_{aj})^{\frac{1}{2}} \delta_a, \quad (2.5a)$$

$$-2D_j \pi^{ij} = \sum_a \gamma_a^{ij} p_{aj} \delta_a. \quad (2.5b)$$

Here R denotes the space curvature of the hypersurface $t = \text{const}$, $\gamma_a^{ij} := \gamma_{\text{reg}}^{ij}(\mathbf{x}_a)$ is the finite part of the inverse metric γ^{ij} ($\gamma^{ij} \gamma_{jk} = \delta_k^i$) evaluated at the particle position (which can be perturbatively and unambiguously defined, see Appendix A 4 of the current paper), and D_j is the d -dimensional covariant derivative (acting on a tensor density of weight one). Let us note that the source terms of both constraint equations are proportional to Dirac δ -functions.

We employ the following ADM transverse-traceless (TT) coordinate conditions, resulting in irreducible canonical field variables,

$$\gamma_{ij} = \left(1 + \frac{d-2}{4(d-1)} \phi \right)^{4/(d-2)} \delta_{ij} + h_{ij}^{\text{TT}}, \quad \pi^{ii} = 0, \quad (2.6)$$

where the metric function h_{ij}^{TT} is a symmetric TT quantity, i.e.

$$h_{ii}^{\text{TT}} = 0, \quad \partial_j h_{ij}^{\text{TT}} = 0, \quad (2.7)$$

and the field momentum π^{ij} is split into its longitudinal and TT parts, respectively

$$\pi^{ij} = \tilde{\pi}^{ij} + \pi_{\text{TT}}^{ij}. \quad (2.8)$$

The longitudinal part of the field momentum can be expressed in terms of a vectorial function V^i ,

$$\tilde{\pi}^{ij} = \partial_i V^j + \partial_j V^i - \frac{2}{d} \delta^{ij} \partial_k V^k, \quad (2.9)$$

and the TT part satisfies conditions analogous to (2.7),

$$\pi_{\text{TT}}^{ii} = 0, \quad \partial_j \pi_{\text{TT}}^{ij} = 0. \quad (2.10)$$

After solving [with the usage of the coordinate conditions (2.6) and by a perturbative expansion] the constraints (2.5) with respect to the longitudinal variables ϕ and $\tilde{\pi}^{ij}$, we plug these solutions (expressed in terms of \mathbf{x}_a , \mathbf{p}_a and $h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}$) into the right-hand side of Eq. (2.1). This way we get the reduced ADM Hamiltonian of the total matter-plus-field system, which can be written in the form

$$H_{\text{red}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] = - \int d^d x \Delta_d \phi[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}]. \quad (2.11)$$

This Hamiltonian describes the evolution of the matter and independent gravitational field variables. The equations of motion of the bodies read

$$\dot{\mathbf{x}}_a = \frac{\partial H_{\text{red}}}{\partial \mathbf{p}_a}, \quad \dot{\mathbf{p}}_a = - \frac{\partial H_{\text{red}}}{\partial \mathbf{x}_a}, \quad (2.12)$$

and the field equations for the independent degrees of freedom have the form

$$\frac{\partial}{\partial t} h_{ij}^{\text{TT}} = \delta_{ij}^{\text{TT}kl} \frac{\delta H_{\text{red}}}{\delta h_{kl}^{\text{TT}}}, \quad \frac{\partial}{\partial t} \pi_{\text{TT}}^{ij} = - \delta_{kl}^{\text{TT}ij} \frac{\delta H_{\text{red}}}{\delta h_{kl}^{\text{TT}}}, \quad (2.13)$$

where the d -dimensional TT-projection operator is defined by

$$\delta_{kl}^{\text{TT}ij} := \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{d-1} \delta_{ij} \delta_{kl} - \frac{1}{2} (\delta_{ik} \partial_{jl} + \delta_{jl} \partial_{ik} + \delta_{il} \partial_{jk} + \delta_{jk} \partial_{il}) \Delta_d^{-1} + \frac{1}{d-1} (\delta_{ij} \partial_{kl} + \delta_{kl} \partial_{ij}) \Delta_d^{-1} + \frac{d-2}{d-1} \partial_{ijkl} \Delta_d^{-2}. \quad (2.14)$$

In the current paper we are interested only in *conservative* dynamics of the matter-plus-field system and we want to describe this dynamics in terms of *only matter variables* \mathbf{x}_a and \mathbf{p}_a . An autonomous (thus conservative) matter Hamiltonian can be obtained through

the transition to a Routhian description. By means of the first field equation (2.13) one expresses the TT part π_{TT}^{ij} of the field momentum as a function of matter variables \mathbf{x}_a , \mathbf{p}_a and field variables h_{ij}^{TT} , \dot{h}_{ij}^{TT} . Then the Routhian is defined as

$$R[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}] := H_{\text{red}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] - \int d^d x \pi_{\text{TT}}^{ij} \dot{h}_{ij}^{\text{TT}}. \quad (2.15)$$

Finally the matter Hamiltonian reads

$$H(\mathbf{x}_a, \mathbf{p}_a) := R[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}(\mathbf{x}_a, \mathbf{p}_a), \dot{h}_{ij}^{\text{TT}}(\mathbf{x}_a, \mathbf{p}_a)], \quad (2.16)$$

where the field variables h_{ij}^{TT} , \dot{h}_{ij}^{TT} are replaced by solutions of field equations, i.e. they are expressed in terms of the matter variables and eventually their time derivatives, which are in turn eliminated through lower-order equations of motion (this procedure is equivalent to performing a higher-order contact transformation [70,71]).

The dependence in the right-hand side of Eq. (2.16) of the field functions on the same position and momentum variables as those being located outside the field expressions vitiates dissipation. For dissipation (radiation damping) to be obtained, another set of (primed, say) variables has to be used in the field expressions, cf. [14]. Because of our restriction to the conservative dynamics, we can iteratively solve the field equations by means of the *time-symmetric*

half-retarded plus half-advanced Green function (see Sec. IV below), which from the very beginning excludes dissipation.

III. THE 4PN-ACCURATE REDUCED HAMILTONIAN

To compute the 4PN-accurate reduced Hamiltonian given by Eq. (2.11) we have to perturbatively solve the constraint equations (2.5) through 4PN order. To do this we expand these equations into the PN series, i.e., into series in powers in the inverse velocity of light, $\epsilon := 1/c$. We take into account that

$$\begin{aligned} m_a &\sim \mathcal{O}(\epsilon^2), & \mathbf{p}_a &\sim \mathcal{O}(\epsilon^3), \\ \phi &\sim \mathcal{O}(\epsilon^2), & h_{ij}^{\text{TT}} &\sim \mathcal{O}(\epsilon^4), \\ \tilde{\pi}^{ij} &\sim \mathcal{O}(\epsilon^3), & \pi_{\text{TT}}^{ij} &\sim \mathcal{O}(\epsilon^5). \end{aligned} \quad (3.1)$$

To compute the 4PN-accurate Hamiltonian we need to expand the Hamiltonian constraint equation (2.5a) up to the order ϵ^{12} . It is convenient to put this equation in the form solved for the (flat) Laplacian of the metric function ϕ . After long calculation we obtain (from here the numbers written in subscripts within parentheses denote the formal orders in ϵ)

$$\Delta_d \phi = \sum_{n=1}^6 \Phi_{(2n)} + \mathcal{O}(\epsilon^{14}), \quad (3.2)$$

where $\Phi_{(2)}, \dots, \Phi_{(12)}$ are given by

$$\Phi_{(2)} = -\sum_a m_a \delta_a, \quad (3.3a)$$

$$\Phi_{(4)} = -\sum_a \frac{\mathbf{p}_a^2}{2m_a} \delta_a - \frac{d-2}{4(d-1)} \phi \Delta_d \phi, \quad (3.3b)$$

$$\Phi_{(6)} = \sum_a \left(\frac{(\mathbf{p}_a^2)^2}{8m_a^3} + \frac{\mathbf{p}_a^2}{2(d-1)m_a} \phi \right) \delta_a - (\tilde{\pi}^{ij})^2 + \frac{d-2}{d-1} \phi_{,ij} h_{ij}^{\text{TT}}, \quad (3.3c)$$

$$\begin{aligned} \Phi_{(8)} = & \sum_a \left(-\frac{(\mathbf{p}_a^2)^3}{16m_a^5} - \frac{(\mathbf{p}_a^2)^2}{4(d-1)m_a^3} \phi - \frac{(d+2)\mathbf{p}_a^2}{16(d-1)^2 m_a} \phi^2 \right) \delta_a - \frac{4-d}{2(d-1)} \phi (\tilde{\pi}^{ij})^2 \\ & + \left\{ \sum_a \frac{p_{ai} p_{aj}}{2m_a} \delta_a - \frac{d-2}{4(d-1)^2} ((6-d)\phi \phi_{,ij} + 3\phi_{,i} \phi_{,j}) \right\} h_{ij}^{\text{TT}} + \frac{3}{4} (h_{ij,k}^{\text{TT}})^2 - \frac{1}{2} h_{ij,k}^{\text{TT}} h_{ik,j}^{\text{TT}} + h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} - 2\tilde{\pi}^{ij} \pi_{\text{TT}}^{ij}, \end{aligned} \quad (3.3d)$$

$$\begin{aligned} \Phi_{(10)} = & \sum_a \left(\frac{5(\mathbf{p}_a^2)^4}{128m_a^7} + \frac{3(\mathbf{p}_a^2)^3}{16(d-1)m_a^5} \phi + \frac{(d+6)(\mathbf{p}_a^2)^2}{32(d-1)^2 m_a^3} \phi^2 + \frac{(d+2)d\mathbf{p}_a^2}{96(d-1)^3 m_a} \phi^3 \right) \delta_a - \frac{(10-3d)(4-d)}{16(d-1)^2} \phi^2 (\tilde{\pi}^{ij})^2 \\ & + \left\{ \sum_a \left(-\frac{\mathbf{p}_a^2}{4m_a^3} - \frac{1}{(d-1)m_a} \phi \right) p_{ai} p_{aj} \delta_a + \frac{3(d-2)}{4(d-1)^3} \phi \phi_{,i} \phi_{,j} + \frac{(d-2)(6-d)}{8(d-1)^3} \phi^2 \phi_{,ij} - 2\tilde{\pi}^{ik} \tilde{\pi}^{jk} \right\} h_{ij}^{\text{TT}} \\ & - \frac{5-d}{4(d-1)} (\Delta_d \phi) (h_{ij}^{\text{TT}})^2 - \frac{d-3}{d-1} \phi_{,ij} h_{ik}^{\text{TT}} h_{jk}^{\text{TT}} - \frac{8-d}{2(d-1)} \phi_{,k} h_{ij}^{\text{TT}} h_{ij,k}^{\text{TT}} + \frac{4-d}{d-1} \phi_{,k} h_{ij}^{\text{TT}} h_{ik,j}^{\text{TT}} \\ & - \frac{6-d}{2(d-1)} \phi \left(\frac{3}{4} (h_{ij,k}^{\text{TT}})^2 - \frac{1}{2} h_{ij,k}^{\text{TT}} h_{ik,j}^{\text{TT}} + h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} \right) - \frac{4-d}{d-1} \phi \tilde{\pi}^{ij} \pi_{\text{TT}}^{ij} - (\pi_{\text{TT}}^{ij})^2, \end{aligned} \quad (3.3e)$$

$$\begin{aligned}
 \Phi_{(12)} = & \sum_a \left(-\frac{7(\mathbf{p}_a^2)^5}{256m_a^9} - \frac{5(\mathbf{p}_a^2)^4}{32(d-1)m_a^7} \phi - \frac{3(d+10)(\mathbf{p}_a^2)^3}{128(d-1)^2 m_a^5} \phi^2 - \frac{(d+6)(d+2)(\mathbf{p}_a^2)^2}{192(d-1)^3 m_a^3} \phi^3 - \frac{(d+2)d(3d-2)\mathbf{p}_a^2}{1536(d-1)^4 m_a} \phi^4 \right) \delta_a \\
 & + \frac{(d-3)(10-3d)(4-d)}{48(d-1)^3} \phi^3 (\tilde{\pi}^{ij})^2 + \left\{ \sum_a \left(\frac{3(\mathbf{p}_a^2)^2}{16m_a^5} + \frac{3\mathbf{p}_a^2}{4(d-1)m_a^3} \phi + \frac{d+6}{8(d-1)^2 m_a} \phi^2 \right) p_{ai} p_{aj} \delta_a + \frac{d-2}{d-1} \phi \tilde{\pi}^{ik} \tilde{\pi}^{jk} \right. \\
 & - \frac{d^2-4}{32(d-1)^4} \phi^2 \left(\frac{1}{3} (6-d) \phi \phi_{,ij} + 3 \phi_{,i} \phi_{,j} \right) \left. \right\} h_{ij}^{\text{TT}} + \left(\frac{(5-d)(10-d)}{16(d-1)^2} \phi \Delta_d \phi + \frac{7-d}{4(d-1)^2} (\phi_{,k})^2 \right) (h_{ij}^{\text{TT}})^2 \\
 & + \left\{ -\sum_a \frac{p_{ai} p_{aj}}{2m_a} \delta_a + \frac{d-3}{2(d-1)^2} \left(3 \phi_{,i} \phi_{,j} + \frac{1}{2} (10-d) \phi \phi_{,ij} \right) \right\} h_{ik}^{\text{TT}} h_{jk}^{\text{TT}} \\
 & + \frac{10-d}{4(d-1)^2} \phi \phi_{,k} h_{ij}^{\text{TT}} \left(\frac{1}{2} (8-d) h_{ij,k}^{\text{TT}} - (4-d) h_{ik,j}^{\text{TT}} \right) + \frac{(6-d)(10-d)}{16(d-1)^2} \phi^2 \left(\frac{3}{4} (h_{ij,k}^{\text{TT}})^2 - \frac{1}{2} h_{ij,k}^{\text{TT}} h_{ik,j}^{\text{TT}} + h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} \right) \\
 & + h_{ij}^{\text{TT}} \left(\frac{1}{2} h_{ik,l}^{\text{TT}} h_{jl,k}^{\text{TT}} - \frac{3}{2} h_{ik,l}^{\text{TT}} h_{jk,l}^{\text{TT}} + h_{ik,l}^{\text{TT}} h_{kl,j}^{\text{TT}} - \frac{3}{4} h_{kl,i}^{\text{TT}} h_{kl,j}^{\text{TT}} \right) - h_{ik}^{\text{TT}} h_{kj}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} + h_{ij}^{\text{TT}} h_{kl}^{\text{TT}} (h_{ik,jl}^{\text{TT}} - h_{ij,kl}^{\text{TT}}) \\
 & - \frac{(10-3d)(4-d)}{8(d-1)^2} \phi^2 \tilde{\pi}^{ij} \pi_{\text{TT}}^{ij} - 4 \tilde{\pi}^{ik} h_{ij}^{\text{TT}} \pi_{\text{TT}}^{jk} - \frac{4-d}{2(d-1)} \phi (\pi_{\text{TT}}^{ij})^2. \tag{3.3f}
 \end{aligned}$$

We also need to expand the momentum constraint equation (2.5b) up to the order ϵ^9 . This expansion reads

$$\tilde{\pi}^{ij}{}_{,j} = \Pi_{(3)}^i + \Pi_{(5)}^i + \Pi_{(7)}^i + \Pi_{(9)}^i + \mathcal{O}(\epsilon^{11}), \tag{3.4}$$

where $\Pi_{(3)}^i, \dots, \Pi_{(9)}^i$ are equal to

$$\Pi_{(3)}^i = -\frac{1}{2} \sum_a p_{ai} \delta_a, \tag{3.5a}$$

$$\Pi_{(5)}^i = \frac{1}{d-1} \left(\frac{1}{2} \phi \sum_a p_{ai} \delta_a - \phi_{,j} \tilde{\pi}^{ij} \right), \tag{3.5b}$$

$$\Pi_{(7)}^i = -\frac{d+2}{16(d-1)^2} \phi^2 \sum_a p_{ai} \delta_a + \frac{d-2}{4(d-1)^2} \phi \phi_{,j} \tilde{\pi}^{ij} + \frac{1}{2} h_{ij}^{\text{TT}} \sum_a p_{aj} \delta_a + \tilde{\pi}^{jk} \left(\frac{1}{2} h_{jk,i}^{\text{TT}} - h_{ij,k}^{\text{TT}} \right) - \frac{1}{d-1} \phi_{,j} \pi_{\text{TT}}^{ij}, \tag{3.5c}$$

$$\begin{aligned}
 \Pi_{(9)}^i = & \frac{(d+2)d}{96(d-1)^3} \phi^3 \sum_a p_{ai} \delta_a - \frac{(d-2)^2}{16(d-1)^3} \phi^2 \phi_{,j} \tilde{\pi}^{ij} - \frac{1}{d-1} \phi h_{ij}^{\text{TT}} \sum_a p_{aj} \delta_a \\
 & + \frac{1}{d-1} \tilde{\pi}^{jk} \left\{ \phi_{,k} h_{ij}^{\text{TT}} + \phi \left(h_{ij,k}^{\text{TT}} - \frac{1}{2} h_{jk,i}^{\text{TT}} \right) \right\} + \frac{d-2}{4(d-1)^2} \phi \phi_{,j} \pi_{\text{TT}}^{ij} + \left(\frac{1}{2} h_{jk,i}^{\text{TT}} - h_{ij,k}^{\text{TT}} \right) \pi_{\text{TT}}^{jk}. \tag{3.5d}
 \end{aligned}$$

We use the ADM canonical approach in an asymptotically flat space-time and we employ asymptotically Minkowskian coordinates. Therefore we have to assume that the functions which enter the formalism have the following asymptotic behavior for $r \rightarrow \infty$ (see [69] for discussion of asymptotics in the $d=3$ case):

$$\phi \sim \frac{1}{r^{d-2}}, \quad h_{ij}^{\text{TT}} \sim \frac{1}{r^{d-2}}, \quad \tilde{\pi}^{ij} \sim \frac{1}{r^{d-1}}, \quad \pi_{\text{TT}}^{ij} \sim \frac{1}{r^{d-1}}. \tag{3.6}$$

Making use of the above asymptotics and the expansion (3.2)–(3.3) of the Laplacian $\Delta_d \phi$, after dropping many total divergences which decay fast enough

at spatial infinity (so they do not contribute to the Hamiltonian), the 4PN-accurate reduced Hamiltonian can be written as

$$\begin{aligned}
 H_{\leq 4\text{PN}}^{\text{red}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] \\
 = \int d^d x \left(\sum_a m_a \delta_a + \sum_{n=2}^6 h_{(2n)}^{\text{red}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}) \right), \tag{3.7}
 \end{aligned}$$

where the Hamiltonian densities $h_{(4)}^{\text{red}}, h_{(6)}^{\text{red}}, \dots, h_{(12)}^{\text{red}}$ read

$$h_{(4)}^{\text{red}} = \sum_a \frac{\mathbf{p}_a^2}{2m_a} \delta_a + \frac{d-2}{4(d-1)} \phi \Delta_d \phi, \quad (3.8a)$$

$$h_{(6)}^{\text{red}} = \sum_a \left(-\frac{(\mathbf{p}_a^2)^2}{8m_a^3} - \frac{\mathbf{p}_a^2}{2(d-1)m_a} \phi \right) \delta_a + (\tilde{\pi}^{ij})^2, \quad (3.8b)$$

$$h_{(8)}^{\text{red}} = \sum_a \left(\frac{(\mathbf{p}_a^2)^3}{16m_a^5} + \frac{(\mathbf{p}_a^2)^2}{4(d-1)m_a^3} \phi + \frac{(d+2)\mathbf{p}_a^2}{16(d-1)^2 m_a} \phi^2 \right) \delta_a + \frac{4-d}{2(d-1)} \phi (\tilde{\pi}^{ij})^2 \\ + \left(-\sum_a \frac{p_{ai} p_{aj}}{2m_a} \delta_a + \frac{(d-2)(d-3)}{4(d-1)^2} \phi_{,i} \phi_{,j} \right) h_{ij}^{\text{TT}} + \frac{1}{4} (h_{ij,k}^{\text{TT}})^2, \quad (3.8c)$$

$$h_{(10)}^{\text{red}} = \sum_a \left(-\frac{5(\mathbf{p}_a^2)^4}{128m_a^7} - \frac{3(\mathbf{p}_a^2)^3}{16(d-1)m_a^5} \phi - \frac{(d+6)(\mathbf{p}_a^2)^2}{32(d-1)^2 m_a^3} \phi^2 - \frac{(d+2)d\mathbf{p}_a^2}{96(d-1)^3 m_a} \phi^3 \right) \delta_a + \frac{(10-3d)(4-d)}{16(d-1)^2} \phi^2 (\tilde{\pi}^{ij})^2 \\ + \left\{ \sum_a \left(\frac{\mathbf{p}_a^2}{4m_a^3} + \frac{1}{(d-1)m_a} \phi \right) p_{ai} p_{aj} \delta_a - \frac{(d-2)(d-3)}{4(d-1)^3} \phi \phi_{,i} \phi_{,j} + 2\tilde{\pi}^{ik} \tilde{\pi}^{jk} \right\} h_{ij}^{\text{TT}} \\ - \frac{d-3}{4(d-1)} (\Delta_d \phi) (h_{ij}^{\text{TT}})^2 + \frac{d-2}{4(d-1)} \phi_{,ij} h_{ik}^{\text{TT}} h_{jk}^{\text{TT}} - \frac{6-d}{8(d-1)} \phi (h_{ij,k}^{\text{TT}})^2 + \frac{4-d}{d-1} \phi \tilde{\pi}^{ij} \pi_{\text{TT}}^{ij} + (\pi_{\text{TT}}^{ij})^2, \quad (3.8d)$$

$$h_{(12)}^{\text{red}} = \sum_a \left(\frac{7(\mathbf{p}_a^2)^5}{256m_a^9} + \frac{5(\mathbf{p}_a^2)^4}{32(d-1)m_a^7} \phi + \frac{3(d+10)(\mathbf{p}_a^2)^3}{128(d-1)^2 m_a^5} \phi^2 + \frac{(d+6)(d+2)(\mathbf{p}_a^2)^2}{192(d-1)^3 m_a^3} \phi^3 + \frac{(d+2)d(3d-2)\mathbf{p}_a^2}{1536(d-1)^4 m_a} \phi^4 \right) \delta_a \\ - \frac{(d-3)(10-3d)(4-d)}{48(d-1)^3} \phi^3 (\tilde{\pi}^{ij})^2 + \left\{ \sum_a \left(-\frac{3(\mathbf{p}_a^2)^2}{16m_a^5} - \frac{3\mathbf{p}_a^2}{4(d-1)m_a^3} \phi - \frac{d+6}{8(d-1)^2 m_a} \phi^2 \right) p_{ai} p_{aj} \delta_a \right. \\ \left. + \frac{(d+2)(d-2)(d-3)}{32(d-1)^4} \phi^2 \phi_{,i} \phi_{,j} - \frac{d-2}{d-1} \phi \tilde{\pi}^{ik} \tilde{\pi}^{jk} \right\} h_{ij}^{\text{TT}} + \left(\frac{(d-3)(10-d)}{16(d-1)^2} \phi \Delta_d \phi - \frac{4-d}{8(d-1)^2} (\phi_{,k})^2 \right) (h_{ij}^{\text{TT}})^2 \\ + \left(\sum_a \frac{p_{ai} p_{aj}}{2m_a} \delta_a + \frac{3(4+4d-d^2)}{16(d-1)^2} \phi_{,i} \phi_{,j} + \frac{(6-d)(10-d)}{16(d-1)^2} \phi \phi_{,ij} \right) h_{ik}^{\text{TT}} h_{jk}^{\text{TT}} + \frac{(6-d)(10-d)}{64(d-1)^2} \phi^2 (h_{ij,k}^{\text{TT}})^2 \\ - \frac{1}{2} h_{ij}^{\text{TT}} \left(h_{ik,l}^{\text{TT}} (h_{jk,l}^{\text{TT}} + h_{jl,k}^{\text{TT}}) + \frac{1}{2} h_{kl,i}^{\text{TT}} h_{kl,j}^{\text{TT}} \right) + 4\tilde{\pi}^{ik} h_{ij}^{\text{TT}} \pi_{\text{TT}}^{jk} + \frac{(10-3d)(4-d)}{8(d-1)^2} \phi^2 \tilde{\pi}^{ij} \pi_{\text{TT}}^{ij} + \frac{4-d}{2(d-1)} \phi (\pi_{\text{TT}}^{ij})^2. \quad (3.8e)$$

In the next step we perform the PN expansion of the field functions ϕ and $\tilde{\pi}^{ij}$. From Eqs. (3.8) it follows that to get the 4PN-accurate Hamiltonian we need to expand the function ϕ up to $\mathcal{O}(\epsilon^{10})$ and the function $\tilde{\pi}^{ij}$ up to $\mathcal{O}(\epsilon^9)$,

$$\phi = \phi_{(2)} + \phi_{(4)} + \bar{\phi}_{(6)} + \bar{\phi}_{(8)} + \bar{\phi}_{(10)} + \mathcal{O}(\epsilon^{12}), \quad (3.9a)$$

$$\tilde{\pi}^{ij} = \tilde{\pi}_{(3)}^{ij} + \tilde{\pi}_{(5)}^{ij} + \tilde{\pi}_{(7)}^{ij} + \tilde{\pi}_{(9)}^{ij} + \mathcal{O}(\epsilon^{11}). \quad (3.9b)$$

Equations (3.3) imply that the functions $\phi_{(2)}$ and $\phi_{(4)}$ depend only on matter variables \mathbf{x}_a , \mathbf{p}_a , the function $\bar{\phi}_{(6)}$ depends on matter variables and on h_{ij}^{TT} , whereas the functions $\bar{\phi}_{(8)}$ and $\bar{\phi}_{(10)}$ depend both on \mathbf{x}_a , \mathbf{p}_a and on h_{ij}^{TT} , π_{TT}^{ij} . The leading-order and next-to-leading-order functions $\tilde{\pi}_{(3)}^{ij}$ and $\tilde{\pi}_{(5)}^{ij}$ depend on matter variables only; the functions $\tilde{\pi}_{(7)}^{ij}$ and $\tilde{\pi}_{(9)}^{ij}$ depend on both matter and TT field variables. The overbar in the functions $\bar{\phi}_{(6)}$, $\bar{\phi}_{(8)}$, $\bar{\phi}_{(10)}$,

$\tilde{\pi}_{(7)}^{ij}$, and $\tilde{\pi}_{(9)}^{ij}$ means that they depend on the non-PN-expanded TT variables h_{ij}^{TT} , π_{TT}^{ij} .

To obtain the equations fulfilled by the functions $\phi_{(2)}$ up to $\bar{\phi}_{(10)}$ we substitute Eqs. (3.9) into Eqs. (3.3) and reexpand them with respect to ϵ . This way we first obtain the Poisson equations for the functions $\phi_{(2)}$ and $\phi_{(4)}$, which read

$$\Delta_d \phi_{(2)} = -\sum_a m_a \delta_a, \quad (3.10a)$$

$$\Delta_d \phi_{(4)} = \sum_a \left(-\frac{\mathbf{p}_a^2}{2m_a} + \frac{(d-2)m_a}{4(d-1)} \phi_{(2)} \right) \delta_a, \quad (3.10b)$$

where in the right-hand side of (3.10b) we have used (3.10a). We have found it useful to split the function $\phi_{(4)}$ into two pieces,

$$\phi_{(4)} = -\frac{1}{2}S_{(4)1} + \frac{d-2}{4(d-1)}S_{(4)2}, \quad (3.11)$$

where $S_{(4)1}$ and $S_{(4)2}$ fulfill the following Poisson equations

$$\Delta_d S_{(4)1} = \sum_a \frac{\mathbf{p}_a^2}{m_a} \delta_a, \quad \Delta_d S_{(4)2} = \phi_{(2)} \sum_a m_a \delta_a. \quad (3.12)$$

In the case of the function $\bar{\phi}_{(6)}$ it is also convenient to split it into two pieces,

$$\bar{\phi}_{(6)}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}] = \phi_{(6)1}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \bar{\phi}_{(6)2}[\mathbf{x}; \mathbf{x}_a, h_{ij}^{\text{TT}}], \quad (3.13)$$

where the first piece $\phi_{(6)1}$ depends only on matter variables $\mathbf{x}_a, \mathbf{p}_a$, whereas the second piece $\bar{\phi}_{(6)2}$ depends on \mathbf{x}_a and (functionally) on h_{ij}^{TT} . The Poisson equation for the functions $\phi_{(6)1}$ and $\bar{\phi}_{(6)2}$ read [in their source terms the

Laplacians $\Delta_d \phi_{(2)}$ and $\Delta_d \phi_{(4)}$ were eliminated by means of Eqs. (3.10)]

$$\Delta_d \phi_{(6)1} = \sum_a \left\{ \frac{(\mathbf{p}_a^2)^2}{8m_a^3} + \frac{(d+2)\mathbf{p}_a^2}{8(d-1)m_a} \phi_{(2)} - \frac{(d-2)m_a}{8(d-1)} \times \left(S_{(4)1} + \frac{d-2}{2(d-1)}(\phi_{(2)}^2 - S_{(4)2}) \right) \right\} \delta_a - (\tilde{\pi}_{(3)}^{ij})^2, \quad (3.14a)$$

$$\Delta_d \bar{\phi}_{(6)2} = \frac{d-2}{d-1} \phi_{(2),ij} h_{ij}^{\text{TT}}. \quad (3.14b)$$

The Poisson equations fulfilled by the functions $\bar{\phi}_{(8)}$ and $\bar{\phi}_{(10)}$ we present in the form where the lower-order Laplacians $\Delta_d \phi_{(2)}$, $\Delta_d \phi_{(4)}$, and $\Delta_d \bar{\phi}_{(6)}$ are not replaced by their source terms. The equations read

$$\begin{aligned} \Delta_d \bar{\phi}_{(8)} = & -\sum_a \left\{ \frac{(\mathbf{p}_a^2)^3}{16m_a^5} + \frac{(\mathbf{p}_a^2)^2}{4(d-1)m_a^3} \phi_{(2)} + \frac{\mathbf{p}_a^2}{2(d-1)m_a} \left(\frac{d+2}{8(d-1)} \phi_{(2)}^2 - \phi_{(4)} \right) \right\} \delta_a - \frac{4-d}{2(d-1)} \phi_{(2)} (\tilde{\pi}_{(3)}^{ij})^2 \\ & - 2\tilde{\pi}_{(3)}^{ij} \tilde{\pi}_{(5)}^{ij} + \left\{ \sum_a \frac{P_{ai} P_{aj}}{2m_a} \delta_a - \frac{d-2}{4(d-1)^2} (3\phi_{(2),i} \phi_{(2),j} + (6-d)\phi_{(2)} \phi_{(2),ij}) + \frac{d-2}{d-1} \phi_{(4),ij} \right\} h_{ij}^{\text{TT}} \\ & + \frac{3}{4} (h_{ij,k}^{\text{TT}})^2 - \frac{1}{2} h_{ij,k}^{\text{TT}} h_{ik,j}^{\text{TT}} + h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} - 2\tilde{\pi}_{(3)}^{ij} \pi_{\text{TT}}^{ij} - \frac{d-2}{4(d-1)} (\phi_{(2)} \Delta_d \bar{\phi}_{(6)} + \phi_{(4)} \Delta_d \phi_{(4)} + \bar{\phi}_{(6)} \Delta_d \phi_{(2)}), \end{aligned} \quad (3.15a)$$

$$\begin{aligned} \Delta_d \bar{\phi}_{(10)} = & \sum_a \left\{ \frac{5(\mathbf{p}_a^2)^4}{128m_a^7} + \frac{3(\mathbf{p}_a^2)^3}{16(d-1)m_a^5} \phi_{(2)} + \frac{(\mathbf{p}_a^2)^2}{4(d-1)m_a^3} \left(\frac{d+6}{8(d-1)} \phi_{(2)}^2 - \phi_{(4)} \right) \right. \\ & \left. + \frac{\mathbf{p}_a^2}{2(d-1)m_a} \left(\frac{(d+2)d}{48(d-1)^2} \phi_{(2)}^3 - \frac{d+2}{4(d-1)} \phi_{(2)} \phi_{(4)} + \bar{\phi}_{(6)} \right) \right\} \delta_a \\ & - \frac{4-d}{d-1} \left(\frac{10-3d}{16(d-1)} \phi_{(2)}^2 (\tilde{\pi}_{(3)}^{ij})^2 + \frac{1}{2} \phi_{(4)} (\tilde{\pi}_{(3)}^{ij})^2 + \phi_{(2)} \tilde{\pi}_{(3)}^{ij} \tilde{\pi}_{(5)}^{ij} \right) - (\tilde{\pi}_{(5)}^{ij})^2 - 2\tilde{\pi}_{(3)}^{ij} \tilde{\pi}_{(7)}^{ij} \\ & + \left\{ -\sum_a \frac{P_{ai} P_{aj}}{m_a} \left(\frac{\mathbf{p}_a^2}{4m_a^2} + \frac{1}{d-1} \phi_{(2)} \right) \delta_a + \frac{d-2}{d-1} \left(\frac{3}{4(d-1)} \left(\frac{1}{d-1} \phi_{(2)} \phi_{(2),i} \phi_{(2),j} - \phi_{(2),i} \phi_{(4),j} - \phi_{(2),j} \phi_{(4),i} \right) \right. \right. \\ & \left. \left. + \frac{6-d}{4(d-1)} \left(\frac{1}{2(d-1)} \phi_{(2)}^2 \phi_{(2),ij} - \phi_{(2),ij} \phi_{(4)} - \phi_{(2)} \phi_{(4),ij} \right) + \bar{\phi}_{(6),ij} \right) - 2\tilde{\pi}_{(3)}^{ik} \tilde{\pi}_{(3)}^{jk} \right\} h_{ij}^{\text{TT}} \\ & - \frac{5-d}{4(d-1)} (\Delta_d \phi_{(2)}) (h_{ij}^{\text{TT}})^2 - \frac{d-3}{d-1} \phi_{(2),ij} h_{ik}^{\text{TT}} h_{jk}^{\text{TT}} - \frac{8-d}{2(d-1)} \phi_{(2),k} h_{ij}^{\text{TT}} h_{ij,k}^{\text{TT}} + \frac{4-d}{d-1} \phi_{(2),k} h_{ij}^{\text{TT}} h_{ik,j}^{\text{TT}} \\ & + \frac{6-d}{2(d-1)} \phi_{(2)} \left(\frac{1}{2} h_{ij,k}^{\text{TT}} h_{ik,j}^{\text{TT}} - \frac{3}{4} (h_{ij,k}^{\text{TT}})^2 - h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} \right) - \left(\frac{4-d}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{ij} + 2\tilde{\pi}_{(5)}^{ij} \right) \pi_{\text{TT}}^{ij} - (\pi_{\text{TT}}^{ij})^2 \\ & - \frac{d-2}{4(d-1)} (\phi_{(2)} \Delta_d \bar{\phi}_{(8)} + \phi_{(4)} \Delta_d \bar{\phi}_{(6)} + \bar{\phi}_{(6)} \Delta_d \phi_{(4)} + \bar{\phi}_{(8)} \Delta_d \phi_{(2)}). \end{aligned} \quad (3.15b)$$

