Generalization of coherent state quantization in higher dimensional de Sitter space

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We have shown coherent state quantization of a particle in a maximally symmetric curved space-time i.e. in de Sitter space. As the coherent states are eigenvectors of the lowering operators, we have constructed the raising and lowering operators with the help of recurrence relation satisfied by the associated Legendre polynomial. These lowering operators have been used to describe an explicit form of coherent states followed by coherent states quantization in two-dimensional de Sitter space. Coherent states and their quantization are also generalized in D-dimensional de Sitter space.

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I. INTRODUCTION

De Sitter space is a unique maximally symmetric curved space-time and it has been widely discussed [1–6]. This space is the simplest vacuum solution to the Einstein equation with a positive cosmological constant and it is characterized by constant positive curvature (cosmological constant). Quantization of a particle in curved space [7–9] is not simple because relativistic quantum theory is not constructed as one particle theory. In fact, it is a theory of one or more quantized fields for the system of many particles. Various approaches for solving the quantization problem have been proposed [10–13]. Positive frequency solutions of the Klein-Gordon equation in the de Sitter space-time in arbitrary dimensions for massless scalar field and quantization using field theory approach have been shown by Lokas [14].

Quantization associates an algebra of classical observables with an algebra of quantum observables. There exists superposition of quantum states which have many features analogous to their classical counterparts called coherent states, studied by Schrödinger [15]. Later on, further investigations of coherent states were undertaken [16,17]. In a seminal work, Glauber [18,19] has shown that coherent states provide an adequate means for a quantum description of coherent laser light beams. Coherent states in quantum physics, generalized coherent states and their applications are widely discussed [20–22]. Quantization through coherent states and its various generalizations have also been discussed [23–27]. In the present study, we derived coherent states by using the properties of the associated Legendre polynomial and generalized these states and their quantization in D-dimensional de Sitter space. The coherent state quantization we present is equivalent to the usual construction with the so-called Bunch-Davies vacuum [1].

The paper is organized as follows. In Sec. II, we review the characteristics of a higher dimensional de Sitter space. Section III is devoted to the solution of the Klein-Gordon equation in D-dimensional space. We obtain this equation as the associated Legendre differential equation i.e. a hypergeometric equation. The study of the hypergeometric equation is important because the hypergeometric function contains all of the orthogonal polynomials. Since each of the orthogonal polynomials defines a complete basis in which the wave function of any quantum mechanical system can be expanded, the applications of these polynomials are quite important in any algebraic study of a quantum mechanical system. In Sec. IV, some recurrence relations of the associated Legendre polynomial have been used to construct the raising and lowering operator. The analysis has been carried out to obtain raising and lowering operators for the associated Legendre polynomial much like the creation and annihilation operators for the harmonic oscillator. Since coherent states are the eigenstates of the lowering operator, it can be expanded in terms of a complete set of basis. Therefore, we have used the lowering operator to obtain coherent states for two-dimensional de Sitter space given by the $SO(2,1)$ symmetry group. Explicit systems of associated Legendre polynomials and corresponding coherent states have been shown in Sec. V. Lorentz boosts, rotations and their coherent state quantization have been described in Sec. VI. In Sec. VII, we have obtained coherent states and quantization in three-dimensional de Sitter space. Generalization of coherent states and quantization in D-dimensional space is presented in Sec. VIII.

