Magnetic structures and Z_2 vortices in a non-Abelian gauge model

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The magnetic order of the triangular lattice with antiferromagnetic interactions is described by an $SO(3)$ field and allows for the presence of Z_2 magnetic vortices as defects. In this work we show how these Z_2 vortices can be fitted into a local $SU(2)$ gauge theory. We propose simple *Ansätze* for vortex configurations and calculate their energies using well-known results of the Abelian gauge model. We comment on how Dzyaloshinskii-Moriya interactions could be derived from a non-Abelian gauge theory and speculate on their effect on nontrivial configurations.

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I. INTRODUCTION

Vortices play a central role in explaining many fundamental phenomena in condensed matter and high energy physics. The first example of vortices in theories with *local* gauge invariance was put forward by Abrikosov [1], who showed that for an intermediate range of external magnetic fields and in a certain region of the parameter space (corresponding to type II superconductors) the Ginzburg-Landau theory of superconductivity admits solutions representing a lattice of vortices.

More than 15 years later, Nielsen and Olesen discussed the role of vortices in a relativistic $U(1)$ Higgs model and pointed out their relevance in string theory in the context of high energy physics [2]. The first attempt to extend the study of vortices to the case of non-Abelian gauge groups can be found already in the pioneering paper of Nielsen and Olesen, who showed how to embed the Abelian solution in the $SU(2)$ non-Abelian case using two noncollinear scalar fields in the (three-dimensional) adjoint representation. It was soon realized that the correct topological characterization of these configurations implies that the topological charge is Z_2, unlike the Abelian case where this charge is Z. Topologically stable non-Abelian vortex solutions were found in [3–5].

Many investigations followed these ideas, exploring the properties of these types of solutions for generic $SU(N)$ groups and generalizing the Yang-Mills gauge field dynamics to include Chern-Simon-like terms which are important in connection with the statistics of elementary excitations [4,6]. A second wave of attention to vortices in theories with non-Abelian gauge invariance arose after the work in Refs. [7–9] where the authors studied the role of vortexlike solutions in models that arise from the bosonic sector of $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $SU(N) \times U(1)$ and N_f flavors of the fundamental matter multiplets (see [9] for a complete list of references).

Vortices also play a prominent role as excitations in magnetic systems. In a field theoretical language, these vortices correspond to nontrivial configurations (defects) in theories with *global* gauge invariance. The best-known example is that of vortices in the XY model which correspond to topologically nontrivial solutions of $U(1)$ sigma models and play a fundamental role in the Kosterlitz-Thouless transition in two-dimensional XY magnetic systems.

It is also known that vortices with Z_2 topological charge can appear as defects in *antiferromagnetic* (AF) spin systems in the triangular lattice since in that case the order parameter manifold is $SO(3)$ [10]. The magnetic behavior in the AF triangular lattice can be described by *three* order parameters (one for each of the three sublattices in which the triangular lattice can be partitioned) which are themselves triplets. In a way that resembles what happens in the XY model, a Kosterlitz-Thouless transition was shown to take place, although in the triangular $SO(3)$ case both low and high temperature phases have exponentially decaying correlations.

More recently, different studies of the triangular AF model with extra interactions, including Kitaev-like [11] or Dzyaloshinskii-Moriya (DM) terms [12], have shown ordered phases in a magnetic field that resemble the much-celebrated $U(1)$ Abrikosov vortex lattice, but with Z_2 vortices instead. From a field theoretical point of view, the appearance of Z_2 vortices can be understood since the magnetic behavior of the AF triangular lattice at low energies can be described by a nonlinear sigma model of an order parameter field in $SO(3)$ [13].

The questions that naturally arise are: are these vortices allowed in a (corresponding) *local* gauge theory? How are they related to the Z_2 vortices that were considered in the high energy literature?

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In this work we show that the vortices of the AF magnetic system can be easily accommodated into a local gauge theory and that, in analogy with the minimal model containing two triplets, there are two *Ansätze* that can be reduced to embeddings of the Abelian model. Using results on the energetics of vortices of the Abelian model we are able to identify the lowest energy one.