The equations satisfied by the longitudinal field momenta $\tilde{\pi}_{(3)}^{ij}$, $\tilde{\pi}_{(5)}^{ij}$, $\tilde{\pi}_{(7)}^{ij}$, and $\tilde{\pi}_{(9)}^{ij}$ we obtain by substituting expansions (3.9) into Eqs. (3.4)–(3.5) and reexpanding them with respect to ϵ . The result is

$$\tilde{\pi}_{(3),j}^{ij} = -\frac{1}{2} \sum_a p_{ai} \delta_a, \quad (3.16a)$$

$$\tilde{\pi}_{(5),j}^{ij} = -\frac{1}{d-1} \partial_j (\phi_{(2)} \tilde{\pi}_{(3)}^{ij}), \quad (3.16b)$$

$$\tilde{\pi}_{(7),j}^{ij} = \partial_j \left\{ -\frac{1}{d-1} (\phi_{(4)} \tilde{\pi}_{(3)}^{ij} + \phi_{(2)} \tilde{\pi}_{(5)}^{ij} + \phi_{(2)} \pi_{\text{TT}}^{ij}) - \frac{6-d}{8(d-1)^2} \phi_{(2)}^2 \tilde{\pi}_{(3)}^{ij} - \tilde{\pi}_{(3)}^{jk} h_{ik}^{\text{TT}} + V_{(3)}^k h_{jk,i}^{\text{TT}} \right\}, \quad (3.16c)$$

$$\begin{aligned} \tilde{\pi}_{(9),j}^{ij} = \partial_j \left\{ -\frac{1}{d-1} (\bar{\phi}_{(6)} \tilde{\pi}_{(3)}^{ij} + \phi_{(4)} \tilde{\pi}_{(5)}^{ij} + \phi_{(2)} \tilde{\pi}_{(7)}^{ij} + \phi_{(4)} \pi_{\text{TT}}^{ij}) - \frac{6-d}{4(d-1)^2} \phi_{(2)} \phi_{(4)} \tilde{\pi}_{(3)}^{ij} - \frac{(4-d)(6-d)}{48(d-1)^3} \phi_{(2)}^3 \tilde{\pi}_{(3)}^{ij} \right. \\ \left. - \frac{6-d}{8(d-1)^2} \phi_{(2)}^2 (\tilde{\pi}_{(5)}^{ij} + \pi_{\text{TT}}^{ij}) + \frac{1}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{jk} h_{ik}^{\text{TT}} + V_{(5)}^k h_{jk,i}^{\text{TT}} - h_{ik}^{\text{TT}} \pi_{\text{TT}}^{jk} \right\} \\ \left. - \frac{1}{d-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{jk} + (d-1) \tilde{\pi}_{(5)}^{jk}) h_{ij,k}^{\text{TT}} + \frac{1}{2} h_{jk,i}^{\text{TT}} \pi_{\text{TT}}^{jk}. \right\} \end{aligned} \quad (3.16d)$$

By making use of Eqs. (3.10)–(3.16) and performing very many integrations by parts in space, it is possible to rewrite the 4PN-accurate reduced Hamiltonian (3.7) in the form in which its density depends on momenta \mathbf{p}_a and on the following functions: $\phi_{(2)}$, $S_{(4)1}$, $S_{(4)2}$, $V_{(3)}^i$, $V_{(5)}^i$, $\bar{\phi}_{(6)}$, h_{ij}^{TT} , π_{TT}^{ij} [for convenience we also use $\tilde{\pi}_{(3)}^{ij}$ and $\tilde{\pi}_{(5)}^{ij}$, which can be expressed by $V_{(3)}^i$ and $V_{(5)}^i$, respectively, by means of Eq. (2.9)]. We will display now this form. The 4PN-accurate Hamiltonian is the sum of pieces related with different PN orders,

$$\begin{aligned} H_{\leq 4\text{PN}}^{\text{red}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] \\ = \int d^d x h_{\leq 4\text{PN}}^{\text{red}}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}], \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} h_{\leq 4\text{PN}}^{\text{red}}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] \\ = \sum_a m_a \delta_a + h_{(4)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\ + h_{(6)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + h_{(8)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}) \\ + h_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}) \\ + h_{(12)}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}], \end{aligned} \quad (3.18)$$

where the Newtonian $h_{(4)}$ and the 1PN $h_{(6)}$ densities depend only on matter variables \mathbf{x}_a , \mathbf{p}_a only, the 2PN density $h_{(8)}$ depends on matter variables and on h_{ij}^{TT} , whereas the 3PN $h_{(10)}$ and 4PN $h_{(12)}$ densities depend on matter variables and on field variables h_{ij}^{TT} , π_{TT}^{ij} . The dependence of $h_{(12)}$ on h_{ij}^{TT} is both pointwise and functional and this is why we have used square brackets for arguments of $h_{(12)}$. The explicit forms of the Newtonian $h_{(4)}$ and the 1PN-level $h_{(6)}$ densities are as follows:

$$h_{(4)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) = \sum_a \left(\frac{\mathbf{p}_a^2}{2m_a} - \frac{(d-2)m_a}{4(d-1)} \phi_{(2)} \right) \delta_a, \quad (3.19a)$$

$$\begin{aligned} h_{(6)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) = \sum_a \left(-\frac{(\mathbf{p}_a^2)^2}{8m_a^3} - \frac{(d+2)\mathbf{p}_a^2}{8(d-1)m_a} \phi_{(2)} \right. \\ \left. + \frac{(d-2)m_a}{8(d-1)} \left(S_{(4)1} - \frac{d-2}{2(d-1)} (S_{(4)2} - \phi_{(2)}^2) \right) \right. \\ \left. + V_{(3)}^i p_{ai} \right) \delta_a. \end{aligned} \quad (3.19b)$$

For displaying the 2PN-level density $h_{(8)}$ we define two auxiliary functions which depend on matter variables only,

$$\begin{aligned} \kappa_{(8)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) := \sum_a \left\{ \frac{(\mathbf{p}_a^2)^3}{16m_a^5} + \frac{(d+2)(\mathbf{p}_a^2)^2}{16(d-1)m_a^3} \phi_{(2)} + \frac{(d+2)\mathbf{p}_a^2}{16(d-1)m_a} \left(\phi_{(2)}^2 + S_{(4)1} - \frac{d-2}{2(d-1)} S_{(4)2} \right) \right. \\ \left. - \frac{(d-2)^2 m_a}{32(d-1)^2} \left(\frac{d-2}{d-1} \phi_{(2)}^2 + 3S_{(4)1} - \frac{3(d-2)}{2(d-1)} S_{(4)2} \right) \phi_{(2)} \right\} \delta_a - \phi_{(2)} (\tilde{\pi}_{(3)}^{ij})^2, \end{aligned} \quad (3.20a)$$

$$S_{(4)ij}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) := -\sum_a \frac{p_{ai} p_{aj}}{m_a} \delta_a - \frac{d-2}{2(d-1)} \phi_{(2),i} \phi_{(2),j}. \quad (3.20b)$$

By means of these functions the density $h_{(8)}$ can be written as

$$h_{(8)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}) = \varkappa_{(8)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \frac{1}{2} S_{(4)ij}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) h_{ij}^{\text{TT}} + \frac{1}{4} (h_{ij,k}^{\text{TT}})^2. \quad (3.21)$$

To display the 3PN density $h_{(10)}$ we define another two auxiliary functions

$$\begin{aligned} \varkappa_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) := & \sum_a \left\{ -\frac{5(\mathbf{p}_a^2)^4}{128m_a^7} - \frac{(d+4)(\mathbf{p}_a^2)^3}{32(d-1)m_a^5} \phi_{(2)} \right. \\ & - \frac{(\mathbf{p}_a^2)^2}{32(d-1)m_a^3} \left(\frac{(d+6)d}{2(d-1)} \phi_{(2)}^2 + (d+4)S_{(4)1} - \frac{d(d-2)}{d-1} S_{(4)2} \right) \\ & - \frac{(d+2)\mathbf{p}_a^2}{16(d-1)^2 m_a} \left(\frac{d(3d-4)}{12(d-1)} \phi_{(2)}^2 + dS_{(4)1} - \frac{(3d-4)(d-2)}{4(d-1)} S_{(4)2} \right) \phi_{(2)} \\ & + \frac{(d-2)m_a}{32(d-1)^2} \left(\frac{(d-2)^3}{4(d-1)^2} \phi_{(2)}^4 + \frac{(3d-4)(d-2)}{2(d-1)} S_{(4)1} \phi_{(2)}^2 - \frac{(d-2)^3}{(d-1)^2} S_{(4)2} \phi_{(2)}^2 \right) \\ & + dS_{(4)1}^2 - \frac{(3d-4)(d-2)}{2(d-1)} S_{(4)1} S_{(4)2} + \frac{(d-2)^3}{2(d-1)^2} S_{(4)2}^2 \\ & + \frac{V_{(3)}^i p_{ai}}{2(d-1)} \left(\frac{(3d-2)(3d-4)}{8(d-1)} \phi_{(2)}^2 + dS_{(4)1} - \frac{(3d-4)(d-2)}{4(d-1)} S_{(4)2} \right) \Big\} \delta_a \\ & + \left(\frac{3d-4}{4(d-1)^2} ((d-2)S_{(4)2,i} - (3d-2)\phi_{(2)}\phi_{(2),i}) - \frac{d}{d-1} S_{(4)1,i} \right) V_{(3)}^j \tilde{\pi}_{(3)}^{ij}, \end{aligned} \quad (3.22a)$$

$$\begin{aligned} B_{(6)ij}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) := & \sum_a \left(\frac{\mathbf{p}_a^2 p_{ai} p_{aj}}{4m_a^3} + \frac{(d+2)p_{ai} p_{aj}}{4(d-1)m_a} \phi_{(2)} \right) \delta_a - 4V_{(3),k}^i V_{(3),j}^k - 2V_{(3),k}^i V_{(3),k}^j + 2V_{(3),i}^k V_{(3),j}^k \\ & + \frac{4(d-2)}{d} V_{(3),j}^i V_{(3),k}^k + \frac{(d+1)(d-2)}{4(d-1)^2} \phi_{(2),i} S_{(4)1,j} - \frac{(2d-3)(d-2)^2}{8(d-1)^3} \phi_{(2),i} S_{(4)2,j} \\ & + \frac{(d-2)(3d-4)}{8(d-1)^3} \phi_{(2)} \phi_{(2),i} \phi_{(2),j}. \end{aligned} \quad (3.22b)$$

The 3PN density $h_{(10)}$ then reads

$$\begin{aligned} h_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}) = & \varkappa_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \frac{2(2d-3)}{(d-1)^2} \phi_{(2),i} V_{(3)}^j (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} + B_{(6)ij}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) h_{ij}^{\text{TT}} \\ & + \frac{1}{2(d-1)} \phi_{(2)} h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} - \frac{2(d-2)}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{ij} \pi_{\text{TT}}^{ij} + (\pi_{\text{TT}}^{ij})^2, \end{aligned} \quad (3.23)$$

where, to diminish number of terms, we have introduced the TT projection of the product $\phi_{(2)} \tilde{\pi}_{(3)}^{ij}$, which, by virtue of Eqs. (C9) and (3.16), can be written as

$$(\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} = \phi_{(2)} \tilde{\pi}_{(3)}^{ij} + (d-1) \tilde{\pi}_{(5)}^{ij}. \quad (3.24)$$

This TT projection should be treated as a function of matter variables only.

To display the very large formula for the 4PN-order density $h_{(12)}$ we introduce three auxiliary quantities $\varkappa_{(12)}^1$, $\varkappa_{(12)}^2$, and $\varkappa_{(12)}^3$. We first define the function $\varkappa_{(12)}^1$ which depends only on matter variables $\mathbf{x}_a, \mathbf{p}_a$,

$$\begin{aligned}
 \chi_{(12)}^1(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) := & \sum_a \left\{ \frac{7(\mathbf{p}_a^2)^5}{256m_a^9} + \frac{5(d+6)(\mathbf{p}_a^2)^4}{256(d-1)m_a^7} \phi_{(2)} \right. \\
 & + \frac{(\mathbf{p}_a^2)^3}{64(d-1)m_a^5} \left[\frac{d+1}{d-1} \left(\frac{1}{2}(d+10)\phi_{(2)}^2 - (d-2)S_{(4)2} \right) + (d+6)S_{(4)1} \right] \\
 & + \frac{(d+6)(\mathbf{p}_a^2)^2}{32(d-1)m_a^3} \left[\frac{(d+2)(3d-2)}{24(d-1)^2} \phi_{(2)}^3 + \frac{1}{2(d-1)} \left((d+1)S_{(4)1} - \frac{(3d-2)(d-2)}{4(d-1)} S_{(4)2} \right) \phi_{(2)} + \phi_{(6)1} \right] \\
 & + \frac{(d+2)\mathbf{p}_a^2}{32(d-1)^2 m_a} \left[\frac{d(3d-2)(2d-3)}{48(d-1)^2} \phi_{(2)}^4 + \frac{d}{4(d-1)} \left((3d-2)S_{(4)1} - \frac{(d-2)(2d-3)}{d-1} S_{(4)2} \right) \phi_{(2)}^2 \right. \\
 & + \left. \frac{1}{2} \left((d+1)S_{(4)1}^2 - \frac{(d-2)(3d-2)}{2(d-1)} S_{(4)1}S_{(4)2} + \frac{(d-2)^2(2d-3)}{4(d-1)^2} S_{(4)2}^2 \right) + (3d-2)\phi_{(2)}\phi_{(6)1} \right] \\
 & + \frac{(d-2)m_a}{32(d-1)^2} \left[-\frac{(d-2)^4}{16(d-1)^3} \phi_{(2)}^5 - \frac{(d-2)^2}{4(d-1)^2} \left((2d-3)S_{(4)1} - \frac{5(d-2)^2}{4(d-1)} S_{(4)2} \right) \phi_{(2)}^3 \right. \\
 & - \left. \frac{d-2}{2(d-1)} \left(\frac{1}{2}(3d-2)S_{(4)1}^2 - \frac{(d-2)(2d-3)}{d-1} S_{(4)1}S_{(4)2} + \frac{5(d-2)^3}{8(d-1)^2} S_{(4)2}^2 \right) \phi_{(2)} \right. \\
 & + \left. \left(\frac{5(d-2)^2}{2(d-1)} (S_{(4)2} - \phi_{(2)}^2) - (3d-2)S_{(4)1} \right) \phi_{(6)1} \right] \delta_a - \frac{3d-2}{4(d-1)} \phi_{(6)1} (\tilde{\pi}_{(3)}^{ij})^2 \\
 & - \frac{3d-2}{8(d-1)^2} \left[\frac{1}{3}(2d-3)\phi_{(2)}^3 + \frac{1}{2} \left((3d-2)S_{(4)1} - \frac{(d-2)(2d-3)}{d-1} S_{(4)2} \right) \phi_{(2)} \right] (\tilde{\pi}_{(3)}^{ij})^2. \tag{3.25}
 \end{aligned}$$

The function $\chi_{(12)}^2$ depends on matter variables $\mathbf{x}_a, \mathbf{p}_a$ and on the field function h_{ij}^{TT} (it contains terms linear, quadratic, and cubic in h_{ij}^{TT}). It reads

$$\begin{aligned}
 \chi_{(12)}^2(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}) := & \left\{ -\sum_a \frac{p_{ai}p_{aj}}{8m_a} \left(\frac{3(\mathbf{p}_a^2)^2}{2m_a^4} + \frac{(d+4)\mathbf{p}_a^2}{(d-1)m_a^2} \phi_{(2)} + \frac{(d+6)d}{2(d-1)^2} \phi_{(2)}^2 + \frac{d+4}{d-1} S_{(4)1} - \frac{d(d-2)}{(d-1)^2} S_{(4)2} \right) \delta_a \right. \\
 & + \frac{d(d-2)}{4(d-1)^2} \phi_{(2),ij} \phi_{(6)1} - \frac{d-2}{8(d-1)^3} \left\{ \frac{2d-3}{2(d-1)} \left(\frac{d+2}{2} \phi_{(2)}^2 - (d-2)S_{(4)2} \right) + \frac{3d-2}{2} S_{(4)1} \right\} \phi_{(2),i} \phi_{(2),j} \\
 & + \left(3dS_{(4)1,i} - \frac{(5d-8)(d-2)}{2(d-1)} S_{(4)2,i} \right) \phi_{(2)} \phi_{(2),j} \left. \right\} - \frac{d-2}{16(d-1)^2} \left((d+3)S_{(4)1,i} S_{(4)1,j} \right. \\
 & - \left. \frac{(2d-1)(d-2)}{d-1} S_{(4)1,i} S_{(4)2,j} + \frac{(3d-5)(d-2)^2}{4(d-1)^2} S_{(4)2,i} S_{(4)2,j} \right) + \frac{2(d-2)}{d-1} \left\{ \left[V_{(3),i}^k V_{(3),j}^k \right. \right. \\
 & - \left. \left. V_{(3),k}^i V_{(3),k}^j - 2V_{(3),k}^i V_{(3),j}^k + 2 \left(1 - \frac{2}{d} \right) V_{(3),j}^i V_{(3),k}^k \right] \phi_{(2)} + 2V_{(3)}^k (V_{(3),j}^i \phi_{(2),k} - V_{(3),k}^i \phi_{(2),j}) \right\} \\
 & + 4(d-1) \left[V_{(3),i}^k V_{(5),j}^k - V_{(3),k}^i V_{(5),k}^j - V_{(3),k}^i V_{(5),j}^k - V_{(3),i}^k V_{(5),k}^j + \left(1 - \frac{2}{d} \right) (V_{(3),k}^i V_{(5),j}^k + V_{(3),j}^i V_{(5),k}^k) \right] h_{ij}^{\text{TT}} \\
 & + \left(\sum_a \frac{p_{ai}p_{aj}}{2m_a} \delta_a - \frac{4+8d-3d^2}{16(d-1)^2} \phi_{(2),i} \phi_{(2),j} - \frac{(d-2)^2}{16(d-1)^2} (\phi_{(2)} \phi_{(2),ij} + S_{(4)2,ij}) + \frac{1}{4(d-1)} S_{(4)1,ij} \right) h_{ik}^{\text{TT}} h_{jk}^{\text{TT}} \\
 & + \left(-\sum_a \frac{3\mathbf{p}_a^2}{16(d-1)m_a} \delta_a + \frac{20-d^2}{64(d-1)^2} (\phi_{(2),k})^2 (h_{ij}^{\text{TT}})^2 - \frac{1}{8(d-1)} \left(\frac{(d+2)(10-d)}{8(d-1)} \phi_{(2)}^2 + 3S_{(4)1} \right. \right. \\
 & \left. \left. - \frac{d^2-4}{4(d-1)} S_{(4)2} \right) h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} - \frac{1}{4} (2h_{ik,l}^{\text{TT}} (h_{jk,l}^{\text{TT}} + h_{jl,k}^{\text{TT}}) + h_{kl,i}^{\text{TT}} h_{kl,j}^{\text{TT}}) h_{ij}^{\text{TT}} \right. \tag{3.26}
 \end{aligned}$$

The function $\chi_{(12)}^3$ is proportional to the second part $\bar{\phi}_{(6)2}$ of the function $\bar{\phi}_{(6)}$. It is a function of matter variables and it depends on h_{ij}^{TT} both pointwisely and, through the function $\bar{\phi}_{(6)2}$, functionally. It equals

$$\begin{aligned} \chi_{(12)}^3[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}] := & \left\{ \sum_a \left[\frac{(d+6)(\mathbf{p}_a^2)^2}{32(d-1)m_a^3} + \frac{(d+2)(3d-2)\mathbf{p}_a^2}{32(d-1)^2 m_a} \phi_{(2)} + \frac{(d-2)m_a}{32(d-1)^2} \left(\frac{5(d-2)^2}{2(d-1)} (S_{(4)2} - \phi_{(2)}^2) - (3d-2)S_{(4)1} \right) \right] \delta_a \right. \\ & \left. - \frac{3d-2}{4(d-1)} (\tilde{\pi}_{(3)}^{ij})^2 + \frac{d(d-2)}{4(d-1)^2} \phi_{(2),ij} h_{ij}^{\text{TT}} \right\} \bar{\phi}_{(6)2}. \end{aligned} \quad (3.27)$$

The 4PN-order Hamiltonian density $h_{(12)}$ finally equals

$$\begin{aligned} h_{(12)}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] = & \chi_{(12)}^1(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \chi_{(12)}^2(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}) + \chi_{(12)}^3[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}] \\ & + \left(\frac{d(11d-18)}{8(d-1)^3} \phi_{(2)}^2 + \frac{2}{d-1} S_{(4)1} - \frac{(5d-8)(d-2)}{4(d-1)^3} S_{(4)2} \right) \tilde{\pi}_{(3)}^{ij} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} \\ & - \frac{3d-5}{(d-1)^3} \phi_{(2)} ((\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}})^2 + 2(2\tilde{\pi}_{(3)}^{ik} h_{ij}^{\text{TT}} + V_{(3)}^i (2h_{ij,k}^{\text{TT}} - h_{jk,i}^{\text{TT}})) \left(\pi_{\text{TT}}^{jk} - \frac{d-2}{d-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{jk})^{\text{TT}} \right) \\ & - \frac{d-3}{d-1} \phi_{(2)} (\pi_{\text{TT}}^{ij})^2 + \left(\frac{(9d-14)(4-d)}{4(d-1)^2} \phi_{(2)} \phi_{(2),j} V_{(3)}^i - \frac{d}{d-1} S_{(4)1,j} V_{(3)}^i \right) \\ & + \frac{(3d-4)(d-2)}{4(d-1)^2} S_{(4)2,j} V_{(3)}^i + \frac{2(3d-5)}{d-1} \phi_{(2),j} V_{(5)}^i \pi_{\text{TT}}^{ij}. \end{aligned} \quad (3.28)$$

IV. FIELD EQUATIONS

Dynamical degrees of freedom of gravitational field described by the functions h_{ij}^{TT} and π_{TT}^{ij} are solutions of the field equations (2.13). From the 4PN-accurate reduced Hamiltonian given in Eq. (3.17) one can derive 4PN-accurate approximate field equations. In the rest of the paper we will only need to use field equations which follow from the 3PN-accurate part of the Hamiltonian (3.17). It reads

$$H_{\leq 3\text{PN}}^{\text{red}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] = \int d^d x h_{\leq 3\text{PN}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}), \quad (4.1)$$

where

$$\begin{aligned} h_{\leq 3\text{PN}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}) &= \sum_a m_a \delta_a + h_{(4)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + h_{(6)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\ &+ h_{(8)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}) + h_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}). \end{aligned} \quad (4.2)$$

For this Hamiltonian the field equations (2.13) take the form

$$\dot{h}_{ij}^{\text{TT}} = \delta_{ij}^{\text{TT}kl} \frac{\partial h_{\leq 3\text{PN}}}{\partial \pi_{\text{TT}}^{kl}} + \mathcal{O}(\epsilon^7), \quad (4.3a)$$

$$\begin{aligned} \dot{\pi}_{\text{TT}}^{ij} = & -\delta_{kl}^{\text{TT}ij} \left\{ \frac{\partial h_{\leq 3\text{PN}}}{\partial h_{kl}^{\text{TT}}} - \left(\frac{\partial h_{\leq 3\text{PN}}}{\partial h_{kl,m}^{\text{TT}}} \right)_{,m} \right. \\ & \left. + \left(\frac{\partial h_{\leq 3\text{PN}}}{\partial h_{kl,mn}^{\text{TT}}} \right)_{,mn} \right\} + \mathcal{O}(\epsilon^8), \end{aligned} \quad (4.3b)$$

or, more explicitly,

$$\dot{h}_{ij}^{\text{TT}} = \delta_{ij}^{\text{TT}kl} \left\{ 2\pi_{\text{TT}}^{kl} - \frac{2(d-2)}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{kl} \right\} + \mathcal{O}(\epsilon^7), \quad (4.4a)$$

$$\begin{aligned} \dot{\pi}_{\text{TT}}^{ij} = & -\delta_{ij}^{\text{TT}kl} \left\{ \frac{1}{2} S_{(4)kl} - \frac{1}{2} \Delta_d h_{kl}^{\text{TT}} + B_{(6)kl} \right. \\ & \left. + \frac{1}{2(d-1)} (\phi_{(2)} \Delta_d h_{kl}^{\text{TT}} + \Delta_d (\phi_{(2)} h_{kl}^{\text{TT}})) \right\} + \mathcal{O}(\epsilon^8). \end{aligned} \quad (4.4b)$$

By combining these two equations one gets the equation fulfilled by the function h_{ij}^{TT} . It can be written in the form of the wave equation,

$$\square_{d+1} h_{ij}^{\text{TT}} = S_{ij}^{\text{TT}}, \quad (4.5)$$

where \square_{d+1} is d'Alembertian in $(d+1)$ -dimensional flat space-time,

$$\square_{d+1} := -\partial_t^2 + \Delta_d, \quad (4.6)$$

and where the source term is

$$\begin{aligned} S_{ij}^{\text{TT}} = & \delta_{ij}^{\text{TT}kl} \left\{ S_{(4)kl} + 2B_{(6)kl} \right. \\ & + \frac{1}{d-1} (\phi_{(2)} \Delta_d h_{kl}^{\text{TT}} + \Delta_d (\phi_{(2)} h_{kl}^{\text{TT}})) \\ & \left. + \frac{2(d-2)}{d-1} \partial_t (\phi_{(2)} \tilde{\pi}_{(3)}^{kl}) \right\} + \mathcal{O}(\epsilon^8). \end{aligned} \quad (4.7)$$

After solving field equation (4.5) for h_{ij}^{TT} one can obtain the TT field momentum π_{TT}^{ij} from Eq. (4.4a):

$$\pi_{\text{TT}}^{ij} = \frac{1}{2} \dot{h}_{ij}^{\text{TT}} + \frac{d-2}{d-1} \delta_{ij}^{\text{TT}kl} (\phi_{(2)} \tilde{\pi}_{(3)}^{kl}) + \mathcal{O}(\epsilon^7). \quad (4.8)$$

In the current paper we are interested only in conservative dynamics of the matter-plus-field system; therefore, we use a time-symmetric (half-retarded half-advanced) formal solution of the wave equation (4.5). We then expand this solution with respect to the retardation. This way we obtain the *near-zone* PN expansion of h_{ij}^{TT} of the form

$$h_{ij}^{\text{TT}} = (\Delta_d^{-1} + \Delta_d^{-2} \partial_t^2 + \Delta_d^{-3} \partial_t^4 + \dots) S_{ij}^{\text{TT}}. \quad (4.9)$$

By making the expansion (4.9) we exclude from the 4PN-level near-zone metric the nonlocal-in-time contribution coming from tail effects (what was found in Ref. [61]). Expansion of h_{ij}^{TT} to the order required in our computations reads

$$h_{ij}^{\text{TT}} = h_{(4)ij}^{\text{TT}} + h_{(6)ij}^{\text{TT}} + \mathcal{O}(\epsilon^8); \quad (4.10)$$

we thus omit the non-time-symmetric parts $h_{(5)ij}^{\text{TT}}$ and $h_{(7)ij}^{\text{TT}}$. After plugging the expansion (4.10) into Eq. (4.5) one obtains equations fulfilled by the functions $h_{(4)ij}^{\text{TT}}$ and $h_{(6)ij}^{\text{TT}}$. The leading-order function $h_{(4)ij}^{\text{TT}}$ is the solution of the Poisson equation

$$\Delta_d h_{(4)ij}^{\text{TT}} = S_{(4)ij}^{\text{TT}}, \quad (4.11)$$

where the source term $S_{(4)ij}$ is defined in Eq. (3.20b). The next-to-leading-order function $h_{(6)ij}^{\text{TT}}$ fulfills equation

$$\Delta_d h_{(6)ij}^{\text{TT}} = S_{(6)ij}^{\text{TT}} + \ddot{h}_{(4)ij}^{\text{TT}}, \quad (4.12)$$

where the source function $S_{(6)ij}$ equals

$$S_{(6)ij} = 2B_{(6)ij} + \frac{1}{d-1} (\phi_{(2)} \Delta_d h_{(4)ij}^{\text{TT}} + \Delta_d (\phi_{(2)} h_{(4)ij}^{\text{TT}})) + 2 \frac{d-2}{d-1} \partial_t (\phi_{(2)} \tilde{\pi}_{(3)}^{ij}), \quad (4.13)$$

and the source function $B_{(6)ij}$ is defined in Eq. (3.22).

V. 4PN-ACCURATE ROUTHIAN

Introducing Routhian description is an intermediate step on our route to the Hamiltonian which depends only on matter variables. The 4PN-accurate Routhian $R_{\leq 4\text{PN}}$ we obtain from the 4PN-accurate reduced Hamiltonian $H_{\leq 4\text{PN}}^{\text{red}}$ of Eq. (3.17) by performing the Legendre transformation with respect to the field momentum π_{TT}^{ij} [see Eq. (2.15)]. The result is

$$R_{\leq 4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}] = H_{\leq 4\text{PN}}^{\text{red}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}] - \int d^d x \dot{h}_{ij}^{\text{TT}} \pi_{\text{TT}}^{ij}, \quad (5.1)$$

where on the right-hand side the field momentum π_{TT}^{ij} is expressed in terms of h_{ij}^{TT} , \dot{h}_{ij}^{TT} , and matter variables. The Legendre transformation is to leading order realized by Eq. (4.8). We will show now that this leading-order formula is enough to get the 4PN-accurate Routhian.

Let us split the field momentum π_{TT}^{ij} in the following way [cf. Eq. (4.8)]:

$$\pi_{\text{TT}}^{ij} = \frac{1}{2} \dot{h}_{ij}^{\text{TT}} + \frac{d-2}{d-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} + \delta\pi_{\text{TT}}^{ij}, \quad (5.2a)$$

$$\delta\pi_{\text{TT}}^{ij} = \mathcal{O}(\epsilon^7), \quad \delta\pi_{\text{TT}}^{ij} = \mathcal{O}(r^{1-d}) \quad \text{for } r \rightarrow \infty. \quad (5.2b)$$

We will show that the density of the 4PN-accurate Routhian (5.1) does not depend on $\delta\pi_{\text{TT}}^{ij}$. The only part of the density which could depend on $\delta\pi_{\text{TT}}^{ij}$ [see Eq. (3.23)] consists of three terms,

$$\delta\mathbf{r} := (\pi_{\text{TT}}^{ij})^2 - \frac{2(d-2)}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{ij} \pi_{\text{TT}}^{ij} - \dot{h}_{ij}^{\text{TT}} \pi_{\text{TT}}^{ij}. \quad (5.3)$$

Making use of the splitting (5.2a) and the relation (3.24) we rewrite the quantity $\delta\mathbf{r}$ in the form

$$\delta\mathbf{r} = -\frac{1}{4} (\dot{h}_{ij}^{\text{TT}})^2 - \frac{d-2}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{ij} \dot{h}_{ij}^{\text{TT}} + (\delta\pi_{\text{TT}}^{ij})^2 + 2(d-2) \tilde{\pi}_{(5)}^{ij} \delta\pi_{\text{TT}}^{ij} + \left(\frac{d-2}{d-1}\right)^2 (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} ((d-1) \tilde{\pi}_{(5)}^{ij} - \phi_{(2)} \tilde{\pi}_{(3)}^{ij}). \quad (5.4)$$

By virtue of the representation (2.9) we rewrite Eq. (5.4) as

$$\delta\mathbf{r} = -\frac{1}{4} (\dot{h}_{ij}^{\text{TT}})^2 - \frac{d-2}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{ij} \dot{h}_{ij}^{\text{TT}} - \left(\frac{d-2}{d-1}\right)^2 \phi_{(2)} \tilde{\pi}_{(3)}^{ij} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} + (\delta\pi_{\text{TT}}^{ij})^2 + \partial_k \text{ED}_{3k}, \quad (5.5)$$

with

$$\text{ED}_{3k} := V_{(5)}^i \left(4(d-2) \delta\pi_{\text{TT}}^{ik} + 2 \frac{(d-2)^2}{d-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{ik})^{\text{TT}} \right). \quad (5.6)$$

Because ED_{3k} decays as $1/r^{2d-3}$ for $r \rightarrow \infty$ [the quantity $V_{(5)}^i = \mathcal{O}(r^{2-d})$ when $r \rightarrow \infty$], it does not contribute to the Routhian, and because $(\delta\pi_{\text{TT}}^{ij})^2$ is of the order of ϵ^{14} , only the first three terms on the right-hand side of (5.5) contribute to the 4PN-accurate Routhian.