II. DE SITTER SPACE

De Sitter space is a unique maximally symmetric curved space-time. D-dimensional de Sitter space can be represented as a hyperboloid, which is placed into (D + 1)-dimensional Minkowski space. If $\eta_{pq} = \mathrm{diag}(1, -1, -1, \ldots, -1)$ and a is the de Sitter radius, then we have the following relation,

$$\eta_{pq} X^p X^q = -a^2 \tag{1}$$

with metric

$$ds_D^2 = \eta_{pq} dX^p dX^q. \tag{2}$$

Here ds_D^2 is defined as a de Sitter metric in a D-dimensional hyperboloid embedded in flat $(D+1)$-dimensional spacetime. A convenient choice of coordinates for satisfying Eq. (1) is

$$X^0 = a\sinh\left(\frac{t}{a}\right), \quad X^i = w^i a\cosh\left(\frac{t}{a}\right), \quad i = 1, 2, \ldots, D \tag{3}$$

where $-\infty < t < -\infty$ and w^i satisfies the following property,

$$\sum_{i=1}^{D} (w^i)^2 = 1 \tag{4}$$

which corresponds to

$$\begin{aligned}
w^1 &= \cos\theta_1, \quad 0 \le \theta_1 < \pi; \\
w^2 &= \sin\theta_1 \cos\theta_2, \quad 0 \le \theta_2 < \pi; \\
&\vdots \\
w^{D-2} &= \sin\theta_1 \cos\theta_2 \ldots \sin\theta_{D-3} \cos\theta_{D-3}, \quad 0 \le \theta_{D-3} < \pi; \\
w^{D-1} &= \sin\theta_1 \cos\theta_2 \ldots \sin\theta_{D-2} \cos\theta_{D-1}, \quad 0 \le \theta_{D-2} < \pi; \\
w^{D} &= \sin\theta_1 \cos\theta_2 \ldots \sin\theta_{D-2} \sin\theta_{D-1}, \quad 0 \le \theta_{D-1} < 2\pi.
\end{aligned} \tag{5}$$

Now, in terms of these coordinates, metric (2) becomes

$$ds_D^2 = dt^2 - a^2 \cosh^2\left(\frac{t}{a}\right) d\Omega_{D-1}^2 \tag{6}$$

where $d\Omega_{D-1}^2$ is a $(D-1)$-dimensional solid angle given by

$$\begin{aligned}
d\Omega_{D-1}^2 &= \sum_{j=1}^{D-1} \left(\prod_{i=1}^{j-1} \sin^2\theta_i \right) d\theta_j^2 \\
&= d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \ldots \sin^2\theta_{D-2} d\theta_{D-1}^2.
\end{aligned} \tag{7}$$

In these coordinates, ds_D looks like a $(D-1)$ sphere which starts out infinitely large at $t \to -\infty$, then shrinks to a minimal finite size at $t = 0$ and grows again to infinite size as $t \to +\infty$.

III. SOLUTION OF THE KLEIN-GORDON EQUATION FOR D-DIMENSIONAL DE SITTER SPACE

The Klein-Gordon equation in D-dimensional space [1,28,29] is given by

$$(\Box + m^2)\psi(t, \Omega) = 0 \tag{8}$$

where D'Alembertian $\Box = \frac{\partial^2}{\partial t^2} + \frac{(D-1)}{a}\tanh(\frac{t}{a})\frac{\partial}{\partial t} - \frac{L_{D-1}^2}{\cosh^2(\frac{t}{a})}$ in which L_{D-1}^2 denotes the Casimir operator of $SO(D-1)$, $\psi(t, \Omega) = f(t)Y(\Omega)$, and $Y(\Omega)$ are spherical harmonics [30] on S^{D-1} obeying

$$L_{D-1}^2 Y(\Omega) + n(n + D - 2)Y(\Omega) = 0. \tag{9}$$

where n denotes the eigenvalue of the angular momentum operator. By using separation of variable in Eq. (8) and using Eq. (9), we can write Eq. (8) as

$$\begin{aligned}
&f''(t) + \frac{(D-1)}{a}\tanh\left(\frac{t}{a}\right) f'(t) \\
&\quad + \left(m^2 + \frac{n(n + D - 2)}{a^2 \cosh^2(\frac{t}{a})}\right) f(t) = 0.
\end{aligned} \tag{10}$$