II. Z_2 VORTICES

Let us start by recovering the main results of vortices in the Abelian Higgs model. The Lagrangian density describing the system is

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\Phi)^* D^\mu\Phi - V(\Phi) \tag{1}$$

where Φ is a complex field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative. Here e is the charge of the scalar field (in the Ginzburg-Landau theory of superconductors this is twice the electric charge). The potential can be written as

$$V(\Phi) = c_4(\Phi^*\Phi)^2 - c_2\Phi^*\Phi. \tag{2}$$

As we are working in the symmetry breaking phase, we take $c_2 > 0$. We work with axially symmetric vortices, so we can ignore completely the z-dependence of the fields. We are also interested in static solutions, so we can as well forget about the t-dependence. Then, we look for configurations minimizing the energy density,

$$\mathcal{E} = \frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}(D_i\Phi)^* D_i\Phi + V(\Phi) \tag{3}$$

where latin indices i, j take values x, y. Making the *Ansatz*

$$\Phi = e^{in\phi} f(r) \qquad A(r) = -e_\phi \frac{a(r)}{r} \tag{4}$$

reduces the equations of motion to a system of coupled radial second order equations. The energy (per unit length) functional becomes

$$E_{Ab} = 2\pi \int r dr \left(\frac{1}{2r^2}\left(\frac{da(r)}{dr}\right)^2 + \frac{1}{2}\left(\frac{df(r)}{dr}\right)^2 + \frac{1}{r^2}((n + ea(r))f(r))^2 + \frac{\lambda}{4}(f(r)^2 - \eta^2)^2 \right) \tag{5}$$

where we have introduced $\lambda = 4c_4$ and $\eta^2 = c_2/(2c_4)$. Minimum energy configurations satisfy

$$\frac{d^2a}{dr^2} - \frac{1}{r}\frac{da}{dr} - e(n + ea)f^2 = 0$$
$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{(n + ea)^2}{r^2}f^2 + 2c_2 f - 4c_4 f^2 = 0 \tag{6}$$

with boundary conditions,

$$f(0) = a(0) = 0 \tag{7}$$

$$f(\infty) = \sqrt{\frac{c_2}{2c_4}} \tag{8}$$

$$a(\infty) = -\frac{n}{e}. \tag{9}$$

When the particular relation of coupling constants $8c_4 = e^2$ holds ($\lambda = e^2/2$), known as the Bogomol'nyi point, this set of equations reduces to a simpler set of first order differential equations [14,15]. At this point, the energy can be shown to satisfy a bound $E = 2n\pi\eta^2$. The Bogomol'nyi point corresponds to the case in which the scalar mass $m_H^2 = 2\lambda\eta^2$ (inverse of the coherence length) and the vector mass $m_v^2 = e^2\eta^2$ (inverse of the penetration length) are equal (i.e. the limit between type I and type II superconductors).

The simplest non-Abelian extension of the Abelian Higgs model is that in which the gauge group is $SU(2)$. The gauge fields A_μ then take values in the Lie algebra of $SU(2)$, $A_\mu = \vec{A}_\mu \cdot \vec{\sigma}/2$. In order to have topologically stable vortices, at least *two* noncollinear scalars in the adjoint (three-dimensional) representation need to be included. Thus we consider scalars $\Phi_a = \vec{\Phi}_a \cdot \vec{\sigma}/2$ ($a = 1, M$) where ($M \geq 2$)

$$L = -\frac{1}{4}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu} + \frac{1}{2}D_\mu\vec{\Phi}_a D^\mu\vec{\Phi}_a - V(\vec{\Phi}_a) \tag{10}$$

where

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + e\vec{A}_\mu \times \vec{A}_\nu \tag{11}$$

$$D_\mu\vec{\Phi}_a = \partial_\mu\vec{\Phi}_a + e\vec{A}_\mu \times \vec{\Phi}_a. \tag{12}$$

The choice of the symmetry breaking potential $V(\vec{\Phi}_a)$ will be discussed below.