By virtue of the above result, to get the 4PN-accurate Routhian it is enough to eliminate from the 4PN-accurate reduced Hamiltonian (3.17) the TT part π_{TT}^{ij} of the field momentum by means of the relation (4.8). Making use of Eqs. (3.19)–(3.28) and Eq. (4.8), the 4PN-accurate Routhian reads

$$\begin{aligned} R_{\leq 4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}] \\ = \int d^d x \mathbf{r}_{\leq 4\text{PN}}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}], \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} \mathbf{r}_{\leq 4\text{PN}}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}] \\ = \sum_a m_a \delta_a + h_{(4)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + h_{(6)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\ + h_{(8)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}) + \mathbf{r}_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}) \\ + \mathbf{r}_{(12)}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}]. \end{aligned} \quad (5.8)$$

The Routhian densities $h_{(4)}$, $h_{(6)}$, and $h_{(8)}$ are identical with the corresponding densities of the reduced Hamiltonian (3.17) and they are given in Eqs. (3.19) and (3.21). The 3PN density $\mathbf{r}_{(10)}$, after some more integrations by parts, can be written as

$$\begin{aligned} \mathbf{r}_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}) \\ = \kappa_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + 2\phi_{(2),i} V_{(3)}^j (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} \\ + B_{(6)ij}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) h_{ij}^{\text{TT}} + \frac{1}{2(d-1)} \phi_{(2)} h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} \\ + \frac{2(d-2)}{d-1} \phi_{(2),i} V_{(3)}^j \dot{h}_{ij}^{\text{TT}} - \frac{1}{4} (\dot{h}_{ij}^{\text{TT}})^2. \end{aligned} \quad (5.9)$$

The 4PN-level Routhian density $\mathbf{r}_{(12)}$ can be easily obtained by replacing in the 4PN-level Hamiltonian density $h_{(12)}$ [given in Eq. (3.28)] the field momentum π_{TT}^{ij} by the two first terms from the right-hand side of Eq. (5.2a),

$$\begin{aligned} \mathbf{r}_{(12)}[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}] \\ = h_{(12)} \left[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \frac{1}{2} \dot{h}_{ij}^{\text{TT}} + \frac{d-2}{d-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} \right]. \end{aligned} \quad (5.10)$$

VI. 4PN-ACCURATE CONSERVATIVE MATTER HAMILTONIAN

In this section we consider the 4PN-accurate *conservative matter* Hamiltonian, which depends only on the particle variables $\mathbf{x}_a, \mathbf{p}_a$. To obtain the conservative Hamiltonian, from now on we use only the time-symmetric part of the field function h_{ij}^{TT} . To get the matter Hamiltonian we eliminate h_{ij}^{TT} (and \dot{h}_{ij}^{TT}) by replacing it by (time-symmetric) solution of the

field equation (4.5). We thus treat h_{ij}^{TT} (and its time derivatives) as functions of matter variables $\mathbf{x}_a, \mathbf{p}_a$. All time derivatives of $\mathbf{x}_a, \mathbf{p}_a$ involved in h_{ij}^{TT} (and \dot{h}_{ij}^{TT}) are eliminated by means of the lower-order equations of motion.

We thus start from plugging into the 4PN-accurate Routhian (5.7) the time-symmetric solution of the field equation for the function h_{ij}^{TT} . This way we obtain the 4PN-accurate conservative matter Hamiltonian

$$\begin{aligned} H_{\leq 4\text{PN}}^{\text{matter}}(\mathbf{x}_a, \mathbf{p}_a) \\ := R_{\leq 4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a), \dot{h}_{ij}^{\text{TT}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a)], \end{aligned} \quad (6.1)$$

where we have assumed that all time derivatives of \mathbf{x}_a and \mathbf{p}_a present in h_{ij}^{TT} and \dot{h}_{ij}^{TT} were eliminated by means of lower-order equations of motion (to get the 4PN-accurate results we only need to use Newtonian and 1PN equations of motion). We reorganize now density of the Hamiltonian (6.1) by employing the field equation (4.5) for the function h_{ij}^{TT} .

We first split h_{ij}^{TT} into the leading-order term $h_{(4)ij}^{\text{TT}}$ and the rest $\delta h_{ij}^{\text{TT}}$:

$$\begin{aligned} h_{ij}^{\text{TT}} &= h_{(4)ij}^{\text{TT}} + \delta h_{ij}^{\text{TT}}, \\ \delta h_{ij}^{\text{TT}} &= \mathcal{O}(\epsilon^6), \quad \delta h_{ij}^{\text{TT}} = \mathcal{O}(r^{2-d}) \quad \text{for } r \rightarrow \infty. \end{aligned} \quad (6.2)$$

The part of the 2PN density $h_{(8)}$ which depends on h_{ij}^{TT} reads [see Eq. (3.21)]

$$\delta h_1 := \frac{1}{2} S_{(4)ij} h_{ij}^{\text{TT}} + \frac{1}{4} (h_{ij,k}^{\text{TT}})^2, \quad (6.3)$$

where $S_{(4)ij}$ is the source term for $h_{(4)ij}^{\text{TT}}$ [see Eq. (4.11)]. Making use of the splitting (6.2), Eq. (6.3) can be rewritten as

$$\begin{aligned} \delta h_1 &= \frac{1}{2} S_{(4)ij} h_{(4)ij}^{\text{TT}} + \frac{1}{4} (h_{(4)ij,k}^{\text{TT}})^2 + \frac{1}{2} h_{(4)ij,k}^{\text{TT}} \delta h_{ij,k}^{\text{TT}} \\ &+ \frac{1}{4} (\delta h_{ij,k}^{\text{TT}})^2 + \frac{1}{2} S_{(4)ij} \delta h_{ij}^{\text{TT}}. \end{aligned} \quad (6.4)$$

We still rewrite Eq. (6.4) in the following way:

$$\begin{aligned} \delta h_1 &= \frac{1}{2} S_{(4)ij} h_{(4)ij}^{\text{TT}} + \frac{1}{4} (h_{(4)ij,k}^{\text{TT}})^2 + \frac{1}{2} (\Delta_d h_{(4)ij}^{\text{TT}}) \delta h_{ij}^{\text{TT}} \\ &+ \frac{1}{2} h_{(4)ij,k}^{\text{TT}} \delta h_{ij,k}^{\text{TT}} + \frac{1}{4} (\delta h_{ij,k}^{\text{TT}})^2 \\ &+ \frac{1}{2} (S_{(4)ij} - S_{(4)ij}^{\text{TT}}) \delta h_{ij}^{\text{TT}}, \end{aligned} \quad (6.5)$$

where we have employed the field equation (4.11) for the function $h_{(4)ij}^{\text{TT}}$. Finally, making use of the identity

$$(\Delta_d h_{(4)ij}^{\text{TT}}) \delta h_{ij}^{\text{TT}} = \partial_k (h_{(4)ij,k}^{\text{TT}} \delta h_{ij}^{\text{TT}}) - h_{(4)ij,k}^{\text{TT}} \delta h_{ij,k}^{\text{TT}}, \quad (6.6)$$

equation (6.5) takes the form

$$\delta h_1 = \frac{1}{2} S_{(4)ij} h_{(4)ij}^{\text{TT}} + \frac{1}{4} (h_{(4)ij,k}^{\text{TT}})^2 + \frac{1}{4} (\delta h_{ij,k}^{\text{TT}})^2 + \partial_k \text{ED}_{1k}, \quad (6.7)$$

where the exact divergence ED_{1k} is defined as [we have used here the explicit form (2.14) of the TT projection operator]

$$\text{ED}_{1k} := \frac{1}{2} h_{(4)ij,k}^{\text{TT}} \delta h_{ij}^{\text{TT}} + \delta h_{ik}^{\text{TT}} \left(\Delta_d^{-1} S_{(4)ij,j} - \frac{1}{2(d-1)} \Delta_d^{-1} S_{(4)ij,i} - \frac{d-2}{2(d-1)} \Delta_d^{-2} S_{(4)jl,ji} \right). \quad (6.8)$$

Because ED_{1k} decays as $1/r^{2d-3}$ for $r \rightarrow \infty$, it does not contribute to the Hamiltonian. Therefore the Hamiltonian density δh_1 can be replaced by $\delta h'_1$, where

$$\delta h'_1 := \frac{1}{2} S_{(4)ij} h_{(4)ij}^{\text{TT}} + \frac{1}{4} (h_{(4)ij,k}^{\text{TT}})^2 + \frac{1}{4} (\delta h_{ij,k}^{\text{TT}})^2, \quad (6.9)$$

$$(\delta h_{ij,k}^{\text{TT}})^2 = \mathcal{O}(e^{12}).$$

Because

$$(\delta h_{ij,k}^{\text{TT}})^2 = -\delta h_{ij}^{\text{TT}} (\Delta_d \delta h_{ij}^{\text{TT}}) + \partial_k \text{ED}_{2k}, \quad (6.10)$$

$$\text{ED}_{2k} := \delta h_{ij}^{\text{TT}} \delta h_{ij,k}^{\text{TT}},$$

and $\text{ED}_{2k} = \mathcal{O}(r^{3-2d})$ for $r \rightarrow \infty$, instead of the density (6.9) one can also use

$$\delta h''_1 := \frac{1}{2} S_{(4)ij} h_{(4)ij}^{\text{TT}} + \frac{1}{4} (h_{(4)ij,k}^{\text{TT}})^2 - \frac{1}{4} \delta h_{ij}^{\text{TT}} (\Delta_d \delta h_{ij}^{\text{TT}}). \quad (6.11)$$

The part of the 3PN Routhian density \mathfrak{r}_{10} [see Eq. (5.9)] which depends on h_{ij}^{TT} or \dot{h}_{ij}^{TT} reads

$$B_{(6)ij} h_{ij}^{\text{TT}} + \frac{1}{2(d-1)} \phi_{(2)} h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} + \frac{2(d-2)}{d-1} \phi_{(2),i} V_{(3)}^j \dot{h}_{ij}^{\text{TT}} - \frac{1}{4} (\dot{h}_{ij}^{\text{TT}})^2. \quad (6.12)$$

Let us denote the sum of these terms and of the expression (6.11) by δh_2 ,

$$\delta h_2 := \frac{1}{2} S_{(4)ij} h_{(4)ij}^{\text{TT}} + \frac{1}{4} (h_{(4)ij,k}^{\text{TT}})^2 + B_{(6)ij} h_{ij}^{\text{TT}} + \frac{1}{2(d-1)} \phi_{(2)} h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} - \frac{1}{4} \delta h_{ij}^{\text{TT}} (\Delta_d \delta h_{ij}^{\text{TT}}) + \frac{2(d-2)}{d-1} \phi_{(2),i} V_{(3)}^j \dot{h}_{ij}^{\text{TT}} - \frac{1}{4} (\dot{h}_{ij}^{\text{TT}})^2. \quad (6.13)$$

In the field equation (4.5) we split the source term S_{ij}^{TT} into the leading-order part $S_{(4)ij}^{\text{TT}}$ and the rest $\delta S_{ij}^{\text{TT}}$,

$$\square_{d+1} h_{ij}^{\text{TT}} = S_{(4)ij}^{\text{TT}} + \delta S_{ij}^{\text{TT}}, \quad \delta S_{ij}^{\text{TT}} = \mathcal{O}(e^6). \quad (6.14)$$

After substituting the splitting (6.2) into (6.14) we obtain (remembering that $\square_{d+1} = -\partial_t^2 + \Delta_d$)

$$\Delta_d h_{ij}^{\text{TT}} = \ddot{h}_{ij}^{\text{TT}} + S_{(4)ij}^{\text{TT}} + \delta S_{ij}^{\text{TT}}. \quad (6.15)$$

By virtue of Eqs. (6.2) and (4.11), Eq. (6.15) finally leads to

$$\Delta_d \delta h_{ij}^{\text{TT}} = \ddot{h}_{ij}^{\text{TT}} + \delta S_{ij}^{\text{TT}}. \quad (6.16)$$

Making use of Eqs. (6.2) and (6.16), we can prove the following relation:

$$-\frac{1}{4} (\dot{h}_{ij}^{\text{TT}})^2 - \frac{1}{4} \delta h_{ij}^{\text{TT}} (\Delta_d \delta h_{ij}^{\text{TT}}) = -\frac{d}{dt} \left(\delta h_{ij}^{\text{TT}} \left(\frac{1}{4} \delta h_{ij}^{\text{TT}} + \frac{1}{2} \dot{h}_{(4)ij}^{\text{TT}} \right) \right) - \frac{1}{4} (\dot{h}_{(4)ij}^{\text{TT}})^2 + \frac{1}{4} \ddot{h}_{(4)ij}^{\text{TT}} \delta h_{ij}^{\text{TT}} - \frac{1}{4} \delta h_{ij}^{\text{TT}} \delta S_{ij}^{\text{TT}}. \quad (6.17)$$

With the aid of the above relation, after dropping the total time derivative on the right-hand side of Eq. (6.17), the density δh_2 from (6.13) can be replaced by

$$\delta h'_2 := \frac{1}{2} S_{(4)ij} h_{(4)ij}^{\text{TT}} + \frac{1}{4} (h_{(4)ij,k}^{\text{TT}})^2 + B_{(6)ij} h_{ij}^{\text{TT}} + \frac{1}{2(d-1)} \phi_{(2)} h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} + \frac{2(d-2)}{d-1} \phi_{(2),i} V_{(3)}^j \dot{h}_{ij}^{\text{TT}} - \frac{1}{4} (\dot{h}_{(4)ij}^{\text{TT}})^2 + \frac{1}{4} \ddot{h}_{(4)ij}^{\text{TT}} \delta h_{ij}^{\text{TT}} - \frac{1}{4} \delta h_{ij}^{\text{TT}} \delta S_{ij}^{\text{TT}}. \quad (6.18)$$

The expression (6.18) contains all terms related with the 2PN-level and 3PN-level parts of the Routhian (5.7) which depend on h_{ij}^{TT} or \dot{h}_{ij}^{TT} . Taking into account this expression and all other terms entering the Routhian (5.7), one can write the 4PN-accurate conservative matter Hamiltonian (6.1) in the form

$$H_{\leq 4\text{PN}}^{\text{matter}}(\mathbf{x}_a, \mathbf{p}_a) = \int d^d x h_{\leq 4\text{PN}}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a), \quad (6.19)$$

where

$$\begin{aligned}
h_{\leq 4\text{PN}}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) &= \sum_a m_a \delta_a + h_{(4)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\
&+ h_{(6)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + h_{(8)}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\
&+ h_{(10)}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + h_{(12)}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a).
\end{aligned} \tag{6.20}$$

Here the densities $h_{(4)}$ (at the Newtonian order) and $h_{(6)}$ (at the 1PN order) can be found in Eqs. (3.19) and the densities at the 2PN, 3PN, and 4PN orders are as follows:

$$\begin{aligned}
h_{(8)}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) &= \varkappa_{(8)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \frac{1}{2} S_{(4)ij} h_{(4)ij}^{\text{TT}} \\
&+ \frac{1}{4} (h_{(4)ij,k}^{\text{TT}})^2,
\end{aligned} \tag{6.21a}$$

$$\begin{aligned}
h_{(10)}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) &= \varkappa_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\
&+ 2\phi_{(2),i} V_{(3)}^j (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} + B_{(6)ij} h_{ij}^{\text{TT}} \\
&+ \frac{1}{2(d-1)} \phi_{(2)} h_{ij}^{\text{TT}} \Delta_d h_{ij}^{\text{TT}} \\
&+ \frac{2(d-2)}{d-1} \phi_{(2),i} V_{(3)}^j \dot{h}_{ij}^{\text{TT}} - \frac{1}{4} (\dot{h}_{(4)ij}^{\text{TT}})^2,
\end{aligned} \tag{6.21b}$$

$$\begin{aligned}
h_{(12)}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) &= h_{(12)} \left[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \frac{1}{2} \dot{h}_{ij}^{\text{TT}} + \frac{d-2}{d-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} \right] \\
&+ \frac{1}{4} \dot{h}_{(4)ij}^{\text{TT}} \delta h_{ij}^{\text{TT}} - \frac{1}{4} \delta h_{ij}^{\text{TT}} \delta S_{ij}^{\text{TT}}.
\end{aligned} \tag{6.21c}$$

VII. 4PN-ACCURATE NEAR-ZONE CONSERVATIVE MATTER HAMILTONIAN

We employ now the crucial near-zone (time-symmetric) PN expansion (4.10) of the field variable h_{ij}^{TT} , i.e., we use $h_{ij}^{\text{TT}} = h_{(4)ij}^{\text{TT}} + \delta h_{ij}^{\text{TT}}$ with $\delta h_{ij}^{\text{TT}} = h_{(6)ij}^{\text{TT}} + \mathcal{O}(\epsilon^8)$. We also employ [see Eq. (6.14)]

$$\delta S_{ij}^{\text{TT}} = S_{(6)ij}^{\text{TT}} + \mathcal{O}(\epsilon^8), \tag{7.1}$$

The first part $h_{(12)}^1$ reads

$$\begin{aligned}
h_{(12)}^1(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) &= \varkappa_{(12)}^1(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + \varkappa_{(12)}^2(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{(4)ij}^{\text{TT}}) + \varkappa_{(12)}^3[\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a, h_{(4)ij}^{\text{TT}}] - \phi_{(2)} ((\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}})^2 \\
&+ \frac{1}{d-1} \left(\frac{(3d-2)(3d-4)}{8(d-1)} \phi_{(2)}^2 + d S_{(4)1} - \frac{(3d-4)(d-2)}{4(d-1)} S_{(4)2} \right) \tilde{\pi}_{(3)}^{ij} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} + \left\{ \frac{3(d-2)^2}{4(d-1)^2} S_{(4)2,j} \right. \\
&- S_{(4)1,j} + \frac{7(d-2)^2}{4(d-1)^2} \phi_{(2)} \phi_{(2),j} \left. \right) V_{(3)}^i + 2(d-2) \phi_{(2),j} V_{(5)}^i - \frac{4}{d} V_{(3),k}^k h_{(4)ij}^{\text{TT}} + 2V_{(3),k}^i h_{(4)jk}^{\text{TT}} - V_{(3)}^k h_{(4)ij,k}^{\text{TT}} \left. \right\} [[\dot{h}_{(4)ij}^{\text{TT}}]]_0 \\
&- \frac{d-3}{4(d-1)} \phi_{(2)} ([[\dot{h}_{(4)ij}^{\text{TT}}]])_0^2 + \frac{2(d-2)}{d-1} \phi_{(2),i} V_{(3)}^j [[\dot{h}_{(4)ij}^{\text{TT}}]]_1 - \frac{1}{2} [[\dot{h}_{(4)ij}^{\text{TT}}]]_0 [[\dot{h}_{(4)ij}^{\text{TT}}]]_1.
\end{aligned} \tag{7.6}$$

where the next-to-leading-order source function $S_{(6)ij}$ is defined in Eq. (4.13). After plugging (4.10) and (7.1) into the 4PN-accurate conservative matter Hamiltonian of Eq. (6.19), we obtain the 4PN-accurate near-zone conservative matter Hamiltonian,

$$H_{\leq 4\text{PN}}^{\text{near-zone}}(\mathbf{x}_a, \mathbf{p}_a) = \int d^d x h_{\leq 4\text{PN}}^{\text{near-zone}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a), \tag{7.2}$$

where

$$\begin{aligned}
h_{\leq 4\text{PN}}^{\text{near-zone}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) &= \sum_a m_a \delta_a + h_{(4)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\
&+ h_{(6)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + h_{(8)}^{\text{near-zone}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\
&+ h_{(10)}^{\text{near-zone}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\
&+ h_{(12)}^{\text{near-zone}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a).
\end{aligned} \tag{7.3}$$

Here again the densities $h_{(4)}$ and $h_{(6)}$ can be found in Eqs. (3.19), the 2PN density $h_{(8)}^{\text{near-zone}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) = h_{(8)}^{\text{matter}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a)$, where $h_{(8)}^{\text{matter}}$ is given in Eq. (6.21), and the 3PN/4PN densities $h_{(10)}^{\text{near-zone}}$ and $h_{(12)}^{\text{near-zone}}$ follow from the sum of densities $h_{(10)}^{\text{matter}} + h_{(12)}^{\text{matter}}$ after plugging the expansions (4.10) and (7.1) into the latter. The 3PN-level density reads

$$\begin{aligned}
h_{(10)}^{\text{near-zone}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) &= \varkappa_{(10)}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) \\
&+ 2\phi_{(2),i} V_{(3)}^j (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} + B_{(6)ij} h_{(4)ij}^{\text{TT}} \\
&+ \frac{1}{2(d-1)} \phi_{(2)} h_{(4)ij}^{\text{TT}} \Delta_d h_{(4)ij}^{\text{TT}} \\
&+ \frac{2(d-2)}{d-1} \phi_{(2),i} V_{(3)}^j [[\dot{h}_{(4)ij}^{\text{TT}}]]_0 \\
&- \frac{1}{4} ([[\dot{h}_{(4)ij}^{\text{TT}}]])_0^2,
\end{aligned} \tag{7.4}$$

where the notation $[[\dot{h}_{(4)ij}^{\text{TT}}]]_0$ means elimination of time derivatives $\dot{\mathbf{x}}_a, \dot{\mathbf{p}}_a$ in $\dot{h}_{(4)ij}^{\text{TT}}$ by means of Newtonian equations of motion.

The 4PN-order density we split into two parts,

$$h_{(12)}^{\text{near-zone}}(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) = h_{(12)}^1(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) + h_{(12)}^2(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a). \tag{7.5}$$

The density $h_{(12)}^1$ depends on the following functions: $\phi_{(2)}$, $S_{(4)1}$ and $S_{(4)2}$ [they determine, via Eq. (3.11), the function $\phi_{(4)}$], $V_{(3)}^i$ and $V_{(5)}^i$ [they define, through Eq. (2.9), the functions $\tilde{\pi}_{(3)}^{ij}$ and $\tilde{\pi}_{(5)}^{ij}$], $h_{(4)ij}^{\text{TT}}$, and $\phi_{(6)1}$, $\phi_{(6)2}$. The function $\phi_{(6)1}$ is the solution of Eq. (3.14a); the equation fulfilled by the function $\phi_{(6)2}$ one obtains from Eq. (3.14b) for the function $\bar{\phi}_{(6)2}$ after replacing in the latter h_{ij}^{TT} by the leading-order $h_{(4)ij}^{\text{TT}}$,

$$\Delta_d \phi_{(6)2} = \frac{d-2}{d-1} \phi_{(2),ij} h_{(4)ij}^{\text{TT}}. \quad (7.7)$$

Formulas needed to express in an explicit way all these functions in terms of \mathbf{x}_a , \mathbf{p}_a , and \mathbf{x} are given in Appendix C. The last two terms in Eq. (7.6) are related with the last two terms in Eq. (7.4); $[[\dot{h}_{(4)ij}^{\text{TT}}]]_1$ means here elimination of the time derivatives $\dot{\mathbf{x}}_a$, $\dot{\mathbf{p}}_a$ by means of 1PN equations of motion. In the rest of this section and in Sec. VIII all time derivatives of \mathbf{x}_a , \mathbf{p}_a are eliminated by means of Newtonian equations of motion; therefore from now on we drop the notation $[[\cdot]]_0$.

The second part $h_{(12)}^2$ of the 4PN density $h_{(12)}^{\text{near-zone}}$ is the 4PN-level contribution coming from the density (6.18); it equals

$$\begin{aligned} h_{(12)}^2(\mathbf{x}; \mathbf{x}_a, \mathbf{p}_a) = & -\frac{d-2}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{ij} \dot{h}_{(6)ij}^{\text{TT}} \\ & + \frac{1}{2(d-1)} \phi_{(2)} (h_{(4)ij}^{\text{TT}} \Delta_d h_{(6)ij}^{\text{TT}} \\ & + h_{(6)ij}^{\text{TT}} \Delta_d h_{(4)ij}^{\text{TT}}) \\ & + \left(B_{(6)ij} + \frac{1}{4} \dot{h}_{(4)ij}^{\text{TT}} - \frac{1}{4} S_{(6)ij}^{\text{TT}} \right) h_{(6)ij}^{\text{TT}}. \end{aligned} \quad (7.8)$$

The function $h_{(6)ij}^{\text{TT}}$, according to Eq. (4.12), is the sum of the two terms,

$$h_{(6)ij}^{\text{TT}} = C_{(6)ij}^{\text{TT}} + \Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}}, \quad (7.9)$$

where

$$C_{(6)ij}^{\text{TT}} = \Delta_d^{-1} S_{(6)ij}^{\text{TT}}. \quad (7.10)$$

After plugging Eq. (7.9) into Eq. (7.8) and shifting some time derivatives,¹ the density $h_{(12)}^2$ can be rewritten as

$$h_{(12)}^2 = h_{(12)}^{2,1} + h_{(12)}^{2,2} + h_{(12)}^{2,3}, \quad (7.11)$$

¹Shifting time derivatives means replacing $\dot{A}B$ by $-A\dot{B}$. This implies adding a total time derivative to the Hamiltonian density and on the level of Hamiltonian is equivalent to performing a canonical transformation.

where²

$$\begin{aligned} h_{(12)}^{2,1} = & \frac{1}{2(d-1)} \phi_{(2)} h_{(4)ij}^{\text{TT}} S_{(6)ij}^{\text{TT}} \\ & + \left(B_{(6)ij} + \frac{1}{2(d-1)} \phi_{(2)} (\Delta_d h_{(4)ij}^{\text{TT}}) - \frac{1}{4} S_{(6)ij}^{\text{TT}} \right) C_{(6)ij}^{\text{TT}} \\ & - \frac{d-2}{d-1} \phi_{(2)} \tilde{\pi}_{(3)}^{ij} \dot{C}_{(6)ij}^{\text{TT}}, \end{aligned} \quad (7.12a)$$

$$\begin{aligned} h_{(12)}^{2,2} = & \frac{1}{2(d-1)} \phi_{(2)} h_{(4)ij}^{\text{TT}} \dot{h}_{(4)ij}^{\text{TT}} - \frac{1}{4} \dot{h}_{(4)ij}^{\text{TT}} \Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}} \\ & + \left(\frac{1}{2(d-1)} \phi_{(2)} (\Delta_d h_{(4)ij}^{\text{TT}}) + \frac{d-2}{d-1} \partial_t (\phi_{(2)} \tilde{\pi}_{(3)}^{ij}) \right) \\ & \times \Delta_d^{-1} \dot{h}_{(4)ij}^{\text{TT}} + B_{(6)ij} \Delta_d^{-1} \dot{h}_{(4)ij}^{\text{TT}}, \end{aligned} \quad (7.12b)$$

$$h_{(12)}^{2,3} = \frac{1}{4} (C_{(6)ij}^{\text{TT}} \ddot{h}_{(4)ij}^{\text{TT}} - S_{(6)ij}^{\text{TT}} (\Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}})). \quad (7.12c)$$

VIII. COMPUTATION OF THE 4PN-ACCURATE NEAR-ZONE CONSERVATIVE MATTER HAMILTONIAN

The bulk of computations we did to derive the explicit form of the 4PN-accurate near-zone conservative matter Hamiltonian, i.e. to perform integration in Eq. (7.2), was performed in $d = 3$ dimensions, where our working horse was Riesz-implemented Hadamard regularization supplemented by a Hadamard “partie finie” concept of a function at its singular point. The results of global (i.e., extending to the whole \mathbb{R}^3 space) three-dimensional integrations were then corrected in two different ways: (i) the UV divergences were locally (i.e., within small balls surrounding particles’ positions) recomputed by using dimensional regularization; (ii) the IR divergences were also “locally” (i.e., outside a large ball enclosing particles, or in a neighborhood of $r = \infty$) recomputed by introducing an additional regularization factor $(r/s)^B$ (with a new IR length scale s), which modifies the behavior of the part of the field function $h_{(6)ij}^{\text{TT}}$ which diverges at $r \rightarrow \infty$ [see Eq. (A40)]. The details of regularization procedure are explained in Appendix A.

Dimensional regularization introduces a natural length scale ℓ_0 , which relates gravitational constants in d and 3 dimensions [see Eq. (A26)]. As explained below correction of the UV divergences by means of dimensional regularization produces some poles proportional to $1/(d-3)$ together with ℓ_0 -dependent logarithms $\ln(r_{12}/\ell_0)$. All

²The density $h_{(12)}^{2,2}$, after ignoring its last term, is identical to the density $r_{4\text{PN}}^2$ introduced in Eq. (3.4) of Ref. [51] (the reason of omission of this term in [51] is explained at the end of Sec. VIII C 2).

these poles and logarithms were removed by adding to the Hamiltonian a total time derivative. The terms which contribute to IR divergences depend, after regularization, on the IR length scale s . Moreover, as we also explain below, different ways of regularizing IR divergences lead to different results, so the final result of IR regularization is ambiguous. We were able to express this ambiguity in terms of only one dimensionless constant, which we denote by C .

The UV divergences is not the only problem caused by usage of distributional sources. General relativity uses standard algebraic and differential calculus of ordinary functions, which includes e.g. the Leibniz rule. By introducing distributional sources we violate this framework: in some places we have to use distributional differentiation which does not fulfill the Leibniz rule [see Appendix A 5]. The field equations (3.10), (3.12), and (3.14)–(3.16) tell us that for the second (space and time) derivatives of $\phi_{(n)}$ and for the first (space and time) derivatives of $\tilde{\pi}_{(n)}^{ij}$ the distributional derivative has to be applied. The same also holds for $h_{(n)ij}^{\text{TT}}$ and $\pi_{(n)\text{TT}}^{ij}$, respectively, taking into account the Eqs. (4.11)–(4.12) and (4.8). The differentiation of any product of these functions goes with the Leibniz rule.

The 4PN-accurate near-zone Hamiltonian is the sum of Hamiltonians at different PN orders,

$$\begin{aligned} H_{\leq 4\text{PN}}^{\text{near-zone}(s)}(\mathbf{x}_a, \mathbf{p}_a; C) &= \sum_a m_a + H_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{4\text{PN}}^{\text{near-zone}(s)}(\mathbf{x}_a, \mathbf{p}_a; C), \end{aligned} \quad (8.1)$$

where we have introduced notation which indicates that the 4PN near-zone Hamiltonian depends on the IR regularization scale s and on one dimensionless constant C parametrizing ambiguity in the regularization of IR divergences. Hamiltonians H_{N} through $H_{3\text{PN}}$ were uniquely recomputed by us using three-dimensional Riesz-implemented Hadamard regularization supplemented by dimensional regularization. These Hamiltonians do not develop IR divergences.

A. Computation of the 3PN-accurate Hamiltonian

The three-dimensional 3PN-accurate Hamiltonian,

$$\begin{aligned} H_{\leq 3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) &= \sum_a m_a + H_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a), \end{aligned} \quad (8.2)$$

was computed for the first time in the series of papers [31–35]. We have recomputed it here and got the result

identical with the previously obtained. The explicit formulas for the three-dimensional Hamiltonians H_{N} through $H_{3\text{PN}}$ written in general reference frame are given at the end of the current section.

In the computation of the 3PN/4PN Hamiltonians we eliminate time derivatives of \mathbf{x}_a and \mathbf{p}_a [present in Eqs. (7.4) and (7.6)] by means of lower-order equations of motion: we use Newtonian [in both (7.4) and (7.6)] or 1PN [in (7.6)] equations of motion. To perform dimensional regularization of UV divergences we have to use the d -dimensional version of these equations, which follows from d -dimensional Hamiltonians. We have explicitly computed the Newtonian and 1PN Hamiltonians in d dimensions. They read [notation “ $+(1 \leftrightarrow 2)$ ” used here means adding to each term another term obtained by the exchange of the bodies’ labels]

$$H_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{\kappa(d-2)m_1m_2}{4(d-1)r_{12}^{d-2}} + (1 \leftrightarrow 2), \quad (8.3a)$$

$$\begin{aligned} H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) &= -\frac{(\mathbf{p}_1^2)^2}{8m_1^3} + \frac{\kappa}{4(d-1)} \left(\frac{1}{2}(3d-2)(\mathbf{p}_1 \cdot \mathbf{p}_2) \right. \\ &\quad \left. - d\frac{m_2}{m_1}\mathbf{p}_1^2 + \frac{1}{2}(d-2)^2(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \right) \\ &\quad \times \frac{1}{r_{12}^{d-2}} + \frac{\kappa^2(d-2)^2m_1^2m_2}{8(d-1)^2r_{12}^{2d-4}} + (1 \leftrightarrow 2), \end{aligned} \quad (8.3b)$$

where we have introduced

$$\kappa := \frac{\Gamma(d/2 - 1)}{4\pi^{d/2}}. \quad (8.4)$$

Let us note that $\kappa = 1/(4\pi)$ in $d = 3$ dimensions.

B. Integral of $h_{(12)}^1$

The part $h_{(12)}^1$ of the 4PN density given in Eq. (7.6) is UV divergent but it does not develop IR poles with the exception of the term $\propto \phi_{(2)}(\dot{h}_{(4)ij}^{\text{TT}})^2$. This term contains however the multiplication factor $d-3$ which causes the IR divergence of $\phi_{(2)}(\dot{h}_{(4)ij}^{\text{TT}})^2$ to not contribute to the Hamiltonian (but the UV divergence of this term produces some nonzero contribution). After employing the regularization procedures described in Appendix A we have obtained

$$\begin{aligned} H_{4\text{PN}}^{\text{reg } 1}(\mathbf{x}_a, \mathbf{p}_a; d) &= \int d^d x h_{(12)}^1 \\ &= \chi_{(12)}^{1,1}(\mathbf{x}_a, \mathbf{p}_a) + \chi_{(12)}^{1,2}(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{\ell_0} \\ &\quad + \frac{\chi_{(12)}^{1,3}(\mathbf{x}_a, \mathbf{p}_a)}{d-3} + \mathcal{O}(d-3), \end{aligned} \quad (8.5)$$

where for convenience we have introduced a new dimensional-regularization (DR) scale $\bar{\ell}_0$ related with the original scale ℓ_0 defined in Eq. (A26) by the relation

$$\bar{\ell}_0 = \frac{\ell_0}{2\sqrt{\pi}} e^{-\gamma_E/2}. \quad (8.6)$$

Here γ_E denotes the Euler-Mascheroni constant.