To find the solutions of differential equation (10), we introduce a new independent variable $x = i\sinh(\frac{t}{a})$ and let $f(x)$ denote the solution of the resultant equation; then Eq. (10) changes to

$$\left[(1 - x^2)\frac{\partial^2}{\partial x^2} - Dx\frac{\partial}{\partial x} - m^2 a^2 - \frac{n(n + D - 2)}{(1 - x^2)}\right] f(x) = 0. \tag{11}$$

In order to simplify the solution of the above differential equation, we substitute $f(x) = (1 - x^2)^{\frac{2-D}{4}} y(x)$, which converts the differential equation (11) into an associated Legendre equation as

$$\begin{aligned}
&\left[\frac{d}{dx}(1 - x^2)\frac{d}{dx} + \frac{D(D-2) + 4m^2 a^2}{4} - \frac{(2n + D - 2)^2}{4(1 - x^2)}\right] \\
&\quad \times y(x) = 0.
\end{aligned} \tag{12}$$

The solutions of Eq. (12) are associated Legendre functions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ as

$$y(x) = C_1 P_\nu^\mu(x) + C_2 Q_\nu^\mu(x) \tag{13}$$

with $\nu(\nu + 1) = \frac{1}{4}(D(D-2) + 4m^2 a^2)$, where the coefficients ν and μ are given by

$$\begin{aligned}
\nu &= \frac{1}{2}\left(-1 + \sqrt{(D-1)^2 + 4m^2 a^2}\right); \\
\mu_n &= \mu = \frac{1}{2}(2n + D - 2).
\end{aligned} \tag{14}$$

Equation (13) contains two linearly independent solutions $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$. The $P_\nu^\mu(x)$ functions are bounded within the interval $-1 \le x \le 1$ while the $Q_\nu^\mu(x)$ are unbounded at

$x = \pm 1$. Divergences of $Q_\nu^\mu(x)$ are nonsquare integrable ones. Therefore, we proceed with $P_\nu^\mu(x)$ to obtain raising and lowering operators. Thus, we have seen that the D-dimensional Klein-Gordon equation has the solution in the form of the associated Legendre polynomial. Recently, recurrence relations of hypergeometric function have been used to obtain raising and lowering operators [31]. In what follows, we obtain raising and lowering operators for the associated Legendre polynomial.

IV. RAISING AND LOWERING OPERATORS FOR THE ASSOCIATED LEGENDRE POLYNOMIAL

Raising and lowering operators for associated Legendre polynomials play the same role for this system as the Hermite polynomials for the standard boson oscillator. With the help of recursion relations for the associated Legendre polynomials [32], we can determine the raising and lowering operators. We obtain the following recursion relation for associated Legendre polynomials,

$$\begin{aligned} A_\nu \bar{P}_\nu^\mu &= \sqrt{(\nu+\mu+1)(\nu-\mu+1)}\bar{P}_{\nu+1}^\mu; \\ A_{\nu+2}^\dagger \bar{P}_{\nu+1}^\mu &= \sqrt{(\nu+\mu+1)(\nu-\mu+1)}\bar{P}_\nu^\mu; \end{aligned} \tag{15}$$

where $\bar{P}_\nu^\mu = \sqrt{\frac{\mu(\nu-\mu)!}{(\nu+\mu)!}}P_\nu^\mu$ is the normalized associated Legendre polynomial, and A_ν and $A_{\nu+2}^\dagger$ are raising and lowering operators with respect to ν, defined as

$$\begin{aligned} A_\nu &= (z^2-1)\frac{d}{dz} + (\nu+1)z; \\ A_{\nu+2}^\dagger &= -(z^2-1)\frac{d}{dz} + (\nu+1)z. \end{aligned} \tag{16}$$

The vacuum state of a system is the state with the lowest energy, and can be calculated by using the definition of the annihilation operator. The annihilation operator defined in Eq. (16) yields a differential equation for the vacuum state as

$$\left[-(z^2-1)\frac{d}{dz} + (\nu+1)z\right]\left|0\right\rangle = 0, \tag{17}$$

which gives the following vacuum state

$$|0\rangle = C(z^2-1)^{\frac{(\nu+1)}{2}}, \tag{18}$$

where C is a normalization constant.