The $M = 2$ (two-triplets) case is the best known in the literature. In this case, two possible *Ansätze* are known, the first one, originally proposed in [3], takes the form

$$\vec{\Phi}_1 = f(r)(-\sin n\theta, \cos n\theta, 0)$$
$$\vec{\Phi}_2 = f(r)(\cos n\theta, \sin n\theta, 0)$$
$$\vec{A}_\theta = -\left(0, 0, \frac{a(r)}{r}\right). \tag{13}$$

It was later realized that another simpler *Ansatz* could be made [6,16]:

$$\vec{\Phi}_1 = f(r)(-\sin n\theta, \cos n\theta, 0)$$
$$\vec{\Phi}_2 = f(r)(0, 0, 1)$$
$$\vec{A}_\theta = -\left(0, 0, \frac{a(r)}{r}\right). \quad (14)$$

Although in principle one could consider arbitrary n, only vortices with odd n are topologically nontrivial, this corresponding to a Z_2 homotopy class. Also, vortices with $n = \pm 1$ have lower energies. Moreover, it has been shown in [17] that vortices corresponding to *Ansatz* (14) have lower energy; hence those associated to *Ansatz* (13) are unstable (they will decay into the former ones).

Vortices in the Abelian Higgs model can be considered as the local gauge counterpart of the vortices of the XY model, characterized by the Hamiltonian,

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \quad (15)$$

where the winding of the polar angle of the two-dimensional spin $\vec{S} = S_x \vec{e}_x + S_y \vec{e}_y$ can be associated with the winding of the complex scalar $\Phi = \Phi_1 + i\Phi_2$. Unlike the case in local gauge theories, vortices in the XY model have a logarithmically divergent energy $E \sim \log(L/a)$ where L represents a characteristic size of the system and a is the lattice spacing.

More sophisticated vortex structures can appear in other magnetic systems. That is the case of the antiferromagnetic Heisenberg model in the triangular lattice,

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \quad (16)$$

where now $\vec{S}$ is a *three*-dimensional vector $\vec{S} = S_x \vec{e}_x + S_y \vec{e}_y + S_z \vec{e}_z$ and ij denote neighbors in the triangular lattice. Let us denote by A, B, C the corners of a plaquette Δ; then

$$H_\Delta = J(\vec{S}_A \cdot \vec{S}_B + \vec{S}_B \cdot \vec{S}_C + \vec{S}_C \cdot \vec{S}_A) \quad (17)$$

or

$$H_\Delta = (\vec{S}_A + \vec{S}_B + \vec{S}_C)^2 - |\vec{S}_A|^2 - |\vec{S}_B|^2 - |\vec{S}_C|^2. \quad (18)$$

Then, as the modulus of the spins are fixed, the minimum energy configuration is obtained when

$$\vec{S}_A + \vec{S}_B + \vec{S}_C = 0. \quad (19)$$

Thus the vacuum determines a 120 degree symmetry structure of the spin configuration (see Fig. 1). As shown by Kawamura and Miyashita [10], vortices can appear as magnetic excitations in such systems and they are characterized by a Z_2 topological charge. They also discuss two possible kinds of vortex configurations. In both of them, vortices are coplanar in each point, but

(i) In type I vortices, the three vectors are always in the same plane while they wind around the vortex center (see Fig 2).
(ii) In type II vortices, one of the vectors is constant, while the other two wind around the vortex center (see Fig 3).

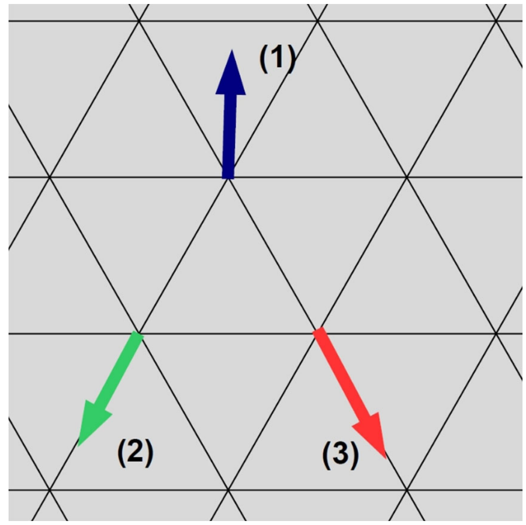

FIG. 1 (color online). Vacuum configuration for the AF triangular lattice. The three order parameters (denoted with different colors and numbers) are coplanar and form a 120 degree structure in the triangular lattice.

The energy of the vortex configurations is presented in [10] where it is shown that type II vortices have lower energies.

