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Singular bouncing cosmology from Gauss-Bonnet modified gravity

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We study how a cosmological bounce, with a type IV singularity at the bouncing point, can be generated by a classical vacuum $F(G)$ gravity. We focus our investigation on the behavior of the vacuum $F(G)$ theory near the type IV singular bouncing point and address the stability of the resulting solution by treating the equations of motion as a dynamical system. In addition, we investigate how the scalar perturbations of the background metric evolve, emphasizing cosmological times near the type IV singular bouncing point. Finally, we also investigate which mimetic vacuum $F(G)$ gravity can describe the singular bounce cosmology.

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I. INTRODUCTION

The observational data that came into play during the last 20 years have consistently described a compelling theory that can harbor all the different observationally verified phenomena. There are two main observations that need to be described theoretically—the late-time acceleration verified in the late 1990s [1] and the early-time acceleration. With regards to the latter, there is much more ground to be covered until we conclude whether inflation ever existed. However, the latest Planck data [2] pose severe constraints on inflationary models and indicate which features should be included in a consistent theory of inflation. Modified gravity theories offer a promising and solid theoretical framework that can consistently describe late-time and early-time acceleration; for reviews on this vast issue, see [3]. Among the most promising are the $F(R)$ theories of gravity, which also offer the possibility to describe simultaneously early and late-time acceleration [4]. The $F(R)$ gravity is the simplest modification of Einstein-Hilbert gravity since instead of having simply the Ricci scalar R in the Lagrangian, a function of R appears. In four dimensions, instead of this simple modification of Einstein-Hilbert gravity, there also exist other theoretical descriptions that are promising, such as the $F(G)$ theories of gravity, with G the Gauss-Bonnet invariant $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, where $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ are the Ricci tensor and the Riemann tensor, respectively. Although, in principle, the resulting equations of motion are expected to have fourth-order derivatives of the metric tensor, it turns out that the theory contains only second-order derivatives, and therefore it is rendered not too complicated to be studied. In particular, it has been shown that within the context of $F(G)$ theory, late-time acceleration can be achieved [5–13]. For informative reviews on the $F(G)$ theory of gravity, see [3].

On the other hand, an appealing alternative to the standard inflationary paradigm comes from the bounce cosmology theories [14–22]. In these kinds of theories, one of the most severe drawbacks of the inflationary paradigm, the initial singularity problem, is absent, and it is conceivable that this feature makes them quite appealing. The initial singularity is a crushing-type, timelike singularity, and at the point it occurs, the geodesics cannot be continuously extended; that is, geodesic incompleteness occurs. The singularity theorems of Hawking and Penrose [23] fully describe these singularities, and much work has been devoted to studying spacelike singularities. In cosmology, however, most of the singularities are timelike, and the most severe of these are the initial singularity and the big rip [24]. With regards to the latter, it is the most severe of the four types of finite singularities classified in [25]. Also sudden singularities were also studied in [26–29]. Among the four types of finite-time singularities, the most mild from a phenomenological point of view, is the type IV singularity, at which no geodesic incompleteness occurs. For recent studies on this kind of singularity see [30–35]. The most phenomenologically interesting feature of theories with type IV finite time singularities is that the Universe may smoothly pass through [34,35] these timelike singularities, without any catastrophic consequences. In some cases, there might be observational evidence or indication of the passage of the Universe through a type IV singularity [34], but it is important that this is not catastrophic and it always is a smooth passage [35].

In view of the aforementioned interesting features of the bouncing cosmologies, in this paper we aim to study a specific bounce which contains a type IV singularity at the bouncing point. In this way, the cosmological system is free from the initial singularity, but at the same time we investigate the implications of such a mild finite time

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singularity. Particularly, we shall investigate how such a cosmology can be described by a classical vacuum $F(G)$ gravity, with special emphasis being given on how the $F(G)$ gravity behaves near the type IV singular bouncing point.[1] Also, we investigate how the resulting solution behaves, by checking the stability of the resulting equations, when these are viewed as a dynamical system. In this case we examine if the solution is the final attractor of the system. It is conceivable of course that since the Universe passed through the singular point smoothly, the resulting $F(G)$ gravity should not be the final attractor of the theory, so in some sense instability of the dynamical system is anticipated. In addition, we shall investigate how the scalar perturbations of the background metric behave for the case of the resulting vacuum $F(G)$, near the type IV singularity. Finally, we shall investigate which vacuum $F(G)$ gravity generates the same singular bounce we are discussing, but in the context of mimetic $F(G)$ theory, which was developed in [36]. Again the focus will be for cosmic times near the singular bouncing point. We believe that our work will provide some new information on the behavior of classical modified theories of gravity near a type IV singular bouncing point, a study which combines bouncing cosmology with a mild singularity at the bounce, and also a classical description with an $F(G)$ gravity.