C. Integral of $h_{(12)}^2$

To compute the integral of $h_{(12)}^2$ we have to study asymptotics of the function $h_{(6)ij}^{\text{TT}}$ for $r \rightarrow \infty$. The function $h_{(6)ij}^{\text{TT}}$ is the sum of two parts, $C_{(6)ij}^{\text{TT}}$ and $\Delta_d^{-1} \tilde{h}_{(4)ij}^{\text{TT}}$ [see Eq. (7.9)]. By virtue of Eq. (4.13) the function $C_{(6)ij}^{\text{TT}}$ [defined in Eq. (7.10)] can be written more explicitly as

$$C_{(6)ij}^{\text{TT}} = \left(\frac{1}{d-1} \phi_{(2)} h_{(4)ij}^{\text{TT}} + \frac{2(d-2)}{d-1} \partial_t \Delta_d^{-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij}) + 2\Delta_d^{-1} B_{(6)ij} + \frac{1}{d-1} \Delta_d^{-1} (\phi_{(2)} \Delta_d h_{(4)ij}^{\text{TT}}) \right)^{\text{TT}}. \quad (8.7)$$

$$\begin{aligned} h_{(12)}^{2,1} = & -\frac{d-2}{(d-1)^2} \partial_t (\phi_{(2)} h_{(4)ij}^{\text{TT}}) (\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} + \frac{1}{4(d-1)^2} \left(2\phi_{(2),i} \phi_{(2),j} h_{(4)ik}^{\text{TT}} h_{(4)jk}^{\text{TT}} - (\phi_{(2),k})^2 (h_{(4)ij}^{\text{TT}})^2 \right. \\ & + \left. \phi_{(2)}^2 h_{(4)ij}^{\text{TT}} \Delta_d h_{(4)ij}^{\text{TT}} + \frac{d-2}{d-1} \phi_{(2),ij} h_{(4)ij}^{\text{TT}} \Delta_d^{-1} (\phi_{(2),kl} h_{(4)kl}^{\text{TT}}) \right) - \frac{(d-2)^2}{(d-1)^2} \phi_{(2)} \tilde{\pi}_{(3)}^{ij} \partial_t^2 (\Delta_d^{-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij}))^{\text{TT}} \\ & + \frac{2(d-2)}{d-1} \left(B_{(6)ij} + \frac{1}{2(d-1)} \phi_{(2)} \Delta_d h_{(4)ij}^{\text{TT}} \right) \partial_t (\Delta_d^{-1} (\phi_{(2)} \tilde{\pi}_{(3)}^{ij}))^{\text{TT}} + h_{(12)}^{2,1,1}, \end{aligned} \quad (8.8)$$

where the last term equals

$$\begin{aligned} h_{(12)}^{2,1,1} = & \frac{1}{d-1} \phi_{(2)} h_{(4)ij}^{\text{TT}} \left(B_{(6)ij} + \frac{1}{2(d-1)} \phi_{(2)} \Delta_d h_{(4)ij}^{\text{TT}} \right)^{\text{TT}} \\ & + \left(B_{(6)ij} + \frac{1}{2(d-1)} \phi_{(2)} \Delta_d h_{(4)ij}^{\text{TT}} \right) \\ & \times \left(\Delta_d^{-1} \left(B_{(6)ij} + \frac{1}{2(d-1)} \phi_{(2)} \Delta_d h_{(4)ij}^{\text{TT}} \right) \right)^{\text{TT}}. \end{aligned} \quad (8.9)$$

It is crucial to single out the last term in Eq. (8.8) and to put it exactly in the form of Eq. (8.9). Due to this it is possible to compute fully explicitly (in $d = 3$ dimensions) all inverse Laplacians involved in Eq. (8.9). Formulas needed to calculate in $d = 3$ all inverse Laplacians and perform TT projections involved in Eqs. (8.8) and (8.9) are given in Appendix C.

After performing regularization by means of the procedures described in Appendix A we have

Analysis of this equation shows that $C_{(6)ij}^{\text{TT}} = \mathcal{O}(r^{2-d})$ for $r \rightarrow \infty$. It is also not difficult to check that the inverse Laplacian $\Delta_d^{-1} \tilde{h}_{(4)ij}^{\text{TT}} = \mathcal{O}(r^{4-d})$ when $r \rightarrow \infty$ (i.e., it diverges linearly in $d = 3$ dimensions). This behavior of $C_{(6)ij}^{\text{TT}}$ and $\Delta_d^{-1} \tilde{h}_{(4)ij}^{\text{TT}}$ for $r \rightarrow \infty$ implies that the term $h_{(12)}^{2,1}$ is convergent at infinity, whereas the terms $h_{(12)}^{2,2}$ and $h_{(12)}^{2,3}$ develop IR divergences.

1. Integral of $h_{(12)}^{2,1}$

The integral of $h_{(12)}^{2,1}$ is convergent at spatial infinity but it is difficult to express its integrand as an explicit function of \mathbf{x} (and of $\mathbf{x}_a, \mathbf{p}_a$). To do this we further transform the density $h_{(12)}^{2,1}$, after employing Eq. (8.7), by making some additional integrations by parts in space and shifting time derivatives. We also replace $\Delta_d (\phi_{(2)} h_{(4)ij}^{\text{TT}})^{\text{TT}}$ [which comes from the first term of Eq. (7.12a)] by more explicit expression using the definition (2.14) of the TT projection operator. We finally obtain

$$\begin{aligned} H_{4\text{PN}}^{\text{reg},2,1}(\mathbf{x}_a, \mathbf{p}_a; d) = & \int d^d x h_{(12)}^{2,1} \\ = & \chi_{(12)}^{2,1,1}(\mathbf{x}_a, \mathbf{p}_a) + \chi_{(12)}^{2,1,2}(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{\bar{\ell}_0} \\ & + \frac{\chi_{(12)}^{2,1,3}(\mathbf{x}_a, \mathbf{p}_a)}{d-3} + \mathcal{O}(d-3). \end{aligned} \quad (8.10)$$

2. Integral of $h_{(12)}^{2,2}$

The integral of the density $h_{(12)}^{2,2}$ is not convergent at spatial infinity and it also develops UV poles. After making use of the procedures described in Appendix A one gets

$$\begin{aligned} \int d^d x h_{(12)}^{2,2} = & \tilde{\chi}_{(12)\text{amb}}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a; C) \\ & + \chi_{(12)}^{2,2,2}(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{\bar{\ell}_0} + \frac{\chi_{(12)}^{2,2,3}(\mathbf{x}_a, \mathbf{p}_a)}{d-3} \\ & + \chi_{(12)}^{2,2,4}(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{s} + \mathcal{O}(d-3). \end{aligned} \quad (8.11)$$

Because of the ambiguity of the results of IR regularization discussed in the Appendix A 3, the term $\bar{\chi}_{(12)\text{amb}}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a; C)$ depends on some undetermined dimensionless constant C which parametrizes this ambiguity. We have found that

$$q(\mathbf{x}_a, \mathbf{p}_a) = \frac{21(m_1 + m_2)m_1m_2^2}{20(16\pi)^4 r_{12}^3} (\mathbf{n}_{12} \cdot \mathbf{p}_1) + \frac{1}{(16\pi)^3 r_{12}^2} \left[\left(\frac{97m_2}{20m_1} - \frac{21}{16} \right) m_2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 \right. \\ \left. + \left(\frac{21m_1}{16} - \frac{687m_2}{40} \right) (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2) + \left(\frac{77}{48} - \frac{237m_2}{40m_1} \right) m_2 (\mathbf{n}_{12} \cdot \mathbf{p}_1) \mathbf{p}_1^2 \right. \\ \left. + \left(\frac{7m_1}{48} + \frac{307m_2}{40} \right) (\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_1^2 + \left(\frac{1597m_2}{120} - \frac{7m_1}{4} \right) (\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{p}_1 \cdot \mathbf{p}_2) \right] + (1 \leftrightarrow 2), \quad (8.12)$$

such that

$$\chi_{(12)}^{2,2,4}(\mathbf{x}_a, \mathbf{p}_a) + \frac{d}{dt} q(\mathbf{x}_a, \mathbf{p}_a) = F(\mathbf{x}_a, \mathbf{p}_a), \quad (8.13)$$

where

$$F(\mathbf{x}_a, \mathbf{p}_a) := \frac{2M}{5(16\pi)^2} (\ddot{I}_{ij})^2. \quad (8.14)$$

Here I_{ij} is the Newtonian quadrupole moment of the binary system,

$$I_{ij} := \sum_a m_a \left(x_a^i x_a^j - \frac{1}{3} \delta^{ij} \mathbf{x}_a^2 \right). \quad (8.15)$$

All time derivatives on the right-hand side of the formula (8.14) were eliminated by means of Newtonian equations of motion. By virtue of Eq. (8.13) the result of Eq. (8.11) we can rewrite in the following form:

$$\int d^d x h_{(12)}^{2,2} + \frac{d}{dt} \left(q(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{s} \right) \\ = \chi_{(12)\text{amb}}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a; C) + \chi_{(12)}^{2,2,2}(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{\ell_0} \\ + \frac{\chi_{(12)}^{2,2,3}(\mathbf{x}_a, \mathbf{p}_a)}{d-3} + F(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{s} + \mathcal{O}(d-3), \quad (8.16)$$

where

$$\chi_{(12)\text{amb}}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a; C) := \bar{\chi}_{(12)\text{amb}}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a; C) \\ + q(\mathbf{x}_a, \mathbf{p}_a) \frac{d}{dt} \left(\ln \frac{r_{12}}{s} \right). \quad (8.17)$$

We have found (see Appendix A 3) that the ambiguity of the term $\chi_{(12)\text{amb}}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a; C)$ can be expressed, up to adding a total time derivative, as a *multiple of the term F* introduced in Eq. (8.14),

$$\chi_{(12)\text{amb}}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a; C) = \chi_{(12)}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a) + CF(\mathbf{x}_a, \mathbf{p}_a) \\ + (\text{total time derivative}). \quad (8.18)$$

the coefficient $\chi_{(12)}^{2,2,4}$ of the logarithm $\ln(r_{12}/s)$ can be written, after adding a total time derivative, in a very specific form. Thus we have shown that there exists a *unique* term $q(\mathbf{x}_a, \mathbf{p}_a)$,

Therefore as the contribution of the density $h_{(12)}^{2,2}$ to the 4PN Hamiltonian we take

$$H_{4\text{PN}}^{\text{reg } 2,2}(\mathbf{x}_a, \mathbf{p}_a; d; s, C) \\ = \chi_{(12)}^{2,2,1}(\mathbf{x}_a, \mathbf{p}_a) + \chi_{(12)}^{2,2,2}(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{\ell_0} + \frac{\chi_{(12)}^{2,2,3}(\mathbf{x}_a, \mathbf{p}_a)}{d-3} \\ + F(\mathbf{x}_a, \mathbf{p}_a) \left(\ln \frac{r_{12}}{s} + C \right) + \mathcal{O}(d-3). \quad (8.19)$$

Let us finally mention that the integral of the last term of the density $h_{(12)}^{2,2}$, $B_{(6)ij} \Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}}$, though IR divergent, can be in fact regularized uniquely. This is so because the difference between results of different methods of regularizing this term is a total time derivative.

3. Integral of $h_{(12)}^{2,3}$

The density $h_{(12)}^{2,3}$ is an exact divergence, it can be written in the form

$$\frac{1}{4} (C_{(6)ij}^{\text{TT}} \ddot{h}_{(4)ij}^{\text{TT}} - S_{(6)ij}^{\text{TT}} (\Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}})) = \frac{1}{4} \partial_k E_k, \quad (8.20)$$

where

$$E_k := C_{(6)ij}^{\text{TT}} (\Delta_d^{-1} \ddot{h}_{(4)ij,k}^{\text{TT}}) - C_{(6)ij,k}^{\text{TT}} (\Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}}). \quad (8.21)$$

The surface integral associated with E_k is not convergent at spatial infinity in $d = 3$ dimensions. However in d dimensions $E_k \sim r^{5-2d}$ for $r \rightarrow \infty$, so the surface integral behaves like r^{4-d} and it vanishes in the limit $r \rightarrow \infty$ when d is large enough. Therefore this term does not contribute to the Hamiltonian.

D. Removing UV poles

As the result of UV and IR regularizations described in the previous subsections, we have obtained the 4PN near-zone Hamiltonian of the following form:

$$H_{4\text{PN}}^{\text{reg}}(\mathbf{x}_a, \mathbf{p}_a; d; s, C) = H_{4\text{PN}}^{\text{reg}1}(\mathbf{x}_a, \mathbf{p}_a; d) + H_{4\text{PN}}^{\text{reg}2,1}(\mathbf{x}_a, \mathbf{p}_a; d) + H_{4\text{PN}}^{\text{reg}2,2}(\mathbf{x}_a, \mathbf{p}_a; d; s, C) \\ = \bar{\chi}_1(\mathbf{x}_a, \mathbf{p}_a) + \bar{\chi}_2(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{\bar{\ell}_0} + \frac{\bar{\chi}_3(\mathbf{x}_a, \mathbf{p}_a)}{d-3} + F(\mathbf{x}_a, \mathbf{p}_a) \left(\ln \frac{r_{12}}{s} + C \right) + \mathcal{O}(d-3), \quad (8.22)$$

where

$$\bar{\chi}_k(\mathbf{x}_a, \mathbf{p}_a) := \chi_{(12)}^{1,k}(\mathbf{x}_a, \mathbf{p}_a) + \chi_{(12)}^{2,1,k}(\mathbf{x}_a, \mathbf{p}_a) + \chi_{(12)}^{2,2,k}(\mathbf{x}_a, \mathbf{p}_a), \quad k = 1, 2, 3. \quad (8.23)$$

In the next step we remove both the DR-related scale $\bar{\ell}_0$ and the pole terms proportional to $1/(d-3)$ by adding a total time derivative. To do this we have found a *unique* function $D(\mathbf{x}_a, \mathbf{p}_a; d)$,

$$D(\mathbf{x}_a, \mathbf{p}_a; d) = \frac{m_1}{(16\pi)^3 r_{12}^2} \left(\left(\frac{39}{80} \{[(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 - \mathbf{p}_1^2](\mathbf{n}_{12} \cdot \mathbf{p}_2) + 2(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{p}_1 \cdot \mathbf{p}_2)\} + \frac{2}{3}(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2 \right) \frac{1}{d-3} \right. \\ \left. + \left(\frac{117}{80} \{[\mathbf{p}_1^2 - (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2](\mathbf{n}_{12} \cdot \mathbf{p}_2) - 2(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{p}_1 \cdot \mathbf{p}_2)\} - 2(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2 \right) \ln \frac{r_{12}}{\bar{\ell}_0} \right) + (1 \leftrightarrow 2), \quad (8.24)$$

such that the sum

$$H_{4\text{PN}}^{\text{reg}}(\mathbf{x}_a, \mathbf{p}_a; d; s, C) + \frac{d}{dt} D(\mathbf{x}_a, \mathbf{p}_a; d)$$

has the finite limit as $d \rightarrow 3$. This limit we take as the regularized value of the 4PN near-zone Hamiltonian:

$$H_{4\text{PN}}^{\text{near-zone}(s)}(\mathbf{x}_a, \mathbf{p}_a; C) := \lim_{d \rightarrow 3} \left(H_{4\text{PN}}^{\text{reg}}(\mathbf{x}_a, \mathbf{p}_a; d; s, C) + \frac{d}{dt} D(\mathbf{x}_a, \mathbf{p}_a; d) \right). \quad (8.25)$$

This Hamiltonian depends on the IR regularization scale s and on the dimensionless constant C and can be rewritten as

$$H_{4\text{PN}}^{\text{near-zone}(s)}(\mathbf{x}_a, \mathbf{p}_a; C) = H_{4\text{PN}}^{\text{local}0}(\mathbf{x}_a, \mathbf{p}_a) + F(\mathbf{x}_a, \mathbf{p}_a) \left(\ln \frac{r_{12}}{s} + C \right), \quad (8.26)$$

where the uniquely computed part $H_{4\text{PN}}^{\text{local}0}$ of the near-zone Hamiltonian reads

$$H_{4\text{PN}}^{\text{local}0}(\mathbf{x}_a, \mathbf{p}_a) := \bar{\chi}_1(\mathbf{x}_a, \mathbf{p}_a) + \lim_{d \rightarrow 3} \left(\bar{\chi}_2(\mathbf{x}_a, \mathbf{p}_a) \ln \frac{r_{12}}{\bar{\ell}_0} + \frac{\bar{\chi}_3(\mathbf{x}_a, \mathbf{p}_a)}{d-3} + \frac{d}{dt} D(\mathbf{x}_a, \mathbf{p}_a; d) \right). \quad (8.27)$$

Let us stress again that $H_{4\text{PN}}^{\text{local}0}$ is without DR-related scale $\bar{\ell}_0$ and poles in $1/(d-3)$.

E. Total 4PN-accurate conservative matter Hamiltonian

Reference [51] showed that the total 4PN conservative matter Hamiltonian is the sum of the local-in-time near-zone Hamiltonian (8.26) and the time-symmetric but non-local-in-time *tail Hamiltonian* $H_{4\text{PN}}^{\text{tail sym}(s)}$,

$$H_{4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a] = H_{4\text{PN}}^{\text{near-zone}(s)}(\mathbf{x}_a, \mathbf{p}_a; C) \\ + H_{4\text{PN}}^{\text{tail sym}(s)}[\mathbf{x}_a, \mathbf{p}_a], \quad (8.28)$$

where we have used brackets $[\cdot, \cdot]$ to emphasize that the tail Hamiltonian $H_{4\text{PN}}^{\text{tail sym}(s)}$ is a *functional* of phase-space trajectories $\mathbf{x}_a(t)$, $\mathbf{p}_a(t)$. Reference [51] also computed the value of the constant C ,

$$C = -\frac{1681}{1536}, \quad (8.29)$$

and showed that the total 4PN Hamiltonian (8.28) does not depend on the scale s and can be written as

$$H_{4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a] = H_{4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) + H_{4\text{PN}}^{\text{nonlocal}}[\mathbf{x}_a, \mathbf{p}_a], \quad (8.30)$$

where the local piece of the 4PN Hamiltonian reads

$$H_{4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) = H_{4\text{PN}}^{\text{local}0}(\mathbf{x}_a, \mathbf{p}_a) + CF(\mathbf{x}_a, \mathbf{p}_a), \quad (8.31)$$

and the nonlocal-in-time piece can be written as (from now on we restore the constants c and G)

$$H_{4\text{PN}}^{\text{nonlocal}}[\mathbf{x}_a, \mathbf{p}_a] = -\frac{1}{5} \frac{G^2 M}{c^8} \overset{\dots}{I}_{ij} \\ \times \text{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \overset{\dots}{I}_{ij}(t+v) \frac{dv}{|v|}. \quad (8.32)$$

Here $\overset{\dots}{I}_{ij}$ denotes a third time derivative of the Newtonian quadrupole moment of the binary [defined in Eq. (8.15)] and $\text{Pf}_{2r_{12}/c}$ is a Hadamard partie finie with time scale $2r_{12}/c$ [see Eq. (4.2) in [51] for the definition of the Pf operation].

The total 4PN-accurate conservative matter Hamiltonian is the sum

$$\begin{aligned}
 H_{\leq 4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a] &= \sum_a m_a c^2 + H_N(\mathbf{x}_a, \mathbf{p}_a) + H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\
 &\quad + H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\
 &\quad + H_{4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a]. \tag{8.33}
 \end{aligned}$$

The explicit formulas for the local-in-time Hamiltonians from the Newtonian up to the 4PN level [the local piece (8.31) of the 4PN Hamiltonian given below incorporates the value (8.29) of the constant C], valid in the generic, i.e., noncenter-of-mass, reference frame, are as follows:

$$H_N(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2} \frac{Gm_1 m_2}{r_{12}} + (1 \leftrightarrow 2), \tag{8.34a}$$

$$\begin{aligned}
 c^2 H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) &= -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8} \frac{Gm_1 m_2}{r_{12}} \left(-12 \frac{\mathbf{p}_1^2}{m_1^2} + 14 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + 2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) + \frac{1}{4} \frac{Gm_1 m_2 G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2), \\
 \tag{8.34b}
 \end{aligned}$$

$$\begin{aligned}
 c^4 H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{1}{16} \frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8} \frac{Gm_1 m_2}{r_{12}} \left(5 \frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2} \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 5 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} - 6 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} \right. \\
 &\quad \left. - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{2 m_1^2 m_2^2} \right) + \frac{1}{4} \frac{G^2 m_1 m_2}{r_{12}^2} \left(m_2 \left(10 \frac{\mathbf{p}_1^2}{m_1^2} + 19 \frac{\mathbf{p}_2^2}{m_2^2} \right) \right. \\
 &\quad \left. - \frac{1}{2} (m_1 + m_2) \frac{27(\mathbf{p}_1 \cdot \mathbf{p}_2) + 6(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) - \frac{1}{8} \frac{Gm_1 m_2 G^2 (m_1^2 + 5m_1 m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2), \tag{8.34c}
 \end{aligned}$$

$$\begin{aligned}
 c^6 H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) &= -\frac{5}{128} \frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32} \frac{Gm_1 m_2}{r_{12}} \left(-14 \frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4 \frac{((\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 4\mathbf{p}_1^2 \mathbf{p}_2^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 6 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^4 m_2^2} \right. \\
 &\quad \left. - 10 \frac{(\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 + \mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2) \mathbf{p}_1^2}{m_1^4 m_2^2} + 24 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^4 m_2^2} + 2 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} \right. \\
 &\quad \left. + \frac{(7\mathbf{p}_1^2 \mathbf{p}_2^2 - 10(\mathbf{p}_1 \cdot \mathbf{p}_2)^2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2^3} + \frac{(\mathbf{p}_1^2 \mathbf{p}_2^2 - 2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2^3} + 15 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^3 m_2^3} \right. \\
 &\quad \left. - 18 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} + 5 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{m_1^3 m_2^3} \right) + \frac{G^2 m_1 m_2}{r_{12}^2} \left(\frac{1}{16} (m_1 - 27m_2) \frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} \right. \\
 &\quad \left. + \frac{1}{48} m_2 \frac{25(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + 371 \mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17 \mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{16 m_1^3} + \frac{5 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{12 m_1^3} \right. \\
 &\quad \left. - \frac{1}{8} m_1 \frac{(15 \mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2) + 11(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_1))}{m_1^3 m_2} - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} \right. \\
 &\quad \left. + \frac{125}{12} m_2 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} - \frac{1}{48} (220m_1 + 193m_2) \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right) \\
 &\quad + \frac{G^3 m_1 m_2}{r_{12}^3} \left(-\frac{1}{48} \left(425m_1^2 + \left(473 - \frac{3}{4} \pi^2 \right) m_1 m_2 + 150m_2^2 \right) \frac{\mathbf{p}_1^2}{m_1^2} + \frac{1}{16} \left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4} \pi^2 \right) m_1 m_2 \right) \right. \\
 &\quad \times \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{1}{16} \left(20m_1^2 - \left(43 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} + \frac{1}{16} \left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4} \pi^2 \right) m_1 m_2 \right) \\
 &\quad \times \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \left. \right) + \frac{1}{8} \frac{G^4 m_1 m_2^3}{r_{12}^4} \left(\left(\frac{227}{3} - \frac{21}{4} \pi^2 \right) m_1 + m_2 \right) + (1 \leftrightarrow 2), \tag{8.34d}
 \end{aligned}$$

$$\begin{aligned}
 c^8 H_{4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) &= \frac{7(\mathbf{p}_1^2)^5}{256m_1^9} + \frac{Gm_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^3 m_1 m_2}{r_{12}^3} (m_1^2 H_{441}(\mathbf{x}_a, \mathbf{p}_a) + m_1 m_2 H_{442}(\mathbf{x}_a, \mathbf{p}_a)) \\
 &\quad + \frac{G^4 m_1 m_2}{r_{12}^4} (m_1^3 H_{421}(\mathbf{x}_a, \mathbf{p}_a) + m_1^2 m_2 H_{422}(\mathbf{x}_a, \mathbf{p}_a)) + \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a, \mathbf{p}_a) + (1 \leftrightarrow 2), \tag{8.34e}
 \end{aligned}$$

$$\begin{aligned}
H_{48}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{16m_1^6m_2^2} \\
& - \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{256m_1^5m_2^3} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^5m_2^3} \\
& + \frac{33(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^5m_2^3} - \frac{85(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^5m_2^3} - \frac{45(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^5m_2^3} \\
& - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^5m_2^3} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^5m_2^3} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64m_1^5m_2^3} \\
& - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{55(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^3} \\
& - \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{23(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\
& - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^4m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^4} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^4m_2^4} \\
& - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^4} + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^4} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^4} \\
& + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^4} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^4m_2^4} - \frac{\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^4} \\
& - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^4}, \tag{8.34f}
\end{aligned}$$

$$\begin{aligned}
H_{46}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{369(\mathbf{n}_{12} \cdot \mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12} \cdot \mathbf{p}_1)^5(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{128m_1^5m_2} \\
& + \frac{67(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5m_2} - \frac{167(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^5m_2} - \frac{851(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{128m_1^5m_2} \\
& + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{256m_1^5m_2} + \frac{3263(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4m_2^2} \\
& - \frac{3571(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{320m_1^4m_2^2} + \frac{3073(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{480m_1^4m_2^2} + \frac{4349(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{1280m_1^4m_2^2} \\
& - \frac{3461\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4m_2^2} - \frac{13(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{8m_1^3m_2^3} \\
& + \frac{191(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^3m_2^3} - \frac{19(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^3m_2^3} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^3m_2^3} \\
& + \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3} - \frac{47(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3} \\
& + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2m_1^2m_2^4} \\
& + \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4} - \frac{133(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4} \\
& - \frac{77(\mathbf{p}_1 \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4} + \frac{197(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6}, \tag{8.34g}
\end{aligned}$$

$$\begin{aligned}
 H_{441}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{5027(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} - \frac{3191(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{640m_1^3 m_2} + \frac{28561(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_1^2}{1920m_1^3 m_2} \\
 & + \frac{8777(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{384m_1^3 m_2} + \frac{752969 \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{28800m_1^3 m_2} - \frac{16481(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{960m_1^2 m_2^2} + \frac{94433(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2}{4800m_1^2 m_2^2} \\
 & - \frac{103957(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{2400m_1^2 m_2^2} + \frac{791(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{400m_1^2 m_2^2} + \frac{26627(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 \mathbf{p}_2^2}{1600m_1^2 m_2^2} - \frac{118261 \mathbf{p}_1^2 \mathbf{p}_2^2}{4800m_1^2 m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^4}, \quad (8.34h)
 \end{aligned}$$

$$\begin{aligned}
 H_{442}(\mathbf{x}_a, \mathbf{p}_a) = & \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200} \right) \frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{m_1^4} \\
 & + \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600} \right) \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} + \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2}{m_1^2 m_2^2} \\
 & + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - \left(\frac{30383}{960} + \frac{36405\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \\
 & + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384} \right) \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2} + \left(\frac{2369}{60} + \frac{35655\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^3 m_2} \\
 & + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_1^2}{m_1^3 m_2} + \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1^3 m_2}, \quad (8.34i)
 \end{aligned}$$

$$H_{421}(\mathbf{x}_a, \mathbf{p}_a) = \frac{64861 \mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1 \cdot \mathbf{p}_2)}{8m_1 m_2} + \frac{105 \mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2m_1 m_2}, \quad (8.34j)$$

$$\begin{aligned}
 H_{422}(\mathbf{x}_a, \mathbf{p}_a) = & \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152} \right) \frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192} \right) \frac{\mathbf{p}_2^2}{m_2^2} \\
 & + \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \\
 & + \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2}, \quad (8.34k)
 \end{aligned}$$

$$H_{40}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400} \right) m_1^3 m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200} \right) m_1^2 m_2^2. \quad (8.34l)$$

For completeness let us supplement the generic 4PN-accurate local Hamiltonian given above by its center-of-mass expression. The center-of-mass reference frame is defined through the condition

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}. \quad (8.35)$$

Let us introduce the reduced mass μ of the system and its symmetric mass ratio ν ,

$$\mu := \frac{m_1 m_2}{M}, \quad \nu := \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}. \quad (8.36)$$

It is convenient to introduce the reduced variables

$$\mathbf{r} := \frac{\mathbf{x}_{12}}{GM}, \quad \mathbf{p} := \frac{\mathbf{p}_1}{\mu} = -\frac{\mathbf{p}_2}{\mu} \quad (8.37)$$

(with $r := |\mathbf{r}|$ and $\mathbf{n} := \mathbf{r}/r$). We also define the reduced 4PN-accurate center-of-mass Hamiltonian

$$\hat{H}_{\leq 4\text{PN}}[\mathbf{r}, \mathbf{p}] := \frac{H_{\leq 4\text{PN}}[\mathbf{r}, \mathbf{p}] - Mc^2}{\mu}, \quad (8.38)$$

which is the sum of different PN contributions,

$$\begin{aligned}
 \hat{H}_{\leq 4\text{PN}}[\mathbf{r}, \mathbf{p}] = & \hat{H}_{\text{N}}(\mathbf{r}, \mathbf{p}) + \hat{H}_{1\text{PN}}(\mathbf{r}, \mathbf{p}) + \hat{H}_{2\text{PN}}(\mathbf{r}, \mathbf{p}) \\
 & + \hat{H}_{3\text{PN}}(\mathbf{r}, \mathbf{p}) + \hat{H}_{4\text{PN}}[\mathbf{r}, \mathbf{p}], \quad (8.39)
 \end{aligned}$$

where the 4PN Hamiltonian is the sum of local- and nonlocal-in-time parts,

$$\hat{H}_{4\text{PN}}[\mathbf{r}, \mathbf{p}] = \hat{H}_{4\text{PN}}^{\text{local}}(\mathbf{r}, \mathbf{p}) + \hat{H}_{4\text{PN}}^{\text{nonlocal}}[\mathbf{r}, \mathbf{p}]. \quad (8.40)$$

The local Hamiltonians from \hat{H}_{N} to $\hat{H}_{4\text{PN}}^{\text{local}}$ are equal to (let us recall that their coefficients depend on masses m_1, m_2 only through the symmetric mass ratio ν)

$$\hat{H}_N(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \quad (8.41a)$$

$$c^2 \hat{H}_{1\text{PN}}(\mathbf{r}, \mathbf{p}) = \frac{1}{8}(3\nu - 1)(\mathbf{p}^2)^2 - \frac{1}{2}\{(3 + \nu)\mathbf{p}^2 + \nu(\mathbf{n} \cdot \mathbf{p})^2\} \frac{1}{r} + \frac{1}{2r^2}, \quad (8.41b)$$

$$c^4 \hat{H}_{2\text{PN}}(\mathbf{r}, \mathbf{p}) = \frac{1}{16}(1 - 5\nu + 5\nu^2)(\mathbf{p}^2)^3 + \frac{1}{8}\{(5 - 20\nu - 3\nu^2)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n} \cdot \mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n} \cdot \mathbf{p})^4\} \frac{1}{r} \\ + \frac{1}{2}\{(5 + 8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n} \cdot \mathbf{p})^2\} \frac{1}{r^2} - \frac{1}{4}(1 + 3\nu) \frac{1}{r^3}, \quad (8.41c)$$

$$c^6 \hat{H}_{3\text{PN}}(\mathbf{r}, \mathbf{p}) = \frac{1}{128}(-5 + 35\nu - 70\nu^2 + 35\nu^3)(\mathbf{p}^2)^4 + \frac{1}{16}\{(-7 + 42\nu - 53\nu^2 - 5\nu^3)(\mathbf{p}^2)^3 + (2 - 3\nu)\nu^2(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 \\ + 3(1 - \nu)\nu^2(\mathbf{n} \cdot \mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n} \cdot \mathbf{p})^6\} \frac{1}{r} + \left\{ \frac{1}{16}(-27 + 136\nu + 109\nu^2)(\mathbf{p}^2)^2 + \frac{1}{16}(17 + 30\nu)\nu(\mathbf{n} \cdot \mathbf{p})^2\mathbf{p}^2 \right. \\ \left. + \frac{1}{12}(5 + 43\nu)\nu(\mathbf{n} \cdot \mathbf{p})^4 \right\} \frac{1}{r^2} + \left\{ \left(-\frac{25}{8} + \left(\frac{\pi^2}{64} - \frac{335}{48} \right) \nu - \frac{23\nu^2}{8} \right) \mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4} \right) \nu(\mathbf{n} \cdot \mathbf{p})^2 \right\} \frac{1}{r^3} \\ + \left\{ \frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2 \right) \nu \right\} \frac{1}{r^4}, \quad (8.41d)$$

$$c^8 \hat{H}_{4\text{PN}}^{\text{local}}(\mathbf{r}, \mathbf{p}) = \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4 \right) (\mathbf{p}^2)^5 + \left\{ \frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^3 \right. \right. \\ \left. \left. - \frac{9}{64}(\mathbf{n} \cdot \mathbf{p})^4(\mathbf{p}^2)^2 \right) \nu^2 + \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n} \cdot \mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n} \cdot \mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n} \cdot \mathbf{p})^8 \right) \nu^3 \right. \\ \left. + \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n} \cdot \mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n} \cdot \mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n} \cdot \mathbf{p})^8 \right) \nu^4 \right\} \frac{1}{r} + \left\{ \frac{13}{8}(\mathbf{p}^2)^3 \right. \\ \left. + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n} \cdot \mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n} \cdot \mathbf{p})^6 \right) \nu + \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 \right. \right. \\ \left. \left. + \frac{9475}{768}(\mathbf{n} \cdot \mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n} \cdot \mathbf{p})^6 \right) \nu^2 + \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n} \cdot \mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n} \cdot \mathbf{p})^4\mathbf{p}^2 \right. \right. \\ \left. \left. + \frac{10353}{1280}(\mathbf{n} \cdot \mathbf{p})^6 \right) \nu^3 \right\} \frac{1}{r^2} + \left\{ \frac{105}{32}(\mathbf{p}^2)^2 + \left(\frac{2749\pi^2}{8192} - \frac{589189}{19200} \right) (\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024} \right) (\mathbf{n} \cdot \mathbf{p})^2\mathbf{p}^2 \right. \\ \left. + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280} \right) (\mathbf{n} \cdot \mathbf{p})^4 \right\} \nu + \left\{ \left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800} \right) (\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048} \right) (\mathbf{n} \cdot \mathbf{p})^2\mathbf{p}^2 \right. \\ \left. + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384} \right) (\mathbf{n} \cdot \mathbf{p})^4 \right\} \nu^2 + \left\{ -\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n} \cdot \mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n} \cdot \mathbf{p})^4 \right\} \nu^3 \left\} \frac{1}{r^3} \\ + \left\{ \frac{105}{32}\mathbf{p}^2 + \left(\frac{185761}{19200} - \frac{21837\pi^2}{8192} \right) \mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576} \right) (\mathbf{n} \cdot \mathbf{p})^2 \right\} \nu + \left\{ \left(\frac{672811}{19200} - \frac{158177\pi^2}{49152} \right) \mathbf{p}^2 \right. \\ \left. + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840} \right) (\mathbf{n} \cdot \mathbf{p})^2 \right\} \nu^2 \left\} \frac{1}{r^4} + \left\{ -\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400} \right) \nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45} \right) \nu^2 \right\} \frac{1}{r^5}. \quad (8.41e)$$

IX. THE POINCARÉ INVARIANCE

There are several possibilities for partial checks of our results: first the test-body limit and second the linear in G part of the 4PN Hamiltonian through comparison with the post-Minkowskian results achieved in Ref. [54]. Both these checks were already performed by us in Ref. [52]. The most important tool however is the Poincaré invariance, discussed for the first time in the context of the ADM Hamiltonian approach to the two-body problem in Ref. [35]. Poincaré symmetry holds

because our (isolated and conservative) two-point-mass system is living in asymptotically flat spacetime with its globally conserved quantities: energy H , linear momentum \mathbf{P} , angular momentum \mathbf{J} , and Lorentz boost vector $\mathbf{K} = -\mathbf{P}t + \mathbf{G}$, where \mathbf{G} denotes the center-of-(mass) energy vector. All these quantities are realized as functions on the two-body phase space $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$, whose usual Poisson brackets,

$$\{f(\mathbf{x}_a, \mathbf{p}_a), g(\mathbf{x}_a, \mathbf{p}_a)\} := \sum_a \left(\frac{\partial f}{\partial x_a^i} \frac{\partial g}{\partial p_{ai}} - \frac{\partial f}{\partial p_{ai}} \frac{\partial g}{\partial x_a^i} \right), \quad (9.1)$$

satisfy the Poincaré algebra relations,

$$\{P_i, P_j\} = 0, \quad \{J_i, J_j\} = \epsilon_{ijk} J_k, \quad (9.2a)$$

$$\{J_i, P_j\} = \epsilon_{ijk} P_k, \quad (9.2b)$$

$$\{P_i, H\} = 0, \quad \{J_i, H\} = 0, \quad (9.2c)$$

$$\{J_i, G_j\} = \epsilon_{ijk} G_k, \quad (9.2d)$$

$$\{G_i, H\} = P_i, \quad (9.2e)$$

$$\{G_i, P_j\} = c^{-2} H \delta_{ij}, \quad (9.2f)$$

$$\{G_i, G_j\} = -c^{-2} \epsilon_{ijk} J_k. \quad (9.2g)$$

These relations have to be fulfilled through 4PN order. The total linear and angular momenta have universal forms,

$$P_i(\mathbf{x}_a, \mathbf{p}_a) = \sum_a p_{ai}, \quad J_i(\mathbf{x}_a, \mathbf{p}_a) = \sum_a \epsilon_{ijk} x_a^j p_{ak}, \quad (9.3)$$

and they exactly satisfy Eqs. (9.2a)–(9.2c). We will construct the boost vector \mathbf{G} as a three-vector from \mathbf{x}_a and \mathbf{p}_a only; therefore, the relation (9.2d) will also be exactly satisfied. Consequently Eqs. (9.2e)–(9.2g) are the only nontrivial relations which have to be satisfied by the vector \mathbf{G} . The Hamiltonian H entering Poincaré algebra (9.2) is the full 4PN-accurate Hamiltonian,

$$H_{\leq 4\text{PN}}[\mathbf{x}_a, \mathbf{p}_a] = H_{\leq 4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) + H_{4\text{PN}}^{\text{nonlocal}}[\mathbf{x}_a, \mathbf{p}_a], \quad (9.4)$$

where the local-in-time part reads

$$\begin{aligned} H_{\leq 4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a) &= \sum_a m_a c^2 + H_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{4\text{PN}}^{\text{local}}(\mathbf{x}_a, \mathbf{p}_a). \end{aligned} \quad (9.5)$$

Because the nonlocal-in-time piece $H_{4\text{PN}}^{\text{nonlocal}}$ is Galileo invariant [see Eq. (5.15) in [51] for the proof], it is enough to restrict the 4PN-accurate Hamiltonian to its local part $H_{\leq 4\text{PN}}^{\text{local}}$ when looking for the 4PN-accurate boost vector \mathbf{G} .