Inspired by these vortices in the antiferromagnetic triangular lattice, we consider the $SU(2)$ gauge model with the three triplets field $\vec{\Phi}_a$, so $M = 3$ and $a = 1, 2, 3$. Each field $\vec{\Phi}_a$ has three components. In the gauge theory language, these are internal indices in the Lie algebra while in the magnetic model they refer to components in space. In order to impose on the vacuum a 120 degree symmetry structure, we take a potential of the form

$$VC = \lambda_1(\vec{\Phi}_1 \cdot \vec{\Phi}_1 - \eta_1^2)^2 + \lambda_2(\vec{\Phi}_2 \cdot \vec{\Phi}_2 - \eta_2^2)^2 + \lambda_3(\vec{\Phi}_1 \cdot \vec{\Phi}_1 - \eta_3^2)^2 + V_{\text{mix}}(\vec{\Phi}_a) \quad (20)$$

where

$$V_{\text{mix}}(\vec{\Phi}_a) = \mu^2(\vec{\Phi}_1 + \vec{\Phi}_2 + \vec{\Phi}_3)^2 + \lambda_4(\vec{\Phi}_1 + \vec{\Phi}_2 + \vec{\Phi}_3)^4. \quad (21)$$

It is clear that if we take $\lambda_i > 0$, $\mu^2 > 0$ and $\eta_1 = \eta_2 = \eta_3 \equiv \eta$, then the vacuum corresponds to $|\Phi_i| = \eta^2$ and $\vec{\Phi}_1 + \vec{\Phi}_2 + \vec{\Phi}_3 = 0$ (for this last condition that ensures the 120° structure we do need $\mu^2 > 0$). The first term in V_{mix} is

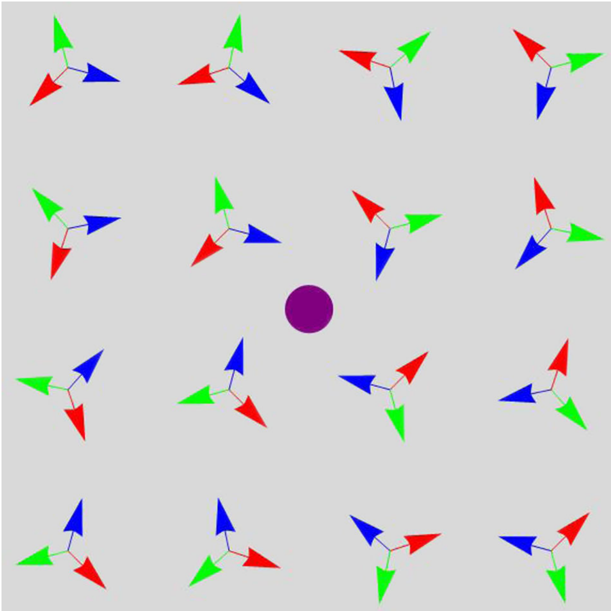

FIG. 2 (color online). Schematic top view of a type I vortex. In this *Ansatz* the three triplets (blue, red, green) are coplanar (in the XY plane) and wind around the core of the vortex (represented as a disk).

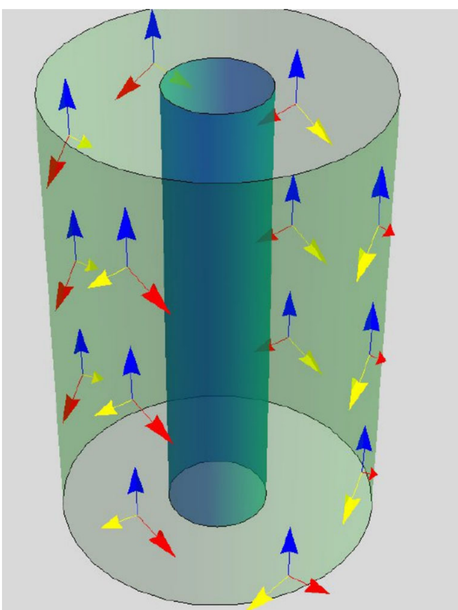

FIG. 3 (color online). Three-dimensional view of a type II vortex. The three triplets (blue, red, yellow) are tangent to cylinders that have the center at the core of the vortex. We have represented the region of intense chromomagnetic field with a darker color.

analogous to the Heisenberg interaction in antiferromagnets. We have included the term with λ_4 as it is compatible with renormalization and does not change the main results of our work. A possible vacuum configuration is illustrated in Fig. 2, which is exactly the same as in the triangular lattice.