Our goal is to describe coherent states with the help of the associated Legendre polynomial. We can see from Eq. (15) that μ is fixed and ν is changing. Therefore, we expand coherent states in terms of μ; to do this we need the lowering operator independent of μ. Therefore, consider operators a_ν and $a_\nu^\dagger$ such that these satisfy the following properties [33,34],

$$a_\nu = A_\nu U_\nu^{-1}, \qquad a_\nu^\dagger = U_\nu A_\nu^\dagger \tag{19}$$

where U_ν is the unitary operator obeying

$$U_\nu \bar{P}_\nu^\mu = e^{-i\epsilon(\lambda_{\nu+1}-\lambda_\nu)}\bar{P}_{\nu+1}^{\mu+1} \tag{20}$$

with $\lambda_\nu = \nu(\nu+1)$ and ϵ being real number. By using Eqs. (19)–(20), we see that the lowering operator given in Eq. (15) satisfies the following relation:

$$a_{\nu+1}\bar{P}_\nu^\mu = \sqrt{(\nu+\mu-1)(\nu-\mu+1)}e^{i\epsilon(\lambda_\nu-\lambda_{\nu-1})}\bar{P}_\nu^{\mu-1}. \tag{21}$$

If we consider $\mu \in N$ and $\bar{P}_\nu^{\mu-\nu} = |\mu\rangle$ then Eq. (21) becomes

$$a_{\nu+1}|\mu\rangle = \sqrt{(\mu-1)(2\nu-\mu+1)}e^{i\epsilon(\lambda_\nu-\lambda_{\nu-1})}|\mu-1\rangle. \tag{22}$$

Here $a_{\nu+1}$ denotes the lowering operator for μ. We use this relation to determine the coherent states for our system in the next section.

V. COHERENT STATES USING THE ASSOCIATED LEGENDRE POLYNOMIAL

Coherent states are defined as eigenvectors of the lowering operator, i.e. for a coherent state $|\alpha\rangle$ we have

$$a_{\nu+1}|\alpha\rangle = \alpha|\alpha\rangle \tag{23}$$

where α can be any complex number. To find the coherent states, i.e. the solution of Eq. (23), let us expand the state $|\alpha\rangle$ in terms of the basis $|\mu\rangle$ as

$$|\alpha\rangle = \sum_{\mu(\mu_n)=\mu_0}^{\infty} c_\mu|\mu\rangle \tag{24}$$

where $c_\mu = \langle\mu|\alpha\rangle$. Applying the annihilation operator $a_{\nu+1}$ to Eq. (24) and using Eq. (23), we get

$$c_\mu = c_0\alpha^\mu e^{-i\epsilon e_\nu}\sqrt{\frac{\Gamma(2\nu-\mu+1)\Gamma(2\nu+1)}{\Gamma(\mu+1)}} \tag{25}$$

where $e_\nu = (\lambda_\nu - \lambda_{\nu-1})$. Thus, we have an eigenstate of annihilation operator $a_{\nu+1}$ as

$$|\alpha\rangle = c_0\sqrt{\Gamma(2\nu+1)}e^{-i\epsilon e_\nu}\sum_{\mu(\mu_n)=\mu_0}^{\infty}\alpha^\mu\sqrt{\frac{\Gamma(2\nu-\mu+1)}{\Gamma(\mu+1)}}\left|\mu\right\rangle. \tag{26}$$

For simplicity let us assume $c_0 = 1$. Coherent states are not orthogonal but it is possible to expand coherent states in terms of a complete basis. The completeness relation condition for coherent state is given by

$$\int d\sigma(\alpha)|\alpha\rangle\langle\alpha| = 1 \tag{27}$$

where $d\sigma(\alpha)$ is the weight function which is determined below. In the coherent state quantization method [35], a quantum operator is defined by

$$f \mapsto A_\epsilon(f) = \int d\sigma(\alpha) f(\alpha)|\alpha\rangle\langle\alpha|. \tag{28}$$