We have found the 4PN-accurate boost vector \mathbf{G} using the method of undetermined coefficients employed at the 3PN level in Ref. [35]. The generic form of the three-vector \mathbf{G} reads

$$\mathbf{G}(\mathbf{x}_a, \mathbf{p}_a) = \sum_a (M_a(\mathbf{x}_b, \mathbf{p}_b) \mathbf{x}_a + N_a(\mathbf{x}_b, \mathbf{p}_b) \mathbf{p}_a), \quad (9.6)$$

where the scalars M_a and N_a possess the following 4PN-accurate expansions:

$$M_a = m_a + M_a^{1\text{PN}} + M_a^{2\text{PN}} + M_a^{3\text{PN}} + M_a^{4\text{PN}}, \quad (9.7a)$$

$$N_a = N_a^{2\text{PN}} + N_a^{3\text{PN}} + N_a^{4\text{PN}}. \quad (9.7b)$$

Let us note that M_a and N_a reduce to m_a and 0, respectively, in the Newtonian approximation. Next we write the most general expressions for the successive PN approximations to the functions M_a and N_a as sums of scalar monomials of the form

$$\begin{aligned} c_{\mathbf{n}} r_{12}^{-n_0} \left(\frac{\mathbf{p}_1^2}{m_1^2} \right)^{n_1} \left(\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{m_1 m_2} \right)^{n_2} \left(\frac{\mathbf{p}_2^2}{m_2^2} \right)^{n_3} \\ \times \left(\frac{\mathbf{n}_{12} \cdot \mathbf{p}_1}{m_1} \right)^{n_4} \left(\frac{\mathbf{n}_{12} \cdot \mathbf{p}_2}{m_2} \right)^{n_5} m_1^{n_6} m_2^{n_7}, \end{aligned} \quad (9.8)$$

where n_0, \dots, n_7 are non-negative integers. To constrain the possible values of n_0, \dots, n_7 we employ dimensional analysis, Euclidean covariance (including parity symmetry), and time reversal symmetry (which imposes that M_a is even and N_a is odd under the operation $\mathbf{p}_a \rightarrow -\mathbf{p}_a$). We also use the $1 \leftrightarrow 2$ relabeling symmetry. At the 4PN level the most general pattern for the functions $M_a^{4\text{PN}}$ and $N_a^{4\text{PN}}$ involves 210 *dimensionless* coefficients $c_{\mathbf{n}}$.

To find the functions $M_a^{1\text{PN}}, \dots, M_a^{4\text{PN}}$ and $N_a^{2\text{PN}}, \dots, N_a^{4\text{PN}}$ it is enough to use Eq. (9.2e) only. The 3PN-accurate functions were constructed in Ref. [35] (for completeness we give below their explicit expressions). At the 4PN level the relation (9.2e) yields 525 equations to be satisfied. We have found a *unique* solution to these equations [and we have then checked that this solution satisfies the remaining Poincaré algebra relations (9.2f) and (9.2g)]. The explicit forms of the functions $M_a^{1\text{PN}}, \dots, M_a^{4\text{PN}}$ and $N_a^{2\text{PN}}, \dots, N_a^{4\text{PN}}$ read

$$c^2 M_1^{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{2} \frac{\mathbf{p}_1^2}{m_1} - \frac{1}{2} \frac{G m_1 m_2}{r_{12}}, \quad (9.9a)$$

$$c^4 M_1^{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8} \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{4} \frac{G m_1 m_2}{r_{12}} \left(-5 \frac{\mathbf{p}_1^2}{m_1^2} - \frac{\mathbf{p}_2^2}{m_2^2} + 7 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right) + \frac{1}{4} \frac{G m_1 m_2 G (m_1 + m_2)}{r_{12}}, \quad (9.9b)$$

$$\begin{aligned} c^6 M_1^{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{1}{16} \frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{16} \frac{G m_1 m_2}{r_{12}} \left(9 \frac{(\mathbf{p}_1^2)^2}{m_1^4} + \frac{(\mathbf{p}_2^2)^2}{m_2^4} - 11 \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{m_1^2 m_2^2} - 2 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} + 3 \frac{\mathbf{p}_1^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \right. \\ & + 7 \frac{\mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2 m_2^2} - 12 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1^2 m_2^2} - 3 \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_1^2 m_2^2} \left. \right) \\ & + \frac{1}{24} \frac{G^2 m_1 m_2}{r_{12}^2} \left((112 m_1 + 45 m_2) \frac{\mathbf{p}_1^2}{m_1^2} + (15 m_1 + 2 m_2) \frac{\mathbf{p}_2^2}{m_2^2} - \frac{1}{2} (209 m_1 + 115 m_2) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} \right. \\ & \left. - (31 m_1 + 5 m_2) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2} \right) - \frac{1}{8} \frac{G m_1 m_2 G^2 (m_1^2 + 5 m_1 m_2 + m_2^2)}{r_{12}^2}, \end{aligned} \quad (9.9c)$$

$$\begin{aligned} c^8 M_1^{4\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{5(\mathbf{p}_1^2)^4}{128 m_1^7} + \frac{G m_1 m_2}{r_{12}} M_{46}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} (m_1 M_{441}(\mathbf{x}_a, \mathbf{p}_a) + m_2 M_{442}(\mathbf{x}_a, \mathbf{p}_a)) \\ & + \frac{G^3 m_1 m_2}{r_{12}^3} (m_1^2 M_{421}(\mathbf{x}_a, \mathbf{p}_a) + m_1 m_2 M_{422}(\mathbf{x}_a, \mathbf{p}_a) + m_2^2 M_{423}(\mathbf{x}_a, \mathbf{p}_a)) + \frac{G^4 m_1 m_2}{r_{12}^4} M_{40}(\mathbf{x}_a, \mathbf{p}_a), \end{aligned} \quad (9.9d)$$

$$\begin{aligned} M_{46}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{13(\mathbf{p}_1^2)^3}{32 m_1^6} - \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{256 m_1^4 m_2^2} + \frac{45(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2}{128 m_1^4 m_2^2} - \frac{91(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1^2)^2}{256 m_1^4 m_2^2} \\ & - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{32 m_1^4 m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{32 m_1^4 m_2^2} + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64 m_1^4 m_2^2} \\ & + \frac{7 \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64 m_1^4 m_2^2} + \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4 \mathbf{p}_2^2}{256 m_1^4 m_2^2} - \frac{47(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2 \mathbf{p}_2^2}{128 m_1^4 m_2^2} + \frac{91(\mathbf{p}_1^2)^2 \mathbf{p}_2^2}{256 m_1^4 m_2^2} \\ & + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{32 m_1^3 m_2^3} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3 \mathbf{p}_1^2}{32 m_1^3 m_2^3} + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{32 m_1^3 m_2^3} \\ & + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{32 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16 m_1^3 m_2^3} - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^3}{16 m_1^3 m_2^3} - \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_2^2}{32 m_1^3 m_2^3} \\ & + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_1^2 \mathbf{p}_2^2}{32 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2}{32 m_1^3 m_2^3} + \frac{\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2}{32 m_1^3 m_2^3} \\ & + \frac{15(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{256 m_1^2 m_2^4} - \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4 \mathbf{p}_1^2}{256 m_1^2 m_2^4} + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{32 m_1^2 m_2^4} - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{64 m_1^2 m_2^4} \\ & - \frac{21(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_2^2}{128 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 \mathbf{p}_2^2}{128 m_1^2 m_2^4} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2}{32 m_1^2 m_2^4} \\ & + \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 \mathbf{p}_2^2}{64 m_1^2 m_2^4} + \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{p}_2^2)^2}{256 m_1^2 m_2^4} + \frac{37 \mathbf{p}_1^2 (\mathbf{p}_2^2)^2}{256 m_1^2 m_2^4} - \frac{(\mathbf{p}_2^2)^3}{32 m_2^6}, \end{aligned} \quad (9.9e)$$

$$\begin{aligned}
M_{441}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{7711(\mathbf{n}_{12} \cdot \mathbf{p}_1)^4}{3840m_1^4} - \frac{2689(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2}{3840m_1^4} + \frac{2683(\mathbf{p}_1^2)^2}{1920m_1^4} - \frac{67(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{30m_1^3 m_2} \\
& + \frac{1621(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_1^2}{1920m_1^3 m_2} - \frac{411(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{1280m_1^3 m_2} - \frac{25021 \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{3840m_1^3 m_2} \\
& + \frac{289(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{128m_1^2 m_2^2} - \frac{259(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2}{128m_1^2 m_2^2} + \frac{689(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{192m_1^2 m_2^2} + \frac{11(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{48m_1^2 m_2^2} \\
& - \frac{147(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 \mathbf{p}_2^2}{64m_1^2 m_2^2} + \frac{283 \mathbf{p}_1^2 \mathbf{p}_2^2}{64m_1^2 m_2^2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{12m_1 m_2^3} + \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{48m_1 m_2^3} \\
& - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_2^2}{6m_1 m_2^3} - \frac{7(\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2}{48m_1 m_2^3} - \frac{9(\mathbf{p}_2^2)^2}{32m_2^4}, \tag{9.9f}
\end{aligned}$$

$$\begin{aligned}
M_{442}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{45(\mathbf{p}_1^2)^2}{32m_1^4} + \frac{7\mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{48m_1^3 m_2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_1^2}{6m_1^3 m_2} - \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{48m_1^3 m_2} \\
& - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3 (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{12m_1^3 m_2} + \frac{7(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{24m_1^2 m_2^2} + \frac{635 \mathbf{p}_1^2 \mathbf{p}_2^2}{192m_1^2 m_2^2} - \frac{983(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 \mathbf{p}_2^2}{384m_1^2 m_2^2} \\
& + \frac{413(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2 (\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{384m_1^2 m_2^2} - \frac{331(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2}{192m_1^2 m_2^2} + \frac{437(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{64m_1^2 m_2^2} \\
& + \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{15m_1 m_2^3} - \frac{1349(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2)}{1280m_1 m_2^3} - \frac{5221(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2) \mathbf{p}_2^2}{1920m_1 m_2^3} \\
& - \frac{2579(\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2}{3840m_1 m_2^3} + \frac{6769(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2 \mathbf{p}_2^2}{3840m_2^4} - \frac{2563(\mathbf{p}_2^2)^2}{1920m_2^4} - \frac{2037(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{1280m_2^4}, \tag{9.9g}
\end{aligned}$$

$$M_{421}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{179843 \mathbf{p}_1^2}{14400m_1^2} + \frac{10223(\mathbf{p}_1 \cdot \mathbf{p}_2)}{1200m_1 m_2} - \frac{15 \mathbf{p}_2^2}{16m_2^2} + \frac{8881(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2400m_1 m_2} + \frac{17737(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{1600m_1^2}, \tag{9.9h}$$

$$\begin{aligned}
M_{422}(\mathbf{x}_a, \mathbf{p}_a) = & \left(\frac{8225\pi^2}{16384} - \frac{12007}{1152} \right) \frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{143}{16} - \frac{\pi^2}{64} \right) \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \left(\frac{655}{1152} - \frac{7969\pi^2}{16384} \right) \frac{\mathbf{p}_2^2}{m_2^2} + \left(\frac{6963\pi^2}{16384} - \frac{40697}{3840} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}{m_1^2} \\
& + \left(\frac{119}{16} + \frac{3\pi^2}{64} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} + \left(\frac{30377}{3840} - \frac{7731\pi^2}{16384} \right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2}, \tag{9.9i}
\end{aligned}$$

$$M_{423}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{35 \mathbf{p}_1^2}{16m_1^2} + \frac{1327(\mathbf{p}_1 \cdot \mathbf{p}_2)}{1200m_1 m_2} + \frac{52343 \mathbf{p}_2^2}{14400m_2^2} - \frac{2581(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2400m_1 m_2} - \frac{15737(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{1600m_2^2}, \tag{9.9j}$$

$$M_{40}(\mathbf{x}_a, \mathbf{p}_a) = \frac{m_1^3}{16} + \left(\frac{3371\pi^2}{6144} - \frac{6701}{1440} \right) m_1^2 m_2 + \left(\frac{20321}{1440} - \frac{7403\pi^2}{6144} \right) m_1 m_2^2 + \frac{m_2^3}{16}, \tag{9.9k}$$

$$c^4 N_1^{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{5}{4} G(\mathbf{n}_{12} \cdot \mathbf{p}_2), \tag{9.9l}$$

$$\begin{aligned}
c^6 N_1^{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{1}{8} \frac{G}{m_1 m_2} (2(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{n}_{12} \cdot \mathbf{p}_2) - \mathbf{p}_2^2 (\mathbf{n}_{12} \cdot \mathbf{p}_1) + 3(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2) \\
& + \frac{1}{48} \frac{G^2}{r_{12}} (19m_2(\mathbf{n}_{12} \cdot \mathbf{p}_1) + (130m_1 + 137m_2)(\mathbf{n}_{12} \cdot \mathbf{p}_2)), \tag{9.9m}
\end{aligned}$$

$$\begin{aligned}
c^8 N_1^{4\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & Gm_2 N_{45}(\mathbf{x}_a, \mathbf{p}_a) + \frac{G^2 m_2}{r_{12}} (m_1 N_{431}(\mathbf{x}_a, \mathbf{p}_a) + m_2 N_{432}(\mathbf{x}_a, \mathbf{p}_a)) \\
& + \frac{G^3 m_2}{r_{12}^2} (m_1^2 N_{411}(\mathbf{x}_a, \mathbf{p}_a) + m_1 m_2 N_{412}(\mathbf{x}_a, \mathbf{p}_a) + m_2^2 N_{413}(\mathbf{x}_a, \mathbf{p}_a)), \tag{9.9n}
\end{aligned}$$

$$\begin{aligned}
N_{45}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{64m_1^3m_2^2} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_1^2}{64m_1^3m_2^2} + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{32m_1^3m_2^2} \\
& - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{32m_1^3m_2^2} + \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{32m_1^3m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3\mathbf{p}_2^2}{64m_1^3m_2^2} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^3m_2^2} \\
& + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{32m_1^2m_2^3} - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3\mathbf{p}_1^2}{32m_1^2m_2^3} + \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2(\mathbf{p}_1 \cdot \mathbf{p}_2)}{16m_1^2m_2^3} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{16m_1^2m_2^3} \\
& - \frac{9(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{32m_1^2m_2^3} + \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^2m_2^3} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{16m_1^2m_2^3} - \frac{11(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^4}{128m_1m_2^4} \\
& + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3(\mathbf{p}_1 \cdot \mathbf{p}_2)}{32m_1m_2^4} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1m_2^4} + \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{p}_2^2}{32m_1m_2^4} - \frac{3(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{p}_2^2)^2}{128m_1m_2^4}, \tag{9.9o}
\end{aligned}$$

$$\begin{aligned}
N_{431}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{387(\mathbf{n}_{12} \cdot \mathbf{p}_1)^3}{1280m_1^3} + \frac{10429(\mathbf{n}_{12} \cdot \mathbf{p}_1)\mathbf{p}_1^2}{3840m_1^3} - \frac{751(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{480m_1^2m_2} + \frac{2209(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{640m_1^2m_2} \\
& - \frac{6851(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{1920m_1^2m_2} + \frac{43(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{192m_1m_2^2} - \frac{125(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{192m_1m_2^2} + \frac{25(\mathbf{n}_{12} \cdot \mathbf{p}_1)\mathbf{p}_2^2}{48m_1m_2^2} \\
& - \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{8m_2^3} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{12m_2^3}, \tag{9.9p}
\end{aligned}$$

$$\begin{aligned}
N_{432}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_1^2}{48m_1^2m_2} + \frac{7(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{24m_1^2m_2} - \frac{49(\mathbf{n}_{12} \cdot \mathbf{p}_1)^2(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{48m_1^2m_2} + \frac{295(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{384m_1m_2^2} \\
& - \frac{5(\mathbf{n}_{12} \cdot \mathbf{p}_2)(\mathbf{p}_1 \cdot \mathbf{p}_2)}{24m_1m_2^2} - \frac{155(\mathbf{n}_{12} \cdot \mathbf{p}_1)\mathbf{p}_2^2}{384m_1m_2^2} - \frac{5999(\mathbf{n}_{12} \cdot \mathbf{p}_2)^3}{3840m_2^3} + \frac{11251(\mathbf{n}_{12} \cdot \mathbf{p}_2)\mathbf{p}_2^2}{3840m_2^3}, \tag{9.9q}
\end{aligned}$$

$$N_{411}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{37397(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{7200m_1} - \frac{12311(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{2400m_2}, \tag{9.9r}$$

$$N_{412}(\mathbf{x}_a, \mathbf{p}_a) = \left(\frac{5005\pi^2}{8192} - \frac{81643}{11520}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1} + \left(\frac{773\pi^2}{2048} - \frac{61177}{11520}\right) \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2}, \tag{9.9s}$$

$$N_{413}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{7073(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{1200m_2}. \tag{9.9t}$$

The fulfillment of the Poincaré algebra does not imply a complete check of the Hamiltonian, but rather a check of all terms besides the purely static ones. Of course, the Poincaré algebra is invariant against canonical transformations, particularly those induced by coordinate-gauge transformations, so the single terms in the Hamiltonian have no direct physical meaning. The reader interested in a representation of a higher-order PN Hamiltonian through center-of-mass and relative coordinates is referred to [72].

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APPENDIX A: REGULARIZATION

In this appendix we describe techniques which we have used to regularize divergent integrals which appear in our paper.

1. Three-dimensional Riesz-implemented Hadamard regularization

The Riesz-implemented Hadamard (RH) regularization was developed in the context of deriving PN equations of motion of binary systems in Refs. [32,41] (see also [73]) to deal with locally divergent integrals computed in 3

dimensions. The RH regularization relies on multiplying the full integrand, say $i(\mathbf{x})$, of the divergent integral by a regularization factor,

$$i(\mathbf{x}) \rightarrow i(\mathbf{x}) \left(\frac{r_1}{s_1}\right)^{\epsilon_1} \left(\frac{r_2}{s_2}\right)^{\epsilon_2}, \quad (\text{A1})$$

and studying the double limit $\epsilon_1 \rightarrow 0$, $\epsilon_2 \rightarrow 0$ (here s_1 and s_2 are arbitrary three-dimensional UV regularization scales). Let us thus consider such an integral performed over the whole space \mathbb{R}^3 and let us assume that it develops *only local poles* (so it is convergent at spatial infinity). The value of the integral, after performing the RH regularization in 3 dimensions, has the structure

$$\begin{aligned} I^{\text{RH}}(3; \epsilon_1, \epsilon_2) &:= \int_{\mathbb{R}^3} i(\mathbf{x}) \left(\frac{r_1}{s_1}\right)^{\epsilon_1} \left(\frac{r_2}{s_2}\right)^{\epsilon_2} d^3x \\ &= A + c_1 \left(\frac{1}{\epsilon_1} + \ln \frac{r_{12}}{s_1}\right) + c_2 \left(\frac{1}{\epsilon_2} + \ln \frac{r_{12}}{s_2}\right) \\ &\quad + \mathcal{O}(\epsilon_1, \epsilon_2). \end{aligned} \quad (\text{A2})$$

In the case of an integral over \mathbb{R}^3 developing poles *only at spatial infinity* (so it is locally integrable) it would be enough to use a regularization factor of the form $(r/r_0)^\epsilon$ (where r_0 is an IR regularization scale), but it is more convenient to use the factor

$$\left(\frac{r_1}{r_0}\right)^{ae} \left(\frac{r_2}{r_0}\right)^{be} \quad (\text{A3})$$

and study the limit $\epsilon \rightarrow 0$. Let us denote the integrand again by $i(\mathbf{x})$. The value of the integral, after performing the RH regularization in 3 dimensions, has the structure

$$\begin{aligned} I^{\text{RH}}(3; a, b, \epsilon) &:= \int_{\mathbb{R}^3} i(\mathbf{x}) \left(\frac{r_1}{r_0}\right)^{ae} \left(\frac{r_2}{r_0}\right)^{be} d^3x \\ &= A - c_\infty \left(\frac{1}{(a+b)\epsilon} + \ln \frac{r_{12}}{r_0}\right) + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{A4})$$

Many integrals appearing in Eqs. (A2) and (A4) we compute using three-dimensional form of the following formula, first derived in d dimensions probably by Riesz [74]:

$$\int r_1^\alpha r_2^\beta d^d x = \pi^{d/2} r_{12}^{\alpha+\beta+d} \frac{\Gamma(\frac{\alpha+d}{2})\Gamma(\frac{\beta+d}{2})\Gamma(-\frac{\alpha+\beta+d}{2})}{\Gamma(-\frac{\alpha}{2})\Gamma(-\frac{\beta}{2})\Gamma(\frac{\alpha+\beta+2d}{2})}. \quad (\text{A5})$$

To compute the 4PN-accurate Hamiltonian one needs to employ a generalization of the three-dimensional version of the Riesz formula (A5) for integrands of the form $r_1^\alpha r_2^\beta \mathfrak{g}^\gamma$, where

$$\mathfrak{g} := r_1 + r_2 + r_{12}. \quad (\text{A6})$$

Such formula was derived in Ref. [32] and it reads

$$\int r_1^\alpha r_2^\beta \mathfrak{g}^\gamma d^3x = R(\alpha, \beta, \gamma) r_{12}^{\alpha+\beta+\gamma+3}, \quad (\text{A7a})$$

where [let us note that the formula given below is invariant under the permutation ($a \leftrightarrow b$)]

$$\begin{aligned} R(\alpha, \beta, \gamma) &:= 2\pi \frac{\Gamma(\alpha+2)\Gamma(\beta+2)\Gamma(-\alpha-\beta-\gamma-4)}{\Gamma(-\gamma)} \\ &\quad \times [I_{1/2}(\alpha+2, -\alpha-\gamma-2) \\ &\quad + I_{1/2}(\beta+2, -\beta-\gamma-2) \\ &\quad - I_{1/2}(\alpha+\beta+4, -\alpha-\beta-\gamma-4) - 1]. \end{aligned} \quad (\text{A7b})$$

The function $I_{1/2}$ in Eq. (A7b) is defined as follows:

$$I_{1/2}(x, y) := \frac{B_{1/2}(x, y)}{B(x, y)}, \quad (\text{A8})$$

where B stands for the beta function (Euler's integral of the first kind) and $B_{1/2}$ is the incomplete beta function; it can be expressed in terms of the Gauss hypergeometric function ${}_2F_1$:

$$B_{1/2}(x, y) = \frac{1}{2^x x} {}_2F_1\left(1-y, x; x+1; \frac{1}{2}\right). \quad (\text{A9})$$

The regularization procedure based on the formulas (A5) and (A7) consists of several steps. We enumerate them now. The most general integrand we have to consider has the form

$$(\mathbf{n}_1 \cdot \mathbf{p}_1)^{q_1} (\mathbf{n}_2 \cdot \mathbf{p}_1)^{q_2} (\mathbf{n}_1 \cdot \mathbf{p}_2)^{q_3} (\mathbf{n}_2 \cdot \mathbf{p}_2)^{q_4} r_1^\alpha r_2^\beta \mathfrak{g}^\gamma, \quad (\text{A10})$$

where q_1, \dots, q_4 are non-negative integers and γ is a negative integer. We first eliminate the unit vector \mathbf{n}_2 by the identity

$$\mathbf{n}_2 = \frac{r_1}{r_2} \mathbf{n}_1 + \frac{r_{12}}{r_2} \mathbf{n}_{12}. \quad (\text{A11})$$

We thus plug Eq. (A11) into (A10) and expand the scalar product $\mathbf{n}_1 \cdot \mathbf{n}_{12}$ by means of the relation

$$\mathbf{n}_1 \cdot \mathbf{n}_{12} = \frac{r_2^2 - r_1^2 - r_{12}^2}{2r_1 r_{12}}. \quad (\text{A12})$$

After this the most general integrand reduces to

$$(\mathbf{n}_1 \cdot \mathbf{p}_1)^{q_1} (\mathbf{n}_1 \cdot \mathbf{p}_2)^{q_2} r_1^\alpha r_2^\beta \mathfrak{g}^\gamma, \quad (\text{A13})$$

where again q_1 and q_2 are non-negative integers.

We perform integration in *prolate spheroidal coordinates*. By using these coordinates it is possible to represent integrand (A13) as a linear combination of integrands of the type $r_1^\alpha r_2^\beta \mathfrak{g}^\gamma$. To show this let us locate the particles in the

focal points of the prolate spheroidal coordinates [they lie along the z axis of the Cartesian coordinate system (x, y, z)], so the particles' position vectors have the following Cartesian coordinates:

$$\mathbf{x}_1 = (0, 0, -r_{12}/2), \quad \mathbf{x}_2 = (0, 0, r_{12}/2). \quad (\text{A14})$$

The Cartesian components of the unit vector $\mathbf{n}_1 := (\mathbf{x} - \mathbf{x}_1)/r_1$ can be expressed by r_1 , r_2 and the azimuthal angle ϕ ,

$$n_1^x = \frac{\sqrt{[(r_1 + r_2)^2 - r_{12}^2][r_{12}^2 - (r_1 - r_2)^2]}}{2r_1 r_{12}} \cos \phi, \quad (\text{A15a})$$

$$n_1^y = \frac{\sqrt{[(r_1 + r_2)^2 - r_{12}^2][r_{12}^2 - (r_1 - r_2)^2]}}{2r_1 r_{12}} \sin \phi, \quad (\text{A15b})$$

$$n_1^z = \frac{r_{12}^2 + r_1^2 - r_2^2}{2r_1 r_{12}}. \quad (\text{A15c})$$

Without loss of generality we can place the vector \mathbf{p}_1 in the (x, z) plane (we can also assume that $p_{1x} > 0$). One can then show that the Cartesian components of the vector \mathbf{p}_1 are as follows:

$$p_{1x} = \sqrt{\mathbf{p}_1^2 - (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}, \quad (\text{A16a})$$

$$p_{1y} = 0, \quad (\text{A16b})$$

$$p_{1z} = -(\mathbf{n}_{12} \cdot \mathbf{p}_1). \quad (\text{A16c})$$

The x and z Cartesian components of the vector \mathbf{p}_2 read

$$p_{2x} = \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2) - (\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{\sqrt{\mathbf{p}_1^2 - (\mathbf{n}_{12} \cdot \mathbf{p}_1)^2}}, \quad (\text{A17a})$$

$$p_{2z} = -(\mathbf{n}_{12} \cdot \mathbf{p}_2); \quad (\text{A17b})$$

the y component of the vector \mathbf{p}_2 can be computed from

$$p_{2y} = \pm \sqrt{\mathbf{p}_2^2 - p_{2x}^2 - p_{2z}^2}. \quad (\text{A17c})$$

We have checked that the result of the procedure described below does not depend on the \pm ambiguity in Eq. (A17c).

Making use of Eqs. (A15)–(A17) we compute scalar products $(\mathbf{n}_1 \cdot \mathbf{p}_1)$ and $(\mathbf{n}_1 \cdot \mathbf{p}_2)$ in Eq. (A13). After this the integrand (A13) becomes a sum of terms of the form $A(r_1, r_2)B(\phi)$, where $B(\phi)$ is a polynomial in $\sin \phi$ and $\cos \phi$. Each of these terms we integrate in the following way:

$$\begin{aligned} \int A(r_1, r_2)B(\phi)d^3x &= \int A(r_1, r_2)d^2x \int_0^{2\pi} B(\phi)d\phi \\ &= \langle B \rangle \int A(r_1, r_2)d^3x, \end{aligned} \quad (\text{A18})$$

where $\langle B \rangle$ is the average over the angle ϕ ,

$$\langle B \rangle := \frac{1}{2\pi} \int_0^{2\pi} B(\phi)d\phi. \quad (\text{A19})$$

After this step the integrand (A13) becomes the linear combination of the type

$$\sum_I c_I r_1^{\alpha_I} r_2^{\beta_I} \mathfrak{g}^{\gamma_I}, \quad (\text{A20})$$

where the constant coefficients c_I may depend only on r_{12} , \mathbf{p}_1^2 , $(\mathbf{p}_1 \cdot \mathbf{p}_2)$, \mathbf{p}_2^2 , $(\mathbf{n}_{12} \cdot \mathbf{p}_1)$, and $(\mathbf{n}_{12} \cdot \mathbf{p}_2)$. The integral of (A20) is computed by means of Eq. (A2) or (A4) and with the usage of formulas (A5) and (A7).

Appendix A 6 of Ref. [28] contains generalization of the above presented procedure to d space dimensions (and it employs prolate spheroidal coordinates in d dimensions).

2. UV corrections

Reference [32] showed that the three-dimensional RH regularization described above used to derive the 3PN two-point-mass Hamiltonian gave ambiguous results. Namely, by means of integration by parts (assuming that all involved integrals are convergent at infinity, so all surface terms can be neglected) one can replace one form of Hamiltonian density (or its part) by some other form. Integration of both equivalent densities should give the same result, but it did not. To correct the result of the three-dimensional RH regularization (i.e., to remove ambiguity), Ref. [31] (see Secs. 3 and 4 there) developed dimensional-regularization (DR) technique,³ which we have also used to make the results of the three-dimensional RH regularization of the locally divergent part of the 4PN-accurate Hamiltonian unique.

The technique of Ref. [31] boils down to the computation of the difference

$$\lim_{d \rightarrow 3} H_{4\text{PN}}^{\text{loc}}(d) - H_{4\text{PN}}^{\text{RHloc}}(3), \quad (\text{A21})$$

where $H_{4\text{PN}}^{\text{RHloc}}(3)$ is the ‘‘local part’’ of the Hamiltonian obtained by means of the three-dimensional RH regularization [it is the sum of all integrals of the type $I^{\text{RH}}(3; \epsilon_1, \epsilon_2)$ introduced in Eq. (A2)]; $H_{4\text{PN}}^{\text{loc}}(d)$ is its d -dimensional counterpart.

³The presentation of this technique given below is an improved and more complete version of the explanations contained in Sec. III of [53].