This paper is organized as follows: In section II we briefly review the basic features of a type IV singular bounce cosmology, and in section III by using known cosmological reconstruction techniques, we investigate which vacuum $F(G)$ can describe the type IV singular bounce, by focusing on the behavior near the bouncing point. Also, we discuss the possible connection of the resulting $F(G)$ theory with other viable $F(G)$ theories. In section IV we study the stability of the resulting solution we found in section III, by treating the system of equations of motion as a dynamical system. To this end, we rewrite the system of equations of motion in terms of new variables, which make the study more clear from a physical point of view. In section V we investigate how the scalar perturbations of the background metric behave, for the vacuum $F(G)$ theory we found in section III, emphasizing again on the behavior near the type IV singular point. Finally, in section VI we investigate which mimetic $F(G)$ gravity can describe the type IV singular bounce cosmology, focusing again on the behavior near the type IV singularity. The conclusions along with a discussion follow in the end of the paper.

II. A BRIEF DESCRIPTION OF THE SINGULAR BOUNCE

As we already mentioned, bouncing cosmology [14–22] is an appealing alternative scenario to the standard inflationary paradigm, with the most attractive feature of bouncing cosmology being the fact that there is no initial singularity, and there exists also the possibility of successfully describing early-time acceleration [20,22]. It has been recently shown, however, that other types of milder singularities [30–33,35] might occur during the cosmological evolution, without having the catastrophic consequences of the crushing-type singularities, such as the big rip [24]. In particular, in Ref. [35] it has been demonstrated that a type IV singularity [25,31–35] may occur at the bounce point of a general bouncing cosmology with interesting consequences. We shall briefly describe this possibility in this section in order to render the presentation self-contained.

First, a cosmological bounce can be separated into two evolution eras—the contraction era, during which the scale factor decreases ($\dot{a} < 0$), and the expansion era, during which the scale factor increases ($\dot{a} > 0$). In between the two eras, and after the contraction era, the Universe reaches a minimal radius, where $\dot{a} = 0$, and this is the reason that the bouncing scenario is free of the initial singularity. In principle, the bouncing point, that is, the point at which the bounce occurs, can freely be chosen, so we assume that the bounce occurs at $t = t_s$, so when $t < t_s$, the Hubble rate $H = \dot{a}/a$ is negative (since $\dot{a} < 0$), while for $t > t_s$ it is positive (since $\dot{a} > 0$). Of course, at the bounce it becomes equal to zero, $H(t_s) = 0$. We shall consider the bouncing cosmology with scale factor,

$$a(t) = e^{f_0(t-t_s)^{2(1+\varepsilon)}}, \tag{1}$$

with $a(t_s) = 1$ and f_0 an arbitrary parameter. In addition, the parameter ε is assumed to be $\varepsilon < 1$, so the bouncing cosmology of Eq. (1) is assumed to be a deformation of the well-known symmetric bounce [37],

$$a(t) = e^{f_0(t-t_s)^2}. \tag{2}$$

From Eq. (1) it easily follows that the Hubble rate is equal to

$$H(t) = 2(1+\varepsilon)f_0(t-t_s)^{2\varepsilon+1}. \tag{3}$$

For simplicity, we introduce the following variables,

$$\beta = 2(1+\varepsilon)f_0, \qquad \alpha = 2\varepsilon + 1, \tag{4}$$

so the Hubble rate becomes

$$H(t) = \beta(t-t_s)^\alpha. \tag{5}$$

According to the classification of finite-time singularities, the following types of singularities occur at $t = t_s$ which is the bouncing point:

(i) $\alpha < -1$ corresponds to the type I singularity
(ii) $-1 < \alpha < 0$ corresponds to the type III singularity

[1] From this point, when we refer to the bouncing point or the singular point, we refer to the same point.

(iii) $0 < \alpha < 1$ corresponds to the type II singularity
(iv) $\alpha > 1$ corresponds to the type IV singularity.

The relevant case here is when $\alpha > 1$, and since we want to render the bounce of Eq. (1) a deformation of the symmetric bounce (2), we further assume that $1 < \alpha < 2$ (or equivalently $0 < \varepsilon < \frac{1}{2}$), something which implies that the second derivative of the Hubble rate (5) diverges. As shown in [34], this can have interesting phenomenological consequences. Finally, we need to make sure that the Hubble rate and the scale factor never become complex, so the parameter α needs to be appropriately chosen, so we make the choice $\alpha = 13/11$, but in general $\alpha = \frac{2n-1}{2m+1}$. For a detailed account on that, see [35].