Let $d\sigma(\alpha) = \sigma(r) dr d\theta$ and $\alpha = re^{i\theta}$. If $|f\rangle$ and $\langle g|$ are two arbitrary vectors, then we can write Eq. (27) as

$$\langle f|g\rangle = \int \sigma(r) dr d\theta \langle f|\alpha\rangle\langle\alpha|g\rangle. \tag{29}$$

Now using $|\alpha\rangle$ from Eq. (26), Eq. (29) becomes

$$\langle f|g\rangle = 2\pi\Gamma(2\nu+1)\sum_{\mu=0}^{\infty}\int_0^\infty r^{2\mu}\sigma(r) dr \frac{\Gamma(2\nu-\mu+1)}{\Gamma(\mu+1)} \times \langle f|\mu\rangle\langle\mu|g\rangle. \tag{30}$$

To determine $\sigma(r)$, we must have

$$2\pi\Gamma(2\nu+1)\int_0^\infty r^{2\mu}\sigma(r) dr = \frac{\Gamma(\mu+1)}{\Gamma(2\nu-\mu+1)}. \tag{31}$$

The explicit form of $\sigma(r)$ is determined with the help of the following relation [32],

$$2\int_0^\infty r^A J_B(2r) dr = \frac{\Gamma\frac{(1+A+B)}{2}}{\Gamma\frac{(1-A+B)}{2}} \tag{32}$$

where $J_B(2r)$ is the Bessel function of the first kind. Choosing $A = 2\mu - 2\nu$ and $B = 2\nu + 1$ in Eq. (32) we get

$$2\int_0^\infty r^{2\mu-2\nu} J_{2\nu+1}(2r) dr = \frac{\Gamma(\mu+1)}{\Gamma(2\nu-\mu+1)}. \tag{33}$$

Thus comparing Eq. (31) and Eq. (33), we obtain the desired function $\sigma(r)$ as

$$\sigma(r) = \frac{J_{2\nu+1}(2r) r^{-2\nu}}{\pi\Gamma(2\nu+1)}. \tag{34}$$

With the help of coherent state $|\alpha\rangle$ given in Eq. (26) and using Eq. (28), we describe quantization in two-dimensional de Sitter space in the next section.

VI. ROTATION AND BOOST IN TWO-DIMENSIONAL DE SITTER SPACE

If we consider two-dimensional de Sitter space, which is embedded in three-dimensional Minkowski space, the symmetry group is $SO(2,1)$ with the following spherical coordinates:

$$\begin{aligned} X^0 &= a\sinh\left(\frac{t}{a}\right); \\ X^1 &= a\cosh\left(\frac{t}{a}\right)\cos\theta; \\ X^2 &= a\cosh\left(\frac{t}{a}\right)\sin\theta. \end{aligned} \tag{35}$$

In this case, the de Sitter generators are given by

$$\begin{aligned} J_1 &= ap_t\cos\theta - \tanh\left(\frac{t}{a}\right)\sin\theta p_\theta; \\ J_2 &= ap_t\sin\theta + \tanh\left(\frac{t}{a}\right)\cos\theta p_\theta; \\ J_0 &= p_\theta. \end{aligned} \tag{36}$$

where J_1 and J_2 represent boosts and J_0 represents rotation in two-dimensional de Sitter space. In this case, the metric tensor has the form

$$ds^2 = dt^2 - a^2\cosh^2\left[\frac{t}{a}\right] d\theta^2. \tag{37}$$

This metric provides the constraints $g^{\mu\nu}p_\mu p_\nu - m^2 = 0$. Therefore, with the help of Eq. (36), this constraint provides the following relation,

$$J_1^2 + J_2^2 - J_0^2 - \lambda^2 = 0 \tag{38}$$

where $\lambda = ma$. If we consider $r = \sqrt{J_0^2 + \lambda^2}$, then we can write Eq. (38) as follows:

$$(ap_t)^2 + \tanh^2\left(\frac{t}{a}\right) p_\theta^2 = J_0^2 + \lambda^2 = r^2. \tag{39}$$