Reference [31] showed that to find the DR correction to the integral $I_1^{\text{RH}}(3; \epsilon_1, \epsilon_2)$ related with the local pole at, say, $\mathbf{x} = \mathbf{x}_1$, it is enough to consider only this part of the integrand $i(\mathbf{x})$ which develops logarithmic singularities, i.e. which locally behaves like $1/r_1^3$,

$$i(\mathbf{x}) = \dots + \tilde{c}_1(\mathbf{n}_1)r_1^{-3} + \dots, \quad \text{when } \mathbf{x} \rightarrow \mathbf{x}_1. \quad (\text{A22})$$

Then the pole part of the integral (A2) (related with the singularity at $\mathbf{x} = \mathbf{x}_1$) we recover by RH regularization of the integral of $\tilde{c}_1(\mathbf{n}_1)r_1^{-3}$ over the ball $B(\mathbf{x}_1, \ell_1)$ of radius ℓ_1 surrounding the particle \mathbf{x}_1 . The RH regularized value of this integral reads

$$\begin{aligned} I_1^{\text{RH}}(3; \epsilon_1) &:= \int_{B(\mathbf{x}_1, \ell_1)} \tilde{c}_1(\mathbf{n}_1)r_1^{-3} \left(\frac{r_1}{s_1}\right)^{\epsilon_1} d^3\mathbf{r}_1 \\ &= c_1 \int_0^{\ell_1} r_1^{-1} \left(\frac{r_1}{s_1}\right)^{\epsilon_1} dr_1, \end{aligned} \quad (\text{A23})$$

where c_1 is the angle-averaged value of the coefficient $\tilde{c}_1(\mathbf{n}_1)$. The expansion of the integral $I_1^{\text{RH}}(3; \epsilon_1)$ around $\epsilon_1 = 0$ equals

$$I_1^{\text{RH}}(3; \epsilon_1) = c_1 \left(\frac{1}{\epsilon_1} + \ln \frac{\ell_1}{s_1} \right) + \mathcal{O}(\epsilon_1). \quad (\text{A24})$$

The idea of the technique developed in [31] relies on replacing the RH-regularized value of the three-dimensional integral $I_1^{\text{RH}}(3; \epsilon_1)$ by the value of its d -dimensional version $I_1(d)$. We thus consider the d -dimensional counterpart of the expansion (A22). It reads

$$i(\mathbf{x}) = \dots + \ell_0^{k(d-3)} \tilde{c}_1(d; \mathbf{n}_1) r_1^{6-3d} + \dots, \quad \text{when } \mathbf{x} \rightarrow \mathbf{x}_1, \quad (\text{A25})$$

where ℓ_0 is the scale which relates the Newtonian G_N and the D -dimensional ($D = d + 1$) G_D gravitational constants,

$$G_D = G_N \ell_0^{d-3}. \quad (\text{A26})$$

The number k in the exponent of $\ell_0^{k(d-3)}$ is related with the momentum-order of the considered term [the term with k is of the order of $\mathcal{O}(p^{10-2k})$, where $k = 1, \dots, 5$]. The integral $I_1(d)$ we define as

$$\begin{aligned} I_1(d) &:= \ell_0^{k(d-3)} \int_{B(\mathbf{x}_1, \ell_1)} \tilde{c}_1(d; \mathbf{n}_1) r_1^{6-3d} d^d\mathbf{r}_1 \\ &= \ell_0^{k(d-3)} c_1(d) \int_0^{\ell_1} r_1^{5-2d} dr_1, \end{aligned} \quad (\text{A27})$$

where $c_1(d)$ is the angle-averaged value of the coefficient $\tilde{c}_1(d; \mathbf{n}_1)$,

$$c_1(d) := \oint_{\mathbb{S}^{d-1}(\mathbf{0}, 1)} \tilde{c}_1(d; \mathbf{n}_1) d\Omega_{d-1}. \quad (\text{A28})$$

One checks that always

$$\lim_{d \rightarrow 3} c_1(d) = c_1(3) = c_1. \quad (\text{A29})$$

The radial integral in Eq. (A27) is convergent if the real part $\Re(d)$ of d fulfills $\Re(d) < 3$. Let us introduce

$$\epsilon := d - 3$$

and let us expand $c_1(d)$ around $\epsilon = 0$,

$$c_1(d) = c_1(3 + \epsilon) = c_1 + c'_1(3)\epsilon + \mathcal{O}(\epsilon^2). \quad (\text{A30})$$

Then the expansion of the integral $I_1(d)$ around $\epsilon = 0$ reads

$$I_1(d) = -\frac{c_1}{2\epsilon} - \frac{1}{2}c'_1(3) + c_1 \ln \frac{\ell_1}{\ell_0} + \mathcal{O}(\epsilon). \quad (\text{A31})$$

In Eqs. (A30)–(A31) we have used (A29). Let us note that the coefficient $c'_1(3)$ usually depends on $\ln r_{12}$ and it has the structure

$$c'_1(3) = c'_{11}(3) + c'_{12}(3) \ln \frac{r_{12}}{\ell_0}. \quad (\text{A32})$$

Therefore the DR correction also changes the terms $\propto \ln r_{12}$.

The DR correction to the RH-regularized value of the integral $I^{\text{RH}}(3; \epsilon_1, \epsilon_2)$ relies on replacing this integral by

$$I^{\text{RH}}(3; \epsilon_1, \epsilon_2) + \Delta I_1 + \Delta I_2, \quad (\text{A33})$$

where

$$\Delta I_a := I_a(d) - I_1^{\text{RH}}(3; \epsilon_1), \quad a = 1, 2. \quad (\text{A34})$$

Then one computes the double limit

$$\begin{aligned} &\lim_{\substack{\epsilon_1 \rightarrow 3 \\ \epsilon_2 \rightarrow 3}} (I^{\text{RH}}(3; \epsilon_1, \epsilon_2) + \Delta I_1 + \Delta I_2) \\ &= A - \frac{c_1 + c_2}{2\epsilon} - \frac{1}{2}(c'_1(3) + c'_2(3)) + (c_1 + c_2) \ln \frac{r_{12}}{\ell_0} + \mathcal{O}(\epsilon) \\ &= A - \frac{c_1 + c_2}{2\epsilon} - \frac{1}{2}(c'_{11}(3) + c'_{21}(3)) \\ &\quad + \left(c_1 - \frac{1}{2}c'_{12}(3) + c_2 - \frac{1}{2}c'_{22}(3) \right) \ln \frac{r_{12}}{\ell_0} + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{A35})$$

Note that all poles $\propto 1/\epsilon_1, 1/\epsilon_2$ and all terms depending on radii ℓ_1, ℓ_2 or scales s_1, s_2 cancel each other. The result (A35) is as if all computations were fully done in d dimensions.

In the DR correcting UV divergences of the 3PN two-point-mass Hamiltonian performed in Ref. [31], after collecting all terms of the type (A35) together, all poles $\propto 1/(d-3)$ cancel each other. This is not the case for the UV divergences of the 4PN two-point-mass Hamiltonian considered in the present paper. As explained in Sec. VIII D of our paper after collecting all terms of the type (A35), one has to add to the Hamiltonian a *unique* total time derivative [given in Eq. (8.24)] to eliminate all poles $\propto 1/(d-3)$ (together with ℓ_0 -dependent logarithms).

3. IR corrections

To regularize IR-divergent integrals which appear in the derivation of the 4PN two-point-mass Hamiltonian, we have originally developed a technique analogous to the one described above and used to compute DR corrections to UV-divergent integrals regularized in 3 dimensions. After completing tedious computations we have obtained IR terms analogous to UV terms described by Eq. (A35). After adding all these terms we have the expression with poles $\propto 1/(d-3)$. Then we have checked that there exists *no* total time derivative by means of which one can eliminate these poles. The conclusion was that even in d -dimensional computation of IR-divergent integrals one has to introduce an additional IR regularization factor $(r/s)^B$ with a new scale s .

To be consistent with DR correction of UV divergences performed in d dimensions, we have developed a d -dimensional version of IR regularization. We have devised two different regularization schemes. We will however see that the results of these regularizations are identical with the results achieved by means of purely three-dimensional computations.

There is a crucial difference between the results of application of UV and IR regularizations: the results of IR regularizations depend on an arbitrary IR regularization scale which we had to introduce, whereas the result of UV regularization does not depend on any scale. Moreover the results of two different IR regularizations developed below are different; therefore, we have to conclude that the result of IR regularization is *ambiguous*. In Appendix A 3 c we show that the ambiguity can be expressed in terms of only one unknown dimensionless parameter.

The basic idea of both IR regularizations is, similarly to what we have done for UV divergences, to replace this part of the three-dimensional integral $I_\infty^{\text{RH}}(3; a, b, \epsilon)$ from Eq. (A4) which is responsible for IR divergences, by its d -dimensional counterpart. Let us thus consider the three-dimensional integral (A4) which is IR divergent. In all considered cases one checks that its IR pole term proportional to $1/\epsilon$ is related to this part of the integrand $i(\mathbf{x})$ of (A4), which, after expansion around $r = \infty$ ($r := |\mathbf{x}|$), is proportional to $1/r^3$,

$$i(\mathbf{x}) = \dots + \tilde{c}_\infty(\mathbf{n})r^{-3} + \dots, \quad \text{when } r \rightarrow \infty, \quad (\text{A36})$$

where $\mathbf{n} := \mathbf{x}/r$. It means that one can reproduce the pole term of Eq. (A4) by means of integration of $\tilde{c}_\infty(\mathbf{n})r^{-3}$ taken over the exterior of the ball $\mathbb{B}(\mathbf{0}, R)$ (with the center at the origin $\mathbf{0}$ of the coordinate system and the radius R). We thus define

$$\begin{aligned} I_\infty^{\text{RH}}(3; a, b, \epsilon) &:= \int_{\mathbb{R}^3 \setminus \mathbb{B}(\mathbf{0}, R)} \tilde{c}_\infty(\mathbf{n})r^{-3} \left(\frac{r}{r_0}\right)^{(a+b)\epsilon} d^3x \\ &= c_\infty \int_R^\infty r^{-3} \left(\frac{r}{r_0}\right)^{(a+b)\epsilon} r^2 dr \\ &= -c_\infty \left(\frac{1}{(a+b)\epsilon} + \ln \frac{R}{r_0} \right) + \mathcal{O}(\epsilon), \end{aligned} \quad (\text{A37})$$

where c_∞ is the angle-averaged value of the coefficient $\tilde{c}_\infty(\mathbf{n})$,

$$c_\infty := \oint_{\mathbb{S}^2(\mathbf{0}, 1)} \tilde{c}_\infty(\mathbf{n}) d\Omega_2. \quad (\text{A38})$$

a. Modifying behavior of the $h_{(6)ij}^{\text{TT}}$ at infinity

All terms contributing to poles at spatial infinity are collected in Eq. (7.12b). They (with the exception of the first term) have the structure

$$f_{ij}(\mathbf{x}) \Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}}, \quad (\text{A39})$$

and the first term in Eq. (7.12b) we treat as

$$\frac{1}{2(d-1)} \phi^{(2)} h_{(4)ij}^{\text{TT}} \Delta_d (\Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}}).$$

To regularize all these terms properly we have made the replacement

$$\Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}} \rightarrow \Delta_d^{-1} \left[\left(\frac{r}{s}\right)^B \ddot{h}_{(4)ij}^{\text{TT}} \right], \quad (\text{A40})$$

where $(r/s)^B$ is an IR regularization factor with s being a new constant (needed to make the regularization factor dimensionless).

After making the replacement (A40) in each term in Eq. (7.12b), we have found in d -dimensions for each term this part of its expansion around $r = \infty$ which contributes to the IR divergence. It is proportional to r^{6-3d+B} and has the structure

$$\begin{aligned} f_{ij}(\mathbf{x}) \Delta_d^{-1} \left[\left(\frac{r}{s}\right)^B \ddot{h}_{(4)ij}^{\text{TT}} \right] &= \dots + \tilde{c}_\infty^1(d, B; \mathbf{n}) \\ &\quad \times s^{-B} r^{6-3d+B} + \dots, \\ &\text{when } r \rightarrow \infty. \end{aligned} \quad (\text{A41})$$

One can make the replacement (A40) directly in $d = 3$ dimensions, then instead of the expansion (A41) one obtains

$$f_{ij}(\mathbf{x})\Delta^{-1}\left[\left(\frac{r}{s}\right)^B \tilde{h}_{(4)ij}^{\text{TT}}\right] = \dots + \tilde{c}_{\infty}^0(B; \mathbf{n}) \times s^{-B} r^{B-3} + \dots, \quad \text{when } r \rightarrow \infty. \quad (\text{A42})$$

We have checked that always

$$\lim_{d \rightarrow 3} \tilde{c}_{\infty}^1(d, B; \mathbf{n}) = \tilde{c}_{\infty}^0(B; \mathbf{n}), \quad (\text{A43a})$$

$$\lim_{B \rightarrow 0} \tilde{c}_{\infty}^0(B; \mathbf{n}) = \tilde{c}_{\infty}(\mathbf{n}), \quad (\text{A43b})$$

where $\tilde{c}_{\infty}(\mathbf{n})$ is the coefficient in the three-dimensional expansion (A36).

For all terms from Eq. (7.12b) we have computed the integral

$$\begin{aligned} I_{\infty}^1(d, B) &:= s^{-B} \int_{\mathbb{R}^d \setminus \mathbb{B}(0, R)} \tilde{c}_{\infty}^1(d, B; \mathbf{n}) r^{6-3d+B} d^d x \\ &= c_{\infty}^1(d, B) s^{-B} \int_R^{\infty} r^{5-2d+B} dr \\ &= -c_{\infty}^1(d, B) s^{-B} \frac{R^{6-2d+B}}{6-2d+B}, \end{aligned} \quad (\text{A44})$$

where $c_{\infty}^1(d, B)$ is the angle-averaged value of the coefficient $\tilde{c}_{\infty}^1(d, B; \mathbf{n})$,

$$c_{\infty}^1(d, B) := \oint_{\mathbb{S}^{d-1}(0,1)} \tilde{c}_{\infty}^1(d, B; \mathbf{n}) d\Omega_{d-1}. \quad (\text{A45})$$

One easily checks that the equalities analogous to (A43) are also fulfilled for the angle-averaged values of the coefficients,

$$\lim_{d \rightarrow 3} c_{\infty}^1(d, B) = c_{\infty}^0(B), \quad (\text{A46a})$$

$$\lim_{B \rightarrow 0} c_{\infty}^0(B) = c_{\infty}, \quad (\text{A46b})$$

where the three-dimensional coefficient $c_{\infty}^0(B)$ is the angle-averaged value of the 3-dimensional coefficient $\tilde{c}_{\infty}^0(B; \mathbf{n})$,

$$c_{\infty}^0(B) := \oint_{\mathbb{S}^2(0,1)} \tilde{c}_{\infty}^0(B; \mathbf{n}) d\Omega_2. \quad (\text{A47})$$

The integral $I_{\infty}^1(d, B)$ can be shortly written as (let us recall that $\varepsilon := d - 3$)

$$I_{\infty}^1(d, B) = I_{\infty}^1(3 + \varepsilon, B) = \frac{N(\varepsilon, B)}{B - 2\varepsilon}, \quad (\text{A48})$$

where we have defined

$$N(\varepsilon, B) := -c_{\infty}^1(3 + \varepsilon, B) s^{-B} R^{B-2\varepsilon}. \quad (\text{A49})$$

As the regularized value of the integral (A48) for $\varepsilon \rightarrow 0$ and $B \rightarrow 0$ we take the finite part (FP) of the pole occurring at $B = 2\varepsilon$ in d dimensions (we follow here Ref. [60]; see especially Sec. VIII there). We thus define

$$\begin{aligned} \text{FPI}_{\infty}^1 &:= \text{FP} \lim_{\varepsilon \rightarrow 0, B \rightarrow 0} \frac{N(\varepsilon, B) - N(\varepsilon, 2\varepsilon)}{B - 2\varepsilon} \\ &= \text{FP} \lim_{\varepsilon \rightarrow 0} \frac{N(\varepsilon, 0) - N(\varepsilon, 2\varepsilon)}{-2\varepsilon} \\ &= \text{FP} \lim_{\varepsilon \rightarrow 0} \left(\frac{\partial N}{\partial B}(0, 0) + \mathcal{O}(\varepsilon) \right) = \frac{\partial N}{\partial B}(0, 0) \\ &= -\frac{\partial c_{\infty}^1}{\partial B}(3, 0) - c_{\infty}^1(3, 0) \ln \frac{R}{s}. \end{aligned} \quad (\text{A50})$$

As the correction to integral $I^{\text{RH}}(3; a, b, \varepsilon)$ we define the difference

$$\Delta I_{\infty}^1 := \text{FPI}_{\infty}^1 - I_{\infty}^{\text{RH}}(3; a, b, \varepsilon), \quad (\text{A51})$$

so the regularized value of the three-dimensional IR-divergent integral over $i(\mathbf{x})$ reads

$$\begin{aligned} \left(\int_{\mathbb{R}^3} i(\mathbf{x}) d^3 x \right)_{\text{reg } 1} &:= \lim_{\varepsilon \rightarrow 0} (I^{\text{RH}}(3; a, b, \varepsilon) + \Delta I_{\infty}^1) \\ &= A - \frac{\partial c_{\infty}^1}{\partial B}(3, 0) - c_{\infty}^1(3, 0) \ln \frac{r_{12}}{s}. \end{aligned} \quad (\text{A52})$$

In view of the equalities (A46) it is clear that the value of the right-hand side of Eq. (A52) would be the same if all computations leading to it were performed in 3 dimensions. We thus have

$$\left(\int_{\mathbb{R}^3} i(\mathbf{x}) d^3 x \right)_{\text{reg } 1} = A - \frac{\partial c_{\infty}^0}{\partial B}(0) - c_{\infty} \ln \frac{r_{12}}{s}. \quad (\text{A53})$$

b. d -dimensional RH regularization

We have also considered another way of regularizing IR divergences. Namely, before integrating an IR-divergent integral over d -dimensional space, we multiply the full integrand by a factor

$$\left(\frac{r_1}{s}\right)^{\alpha_1} \left(\frac{r_2}{s}\right)^{\alpha_2} \quad (\text{A54})$$

and after evaluating it we take the finite part of the IR pole occurring at $\alpha_1 + \alpha_2 = 2(d - 3)$. This recipe means nothing more than the d -dimensional version of the Riesz-implemented Hadamard regularization we performed in 3 dimensions.

In this approach instead of the expansion (A41) one has (we introduce here $\beta := \alpha_1 + \alpha_2$)

$$f_{ij}(\mathbf{x})\Delta_d^{-1}\ddot{h}_{(4)ij}^{\text{TT}} = \dots + \tilde{c}_\infty^2(d; \mathbf{n})s^{-\beta}r^{6-3d+\beta} + \dots, \quad \text{when } r \rightarrow \infty, \quad (\text{A55})$$

and instead of the integral (A44) one considers the integral

$$\begin{aligned} I_\infty^2(d, \beta) &:= s^{-\beta} \int_{\mathbb{R}^d \setminus \mathbb{B}(\mathbf{0}, R)} \tilde{c}_\infty^2(d; \mathbf{n})r^{6-3d+\beta} d^d x \\ &= c_\infty^2(d)s^{-\beta} \int_R^\infty r^{5-2d+\beta} dr \\ &= -c_\infty^2(d)s^{-\beta} \frac{R^{6-2d+\beta}}{6-2d+\beta}, \end{aligned} \quad (\text{A56})$$

where

$$c_\infty^2(d) := \oint_{\mathbb{S}^{d-1}(\mathbf{0}, 1)} \tilde{c}_\infty^2(d; \mathbf{n}) d\Omega_{d-1}. \quad (\text{A57})$$

One checks that always

$$\lim_{d \rightarrow 3} c_\infty^2(d) = c_\infty^2(3) = c_\infty. \quad (\text{A58})$$

The crucial difference between the integral $I_\infty^2(d, \beta)$ and the integral $I_\infty^1(d, B)$ of (A44) is such that in $I_\infty^2(d, \beta)$ the coefficient $\tilde{c}_\infty^2(d; \mathbf{n})$ [and its angle-averaged value $c_\infty^2(d)$] does not depend on β , whereas in $I_\infty^1(d, B)$ the coefficient $\tilde{c}_\infty^1(d, B; \mathbf{n})$ [and its angle-averaged value $c_\infty^1(d; B)$] does depend on B . The integral (A56) can be written as

$$I_\infty^2(d, \beta) = \frac{\eta(\varepsilon, \beta)}{\beta - 2\varepsilon}, \quad (\text{A59a})$$

$$\eta(\varepsilon, \beta) := -c_\infty^2(3 + \varepsilon)s^{-\beta}R^{\beta-2\varepsilon}, \quad (\text{A59b})$$

and its regularized value for $\varepsilon \rightarrow 0$ and $\beta \rightarrow 0$ we define as the finite part

$$\begin{aligned} \text{FPI}_\infty^2 &:= \text{FP} \lim_{\varepsilon \rightarrow 0, \beta \rightarrow 0} \frac{\eta(\varepsilon, \beta) - \eta(\varepsilon, 2\varepsilon)}{\beta - 2\varepsilon} \\ &= \frac{\partial \eta}{\partial \beta}(0, 0) = -c_\infty^2(3) \ln \frac{R}{s}. \end{aligned} \quad (\text{A60})$$

The correction to the integral $I^{\text{RH}}(3; a, b, \varepsilon)$ is again the difference

$$\Delta I_\infty^2 := \text{FPI}_\infty^2 - I_\infty^{\text{RH}}(3; a, b, \varepsilon), \quad (\text{A61})$$

so the regularized value of the three-dimensional IR-divergent integral over $i(\mathbf{x})$ reads

$$\begin{aligned} \left(\int_{\mathbb{R}^3} i(\mathbf{x}) d^3 x \right)_{\text{reg } 2} &:= \lim_{\varepsilon \rightarrow 0} (I^{\text{RH}}(3; a, b, \varepsilon) + \Delta I_\infty^2) \\ &= A - c_\infty^2(3) \ln \frac{r_{12}}{s}. \end{aligned} \quad (\text{A62})$$

By virtue of Eq. (A58) it is clear that the value of the right-hand side of Eq. (A63) would be the same if all computations were performed in 3 dimensions. We thus have

$$\left(\int_{\mathbb{R}^3} i(\mathbf{x}) d^3 x \right)_{\text{reg } 2} = A - c_\infty \ln \frac{r_{12}}{s}. \quad (\text{A63})$$

Let us finally note that the above result can be immediately read off from Eq. (A4) after dropping the pole part and identifying the scales r_0 and s .

c. IR ambiguity

Comparison of the two IR regularization schemes considered above immediately leads to the conclusion that the results of their application to computation of the integral of $h_{(12)}^{2,2}$, Eq. (7.12b), are different. By virtue of Eqs. (A53) and (A63) one gets

$$\begin{aligned} \Delta H_{4\text{PN}}^{\text{reg } 2,2} &:= \left(\int_{\mathbb{R}^3} h_{(12)}^{2,2} d^3 x \right)_{\text{reg } 1} - \left(\int_{\mathbb{R}^3} h_{(12)}^{2,2} d^3 x \right)_{\text{reg } 2} \\ &= - \sum \frac{\partial c_\infty^0}{\partial B}(0), \end{aligned} \quad (\text{A64})$$

where the summation is over all terms of the integrand $h_{(12)}^{2,2}$. We have computed the difference $\Delta H_{4\text{PN}}^{\text{reg } 2,2}$. It turns out that it can be written as

$$\Delta H_{4\text{PN}}^{\text{reg } 2,2} = (\text{total time derivative}) + \frac{137}{768} F, \quad (\text{A65})$$

where F is defined in Eq. (8.14).

To take into account ambiguity (A65) of IR regularization we have introduced in Eq. (8.11) a dimensionless ambiguity parameter C . According to Eq. (A65) this ambiguity can be written (up to adding a total time derivative) as a multiple of the term F , and this is the content of Eq. (8.18).

4. ‘‘Partie finie’’ value of singular function

The concept of ‘‘partie finie’’ value of function at its singular point was previously used e.g. in the calculation of ADM point-particle Hamiltonians at the 2PN and 2.5PN levels (see Appendix B in [75]), and also at the 3PN and 3.5PN orders (see also [41]).

Let f be a smooth real-valued function defined in an open ball $\mathbb{B}(\mathbf{x}_0, E) \subset \mathbb{R}^3$ of radius $E > 0$ and origin at $\mathbf{x}_0 \in \mathbb{R}^3$, excluding the point \mathbf{x}_0 , i.e., $f \in C^\infty(\mathbb{B}(\mathbf{x}_0, E) \setminus \{\mathbf{x}_0\})$. At \mathbf{x}_0 the function f is assumed to be singular. It is enough to

consider functions f which have only rational singularities. We thus assume that there exists positive integer N such that the limit

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) |\mathbf{x} - \mathbf{x}_0|^N \quad (\text{A66})$$

exists and is finite. Let us denote by N_{\min} the smallest positive integer N for which the limit (A66) exists and is finite. We define

$$g(\mathbf{x}) := \begin{cases} f(\mathbf{x}) |\mathbf{x} - \mathbf{x}_0|^{N_{\min}}, & \text{for } \mathbf{x} \neq \mathbf{x}_0, \\ \lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) |\mathbf{x} - \mathbf{x}_0|^{N_{\min}}, & \text{for } \mathbf{x} = \mathbf{x}_0. \end{cases} \quad (\text{A67})$$

We also define two families of auxiliary functions $f_{\mathbf{n}}$ and $g_{\mathbf{n}}$ (labeled by unit vectors \mathbf{n} , $\mathbf{n} \cdot \mathbf{n} = 1$):

$$(0; E) \ni \epsilon \mapsto f_{\mathbf{n}}(\epsilon) := f(\mathbf{x}_0 + \epsilon \mathbf{n}) \in \mathbb{R}, \quad (\text{A68a})$$

$$\langle 0; E \rangle \ni \epsilon \mapsto g_{\mathbf{n}}(\epsilon) := g(\mathbf{x}_0 + \epsilon \mathbf{n}) \in \mathbb{R}. \quad (\text{A68b})$$

Our final assumption is that *for all unit vectors \mathbf{n} the function $g_{\mathbf{n}}$ is smooth in the half-closed interval $\langle 0; E \rangle$* . Using Eqs. (A67) and (A68) we obtain

$$f_{\mathbf{n}}(\epsilon) = \frac{g_{\mathbf{n}}(\epsilon)}{\epsilon^{N_{\min}}}. \quad (\text{A69})$$

From Eq. (A69), after expanding $g_{\mathbf{n}}$ into Taylor series around $\epsilon = 0$, the *formal* expansion of $f_{\mathbf{n}}$ into Laurent series around $\epsilon = 0$ follows:

$$\begin{aligned} f_{\mathbf{n}}(\epsilon) &= \sum_{m=-N_{\min}}^{\infty} a_m(\mathbf{n}) \epsilon^m \\ &= \frac{g_{\mathbf{n}}(0)}{\epsilon^{N_{\min}}} + \frac{g'_{\mathbf{n}}(0)}{\epsilon^{N_{\min}-1}} + \cdots + \frac{g_{\mathbf{n}}^{(N_{\min}-1)}(0)}{(N_{\min}-1)!} \epsilon \\ &\quad + \frac{1}{N_{\min}!} g_{\mathbf{n}}^{(N_{\min})}(0) + \mathcal{O}(\epsilon). \end{aligned} \quad (\text{A70})$$

We define the regularized “partie finie” value of the function f at \mathbf{x}_0 as the coefficient at ϵ^0 in the expansion (A70) averaged over all unit vectors \mathbf{n} :

$$\begin{aligned} f_{\text{reg}}(\mathbf{x}_0) &:= \frac{1}{4\pi} \oint d\Omega_2 a_0(\mathbf{n}) \\ &= \frac{1}{4\pi N_{\min}!} \oint d\Omega_2 g_{\mathbf{n}}^{(N_{\min})}(0). \end{aligned} \quad (\text{A71})$$

Let us consider the function

$$\tilde{g}(\mathbf{x}) := f(\mathbf{x}) |\mathbf{x} - \mathbf{x}_0|^{\tilde{N}},$$

where \tilde{N} is the positive integer $\tilde{N} > N_{\min}$. Then one can define $f_{\text{reg}}(\mathbf{x}_0)$ using the function \tilde{g} instead of g . It is easy to show that the value of $f_{\text{reg}}(\mathbf{x}_0)$ will not change, so the definition (A71) does not depend on the choice of the number \tilde{N} provided $\tilde{N} \geq N_{\min}$.

Definition (A71) is used to give the meaning to integrals of $f(\mathbf{x}) \delta_a$, where the function f is assumed to be singular at $\mathbf{x} = \mathbf{x}_a$. Namely, we define

$$\int d^3x f(\mathbf{x}) \delta_a := f_{\text{reg}}(\mathbf{x}_a). \quad (\text{A72})$$

The definition (A72) is an extension of the notion of “good” Dirac δ -functions introduced by Infeld and Plebański [44]. Their definition assumes that “good” δ -function, besides having the properties of ordinary Dirac δ distributions, also satisfies the condition (cf. Appendix 1 in [44])

$$\int d^3x \frac{\delta_a}{r_a^k} = 0, \quad \text{for } k = 1, 2, \dots, p. \quad (\text{A73})$$

Obviously the definition (A72) entails the fulfillment of the condition (A73).

The important feature of the definition (A71) is that the regularized value of the product of functions is not, in general, equal to the product of the regularized values of the individual functions:

$$(f_1 f_2 \cdots)_{\text{reg}}(\mathbf{x}_0) \neq f_{1\text{reg}}(\mathbf{x}_0) f_{2\text{reg}}(\mathbf{x}_0) \cdots \quad (\text{A74})$$

(the above property but with the equality sign was called “tweedling of products” by Infeld and Plebański; see, e.g., Appendix 2 in [44]). The property (A74) of three-dimensional “partie finie” operation leads to problems. Let us consider some function S singular at $\mathbf{x} = \mathbf{x}_a$. It is natural to demand that

$$S(\mathbf{x}) \delta_a = S_{\text{reg}}(\mathbf{x}_a) \delta_a. \quad (\text{A75})$$

The above rule is always used when solving Poisson equations with singular source terms of the form $S(\mathbf{x}) \delta_a$ [see Eq. (B2) below]. Then, multiplying both sides of Eq. (A75) by another function T which is also singular at $\mathbf{x} = \mathbf{x}_a$, one gets

$$T(\mathbf{x}) S(\mathbf{x}) \delta_a = T(\mathbf{x}) S_{\text{reg}}(\mathbf{x}_a) \delta_a. \quad (\text{A76})$$

The rule (A75) applied to Eq. (A76) implies that

$$(TS)_{\text{reg}}(\mathbf{x}_a) = T_{\text{reg}}(\mathbf{x}_a) S_{\text{reg}}(\mathbf{x}_a), \quad (\text{A77})$$

what in general contradicts Eq. (A74).

To avoid this kind of problem one should employ dimensional regularization. One can check that in the generic d -dimensional case the distributivity is always satisfied,

$$(f_1^{(d)} f_2^{(d)} \dots)_{\text{reg}}(\mathbf{x}_a) = f_{1\text{reg}}^{(d)}(\mathbf{x}_a) f_{2\text{reg}}^{(d)}(\mathbf{x}_a) \dots, \quad (\text{A78})$$

where $f_{\text{reg}}^{(d)}(\mathbf{x}_a)$ is rather not defined by the d -dimensional analog of the definition (A71), but it is directly computed by employing the existence of such region in the complex d -plane where the function $f^{(d)}(\mathbf{x})$ is finite for $\mathbf{x} = \mathbf{x}_a$ (see the example at the end of this appendix). In d dimensions one can thus always use

$$f^{(d)}(\mathbf{x}) \delta^{(d)}(\mathbf{x} - \mathbf{x}_a) = f_{\text{reg}}^{(d)}(\mathbf{x}_a) \delta^{(d)}(\mathbf{x} - \mathbf{x}_a). \quad (\text{A79})$$

Therefore one defines the dimensional-regularization rule,

$$f_{\text{reg}}(\mathbf{x}_a) := \lim_{d \rightarrow 3} (f_{\text{reg}}^{(d)}(\mathbf{x}_a)), \quad (\text{A80})$$

where $f^{(d)}$ is the d -dimensional version of f . In the computation of the 4PN Hamiltonian the usage of the rule (A80) boils down in practice to the usage of the three-dimensional definition (A71) after the usage of distributivity (A78) (in the case of computing regularized value of the product of functions).

As an example of justifying the distributivity (A78) let us consider the following 4PN-related contact integral [here we denote by $\phi_{(2)}^{(d)}$ the function, given by the right-hand side of Eq. (C1a), which is the d -dimensional version of the three-dimensional potential denoted by $\phi_{(2)}$]:

$$\int d^d x (\phi_{(2)}^{(d)}(\mathbf{x}))^5 \delta_1 = \int d^d x \kappa^5 (m_1 r_1^{2-d} + m_2 r_2^{2-d})^5 \delta_1. \quad (\text{A81})$$

Because $\Re(d) < 2 \Rightarrow \lim_{x \rightarrow x_1} r_1^{2-d} = 0$, then

$$\int d^d x (\phi_{(2)}^{(d)}(\mathbf{x}))^5 \delta_1 = \kappa^5 (m_2 r_2^{2-d})^5 = [(\phi_{(2)}^{(d)})_{\text{reg}}(\mathbf{x}_1)]^5. \quad (\text{A82})$$

Therefore the three-dimensional value of the integral (A81) equals

$$\begin{aligned} \int d^3 x (\phi_{(2)}(\mathbf{x}))^5 \delta_1 &= \lim_{d \rightarrow 3} [(\phi_{(2)}^{(d)})_{\text{reg}}(\mathbf{x}_1)]^5 \\ &= [(\phi_{(2)})_{\text{reg}}(\mathbf{x}_1)]^5. \end{aligned} \quad (\text{A83})$$

Computation of the three-dimensional integral of $(\phi_{(2)}(\mathbf{x}))^5 \delta_1$ directly by means of Eq. (A71) leads to the result different from the result of Eq. (A83).

5. Distributional differentiation of homogeneous functions

Appearance of UV divergences is not the only cost of employing Dirac-delta sources. Another consequence is that in our computations we have to differentiate homogeneous functions using an enhanced (or distributional)

rule, which comes from standard distribution theory (see Sec. 3.3 in Chapter III of Ref. [76]).

Let f be a real-valued function defined in a neighborhood of the origin of \mathbb{R}^3 . f is said to be a *positively homogeneous function of degree λ* , if for any number $a > 0$

$$f(a\mathbf{x}) = a^\lambda f(\mathbf{x}). \quad (\text{A84})$$

Let $k := -\lambda - 2$. If λ is an integer and if $\lambda \leq -2$ (i.e. k is a non-negative integer), then the partial derivative of f with respect to the coordinate x^i has to be calculated by means of the formula [cf. Eq. (5.15) in [77]]

$$\begin{aligned} \partial_i f(\mathbf{x}) &= \partial_i f(\mathbf{x}) + \frac{(-1)^k}{k!} \frac{\partial^k \delta(\mathbf{x})}{\partial x^{i_1} \dots \partial x^{i_k}} \\ &\times \oint_{\Sigma} d\sigma_i f(\mathbf{x}') x'^{i_1} \dots x'^{i_k}, \end{aligned} \quad (\text{A85})$$

where $\partial_i f$ means the derivative computed using the standard rules of differentiations, Σ is any smooth close surface surrounding the origin and $d\sigma_i$ is the surface element on Σ .