In order to find vortex configurations in the $SU(2)$ gauge model, we need to solve the field equations arising from Lagrangian (10),

$$D^\alpha \vec{F}_{\alpha\mu} = eD_\mu \vec{\Phi}_i \times \vec{\Phi}_i \tag{22}$$

$$D_\mu D^\mu \vec{\Phi}_i = -\frac{\delta V}{\delta \vec{\Phi}_i}. \tag{23}$$

The idea is to propose an *Ansatz* and determine whether the field equations can be reduced to a simpler, self-consistent system of ordinary differential equations. Inspired by the (global) vortices of the antiferromagnetic triangular lattice and the vortices of the $SU(2)$, $M = 2$ model described above we propose the following type I *Ansatz*,

$$\begin{aligned}
\vec{\Phi}_1 &= f(r)(-\sin n\theta, \cos n\theta, 0)\\
\vec{\Phi}_2 &= f(r)\left(-\sin\left(n\theta + \frac{2\pi}{3}\right), \cos\left(n\theta + \frac{2\pi}{3}\right), 0\right)\\
\vec{\Phi}_3 &= f(r)\left(-\sin\left(n\theta + \frac{4\pi}{3}\right), \cos\left(n\theta + \frac{4\pi}{3}\right), 0\right)\\
\vec{A}_\theta &= -\left(0, 0, \frac{a(r)}{r}\right)
\end{aligned} \tag{24}$$

with $n \in \mathbb{Z}$. Notice that this *Ansatz* implies that

$$\vec{\Phi}_1(r,\theta) + \vec{\Phi}_2(r,\theta) + \vec{\Phi}_3(r,\theta) = 0. \tag{25}$$

At each point, the three triplets are then at 120 degrees and they are always in the same plane while they wind around the origin of the vortex (the origin). The main differences with the type I magnetic vortex are of course that now we have a gauge field which we have chosen in the third direction, and that the moduli of the triplets are constant only at infinity, where they tend to a minimum of the potential. We have made a schematic representation of the solutions in Fig. 2.

Notice that no terms arising from $V_{\rm mix}$ appear in the field equations since $\delta V_{\rm mix}/\delta\Phi_i$ is a polynomial in powers of $(\vec{\Phi}_1 + \vec{\Phi}_2 + \vec{\Phi}_3)$ so that it vanishes. One then has

$$\frac{\delta V}{\delta \vec{\Phi}_a} = 4\lambda f(r)(f^2 - \eta^2)\vec{\Phi}_a \tag{26}$$

so that the equations of motion for the scalar fields

$$\Box\vec{\Phi}_a - 2e\frac{a}{r}\vec{\Phi}_a - e^2\frac{a^2}{r^2}\vec{\Phi}_a = -\frac{\delta V}{\delta\vec{\Phi}_a} \tag{27}$$

reduce to the radial equation

$$f'' + \frac{1}{r}f' - \frac{1}{r^2}(n+ea)^2 f = 4\lambda f(r)(f^2-1) \tag{28}$$

which coincides, apart from a numerical factor in the rhs, with the radial equation for the Abelian Higgs model scalar equation of motion.

Concerning the scalar current, once the *Ansatz* is inserted it takes the simple form

$$\vec{J}_\theta = eD_\theta\vec{\Phi}_a \times \vec{\Phi}_a = -\frac{e}{r}f^2(n+ea)(0,0,1) \tag{29}$$

so that the gauge field radial equation of motion also reduces to the Abelian model one.

The conditions to ensure finite-energy configurations are

$$f(0)=0 \qquad a(0)=0$$

$$\lim_{r\to\infty} f(r) = \eta \qquad \lim_{r\to\infty} a(r) = -\frac{n}{e}. \tag{30}$$

Finally, the energy of static configurations satisfying the *Ansatz* (24) is given by

$$E = \int d^2x\left(\frac{1}{4}\vec{F}_{lm}\cdot\vec{F}_{lm} + \frac{1}{2}D_l\vec{\Phi}_i\cdot D_l\vec{\Phi}_i + V\right) \tag{31}$$

or

$$\begin{aligned} E = \int d^2x &\left(\frac{1}{2r^2}(\partial_r a(r)\right)^2 \\ &+ \frac{3}{2}\left(\partial_r f(r)^2 + \frac{1}{r^2}((n+ea(r))f(r)\right)^2 \\ &+ 3\frac{\lambda}{4}(f^2-\eta^2)^2. \end{aligned} \tag{32}$$