If we define $\cos\beta = \frac{(ap_t)}{\sqrt{(ap_t)^2 + \tanh^2(\frac{t}{a})p_\theta^2}}$, $\cos\theta = \frac{J_0}{\sqrt{J_0^2+\lambda^2}}$, $\theta + \beta = \phi$, it gives Eq. (36) in a simplified form as

$$\begin{aligned} J_1 &= r\cos(\theta+\beta) = r\cos\phi; \\ J_2 &= r\sin(\theta+\beta) = r\sin\phi; \\ J_0 &= \sqrt{r^2 - \lambda^2}. \end{aligned} \tag{40}$$

From Eq. (40), we see that Eq. (38) defines a circle of radius r. J_1, J_2 and J_0 form a hyperboloid and satisfy following relations:

$$J_1^2 + J_2^2 = \lambda^2 + J_0^2 = r^2. \tag{41}$$

Now we quantize J_1, J_2 and J_0 with the help of coherent states given in Eq. (26). By using the definition (28) and taking $\epsilon \mapsto 0$, we can obtain

$$A_\epsilon(re^{i\phi}) = A_\epsilon(J_+) = \sum_{\mu(\mu_n)=\mu_0}^{\infty} \sqrt{(\mu+1)(2\nu-\mu)}\,|\mu\rangle\langle\mu+1|;$$
$$A_\epsilon(re^{-i\phi}) = A_\epsilon(J_+) = \sum_{\mu(\mu_n)=\mu_0}^{\infty} \sqrt{(\mu)(2\nu-\mu+1)}\,|\mu\rangle\langle\mu-1|; \tag{42}$$

where $J_\pm = J_1 \pm J_2$. Equation (42) provides

$$[A_\epsilon(J_+), A_\epsilon(J_-)] = -2\sum_{\mu(\mu_n)=\mu_0}^{\infty} (\mu-\nu)|\mu\rangle\langle\mu| = -2A_\epsilon(J_0), \tag{43}$$

and from Eqs. (26), (28), (42), and (43), one can obtain the following commutation relation of the observables

$$[A_\epsilon(J_1), A_\epsilon(J_2)] = -iA_\epsilon(J_0);$$
$$[A_\epsilon(J_0), A_\epsilon(J_1)] = -iA_\epsilon(J_2);$$
$$[A_\epsilon(J_0), A_\epsilon(J_2)] = iA_\epsilon(J_1). \tag{44}$$

In the next section, we follow the same steps to obtain the quantization in three-dimensional de Sitter space. Subsequently we generalize quantization of rotations and boosts for D-dimensional de Sitter space.

VII. QUANTIZATION IN THREE-DIMENSIONAL DE SITTER SPACE

Three-dimensional de Sitter space embedded in four-dimensional space satisfies the following relation,

$$X_1^2 + X_2^2 + X_3^2 - X_0^2 = a^2 \tag{45}$$

where X_0, X_1, X_2, X_3 represent the coordinates of four-dimensional space. We choose these coordinates as global coordinates given by

$$X_0 = a\sinh\left(\frac{t}{a}\right);$$
$$X_1 = a\cosh\left(\frac{t}{a}\right)\cos\theta_1;$$
$$X_2 = a\cosh\left(\frac{t}{a}\right)\sin\theta_1\cos\zeta;$$
$$X_3 = a\cosh\left(\frac{t}{a}\right)\sin\theta_1\sin\zeta. \tag{46}$$

In these coordinates, the metric tensor has the form

$$ds^2 = dt^2 - a^2\cosh^2\left[\frac{t}{a}\right](d\theta_1^2 + \sin^2\theta_1 d\zeta^2). \tag{47}$$