The rule (A85) should also be applied to differentiation of functions which are homogeneous not with respect to \mathbf{x} , but with respect to $\mathbf{x} - \mathbf{x}_0$, for some constant $\mathbf{x}_0 \in \mathbb{R}^3$. Let f be such function, then there exists another function ϕ for which the relation

$$f(\mathbf{x}) = \phi(\mathbf{x} - \mathbf{x}_0) \quad (\text{A86})$$

is fulfilled and the function $\xi \mapsto \phi(\xi)$ is a positively homogeneous function of degree λ , i.e., for any number $a > 0$

$$\phi(a\xi) = a^\lambda \phi(\xi). \quad (\text{A87})$$

Obviously (here $\xi := \mathbf{x} - \mathbf{x}_0$)

$$\frac{\partial f(\mathbf{x})}{\partial x^i} = \frac{\partial \phi(\xi)}{\partial \xi^i} \quad (\text{A88})$$

and for the derivative $\partial \phi / \partial \xi^i$ the rule (A84) is directly applicable. Therefore the partial derivative with respect to the coordinate x^i of the function f satisfying conditions (A86) and (A87) should, by virtue of (A88) and (A85), be calculated by means of the formula

$$\begin{aligned} \partial_i f(\mathbf{x}) &= \partial_i f(\mathbf{x}) + \frac{(-1)^k}{k!} \frac{\partial^k \delta(\mathbf{x} - \mathbf{x}_0)}{\partial x^{i_1} \dots \partial x^{i_k}} \\ &\times \oint_{\Sigma} d\sigma_i \phi(\xi') \xi'^{i_1} \dots \xi'^{i_k}, \end{aligned} \quad (\text{A89})$$

where $\partial_i f$ means the derivative computed using the standard rules of differentiations, Σ is any smooth close surface surrounding the point \mathbf{x}_0 and $d\sigma_i$ is the surface element on Σ .

As an example let us employ the formula (A89) to calculate first and second partial derivatives of $1/r_a$, $1/r_a^2$, and $1/r_a^3$. For the first partial derivatives we obtain

$$\partial_i \frac{1}{r_a} = \partial_i \frac{1}{r_a}, \quad (\text{A90a})$$

$$\partial_i \frac{1}{r_a^2} = \partial_i \frac{1}{r_a^2}, \quad (\text{A90b})$$

$$\partial_i \frac{1}{r_a^3} = \partial_i \frac{1}{r_a^3} - \frac{4\pi}{3} \partial_i \delta_a. \quad (\text{A90c})$$

Let us note that in Eq. (A90a) there is no need to use the rule (A89), and in Eq. (A90) the term proportional to δ_a [obtained from the usage of (A89)], vanishes. The second partial derivatives read

$$\partial_i \partial_j \frac{1}{r_a} = \partial_i \partial_j \frac{1}{r_a} - \frac{4\pi}{3} \delta_{ij} \delta_a, \quad (\text{A91a})$$

$$\partial_i \partial_j \frac{1}{r_a^2} = \partial_i \partial_j \frac{1}{r_a^2}, \quad (\text{A91b})$$

$$\partial_i \partial_j \frac{1}{r_a^3} = \partial_i \partial_j \frac{1}{r_a^3} - \frac{2\pi}{15} (16 \partial_i \partial_j \delta_a + 3 \delta_{ij} \Delta \delta_a). \quad (\text{A91c})$$

Making use of Eqs. (A91a), (A91b), and (A91c), one gets

$$\Delta \frac{1}{r_a} = -4\pi \delta_a, \quad (\text{A92a})$$

$$\Delta \frac{1}{r_a^2} = \frac{2}{r_a^4}, \quad (\text{A92b})$$

$$\Delta \frac{1}{r_a^3} = \frac{6}{r_a^5} - \frac{10\pi}{3} \Delta \delta_a. \quad (\text{A92c})$$

The distributional derivative (A89) does not obey the Leibniz rule. It can easily be seen by considering the distributional partial derivative of the product $1/r_a$ and $1/r_a^2$. Let us suppose that the Leibniz rule is applicable here:

$$\partial_i \frac{1}{r_a^3} = \partial_i \left(\frac{1}{r_a} \frac{1}{r_a^2} \right) = \frac{1}{r_a^2} \partial_i \frac{1}{r_a} + \frac{1}{r_a} \partial_i \frac{1}{r_a^2}. \quad (\text{A93})$$

By virtue of Eqs. (A90a) and (A90b) the right-hand side of Eq. (A93) can be computed using standard differential calculus (no terms with Dirac deltas), whereas computing the left-hand side of (A93) by means of (A90c) one obtains some term proportional to $\partial_i \delta_a$.

The distributional differentiation is necessary when one differentiates homogeneous functions under the integral sign. Let us consider the following locally divergent integral (here $a \neq b$):

$$p_{ai} p_{aj} \int d^3x \left(\partial_i \partial_j \frac{1}{r_a} \right) \frac{1}{r_b^4}. \quad (\text{A94})$$

We shall regularize this integral in two different ways. We first replace in (A94) differentiations with respect to x^i by those with respect to x_a^i (obviously $\partial_i r_a = -\partial_{ai} r_a$). Then we shift the differentiations before the integral sign and apply directly the Riesz formula (A5). The result is

$$\begin{aligned} p_{ai} p_{aj} \int d^3x \left(\partial_i \partial_j \frac{1}{r_a} \right) \frac{1}{r_b^4} &= p_{ai} p_{aj} \partial_{ai} \partial_{aj} \int \frac{d^3x}{r_a r_b^4} \\ &= p_{ai} p_{aj} \partial_{ai} \partial_{aj} \left(-\frac{2\pi}{r_{ab}^2} \right) \\ &= \frac{4\pi [\mathbf{p}_a^2 - 4(\mathbf{n}_{ab} \cdot \mathbf{p}_a)^2]}{r_{ab}^4}. \end{aligned} \quad (\text{A95})$$

We have obtained (A95) performing integration first and then differentiation. Now we shall regularize the integral (A94) doing differentiation first. To do it we have to use the rule (A89), which gives [cf. Eq. (A91a)]

$$\partial_i \partial_j \frac{1}{r_a} = (3n_a^i n_a^j - \delta_{ij}) \frac{1}{r_a^3} - \frac{4\pi}{3} \delta_{ij} \delta_a. \quad (\text{A96})$$

We substitute (A96) into (A94):

$$\begin{aligned} p_{ai} p_{aj} \int d^3x \left(\partial_i \partial_j \frac{1}{r_a} \right) \frac{1}{r_b^4} \\ = \int d^3x \frac{3(\mathbf{n}_a \cdot \mathbf{p}_a)^2 - \mathbf{p}_a^2}{r_a^3 r_b^4} - \frac{4\pi}{3} \mathbf{p}_a^2 \int d^3x \frac{\delta_a}{r_b^4}. \end{aligned} \quad (\text{A97})$$

The second integral on the right-hand side of (A97) is calculated by means of the definition (A72). The obvious result is

$$\int d^3x \frac{\delta_a}{r_b^4} = \frac{1}{r_{ab}^4}. \quad (\text{A98})$$

To calculate the first integral on the right-hand side of (A97) we apply the procedure described in Appendix A.1. We obtain

$$\int d^3x \frac{3(\mathbf{n}_a \cdot \mathbf{p}_a)^2 - \mathbf{p}_a^2}{r_a^3 r_b^4} = \frac{16\pi [\mathbf{p}_a^2 - 3(\mathbf{n}_{ab} \cdot \mathbf{p}_a)^2]}{3r_{ab}^4}. \quad (\text{A99})$$

Collecting Eqs. (A97)–(A99) together we get the result which coincides with the result (A95) obtained before. The two ways of regularizing the integral (A94), described above, coincide only if we apply formula (A89) when we perform differentiation before integration.

It is not difficult to show that the formula (A85) is also valid (without any change) in the d -dimensional case, i.e. when f is a real-valued and positively homogeneous function of degree λ defined in a neighborhood of the origin of \mathbb{R}^d . The formula (A85) can be applied in d

dimensions when the number $k := -\lambda + d - 1$ is a non-negative integer, e.g., the d -dimensional versions of Eqs. (A90a) and (A91a) read

$$\partial_i \frac{1}{r_a^{d-2}} = \partial_i \frac{1}{r_a^{d-2}}, \quad (\text{A100a})$$

$$\partial_i \partial_j \frac{1}{r_a^{d-2}} = \partial_i \partial_j \frac{1}{r_a^{d-2}} - \frac{1}{\kappa d} \delta_{ij} \delta_a. \quad (\text{A100b})$$

Using the formula (A85) [or (A89)] in d (or 3) dimensions requires averaging of products of unit vectors over the unit sphere. This can be done by means of the following formulas (see Appendix A 2 of [28] for d -dimensional expression and Appendix A 5 of [78] for its three-dimensional version):

$$\oint_{\mathbb{S}^{d-1}} d\Omega_{d-1} n^{i_1} \dots n^{i_{k+1}} = 0, \quad k = 0, 1, 2, \dots, \quad (\text{A101a})$$

$$\begin{aligned} \oint_{\mathbb{S}^{d-1}} d\Omega_{d-1} n^{i_1} \dots n^{i_k} &= \frac{(d-2)!!}{(d+2(k-1))!!} \Omega_{d-1} \\ &\times \delta_{\{i_1 i_2 \dots i_{2k}\}}, \\ k &= 0, 1, 2, \dots \end{aligned} \quad (\text{A101b})$$

Here, for positive integer k , $k!! := k(k-2) \dots 1$ for k odd and $k!! := k(k-2) \dots 2$ for k even, Ω_{d-1} is the area of the unit sphere in \mathbb{R}^d ,

$$\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad (\text{A102})$$

and $A_{\{i_1 i_2 \dots i_k\}} := \sum_{\sigma \in S} A_{\sigma(i_1) \dots \sigma(i_k)}$, where S is the smallest set of permutations of $\{1, \dots, k\}$ making $A_{\{i_1 i_2 \dots i_k\}}$ fully symmetric in i_1, i_2, \dots, i_k . Let us note some simple cases:

$$\oint_{\mathbb{S}^{d-1}} d\Omega_{d-1} n^{i_1} n^{i_2} = \frac{1}{d} \Omega_{d-1} \delta_{i_1 i_2}, \quad (\text{A103a})$$

$$\begin{aligned} \oint_{\mathbb{S}^{d-1}} d\Omega_{d-1} n^{i_1} n^{i_2} n^{i_3} n^{i_4} &= \frac{1}{d(d+2)} \Omega_{d-1} \\ &\times (\delta_{i_1 i_2} \delta_{i_3 i_4} + \delta_{i_1 i_3} \delta_{i_2 i_4} + \delta_{i_1 i_4} \delta_{i_2 i_3}). \end{aligned} \quad (\text{A103b})$$

6. Riesz kernel

To avoid problems related with applying distributional derivatives (even in d dimensions) one can replace Dirac delta distribution δ by some functional representation (“delta sequence”). In d dimensions one can e.g. employ the *Riesz kernel* $K_a(\epsilon_a)$:

$$\delta_a = \lim_{\epsilon_a \rightarrow 0} K_a(\epsilon_a), \quad K_a(\epsilon_a) := \frac{\Gamma((d-\epsilon_a)/2)}{\pi^{d/2} 2^{\epsilon_a} \Gamma(\epsilon_a/2)} r_a^{\epsilon_a-d}. \quad (\text{A104})$$

Then one should replace in the constraint equations (2.5) Dirac-delta sources by Riesz kernels (A104), solve the constraints perturbatively and develop the whole PN scheme (let us stress that then no distributional differentiation is needed). At the end of the calculation, one takes the limits $\epsilon_1 \rightarrow 0$, $\epsilon_2 \rightarrow 0$, and only after this one computes $d \rightarrow 3$ limit.

This procedure was applied by us in [46] to recompute all UV divergent integrals at the 3PN level. It is however too difficult to be performed fully at the 4PN level, so it has not been applied in the main part of the present paper, but it was used for checking some results. It should be mentioned that the Riesz-kernel method of regularization has been applied by Damour in Refs. [79,80]. It also may be pointed out that the dimensional regularization calculations of Ref. [63] have been performed in momentum representation, where also only ordinary (i.e., nondistributional) space-time derivatives show up.

As an example let us first compute the potential $\phi_{(2)}$ for Riesz-kernel sources. Instead of Eq. (3.10a) we thus solve

$$\Delta_d \phi_{(2)} = - \sum_a m_a K_a. \quad (\text{A105})$$

Making use of Eq. (B4) one gets

$$\phi_{(2)} = - \sum_a \frac{2^{-\epsilon_a} \pi^{-d/2} \Gamma((d-\epsilon_a)/2)}{\epsilon_a (2-d+\epsilon_a) \Gamma(\epsilon_a/2)} m_a r_a^{2-d+\epsilon_a}. \quad (\text{A106})$$

Let us next consider the integral of Eq. (A81), which now takes the form

$$\int d^d x \phi_{(2)}^5 K_1. \quad (\text{A107})$$

To compute this integral it is enough to use Eq. (A5). After computing the double limit $\epsilon_1 \rightarrow 0$, $\epsilon_2 \rightarrow 0$ (the order of the limits does not matter) one obtains

$$\frac{\pi^{-5d/2} \Gamma(d/2)^5}{32(d-2)^5} m_2^5 r_{12}^{10-5d}. \quad (\text{A108})$$

The value of the above formula in the limit $d \rightarrow 3$ coincides with the result (A83).

Let us finally mention that the usage of the Riesz kernel directly in 3 dimensions does not resolve ambiguities. For example, three-dimensional computations corresponding to d -dimensional ones presented in the above example would lead to the result which is different from (A83) (so the “tweedling property” of the product would be violated).

APPENDIX B: INVERSE LAPLACIANS

In the present paper we have to consider (both in d and in 3 dimensions) numerous Poisson equations with distributional source terms,

$$\Delta_d f = \sum_a g(\mathbf{x}) \delta_a, \quad (\text{B1})$$

where usually the function g is singular at $\mathbf{x} = \mathbf{x}_a$. Equation of this type we solve as follows:

$$\begin{aligned} f(\mathbf{x}) &= \Delta_d^{-1} \sum_a g(\mathbf{x}) \delta_a = \Delta_d^{-1} \sum_a g_{\text{reg}}(\mathbf{x}_a) \delta_a \\ &= \sum_a g_{\text{reg}}(\mathbf{x}_a) \Delta_d^{-1} \delta_a, \end{aligned} \quad (\text{B2})$$

where the regularized value g_{reg} of the function g is defined in Eq. (A71) (in 3 dimensions) or in Eq. (A80) (when, to avoid ambiguities, one has to employ the d -dimensional version of computing of g_{reg}). The function $\Delta_d^{-1} \delta_a$ is defined below in Eq. (B9) (in d dimensions) or in Eq. (B15) (in 3 dimensions).

1. d -dimensional inverse Laplacians

We start from considering some solutions of equation

$$\Delta_d^n f = r_a^\alpha, \quad n = 1, 2, \dots, \quad (\text{B3})$$

where Δ_d^n denotes n -folded superposition of the d -dimensional flat Laplacian Δ_d . By direct computation one checks that the function

$$\Delta_d^{-n} r_a^\alpha := \frac{\Gamma(\alpha/2 + 1) \Gamma((\alpha + d)/2)}{2^{2n} \Gamma(\alpha/2 + n + 1) \Gamma((\alpha + d)/2 + n)} r_a^{\alpha + 2n} \quad (\text{B4})$$

is the solution of Eq. (B3). The function $\Delta_d^{-n} r_a^\alpha$ we call *n th inverse Laplacian of r_a^α* . Let us note the formulas for the first, second, and third inverse Laplacians of r_a^α :

$$\Delta_d^{-1} n_a^i r_a^\alpha := \frac{n_a^i r_a^{2+\alpha}}{(\alpha + 1)(\alpha + d + 1)}, \quad (\text{B11a})$$

$$\Delta_d^{-1} n_a^i n_a^j r_a^\alpha := \frac{[(2 + \alpha)(d + \alpha) n_a^i n_a^j - 2\delta_{ij}] r_a^{2+\alpha}}{\alpha(2 + \alpha)(d + \alpha)(2 + d + \alpha)}, \quad (\text{B11b})$$

$$\Delta_d^{-1} n_a^i n_a^j n_a^k r_a^\alpha := \frac{[2(\delta_{jk} n_a^i + \delta_{ik} n_a^j + \delta_{ij} n_a^k) - (1 + \alpha)(1 + d + \alpha) n_a^i n_a^j n_a^k] r_a^{2+\alpha}}{(1 - \alpha)(1 + \alpha)(1 + d + \alpha)(3 + d + \alpha)}, \quad (\text{B11c})$$

$$\Delta_d^{-2} n_a^i r_a^\alpha := \frac{n_a^i r_a^{4+\alpha}}{(1 + \alpha)(3 + \alpha)(1 + d + \alpha)(3 + d + \alpha)}, \quad (\text{B11d})$$

$$\Delta_d^{-2} n_a^i n_a^j r_a^\alpha := \frac{[(4 + \alpha)(d + \alpha) n_a^i n_a^j - 4\delta_{ij}] r_a^{4+\alpha}}{\alpha(2 + \alpha)(4 + \alpha)(d + \alpha)(2 + d + \alpha)(4 + d + \alpha)}, \quad (\text{B11e})$$

$$\Delta_d^{-1} r_a^\alpha := \frac{r_a^{\alpha+2}}{(\alpha + 2)(\alpha + d)}, \quad (\text{B5a})$$

$$\Delta_d^{-2} r_a^\alpha := \frac{r_a^{4+\alpha}}{(2 + \alpha)(4 + \alpha)(d + \alpha)(2 + d + \alpha)}, \quad (\text{B5b})$$

$$\Delta_d^{-3} r_a^\alpha := \frac{r_a^{6+\alpha}}{(2 + \alpha)(4 + \alpha)(6 + \alpha)(d + \alpha)(2 + d + \alpha)(4 + d + \alpha)}. \quad (\text{B5c})$$

We also have to consider solutions of equation

$$\Delta_d^n f = \delta_a, \quad n = 1, 2, \dots \quad (\text{B6})$$

Making use of Eq. (B4) and the formula

$$\Delta_d r_a^{2-d} = -\kappa^{-1} \delta_a, \quad (\text{B7})$$

one checks that the function

$$\Delta_d^{-n} \delta_a := -\frac{\kappa \Gamma(2 - d/2)}{4^{n-1} (n-1)! \Gamma(n + 1 - d/2)} r_a^{2n-d} \quad (\text{B8})$$

solves Eq. (B6). The function $\Delta_d^{-n} \delta_a$ we call the *n th inverse Laplacian of δ_a* . The first inverse Laplacian of δ_a thus reads

$$\Delta_d^{-1} \delta_a := -\kappa r_a^{2-d}. \quad (\text{B9})$$

We also need solutions of equations of a more general type than Eq. (B3). It reads

$$\Delta_d^n f = r_a^\alpha n_a^{i_1} \dots n_a^{i_k}, \quad n = 1, 2, \dots \quad (\text{B10})$$

More precisely, we need to consider Eq. (B10) for $n = 1, 2, 3$ and for each of these values of n we need to take $k = 1, \dots, 6$. To save space we list below solutions (inverse Laplacians) related to Eq. (B10) only for $n = 1, 2, 3$ and $k = 1, 2, 3$:

$$\Delta_d^{-2} n_a^i n_a^j n_a^k r_a^\alpha := \frac{[(3+\alpha)(1+d+\alpha)n_a^i n_a^j n_a^k - 4(\delta_{ij} n_a^k + \delta_{jk} n_a^i + \delta_{ik} n_a^j)] r_a^{4+\alpha}}{(\alpha-1)(1+\alpha)(3+\alpha)(1+d+\alpha)(3+d+\alpha)(5+d+\alpha)}, \quad (\text{B11f})$$

$$\Delta_d^{-3} n_a^i r_a^\alpha := \frac{n_a^i r_a^{6+\alpha}}{(1+\alpha)(3+\alpha)(5+\alpha)(1+d+\alpha)(3+d+\alpha)(5+d+\alpha)}, \quad (\text{B11g})$$

$$\Delta_d^{-3} n_a^i n_a^j r_a^\alpha := \frac{[(6+\alpha)(d+\alpha)n_a^i n_a^j - 6\delta_{ij}] r_a^{6+\alpha}}{\alpha(2+\alpha)(4+\alpha)(6+\alpha)(d+\alpha)(2+d+\alpha)(4+d+\alpha)(6+d+\alpha)}, \quad (\text{B11h})$$

$$\Delta_d^{-3} n_a^i n_a^j n_a^k r_a^\alpha := \frac{[(5+\alpha)(1+d+\alpha)n_a^i n_a^j n_a^k - 6(\delta_{ij} n_a^k + \delta_{jk} n_a^i + \delta_{ik} n_a^j)] r_a^{6+\alpha}}{(\alpha-1)(1+\alpha)(3+\alpha)(5+\alpha)(1+d+\alpha)(3+d+\alpha)(5+d+\alpha)(7+d+\alpha)}. \quad (\text{B11i})$$

2. Three-dimensional inverse Laplacians

For convenience let us start from rewriting some d -dimensional results of Appendix B 1 in $d=3$ dimensions. The function

$$\Delta^{-n} r_a^\alpha := \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+2n+2)} r_a^{\alpha+2n} \quad (\text{B12})$$

is the n th inverse Laplacian of r_a^α [it is thus the solution of Eq. (B3) in $d=3$ dimensions], so the first inverse Laplacian of r_a^α reads

$$\Delta^{-1} r_a^\alpha := \frac{r_a^{\alpha+2}}{(\alpha+2)(\alpha+3)}. \quad (\text{B13})$$

The function

$$\Delta^{-n} \delta_a := -\frac{1}{4\pi(2n-2)!} r_a^{2n-3} \quad (\text{B14})$$

solves Eq. (B6) in $d=3$ dimensions and is the n th inverse Laplacian of δ_a . The first inverse Laplacian of δ_a thus equals

$$\Delta^{-1} \delta_a := -\frac{1}{4\pi r_a}. \quad (\text{B15})$$

In the derivations of the field functions needed to calculate the 4PN-accurate two-point-mass Hamiltonian we have used some special solutions to the partial differential equations of the form

$$\Delta^n f = r_a^\alpha r_b^\beta \quad (a \neq b), \quad n = 1, 2, \dots \quad (\text{B16})$$

This special solution to Eq. (B16) is defined below, we denote it by $I(n; \alpha, \beta)$ and call the n th inverse Laplacian of $r_a^\alpha r_b^\beta$.

Let us mention that in Ref. [81] one can find formulas [see Eq. (B13) there and equations below it], which describe the formal solution to Eq. (B16) in d dimensions

for any complex values of n , α , and β . The solution is expressed in terms of the Appell hypergeometric function F_4 of two variables. We have tried to use it in our 4PN-related computations but, in general, this has led to calculations too complicated to be useful.

The large family of three-dimensional inverse Laplacians of $r_a^\alpha r_b^\beta$ is based on the solution to the Poisson equation

$$\Delta f = \frac{1}{r_a r_b}. \quad (\text{B17})$$

One immediately checks that $\ln(r_a + r_b \pm r_{ab})$ solves Eq. (B17). Because $\ln(r_a + r_b - r_{ab})$ is singular along the segment joining the points \mathbf{x}_a and \mathbf{x}_b , the first inverse Laplacian of $r_a^{-1} r_b^{-1}$ we define to be

$$I(1; -1, -1) := \ln(r_a + r_b + r_{ab}). \quad (\text{B18})$$

Let us apply the operator Δ_a (Δ_a contains differentiations with respect to x_a^i only) to the both sides of Eq. (B17). Assuming commutativity of the operators Δ_a and Δ one obtains

$$\Delta_a(\Delta f) = \Delta(\Delta_a f) = \frac{1}{r_b} \left(\Delta_a \frac{1}{r_a} \right) = -4\pi \frac{\delta_a}{r_b}, \quad (\text{B19})$$

where we have used

$$\Delta_a \frac{1}{r_a} = \Delta \frac{1}{r_a} = -4\pi \delta_a. \quad (\text{B20})$$

After applying the rule [see Eq. (A75) and the discussion around this equation]

$$f(\mathbf{x}) \delta_a = f_{\text{reg}}(\mathbf{x}_a) \delta_a \quad (\text{B21})$$

to the right-hand side of Eq. (B19), one gets

$$\Delta(\Delta_a f) = -4\pi \frac{\delta_a}{r_{ab}}. \quad (\text{B22})$$

Using Eq. (B15) from (B22) one obtains

$$\Delta_a f = -\frac{4\pi}{r_{ab}} \Delta^{-1} \delta_a = \frac{1}{r_a r_{ab}}. \quad (\text{B23})$$

Analogously one can derive the equation

$$\Delta_b f = \frac{1}{r_b r_{ab}}. \quad (\text{B24})$$

One checks that the solution (B18) fulfills Eqs. (B23) and (B24), whereas the function $\ln(r_a + r_b - r_{ab})$ does not.

Using the result (B18) it is possible to calculate all multiple inverse Laplacians of the type⁴

$$I(n; 2k-1, 2l-1), \quad n=1, 2, \dots, \quad k, l=0, 1, 2, \dots \quad (\text{B25})$$

To obtain $I(n; 2k-1, 2l-1)$ (for $n=1, 2, \dots, k, l=0, 1, 2, \dots$) one assumes that it has the following structure:

$$I(n; 2k-1, 2l-1) := W_1(r_a, r_b, r_{ab}) + W_2(r_a, r_b, r_{ab}) \ln(r_a + r_b + r_{ab}), \quad (\text{B26})$$

where W_1 and W_2 are polynomials of variables r_a, r_b , and r_{ab} , consisting of terms of only $2(k+l+n-1)$ -order in these variables. The form of Eq. (B26) can be inferred from dimensional analysis which takes into account the first inverse Laplacian of $r_a^{-1} r_b^{-1}$ given in Eq. (B18). To fix the coefficients of W_1 and W_2 uniquely one requires that the following conditions are fulfilled:

$$\Delta I(n; 2k-1, 2l-1) = I(n-1; 2k-1, 2l-1), \quad (\text{B27a})$$

$$\Delta_a I(n; 2k-1, 2l-1) = 2k(2k-1) I(n; 2k-3, 2l-1), \quad (\text{B27b})$$

$$\Delta_b I(n; 2k-1, 2l-1) = 2l(2l-1) I(n; 2k-1, 2l-3). \quad (\text{B27c})$$

Equation (B27b) and (B27c) corresponds to Eq. (B23) and (B24), respectively. In the case of $k=0$, Eq. (B27b) should be replaced by

$$\Delta_a I(n; -1, 2l-1) = \frac{r_{ab}^{2l-1}}{(2n-2)!} r_a^{2n-3}, \quad (\text{B27d})$$

analogously for $l=0$ instead of Eq. (B27c) one uses

$$\Delta_b I(n; 2k-1, -1) = \frac{r_{ab}^{2k-1}}{(2n-2)!} r_b^{2n-3}. \quad (\text{B27e})$$

To illustrate the above procedure let us consider the second inverse Laplacian $I(2; -1, -1)$. We are thus looking for a solution f of the partial differential equation $\Delta^2 f = 1/(r_a r_b)$. We assume that the solution f is of the form $W_1(r_a, r_b, r_{ab}) + W_2(r_a, r_b, r_{ab}) \ln(r_a + r_b + r_{ab})$, where W_1 and W_2 are polynomials of indicated variables consisting of only quadratic terms in these variables (i.e. the most general polynomial of this type is $a_1 r_a^2 + a_2 r_b^2 + a_3 r_{ab}^2 + a_4 r_a r_b + a_5 r_a r_{ab} + a_6 r_b r_{ab}$). We need to fix the values of 12 coefficients (in fact less because of symmetry with respect to interchanging the labels a and b of the particles) defining the polynomials W_1 and W_2 . To do this we employ Eqs. (B27a), (B27d), and (B27e), which take the form (1) $\Delta f = \ln(r_a + r_b + r_{ab})$; (2) $\Delta_a f = r_a/(2r_{ab})$; (3) $\Delta_b f = r_b/(2r_{ab})$. The equations 1–3 fix the 12 coefficients of the polynomials W_1 and W_2 uniquely. The result is given in Eq. (B28) below.

Below we list explicit formulas for the inverse Laplacians which have been used throughout this paper (here $a \neq b$ and $s_{ab} := r_a + r_b + r_{ab}$):

$$\Delta^{-1} \frac{1}{r_a r_b} := \ln s_{ab}, \quad (\text{B28a})$$

$$\Delta^{-2} \frac{1}{r_a r_b} := \frac{1}{36} (-r_a^2 + 3r_a r_{ab} + r_{ab}^2 - 3r_a r_b + 3r_{ab} r_b - r_b^2) + \frac{1}{12} (r_a^2 - r_{ab}^2 + r_b^2) \ln s_{ab}, \quad (\text{B28b})$$

$$\begin{aligned} \Delta^{-3} \frac{1}{r_a r_b} := & \frac{1}{28800} (-63r_a^4 + 150r_a^3 r_{ab} + 126r_a^2 r_{ab}^2 - 90r_a r_{ab}^3 - 63r_{ab}^4 - 90r_a^3 r_b + 90r_a^2 r_{ab} r_b + 90r_a r_{ab}^2 r_b - 90r_{ab}^3 r_b \\ & - 2r_a^2 r_b^2 + 90r_a r_{ab} r_b^2 + 126r_{ab}^2 r_b^2 - 90r_a r_b^3 + 150r_{ab} r_b^3 - 63r_b^4) \\ & + \frac{1}{960} (3r_a^4 - 6r_a^2 r_{ab}^2 + 3r_{ab}^4 + 2r_a^2 r_b^2 - 6r_{ab}^2 r_b^2 + 3r_b^4) \ln s_{ab}, \end{aligned} \quad (\text{B28c})$$

$$\Delta^{-1} \frac{r_a}{r_b} := \frac{1}{18} (-r_a^2 - 3r_a r_{ab} - r_{ab}^2 + 3r_a r_b + 3r_{ab} r_b + r_b^2) + \frac{1}{6} (r_a^2 + r_{ab}^2 - r_b^2) \ln s_{ab}, \quad (\text{B28d})$$

⁴The method presented below was devised in Ref. [14]; see Appendix C there.

$$\begin{aligned} \Delta^{-2} \frac{r_a}{r_b} := & \frac{1}{7200} (-r_a^4 - 30r_a^3 r_{ab} - 62r_a^2 r_{ab}^2 + 90r_a r_{ab}^3 + 63r_{ab}^4 - 30r_a^3 r_b + 30r_a^2 r_{ab} r_b \\ & - 90r_a r_{ab}^2 r_b + 90r_{ab}^3 r_b - 62r_a^2 r_b^2 - 90r_a r_{ab} r_b^2 - 126r_{ab}^2 r_b^2 + 90r_a r_b^3 \\ & + 90r_{ab} r_b^3 + 63r_b^4) + \frac{1}{240} (r_a^4 + 2r_a^2 r_{ab}^2 - 3r_{ab}^4 + 2r_a^2 r_b^2 + 6r_{ab}^2 r_b^2 - 3r_b^4) \ln s_{ab}, \end{aligned} \quad (\text{B28e})$$

$$\begin{aligned} \Delta^{-3} \frac{r_a}{r_b} := & \frac{1}{2822400} (-37r_a^6 - 210r_a^5 r_{ab} - 951r_a^4 r_{ab}^2 + 1540r_a^3 r_{ab}^3 + 2013r_a^2 r_{ab}^4 - 1050r_a r_{ab}^5 - 1025r_{ab}^6 - 210r_a^5 r_b \\ & + 210r_a^4 r_{ab} r_b - 840r_a^3 r_{ab}^2 r_b + 840r_a^2 r_{ab}^3 r_b + 1050r_a r_{ab}^4 r_b - 1050r_{ab}^5 r_b - 37r_a^4 r_b^2 - 420r_a^3 r_{ab} r_b^2 - 1062r_a^2 r_{ab}^2 r_b^2 \\ & + 2100r_a r_{ab}^3 r_b^2 + 3075r_{ab}^4 r_b^2 - 280r_a^3 r_b^3 + 420r_a^2 r_{ab} r_b^3 - 2100r_a r_{ab}^2 r_b^3 + 2800r_{ab}^3 r_b^3 - 951r_a^2 r_b^4 - 1050r_a r_{ab} r_b^4 \\ & - 3075r_{ab}^2 r_b^4 + 1050r_a r_b^5 + 1050r_{ab} r_b^5 + 1025r_b^6) + \frac{1}{13440} (r_a^6 + 3r_a^4 r_{ab}^2 - 9r_a^2 r_{ab}^4 + 5r_{ab}^6 + r_a^4 r_b^2 + 6r_a^2 r_{ab}^2 r_b^2 \\ & - 15r_{ab}^4 r_b^2 + 3r_a^2 r_b^4 + 15r_{ab}^2 r_b^4 - 5r_b^6) \ln s_{ab}, \end{aligned} \quad (\text{B28f})$$

$$\begin{aligned} \Delta^{-1} r_a r_b := & \frac{1}{3600} (63r_a^4 + 90r_a^3 r_{ab} - 62r_a^2 r_{ab}^2 - 30r_a r_{ab}^3 - r_{ab}^4 + 90r_a^3 r_b - 90r_a^2 r_{ab} r_b + 30r_a r_{ab}^2 r_b \\ & - 30r_{ab}^3 r_b - 126r_a^2 r_b^2 - 90r_a r_{ab} r_b^2 - 62r_{ab}^2 r_b^2 + 90r_a r_b^3 + 90r_{ab} r_b^3 + 63r_b^4) \\ & + \frac{1}{120} (-3r_a^4 + 2r_a^2 r_{ab}^2 + r_{ab}^4 + 6r_a^2 r_b^2 + 2r_{ab}^2 r_b^2 - 3r_b^4) \ln s_{ab}, \end{aligned} \quad (\text{B28g})$$

$$\begin{aligned} \Delta^{-2} r_a r_b := & \frac{1}{705600} (531r_a^6 + 630r_a^5 r_{ab} - 531r_a^4 r_{ab}^2 - 420r_a^3 r_{ab}^3 - 531r_a^2 r_{ab}^4 + 630r_a r_{ab}^5 + 531r_{ab}^6 + 630r_a^5 r_b \\ & - 630r_a^4 r_{ab} r_b - 630r_a^3 r_{ab}^2 r_b + 630r_a^2 r_{ab}^3 r_b - 531r_a r_{ab}^4 r_b - 914r_a^2 r_{ab}^2 r_b^2 - 531r_a r_{ab}^3 r_b^2 - 420r_a^3 r_b^3 - 420r_{ab}^3 r_b^3 \\ & - 531r_a^2 r_b^4 - 630r_a r_{ab} r_b^4 - 531r_{ab}^2 r_b^4 + 630r_a r_b^5 + 630r_{ab} r_b^5 + 531r_b^6) \\ & + \frac{1}{3360} (-3r_a^6 + 3r_a^4 r_{ab}^2 + 3r_a^2 r_{ab}^4 - 3r_{ab}^6 + 3r_a^4 r_b^2 + 2r_a^2 r_{ab}^2 r_b^2 + 3r_{ab}^4 r_b^2 + 3r_a^2 r_b^4 + 3r_{ab}^2 r_b^4 - 3r_b^6) \ln s_{ab}, \end{aligned} \quad (\text{B28h})$$

$$\begin{aligned} \Delta^{-3} r_a r_b := & \frac{1}{1219276800} (18975r_a^8 + 18900r_a^7 r_{ab} - 21100r_a^6 r_{ab}^2 - 18900r_a^5 r_{ab}^3 - 50550r_a^4 r_{ab}^4 + 65100r_a^3 r_{ab}^5 + 88500r_a^2 r_{ab}^6 \\ & - 31500r_a r_{ab}^7 - 35825r_{ab}^8 + 18900r_a^7 r_b - 18900r_a^6 r_{ab} r_b - 6300r_a^5 r_{ab}^2 r_b + 6300r_a^4 r_{ab}^3 r_b - 44100r_a^3 r_{ab}^4 r_b \\ & + 44100r_a^2 r_{ab}^5 r_b + 31500r_a r_{ab}^6 r_b - 31500r_{ab}^7 r_b - 18204r_a^6 r_b^2 + 3780r_a^5 r_{ab} r_b^2 - 30804r_a^4 r_{ab}^2 r_b^2 - 7560r_a^3 r_{ab}^3 r_b^2 \\ & - 39492r_a^2 r_{ab}^4 r_b^2 + 44100r_a r_{ab}^5 r_b^2 + 88500r_{ab}^6 r_b^2 - 8820r_a^5 r_b^3 - 3780r_a^4 r_{ab} r_b^3 - 840r_a^3 r_{ab}^2 r_b^3 - 7560r_a^2 r_{ab}^3 r_b^3 \\ & - 44100r_a r_{ab}^4 r_b^3 + 65100r_{ab}^5 r_b^3 - 1542r_a^4 r_b^4 - 3780r_a^3 r_{ab} r_b^4 - 30804r_a^2 r_{ab}^2 r_b^4 + 6300r_a r_{ab}^3 r_b^4 - 50550r_{ab}^4 r_b^4 \\ & - 8820r_a^3 r_b^5 + 3780r_a^2 r_{ab} r_b^5 - 6300r_a r_{ab}^2 r_b^5 - 18900r_{ab}^3 r_b^5 - 18204r_a^2 r_b^6 - 18900r_a r_{ab} r_b^6 - 21100r_{ab}^2 r_b^6 \\ & + 18900r_a r_b^7 + 18900r_{ab} r_b^7 + 18975r_b^8) + \frac{1}{967680} (-15r_a^8 + 20r_a^6 r_{ab}^2 + 30r_a^4 r_{ab}^4 - 60r_a^2 r_{ab}^6 + 25r_{ab}^8 + 12r_a^6 r_b^2 \\ & + 12r_a^4 r_{ab}^2 r_b^2 + 36r_a^2 r_{ab}^4 r_b^2 - 60r_{ab}^6 r_b^2 + 6r_a^4 r_b^4 + 12r_a^2 r_{ab}^2 r_b^4 + 30r_{ab}^4 r_b^4 + 12r_a^2 r_b^6 + 20r_{ab}^2 r_b^6 - 15r_b^8) \ln s_{ab} \quad (\text{B28i}) \end{aligned}$$

$$\begin{aligned} \Delta^{-1} \frac{r_a^3}{r_b} := & \frac{1}{1200} (-63r_a^4 - 90r_a^3 r_{ab} - 2r_a^2 r_{ab}^2 - 90r_a r_{ab}^3 - 63r_{ab}^4 + 150r_a^3 r_b + 90r_a^2 r_{ab} r_b + 90r_a r_{ab}^2 r_b + 150r_{ab}^3 r_b \\ & + 126r_a^2 r_b^2 + 90r_a r_{ab} r_b^2 + 126r_{ab}^2 r_b^2 - 90r_a r_b^3 - 90r_{ab} r_b^3 - 63r_b^4) \\ & + \frac{1}{40} (3r_a^4 + 2r_a^2 r_{ab}^2 + 3r_{ab}^4 - 6r_a^2 r_b^2 - 6r_{ab}^2 r_b^2 + 3r_b^4) \ln s_{ab}. \end{aligned} \quad (\text{B28j})$$

APPENDIX C: EXPLICIT RESULTS FOR THE FIELD FUNCTIONS

In this appendix we give the explicit formulas for the PN approximate solutions of both the constraint (2.5) and the field (4.3) equations which can be found in the literature or were computed by us for the first time. The different inverse Laplacians needed to compute field functions presented below are given in Appendix B.