Redefining $r=\kappa\rho$ we end up with

$$\begin{aligned} E = 2\pi\frac{1}{\kappa^2}\int &\left(\frac{1}{2\rho^2}(\partial_\rho a(\rho))^2\right. \\ &+ \frac{3\kappa^2}{2}\left(\partial_\rho h(\rho)^2 + \frac{1}{\rho^2}(n+ea(\rho))^2 h(\rho)\right)^2 \\ &+ \frac{3\kappa^4\lambda}{4}(f^2-\eta^2)^2. \end{aligned} \tag{33}$$

Thus, choosing $\kappa^2 = 1/3$, the energy functional which we shall denote $E^{(1)}$ becomes, apart from a factor, identical to the Abelian one, Eq. (5), but with $\lambda \to \lambda/3$:

$$E^{(I)} = 3E_{Ab}(\lambda/3, e, n, \eta). \tag{34}$$

We can next take an *Ansatz* inspired in the type II vortices. We then consider

$$\begin{aligned} \vec{\Phi}_1 &= (0,0,\eta) \\ \vec{\Phi}_2 &= \frac{1}{2}(\sqrt{3}f(r)\sin(n\theta), \sqrt{3}f(r)\cos(n\theta)), -\eta) \\ \vec{\Phi}_3 &= \frac{1}{2}(-\sqrt{3}f(r)\sin(n\theta), -\sqrt{3}f(r)\cos(n\theta)), -\eta) \\ \vec{A}_\theta &= -\left(0,0,\frac{a(r)}{r}\right). \end{aligned} \tag{35}$$

As before, Eq. (25) holds and the triplets are at 120 degrees. In this case the field $\vec{\Phi}_1$ does not contribute to the energy and $D_\mu\vec{\Phi}_a$ is projected into the (1,2) plane. Within this *Ansatz*, for each point, the triplets live in the tangent plane of a cylinder with its center at the vortex core, and one of the triplets is everywhere constant (see Fig. 2). As before, one can see that inserting the *Ansatz* in the equations of motion reduces them to a system of coupled ordinary differential equations, which after scaling can be related to the Abelian model ones. As for the energy, plugging the *Ansatz* into the energy functional one obtains

$$\begin{aligned} E = \int d^2x &\left(\frac{1}{2r^2}(\partial_r a(r))^2\right. \\ &+ \frac{3}{4}(\partial_r f(r)^2 + \frac{1}{r^2}((n+ea(r))f(r))^2 \\ &+ 2\frac{\lambda}{4}(f^2-\eta^2)^2 \end{aligned} \tag{36}$$

which, after the rescaling $r=\kappa\rho$, becomes

$$\begin{aligned} E = 2\pi\frac{1}{\kappa^2}\int &\left(\frac{1}{2\rho^2}(\partial_\rho a(\rho)\right)^2 \\ &+ \frac{3\kappa^2}{4}\left(\partial_\rho h(\rho)^2 + \frac{1}{\rho^2}(n+ea(\rho))^2 h(\rho)\right)^2 \\ &+ \frac{2\kappa^4\lambda}{4}(f^2-\eta^2)^2. \end{aligned} \tag{37}$$

Thus, choosing,

$$\kappa^2 = 2/3 \tag{38}$$

one finally gets

$$\begin{aligned} E = 2\pi\frac{3}{2}\int &\left(\frac{1}{2\rho^2}(\partial_\rho a(\rho)\right)^2 \\ &+ \frac{1}{2}\left(\partial_\rho h(\rho)^2 + \frac{1}{\rho^2}(n+ea(\rho))^2 h(\rho)\right)^2 \\ &+ \frac{2\lambda}{9}(f^2-\eta^2)^2 \end{aligned} \tag{39}$$

leading to

$$E^{(II)} = \frac{3}{2} E_{Ab}\left(\frac{8}{9}\lambda, e, n, \eta\right). \tag{40}$$