Boosts and rotations are given by

$$M_{01} = a\cos\theta_1 p_t - \sin\theta_1\tanh\left(\frac{t}{a}\right)p_{\theta_1};$$
$$M_{02} = a\sin\theta_1\cos\zeta p_t - \tanh\left(\frac{t}{a}\right) \times \left(\frac{\sin\zeta}{\sin\theta_1}p_\zeta - \cos\theta_1\cos\zeta p_{\theta_1}\right);$$
$$M_{03} = a\sin\theta_1\sin\zeta p_t + \tanh\left(\frac{t}{a}\right) \times \left(\frac{\cos\zeta}{\sin\theta_1}p_\zeta + \cos\theta_1\sin\zeta p_{\theta_1}\right);$$
$$M_{12} = \cos\zeta p_{\theta_1} - \cot\theta_1\sin\zeta p_\zeta;$$
$$M_{13} = \sin\zeta p_{\theta_1} + \cot\theta_1\cos\zeta p_\zeta;$$
$$M_{23} = p_\zeta; \tag{48}$$

where M_{12}, M_{13} and M_{23} are rotations and M_{01}, M_{02}, M_{03} are boosts. The metric equation (47) provides the following constraint,

$$a^2p_t^2 - p_{\theta_1}^2\mathrm{sech}^2\left(\frac{t}{a}\right) - p_\zeta^2\mathrm{sech}^2\left(\frac{t}{a}\right)\mathrm{cosec}^2\theta_1 - \lambda^2 = 0 \tag{49}$$

where $\lambda = ma$. From Eq. (47) and Eq. (49), we can see that

$$M_{01}^2 + M_{02}^2 + M_{03}^2 - (M_{12}^2 + M_{13}^2 + M_{23}^2) = \lambda^2. \tag{50}$$

Equation (50) can be viewed as a sphere whose radius is $\lambda^2 + (M_{12}^2 + M_{13}^2 + M_{23}^2)$. If we redefine M_{01}, M_{02} by Y_1, Y_2 and $M_{12}^2 + M_{13}^2 + M_{23}^2 - M_{03}^2$ by Y_0^2 then Eq. (50) becomes

$$Y_1^2 + Y_2^2 - Y_0^2 = \lambda^2. \tag{51}$$

If Y_1, Y_2 and Y_0 are defined in such a way,

$$Y_1 = R\sin\vartheta_1\sin\varphi;$$
$$Y_2 = R\sin\vartheta_1\cos\varphi;$$
$$Y_0 = \sqrt{R^2 - \lambda^2}. \tag{52}$$

This represents a circle of radius $R\sin\vartheta_1$ and we can write this circle in complex form as $\alpha = R\sin\theta_1 e^{i\varphi}$. With the help of Eq. (27) and using the value of α, we obtain the weight factor $d\sigma(\alpha)$ as

$$d\sigma(\alpha) = \frac{J_{2\nu+\frac{3}{2}}(2R)R^{-2\nu+\frac{1}{2}} \sin\vartheta_1 d\vartheta_1 dR d\varphi}{(\pi)^{3/2}\Gamma(2\nu+1)}. \quad (53)$$

Using Eqs. (52)–(53), the quantization in three-dimensional de Sitter space is done by following the same procedure as defined in Sec. VI and we get

$$A_\epsilon(R\sin\vartheta_1 e^{i\varphi}) = \sum_{\mu(\mu_n)=\mu_0}^{\infty} \sqrt{(2\nu-\mu)(\mu+1)}\,|\mu\rangle\langle\mu+1|;$$
$$A_\epsilon(R\sin\vartheta_1 e^{-i\varphi}) = \sum_{\mu(\mu_n)=\mu_0}^{\infty} \sqrt{\mu(2\nu-\mu+1)}\,|\mu\rangle\langle\mu-1|. \quad (54)$$

We can see that this is the same quantization as obtained in two-dimensional de Sitter space. Furthermore, we generalize such a quantization in D-dimensional de Sitter space in the next section.