1. Potentials $\phi_{(n)}$

The solutions of Eqs. (3.10) for the potentials $\phi_{(2)}$ and $\phi_{(4)}$ are known in d dimensions. They can be obtained by means of Eqs. (B2) and (B9) using the result $(\phi_{(2)})_{\text{reg}}(\mathbf{x}_a) = \kappa \sum_{b \neq a} m_b r_{ab}^{2-d}$. They read

$$\phi_{(2)} = \kappa \sum_a m_a r_a^{2-d}, \quad (\text{C1a})$$

$$\phi_{(4)} = -\frac{1}{2} S_{(4)1} + \frac{d-2}{4(d-1)} S_{(4)2}, \quad (\text{C1b})$$

$$S_{(4)1} = -\kappa \sum_a \frac{\mathbf{p}_a^2}{m_a} r_a^{2-d}, \quad (\text{C1c})$$

$$S_{(4)1} = -\kappa^2 \sum_a \sum_{b \neq a} m_a m_b r_{ab}^{2-d} r_a^{2-d}, \quad (\text{C1d})$$

where the coefficient κ is defined in Eq. (8.4). These solutions are valid also for general n -body, i.e., not only for two-body, point-mass systems.

The potential $\phi_{(6)}$ is split into two parts, $\phi_{(6)} = \phi_{(6)1} + \phi_{(6)2}$, where $\phi_{(6)1}$ is the solution of the Poisson equation (3.14a) and $\phi_{(6)2}$ fulfills the equation (7.7). The solutions to these equations are fully known only for two-body point-mass systems and in $d = 3$ dimensions (they can be found in an implicit form e.g. in Refs. [82,83] and [75]). They can be symbolically written as

$$\phi_{(6)1} = \Delta^{-1} \left(\sum_a \left(-\frac{1}{64} \phi_{(2)}^2 + \frac{1}{8} \phi_{(4)} + \frac{5}{16} \phi_{(2)} \frac{\mathbf{p}_a^2}{m_a^2} + \frac{1}{8} \frac{(\mathbf{p}_a^2)^2}{m_a^4} \right) m_a \delta_a - (\tilde{\pi}_{(3)}^{ij})^2 \right), \quad (\text{C2a})$$

$$\phi_{(6)2} = \frac{1}{2} \Delta^{-1} (\phi_{(2),ij} h_{(4)ij}^{\text{TT}}) = \frac{1}{8\pi} \sum_a m_a \Delta^{-1} \left(\left(\frac{1}{r_a} \right)_{,ij} h_{(4)ij}^{\text{TT}} \right). \quad (\text{C2b})$$

A more explicit form of the function $\phi_{(6)1}$ can be written as⁵

$$\begin{aligned} \phi_{(6)1} = & -\frac{1}{32\pi} \sum_a \frac{(\mathbf{p}_a^2)^2}{m_a^3 r_a} - \frac{1}{(16\pi)^2} \sum_a \sum_{b \neq a} \frac{m_b \mathbf{p}_a^2}{m_a r_{ab}} \left(\frac{5}{r_a} + \frac{1}{r_b} \right) + \frac{1}{(16\pi)^3} \sum_a \sum_{b \neq a} \frac{m_a^2 m_b}{r_{ab}^2} \left(\frac{1}{r_a} + \frac{1}{r_b} \right) - \frac{9}{4} \frac{1}{(16\pi)^2} \sum_a \frac{2\mathbf{p}_a^2 - (\mathbf{n}_a \cdot \mathbf{p}_a)^2}{r_a^2} \\ & + \frac{1}{(16\pi)^2} \sum_a \sum_{b \neq a} \left\{ -\frac{1}{4} (\mathbf{p}_a \cdot \nabla_a) (\mathbf{p}_b \cdot \nabla_b) (\nabla_a \cdot \nabla_b)^2 (\Delta^{-1} r_a r_b) + [-8(\mathbf{p}_a \cdot \mathbf{p}_b) (\nabla_a \cdot \nabla_b) + 3(\mathbf{p}_a \cdot \nabla_a) (\mathbf{p}_b \cdot \nabla_b) \right. \\ & \left. - 8(\mathbf{p}_a \cdot \nabla_b) (\mathbf{p}_b \cdot \nabla_a)] \left(\Delta^{-1} \frac{1}{r_a r_b} \right) + 4(\mathbf{p}_a \cdot \nabla_a) (\mathbf{p}_b \cdot \nabla_a) (\nabla_a \cdot \nabla_b) \left(\Delta^{-1} \frac{r_a}{r_b} \right) \right\}, \quad (\text{C3}) \end{aligned}$$

where the inverse Laplacians $\Delta^{-1}(r_a r_b)$, $\Delta^{-1}(r_a^{-1} r_b^{-1})$, and $\Delta^{-1}(r_a r_b^{-1})$ can be found in Eqs. (B28). The Poisson integral needed to calculate the function $\phi_{(6)2}$ reads (here $b \neq a$)

$$\begin{aligned} \Delta^{-1} \left(\left(\frac{1}{r_a} \right)_{,ij} h_{(4)ij}^{\text{TT}} \right) = & \frac{1}{32(16\pi)^2} \frac{m_a m_b}{r_a^3 r_b r_{ab}^3} (3r_a^4 - 12r_a^3 r_{ab} + 18r_a^2 r_{ab}^2 - 12r_a r_{ab}^3 + 3r_{ab}^4 + 28r_a^3 r_b - 12r_a^2 r_{ab} r_b - 12r_a r_{ab}^2 r_b \\ & + 28r_{ab}^3 r_b - 30r_a^2 r_b^2 + 60r_a r_{ab} r_b^2 - 30r_{ab}^2 r_b^2 - 36r_a r_b^3 - 36r_{ab} r_b^3 + 35r_b^4) + \frac{3}{64\pi} \frac{\mathbf{p}_a^2 - 3(\mathbf{n}_a \cdot \mathbf{p}_a)^2}{m_a r_a^2} \\ & + \frac{1}{32\pi m_b} \left\{ 2(\mathbf{p}_b \cdot \nabla_b)^2 \frac{r_b}{r_a} - 4 \frac{\mathbf{p}_b^2}{r_a r_b} + [2\mathbf{p}_b^2 (\nabla \cdot \nabla_b)^2 - 4(\mathbf{p}_b \cdot \nabla) (\mathbf{p}_b \cdot \nabla_b) (\nabla \cdot \nabla_b) \right. \\ & \left. - 4(\mathbf{p}_b \cdot \nabla) (\mathbf{p}_b \cdot \nabla_b) (\nabla \cdot \nabla_b)] \left(\Delta^{-1} \frac{r_b}{r_a} \right) + 8(\mathbf{p}_b \cdot \nabla)^2 \left(\Delta^{-1} \frac{1}{r_a r_b} \right) \right. \\ & \left. + \frac{1}{6} (\mathbf{p}_b \cdot \nabla_b)^2 (\nabla \cdot \nabla_b)^2 \left(\Delta^{-1} \frac{r_b^3}{r_a} \right) \right\}, \quad (\text{C4}) \end{aligned}$$

where the inverse Laplacians $\Delta^{-1}(r_b r_a^{-1})$, $\Delta^{-1}(r_a^{-1} r_b^{-1})$, and $\Delta^{-1}(r_b^3 r_a^{-1})$ are given in Eqs. (B28).

⁵In Appendix C we employ the following notation: $\nabla = (\partial_i)$, $\nabla_a = (\partial_{ai})$, $(\mathbf{p}_a \cdot \nabla) = p_{ai} \partial_i$, $(\mathbf{p}_a \cdot \nabla_b) = p_{ai} \partial_{bi}$, $(\nabla_a \cdot \nabla_b) = \partial_{ai} \partial_{bi}$.

2. Longitudinal field momenta $\tilde{\pi}_{(n)}^{ij}$

Equations (3.16) fulfilled by the longitudinal field momenta $\tilde{\pi}_{(n)}^{ij}$ can be written in the form

$$\tilde{\pi}_{(n),j}^{ij} = S_{(n)}^i, \quad (\text{C5})$$

where $S_{(n)}^i$ stands for the source term. Making use of the decomposition (2.9) one rewrites Eq. (C5) in terms of the vectorial function $V_{(n)}^i$,

$$\Delta_d V_{(n)}^i + \frac{d-2}{d} \partial_{ij} V_{(n)}^j = S_{(n)}^i. \quad (\text{C6})$$

It is not difficult to find the formal solution of Eq. (C6) in terms of the first and the second inverse Laplacian of the source $S_{(n)}^i$. It reads

$$V_{(n)}^i = \Delta_d^{-1} S_{(n)}^i - \frac{d-2}{2(d-1)} \partial_{ij} \Delta_d^{-2} S_{(n)}^j. \quad (\text{C7})$$

Let us now assume that the source term in Eq. (C5) has the form of a divergence of a symmetric and trace-free quantity,

$$S_{(n)}^i = \partial_j T_{(n)}^{ij}, \quad (\text{C8})$$

where $T_{(n)}^{ij} = T_{(n)}^{ji}$ and $T_{(n)}^{ii} = 0$. Then, making use of Eq. (2.9) and the definition (2.14) of the TT operator in d dimensions, one can show that

$$\tilde{\pi}_{(n)}^{ij} = T_{(n)}^{ij} - (T_{(n)}^{ij})^{\text{TT}}. \quad (\text{C9})$$

Using the source term from Eq. (3.16a) we have obtained the explicit solution (C7) for the leading-order function $V_{(3)}^i$ in d dimensions,

$$V_{(3)}^i = \frac{\kappa}{8(d-1)} \sum_a ((3d-2)p_{ai} + (d-2)^2 (\mathbf{n}_a \cdot \mathbf{p}_a) n_a^i) r_a^{2-d}. \quad (\text{C10})$$

This solution is also valid for n -body point-mass systems. The leading-order longitudinal field momentum $\tilde{\pi}_{(3)}^{ij}$ can be constructed from the function $V_{(3)}^i$ by means of the formula (2.9). Let us also quote the following useful form of the field momentum $\tilde{\pi}_{(3)}^{ij}$ in $d = 3$ dimensions,

$$\tilde{\pi}_{(3)}^{ij} = \frac{1}{16\pi} \sum_a p_{ak} \left\{ 2 \left[\delta_{ik} \left(\frac{1}{r_a} \right)_{,j} + \delta_{jk} \left(\frac{1}{r_a} \right)_{,i} \right] - \delta_{ij} \left(\frac{1}{r_a} \right)_{,k} - \frac{1}{2} r_{a,ijk} \right\}. \quad (\text{C11})$$

The next-to-leading-order function $V_{(5)}^i$ has been calculated in $d = 3$ dimensions and for two-point-mass systems only. It reads

$$V_{(5)}^i = -\frac{3}{16(16\pi)^2} \sum_a \frac{m_a}{r_a^2} (9p_{ai} + 2(\mathbf{n}_a \cdot \mathbf{p}_a) n_a^i) + \frac{1}{(16\pi)^2} \sum_a \sum_{b \neq a} m_b \left\{ \left[4\partial_{ai} (\mathbf{p}_a \cdot \nabla) - \frac{3}{2} \partial_i (\mathbf{p}_a \cdot \nabla_a) + 4p_{ai} (\nabla \cdot \nabla_a) \right] \left(\Delta^{-1} \frac{1}{r_a r_b} \right) - 2\partial_i (\mathbf{p}_a \cdot \nabla) (\nabla \cdot \nabla_a) \left(\Delta^{-2} \frac{1}{r_a r_b} \right) - \partial_{ai} (\mathbf{p}_a \cdot \nabla_a) (\nabla \cdot \nabla_a) \left(\Delta^{-1} \frac{r_a}{r_b} \right) + \frac{1}{4} \partial_i (\mathbf{p}_a \cdot \nabla_a) (\nabla \cdot \nabla_a)^2 \left(\Delta^{-2} \frac{r_a}{r_b} \right) \right\}, \quad (\text{C12})$$

where the inverse Laplacians $\Delta^{-1}(r_a^{-1} r_b^{-1})$, $\Delta^{-2}(r_a^{-1} r_b^{-1})$, $\Delta^{-1}(r_a r_b^{-1})$, and $\Delta^{-2}(r_a r_b^{-1})$ are given in Eqs. (B28). The field momentum $\tilde{\pi}_{(5)}^{ij}$ can be constructed from the vectorial function $V_{(5)}^i$ by means of the formula (2.9).

3. $h_{(4)ij}^{\text{TT}}$ and $h_{(6)ij}^{\text{TT}}$ related terms

The leading-order TT part $h_{(4)ij}^{\text{TT}}$ of the metric is the solution of Eq. (4.11). It is the sum

$$h_{(4)ij}^{\text{TT}} = h_{(4)0ij}^{\text{TT}} + h_{(4)2ij}^{\text{TT}}, \quad (\text{C13})$$

where $h_{(4)0ij}^{\text{TT}}$ does not depend on particles' momenta and $h_{(4)2ij}^{\text{TT}}$ is quadratic in momenta. The quadratic-in-momenta part of $h_{(4)2ij}^{\text{TT}}$ can be computed in d dimensions and for general n -body systems. It reads

$$h_{(4)2ij}^{\text{TT}} = \frac{\kappa(d-2)}{8(d-1)} \sum_a \frac{1}{m_a} \{ [\mathbf{p}_a^2 - (d+2)(\mathbf{n}_a \cdot \mathbf{p}_a)^2] \delta_{ij} + [d(d-2)(\mathbf{n}_a \cdot \mathbf{p}_a)^2 - (d+2)\mathbf{p}_a^2] n_a^i n_a^j + 2d(\mathbf{n}_a \cdot \mathbf{p}_a) (n_a^i p_{aj} + n_a^j p_{ai}) + 2p_{ai} p_{aj} \} r_a^{2-d}. \quad (\text{C14})$$

The momentum independent part $h_{(4)0ij}^{\text{TT}}$ is known only in $d = 3$ dimensions and for two-body systems. Its fully explicit form reads (here $s_{ab} := r_a + r_b + r_{ab}$ for $a \neq b$; see, e.g., Ref. [32])

$$\begin{aligned}
 h_{(4)0ij}^{\text{TT}} = & \frac{1}{8} \frac{1}{(16\pi)^2} \sum_a \sum_{b \neq a} m_a m_b \left\{ -\frac{32}{s_{ab}} \left(\frac{1}{r_{ab}} + \frac{1}{s_{ab}} \right) n_{ab}^i n_{ab}^j + 2 \left(\frac{r_a + r_b}{r_{ab}^3} + \frac{12}{s_{ab}^2} \right) n_a^i n_b^j \right. \\
 & + 16 \left(\frac{2}{s_{ab}^2} - \frac{1}{r_{ab}^2} \right) (n_a^i n_{ab}^j + n_b^j n_{ab}^i) + \left[\frac{5}{r_{ab} r_a} - \frac{1}{r_{ab}^3} \left(\frac{r_b^2}{r_a} + 3r_a \right) - \frac{8}{s_{ab}} \left(\frac{1}{r_a} + \frac{1}{s_{ab}} \right) \right] n_a^i n_b^j \\
 & \left. + \left[5 \frac{r_a}{r_{ab}^3} \left(\frac{r_a}{r_b} - 1 \right) - \frac{17}{r_{ab} r_a} + \frac{4}{r_a r_b} + \frac{8}{s_{ab}} \left(\frac{1}{r_a} + \frac{4}{r_{ab}} \right) \right] \delta_{ij} \right\}. \quad (\text{C15})
 \end{aligned}$$

The next-to-leading-order conservative TT part $h_{(6)ij}^{\text{TT}}$ of the metric function h_{ij}^{TT} is very complicated. It fulfills equation (4.12). The piece of $h_{(6)ij}^{\text{TT}}$ which diverges linearly at infinity in $d = 3$ dimensions equals $\Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}}$. The part of this inverse Laplacian which corresponds to the function $h_{(4)2ij}^{\text{TT}}$ can be computed in d dimensions (and for general n -point-mass systems). It reads

$$\begin{aligned}
 \Delta_d^{-1} \ddot{h}_{(4)2ij}^{\text{TT}} = & \frac{\kappa}{48(4-d)(d-1)} \frac{\partial^2}{\partial t^2} \sum_a \frac{r_a^{4-d}}{m_a} \{ [(16-d^2)(\mathbf{n}_a \cdot \mathbf{p}_a)^2 - (14-d)\mathbf{p}_a^2] \delta_{ij} + 2(7d-8)p_{ai}p_{aj} \\
 & + (4-d)[(4+d)\mathbf{p}_a^2 - (d-2)^2(\mathbf{n}_a \cdot \mathbf{p}_a)^2] n_a^i n_a^j - 2(4-d)(2d-1)(\mathbf{n}_a \cdot \mathbf{p}_a)(n_a^i p_{aj} + n_a^j p_{ai}) \}. \quad (\text{C16})
 \end{aligned}$$

The part of the inverse Laplacian $\Delta_d^{-1} \ddot{h}_{(4)ij}^{\text{TT}}$ corresponding to the function $h_{(4)0ij}^{\text{TT}}$ can be computed only in 3 dimensions and for two-point-mass systems. It reads

$$\begin{aligned}
 \Delta^{-1} \ddot{h}_{(4)0ij}^{\text{TT}} = & -\frac{1}{(16\pi)^2} \frac{\partial^2}{\partial t^2} \sum_a \sum_{b \neq a} m_a m_b \left\{ \frac{1}{8} \left[\frac{1}{r_{ab}^3} (r_a^3 - r_a^2 r_b) + \frac{5r_a}{r_{ab}} \right] \delta_{ij} + \frac{1}{12} (\partial_i \partial_{bj} + \partial_j \partial_{bi} - \partial_i \partial_{aj} - \partial_j \partial_{ai}) \frac{r_a^3}{r_{ab}} \right. \\
 & \left. + \frac{1}{144r_{ab}^3} \partial_i \partial_j [r_a^5 - r_a^3 (r_b^2 + 17r_{ab}^2)] - \frac{1}{2} \delta_{ij} \left(\Delta^{-1} \frac{1}{r_a r_b} \right) + \frac{1}{2} (8\partial_{ai} \partial_{bj} - \partial_i \partial_j) \left(\Delta^{-2} \frac{1}{r_a r_b} \right) \right\}. \quad (\text{C17})
 \end{aligned}$$

The inverse Laplacians $\Delta^{-1}(r_a^{-1} r_b^{-1})$ and $\Delta^{-2}(r_a^{-1} r_b^{-1})$ can be found in Eqs. (B28).

Other parts of the function $h_{(6)ij}^{\text{TT}}$ are determined by $\Delta_d^{-1} C_{(6)ij}^{\text{TT}}$ and they enter the density $h_{(12)}^{2,1}$ from Eq. (8.8). We list below formulas by which, together with the inverse Laplacians enumerated in Appendix B 2, one can compute the density $h_{(12)}^{2,1}$ explicitly in 3 dimensions. These formulas read

$$(\phi_{(2)} \tilde{\pi}_{(3)}^{ij})^{\text{TT}} = \frac{4m_2}{(16\pi)^2} \left(-2(p_{1i} \partial_{1j} + p_{1j} \partial_{1i}) \frac{1}{r_1 r_2} + \frac{1}{2} \partial_{1i} \partial_{1j} (\mathbf{p}_1 \cdot \nabla_1) \frac{r_1}{r_2} \right)^{\text{TT}} + (1 \leftrightarrow 2), \quad (\text{C18a})$$

$$\begin{aligned}
 (\phi_{(2)} \Delta h_{(4)ij}^{\text{TT}})^{\text{TT}} = & \frac{4m_1^2 m_2}{(16\pi)^3} \left(\frac{3}{2r_{12}} \partial_{1i} \partial_{2j} \frac{1}{r_1 r_2} + \frac{1}{2r_{12}^3} \left(13\partial_{1i} \partial_{1j} \frac{r_1}{r_2} + 15\partial_{2i} \partial_{2j} \frac{r_2}{r_1} \right) + \frac{35}{8} \partial_{1i} \partial_{1j} \frac{1}{r_1^2 r_2} \right)^{\text{TT}} \\
 & + \frac{4}{(16\pi)^2} \frac{m_1}{m_2} \left((2\mathbf{p}_2^2 \partial_{2i} \partial_{2j} - 4(p_{2j} \partial_{2i} + p_{2i} \partial_{2j})(\mathbf{p}_2 \cdot \nabla_2)) \frac{1}{r_1 r_2} + \partial_{2i} \partial_{2j} (\mathbf{p}_2 \cdot \nabla_2)^2 \frac{r_2}{r_1} \right)^{\text{TT}} \\
 & + \frac{1}{12\pi} \frac{m_1}{m_2} \left(\frac{\mathbf{p}_2^2 \delta_{ij} - 3p_{2i} p_{2j}}{r_{12}} \delta_2 \right)^{\text{TT}} + (1 \leftrightarrow 2), \quad (\text{C18b})
 \end{aligned}$$

$$\begin{aligned}
B_{(6)ij}^{\text{TT}} = & \frac{m_1^2 m_2}{(16\pi)^3} \left(\frac{3}{r_{12}} \partial_{1i} \partial_{2j} \frac{1}{r_1 r_2} - \frac{35}{8} \partial_{1i} \partial_{1j} \frac{1}{r_1^2 r_2} \right)^{\text{TT}} - \frac{4}{(16\pi)^2} \frac{m_2}{m_1} \mathbf{p}_1^2 \left(\partial_{1j} \partial_{2i} \frac{1}{r_1 r_2} \right)^{\text{TT}} \\
& + \frac{1}{(16\pi)^2} \left(\left(8(\mathbf{p}_1 \cdot \mathbf{p}_2) \partial_{1i} \partial_{2j} + 16 p_{1i} \left(\frac{1}{3} \partial_{1j} (\mathbf{p}_2 \cdot \nabla_2) - \partial_{2j} (\mathbf{p}_2 \cdot \nabla_1) \right) - 8 p_{1i} p_{2j} (\nabla_1 \cdot \nabla_2) \right) \frac{1}{r_1 r_2} \right. \\
& + \left(\partial_{1i} (\mathbf{p}_1 \cdot \nabla_1) \left(\partial_{2j} (\mathbf{p}_2 \cdot \nabla_1) - \frac{2}{3} \partial_{1j} (\mathbf{p}_2 \cdot \nabla_2) \right) + p_{2j} \partial_{1i} (\mathbf{p}_1 \cdot \nabla_1) (\nabla_1 \cdot \nabla_2) \right) \frac{r_1}{r_2} \\
& + \left(p_{1i} (\mathbf{p}_2 \cdot \nabla_2) \left(3 \partial_{2j} (\nabla_1 \cdot \nabla_2) - \frac{2}{3} \partial_{1j} \Delta_2 \right) - \partial_{1i} \partial_{2j} (\mathbf{p}_1 \cdot \nabla_2) (\mathbf{p}_2 \cdot \nabla_2) \right) \frac{r_2}{r_1} \\
& \left. + \frac{1}{4} \partial_{1i} (\mathbf{p}_1 \cdot \nabla_1) (\mathbf{p}_2 \cdot \nabla_2) \left(\frac{1}{3} \partial_{1j} \Delta_2 - \partial_{2j} (\nabla_1 \cdot \nabla_2) \right) r_1 r_2 \right)^{\text{TT}} + \left(\frac{5}{32\pi m_1} \frac{m_2 p_{1i} p_{1j}}{r_{12}} \delta_1 + \frac{1}{4} \frac{\mathbf{p}_1^2 p_{1i} p_{1j}}{m_1^3} \delta_1 \right)^{\text{TT}} + (1 \leftrightarrow 2).
\end{aligned} \tag{C18c}$$

To compute the density $h_{(12)}^{2,1}$ explicitly one has to employ Eqs. (C18) together with the following inverse Laplacians, which can be found in Eqs. (B14) and (B28):

$$\begin{aligned}
& \Delta^{-1} \delta_a, & \Delta^{-2} \delta_a, & \Delta^{-3} \delta_a, & \Delta^{-1} \frac{1}{r_1 r_2}, & \Delta^{-2} \frac{1}{r_1 r_2}, & \Delta^{-3} \frac{1}{r_1 r_2}, \\
& \Delta^{-1} \frac{r_1}{r_2}, & \Delta^{-2} \frac{r_1}{r_2}, & \Delta^{-3} \frac{r_1}{r_2}, & \Delta^{-1} r_1 r_2, & \Delta^{-2} r_1 r_2, & \Delta^{-3} r_1 r_2.
\end{aligned} \tag{C19}$$

Let us also note that after combining $B_{(6)ij}^{\text{TT}}$ and $(\phi_{(2)} \Delta h_{(4)ij}^{\text{TT}})^{\text{TT}}$ into $(B_{(6)ij} + (1/4) \phi_{(2)} \Delta h_{(4)ij}^{\text{TT}})^{\text{TT}}$, see Eq. (8.9), there is no need to compute any inverse Laplacians of $(r_1^2 r_2)^{-1}$. The inverse Laplacian $\Delta^{-1}(\phi_{(2),kl} h_{(4)kl}^{\text{TT}})$ also needed to compute $h_{(12)}^{2,1}$ explicitly, can be inferred from Eqs. (C2) and (C4).

4. Local d -dimensional UV analysis

In Sec. 3 of Ref. [31] a method of local analysis of UV divergences in d dimensions was proposed. This method was essentially used in [31] for studying the local behavior of only one specific function, namely, the momentum-independent part $h_{(4)0ij}^{\text{TT}}$ of the field function $h_{(4)ij}^{\text{TT}}$. At the 4PN level we have had to use it in many more cases; therefore, we present this method here in more detail.

The problem is to find the local behavior in d dimensions and, say, around $\mathbf{x} = \mathbf{x}_1$, of the solution of equation of the type (for some cases we have to consider the Poisson equation without TT projection)

$$(\Delta_d f)^{\text{TT}} = g^{\text{TT}}. \tag{C20}$$

We assume that the full analytic solution of this equation is not available. The source function g depends on the field point \mathbf{x} only through $\mathbf{x} - \mathbf{x}_1$ and $\mathbf{x} - \mathbf{x}_2$; therefore, one defines the auxiliary function \bar{g} as follows:

$$\bar{g}(r_1) := g(\mathbf{x} - \mathbf{x}_1, \mathbf{x} - \mathbf{x}_2) = g(r_1 \mathbf{n}_1, r_1 \mathbf{n}_1 + r_{12} \mathbf{n}_{12}), \tag{C21}$$

where we have used $\mathbf{x} - \mathbf{x}_1 = r_1 \mathbf{n}_1$ and $\mathbf{x} - \mathbf{x}_2 = r_1 \mathbf{n}_1 + r_{12} \mathbf{n}_{12}$. The method of Ref. [31] relies on expansion of the source function \bar{g} around $r_1 = 0$ and applying the operator Δ_d^{-1} to each term of the expansion. This expansion has the form

$$\bar{g}(r_1) = \sum_{k=-m}^{\infty} \frac{\bar{g}^{(k)}(\mathbf{n}_1)}{k!} r_1^k, \tag{C22}$$

where $m \geq 0$ is some nonnegative integer. With the expansion (C22) of the source term one can relate the following expansion of the solution to Eq. (C20) near $\mathbf{x} = \mathbf{x}_1$:

$$f_{\text{nonhom}}^{\text{TT}}(\mathbf{x}) = \sum_{k=-m}^{\infty} \frac{1}{k!} \Delta_d^{-1} (\bar{g}^{(k)}(\mathbf{n}_1) r_1^k)^{\text{TT}}. \tag{C23}$$

The above formula does not contain all terms of the expansion of the physically acceptable solution of Eq. (C20). The missing terms are solutions of homogeneous Poisson equation. We have devised a method to compute these terms.

Let us write the formal solution of Eq. (C20) in the form of the integral

$$f^{\text{TT}}(\mathbf{x}) = -\kappa \int d^d x' g(\mathbf{x}') (|\mathbf{x} - \mathbf{x}'|^{2-d})^{\text{TT}}. \tag{C24}$$

The crucial element of the method is expansion of the kernel of the integral (C24) around $\mathbf{x} = \mathbf{x}_1$. To do this one introduces the auxiliary function

$$K(r_1) = |\mathbf{x} - \mathbf{x}'|^{2-d} = |(\mathbf{x} - \mathbf{x}_1) - (\mathbf{x}' - \mathbf{x}_1)|^{2-d} \\ = |r_1 \mathbf{n}_1 - r'_1 \mathbf{n}'_1|^{2-d}, \quad (\text{C25})$$

and expands it around $r_1 = 0$:

$$K(r_1) = \sum_{\ell=0}^{\infty} \frac{K^{(\ell)}(\mathbf{n}_1; \mathbf{n}'_1, r'_1)}{\ell!} r_1^\ell. \quad (\text{C26})$$

One then substitutes the expansion (C26) into the integral (C24) and integrates the sum term by term. Consequently one gets the expansion of the form

$$f_{\text{hom}}^{\text{TT}}(\mathbf{x}) = -\kappa \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int d^d x' g(\mathbf{x}') (K^{(\ell)}(\mathbf{n}_1; \mathbf{n}'_1, r'_1) r_1^\ell)^{\text{TT}}. \quad (\text{C27})$$

As an example of applying the above formula let us compute the leading-order term in the expansion of the momentum-independent part $h_{(4)0ij}^{\text{TT}}$ of the function $h_{(4)ij}^{\text{TT}}$ near $\mathbf{x} = \mathbf{x}_1$. Using Eq. (C15) one easily checks that the function $h_{(4)0ij}^{\text{TT}}$ is finite for $\mathbf{x} = \mathbf{x}_1$ in $d = 3$ dimensions. This finite value cannot be obtained in d dimensions from the expansion (C23), but it follows from Eq. (C27) (with $\ell = 0$). After making use of Eq. (3.20b) [which is the source term for the function $h_{(4)ij}^{\text{TT}}$, see Eq. (4.11)] one gets

$$h_{(4)0ij}^{\text{TT}}(\mathbf{x} = \mathbf{x}_1) = \frac{(d+1)d(d-2)^3 \kappa^2}{16(d-1)^3(4-d)} m_1 m_2 \\ \times (\delta_{ij} - dn_{12}^i n_{12}^j) r_{12}^{4-2d}. \quad (\text{C28})$$

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