In order to compare the energies of these two *Ansätze* we can borrow some results from the Abelian model. First, the energy is an increasing function of n. As in the $M = 2$ case, there is only one class of topologically nontrivial configurations that we consider, $n = \pm 1$. Second, a simple dimensional analysis shows that

$$E_{Ab}(\lambda, e, \eta) = \eta^2 \epsilon(\lambda/e^2). \tag{41}$$

Now, as is well known, for generic values of λ/e^2 there is no analytical result for $\epsilon(\lambda/e^2)$ except at the Bogomol'nyi point, for which $\epsilon(1/2) = 2\pi$. For other λ/e^2 values, a numerical calculation is required. We can use the accurate result of the variational calculation presented in [18]

$$\epsilon = 2.38\pi\left(\frac{\lambda}{e^2}\right)^{\alpha} \tag{42}$$

with $\alpha = 0.195\ldots$. Using this value for the case at hand we find for the ratio of E^I and E^{II} energies as given by Eqs. (34), (40)

$$\frac{E^{(I)}}{E^{(II)}} = 1.65\ldots \tag{43}$$

Then, as in the global case [19] and in the $SU(2)$ gauge theory with the two Higgs scalar model discussed in [16], the *Ansatz* containing one "constant" Higgs scalar leads to the solution having the lowest energy.

Note that the first term in the energy integral (33) can be interpreted as the radial component of magnetic field defined as

$$B \equiv \frac{1}{2}\varepsilon_{ijk}\frac{\vec{\Phi}_3}{\eta}\cdot \vec{F}_{jk} = \partial_r a(r)/r. \tag{44}$$

In view of the boundary conditions (9) the resulting vortex magnetic magnetic flux $\mathcal{F}_B$ is quantized in units of $2\pi/e$,

$$\mathcal{F}_B \equiv \int d^2x B = -\frac{2\pi}{e} n, \qquad n \in \mathbb{Z}. \tag{45}$$

Since the invariant group of the vacuum associated to *Ansätze* (24) and (35) is Z_2, the relevant homotopy group is $\Pi_1(SU(2)/Z_2) = Z_2$. The corresponding topological charges can be calculated via the Wilson loop

$$Q = \frac{1}{2}\,\mathrm{Tr}\exp\left(i\oint_{C_\infty} A_\mu dx^\mu\right) \tag{46}$$

with Tr the $SU(2)$ trace and C_∞ a closed curve at infinity. Both in the case of type I and type II vortices this gives

$$Q = \frac{1}{2}\,\mathrm{Tr}\exp\left(\frac{i}{2}\oint_{S^1} d\theta a(r=\infty,\theta)\sigma_3\right) = \frac{1}{2}\,\mathrm{Tr}\exp(i\pi n\sigma_3) = (-1)^n. \tag{47}$$

Hence we conclude that there are two topologically inequivalent configurations, the "trivial" $Q = 1$ ones ($n = 2k$) and those with $Q = -1$ for the "non-trivial" ones ($n = 2k + 1$). Notice that the fact of being topologically nontrivial does not ensure stability. Indeed we have shown that the type I *Ansatz* is topologically nontrivial but unstable towards decay into a type II *Ansatz*.

Following Kawamura and Miyashita [10] we can also define the vector chirality

$$\vec{\kappa} = \frac{2}{3\sqrt{3}}(\check{\Phi}_1 \times \check{\Phi}_2 + \check{\Phi}_2 \times \check{\Phi}_3 + \check{\Phi}_3 \times \check{\Phi}_1) \tag{48}$$

with

$$\check{\Phi}_i = \frac{\Phi_i}{|\Phi_i|}. \tag{49}$$

For type I vortices, this gives

$$\vec{\kappa} = (0, 0, 1) \tag{50}$$

while for type II vortices one has

$$\vec{\kappa} = (-\cos\theta, \sin\theta, 0). \tag{51}$$

We see that in the type I vortex the chirality vector is fixed and perpendicular to the plane where the 120 degree structure lies while in the type II case vector $\vec{\kappa}$ rotates around the vortex core.

III. CONCLUSIONS

In this work we have analyzed vortex solutions in a non-Abelian $SU(2)$ gauge model with three matter fields in the adjoint representation (triplets). Our original motivation was to determine whether the global vortices of the triangular antiferromagnetic lattice [10] can be conveniently fitted into a local gauge theory. We have shown that this is indeed the case and the vortex solutions that we have constructed share many similarities to those of the minimal theory with *two* triplets that have been considered for many years in the context of high energy models.

Vortices in a local $SU(2)$ gauge theory with $M = 3$ matter fields have also been recently considered in the context of QCD [20]. Notice however that our model and *Ansätze* are different: in [20] the triplets determine a frame of three orthogonal vectors and a different potential is chosen. Our model bears also many similarities to those discussed in the case of three-component superconductors, although in those systems the three order parameters are complex fields (rather than triplets) and the gauge field is Abelian [21].