VIII. GENERALIZATION OF QUANTIZATION IN D-DIMENSIONAL DE SITTER SPACE

Let us consider a D-dimensional de sitter space embedded in $(D+1)$-dimensional Minkowski space which has the symmetry group $SO(D,1)$. In this space we have $\frac{D(D-1)}{2}$ rotations and D boosts. Keeping in mind the metric given in Eq. (6) and constraint $g^{\mu\nu}p_\mu p_\nu - m^2 = 0$, boosts and rotations in D-dimensional de Sitter space form a circle with radius $(R\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{D-1})$. Such a generalization in D-dimensional space is represented by

$${Y_1}^D = R\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{D-1}\sin\varphi;$$
$${Y_2}^D = R\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{D-1}\cos\varphi;$$
$${Y_0}^D = \sqrt{(R\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{D-1})^2 - \lambda^2}, \quad (55)$$

which forms a circle as

$$\left({Y_1}^D\right)^2 + \left({Y_2}^D\right)^2 = (R\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{D-1})^2, \quad (56)$$

or $\alpha = (R\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{D-1})e^{i\varphi}$. In D-dimensional space we have weight factor

$$d\sigma(\alpha) = \sigma(R)\sin\vartheta_{D-2}\ldots\sin^{D-2}\vartheta_1 d\vartheta_{D-2}\ldots d\vartheta_2 d\vartheta_1 d\varphi dR. \quad (57)$$

Now with the definition of coherent states given in Eq. (26) and the weight factor given in Eq. (57), we can determine the $\sigma(R)$ by the following relation:

$$2(\pi)^{1+\frac{D-2}{2}}\Gamma(2\nu+1)\int_0^\infty R^{2\mu}\sigma(R)dR = \frac{\Gamma(\mu+\frac{D}{2})}{\Gamma(2\nu-\mu+1)}. \quad (58)$$

With the help of Eqs. (32) and (58), we get the generalized $\sigma(R)$ in D-dimensional de Sitter space as

$$\sigma(R) = \frac{J_{2\nu+\frac{D}{2}}(2R)R^{-2\nu+\frac{(D-2)}{2}}}{(\pi)^{1+\frac{D-2}{2}}\Gamma(2\nu+1)}. \quad (59)$$

Now, using the definition of quantization given in Eq. (28), and the coherent states in Eq. (26) along with Eqs. (58)–(59) we get

$$A_\epsilon(R\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{D-2}e^{i\varphi})$$
$$= A_\epsilon(Y_+^D) = \sum_{\mu(\mu_n)=\mu_0}^{\infty} \sqrt{(2\nu-\mu)(\mu+1)}\,|\mu\rangle\langle\mu+1|;$$
$$A_\epsilon(R\sin\vartheta_1\sin\vartheta_2\ldots\sin\vartheta_{D-2}e^{-i\varphi})$$
$$= A_\epsilon(Y_-^D) = \sum_{\mu(\mu_n)=\mu_0}^{\infty} \sqrt{\mu(2\nu-\mu+1)}\,|\mu\rangle\langle\mu-1|, \quad (60)$$

where $Y_+^D = Y_1 + iY_2$ and $Y_-^D = Y_1 - iY_2$. Thus we can see that these relations are the same as we obtained in two- and three-dimensional de Sitter space. It shows that we have the same quantization in any arbitrary dimensional space.

To conclude, starting with some recurrence relations satisfied by the associated Legendre Polynomial, we have described the raising and lowering operator. Since coherent states are the eigenstates of the annihilation operator, with the help of these operators we have obtained coherent states in two-dimensional de Sitter space which is embedded in three-dimensional Minkowski space and generalized in D-dimensional de Sitter space. Coherent state quantization in D-dimensional space has also been generalized.

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