Coming back to the original motivation of the magnetic analogy, let us point out that the antiferromagnetic triangular lattice has been recently the focus of attention in connection to the existence of vortex and Skyrmion lattices [11,12]. As shown in numerical simulations of the Heisenberg model in this lattice, the inclusion of Kitaev type and DM interactions can induce vortex and Skyrmion lattice phases in some region of the parameter space. In the continuum description of the Heisenberg model in the square lattice and for certain choices of the DM vectors, the DM interaction corresponds to a term in the energy of the form

$$\mathcal{E}_M = D\epsilon_{ijk}\Phi_i\nabla_j\Phi_k \tag{52}$$

where in the magnetic case, Φ_i corresponds to the components of the order parameter living in S^2 [22].

Notice that a term of this kind naturally arises in a theory with non-Abelian gauge fields. Indeed, consider the covariant derivative energy density part of the energy functional,

$$\mathcal{E}_{cd} = |(\nabla_i\vec{\Phi} + e\vec{A}_i \times \vec{\Phi})|^2 \tag{53}$$

$$\mathcal{E}_{cd} = |\nabla_i\vec{\Phi}|^2 + |e\vec{A}_i \times \vec{\Phi}|^2 + 2e\partial_i\vec{\Phi}\cdot(\vec{A}_i \times \vec{\Phi}). \tag{54}$$

The last term is

$$2e\nabla_i\vec{\Phi}\cdot(\vec{A}_i \times \vec{\Phi}) = 2e\nabla_i\Phi_l\epsilon_{lmn}A_{im}\Phi_n. \tag{55}$$

Thus, if we choose, $A_{im} = \gamma\delta_{im}$ (in A_{im} the first subindex denotes the space index and the second the Lie algebra component),

$$2e\nabla_i\vec{\Phi}\cdot(\vec{A}_i \times \vec{\Phi}) = -2e\gamma\nabla_i\Phi_l\epsilon_{nil}\Phi_n. \tag{56}$$

This is exactly the Moriya term with $D = -2\gamma$. The second term in (54) is just an irrelevant quadratic factor $e^2\gamma^2\Phi_l\Phi_l$.

Thus, the Moriya term appears as a result of a constant $SU(2)$ background vector potential. Notice that a constant vector potential is *not* trivial in a non-Abelian theory. Indeed, for our purposes it is enough to take $A_{11} = \gamma = A_{22}$, thus giving it a constant magnetic field $B_{33} = e\gamma^2$ which is in the third direction in the Lie algebra and in the z-direction of space. Thus a Moriya term can be incorporated by choosing a constant chromomagnetic field (it is the equivalent of the Landau problem for a non-Abelian theory). Interestingly, a similar argument is used to introduce Rashba interactions for triplets in the context of cold atoms [23] where non-Abelian gauge fields can be engineered using laser beams.

Notice that the Moriya type term can be easily generalized to the case of a theory containing three triplets ($a = 1, 2, 3$),

$$\mathcal{E}_M^{(1)} = D_1\epsilon_{ijk}\Phi_{ia}\nabla_j\Phi_{ka}. \tag{57}$$

One could also include a term of the form

$$\mathcal{E}_M^{(2)} = D_2\epsilon_{ijk}\epsilon_{abc}\Phi_{ia}\Phi_{jb}\Phi_{kc} \tag{58}$$

which preserves the global $SO(3)$ invariance of the theory. Of course, one could combine these two terms in a non-Abelian Chern-Simons type term if one were willing to interpret $\vec{\Phi}_a$ as a vector field.

Our work could be extended in many directions. One direct generalization would be to include a non-Abelian Chern-Simons term for the $\vec{A}_i$ field. Such a term is interesting since it alters the statistics of excitations and binds "chromoelectric" charge to the vortices. Is is easy to show that the *Ansätze* that we have presented work equally fine for this case, although the analysis of the energetics of the different *Ansätze* might require some numerical work.

Another interesting issue is to determine the role of terms like those in Eqs. (57)–(58) in the properties of the solutions. If the magnetic analogy would carry through the local gauge theory, then one would expect that these types of terms play a fundamental role in the appearance of vortex and Skyrmion lattices. We expect to report on these issues on a future work.

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