III. SINGULAR BOUNCE FROM $F(G)$ GRAVITY

Having described the cosmological bounce with a type IV singularity at the bouncing point, in this section we shall investigate how this type of cosmological evolution can be described in terms of a vacuum $F(G)$ gravity. Special emphasis shall be given in the $F(G)$ gravity that describes the bounce near the type IV singularity since the general problem is rather difficult to address due to the lack of analytic solutions of the corresponding differential equations.

In order to find the $F(G)$ gravity that describes the bounce near the singular point, we shall use some very well-known reconstruction techniques for $F(G)$ theories of gravity [9–11]. For a similar technique to the one we shall use here, see [12]. The Jordan frame $F(G)$ gravity action in the absence of matter fluids [vacuum $F(G)$] is equal to [3,9–11],

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}(R + F(G)), \tag{6}$$

where $\kappa^2 = 1/M_{\mathrm{pl}}^2$, with $M_{\mathrm{pl}} = 1.22 \times 10^{19}$ GeV. By varying with respect to the metric, the gravitational equations read

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}F(G) - (-2RR_{\mu\nu} + 4R_{\mu\rho}R^{\rho}_{\nu} - 2R_{\mu}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau} + 4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}R_{\rho\sigma})F'(G) \\ &\quad - 2(\nabla_\mu\nabla_\nu F'(G))R + 2g_{\mu\nu}(\Box F'(G))R - 4(\Box F'(G))R_{\mu\nu} + 4(\nabla_\mu\nabla_\nu F'(G))R^{\rho}_{\nu} \\ &\quad + 4(\nabla_\rho\nabla_\nu F'(G))R^{\rho}_{\mu} - 4g_{\mu\nu}(\nabla_\rho\nabla_\sigma F'(G))R^{\rho\sigma} + 4(\nabla_\rho\nabla_\sigma F'(G))g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta} = 0, \end{aligned} \tag{7}$$

where the Gauss-Bonnet invariant expressed in terms of the Hubble rate equals

$$G = 24H^2(\dot{H} + H^2). \tag{8}$$

Assuming a flat Friedmann-Robertson-Walker (FRW) background, with line element,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\sum_i \mathrm{d}x_i^2, \tag{9}$$

the gravitational equations of Eq. (7) take the following form:

$$\begin{aligned} &6H^2 + F(G) - GF'(G) + 24H^3\dot{G}F''(G) = 0 \\ &4\dot{H} + 6H^2 + F(G) - GF'(G) + 16H\dot{G}(\dot{H} + H^2)F''(G) \\ &\quad + 8H^2\ddot{G}F''(G) + 8H^2\dot{G}^2F'''(G) = 0. \end{aligned} \tag{10}$$

The reconstruction technique presented in [9,11] employs the use of an auxiliary field denoted as ϕ, which as was shown can be identified with the cosmic time t, that is, $\phi = t$. By introducing two proper functions of t—namely, $P(t)$ and $Q(t)$—the Jordan frame action of Eq. (6) becomes

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}(R + P(t)G + Q(t)), \tag{11}$$

and by varying it with respect to t, we obtain

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t}G + \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = 0. \tag{12}$$

Having Eq. (12) at hand, solving it with respect to $t = t(G)$, and substituting this into the following expression,

$$F(G) = P(t)G + Q(t), \tag{13}$$

we finally have the $F(G)$ gravity. It is, therefore, obvious that the functions $P(t)$ and $Q(t)$ play an important role in the determination of the $F(G)$ gravity, so we now derive the differential equations that yield these functions. By combining Eq. (13) and the first of the equations appearing in Eq. (10), we obtain the following differential equation,

$$Q(t) = -6H^2(t) - 24H^3(t)\frac{\mathrm{d}P}{\mathrm{d}t}, \tag{14}$$

and finally combining Eqs. (13) and (14), we obtain the following differential equation:

$$2H^2(t)\frac{\mathrm{d}^2P}{\mathrm{d}t^2} + 2H(t)(2\dot{H}(t) - H^2(t))\frac{\mathrm{d}P}{\mathrm{d}t} + \dot{H}(t) = 0. \tag{15}$$

When solved, Eq. (15) determines the function $P(t)$, and therefore $Q(t)$, so upon substitution of $P(t)$ in Eq. (14), we get $Q(t)$. Finally, from Eq. (12) we get the function $t = t(G)$, and by substituting that in Eq. (13), we obtain the final form of the $F(G)$ gravity. We now apply this method in order to find the $F(G)$ gravity which describes the bounce near the bouncing point.

A. $F(G)$ gravity near the bouncing point

The general problem of finding the $F(G)$ gravity for the singular bounce with Hubble rate (5) is rather difficult to deal with analytically, so we shall focus on finding the $F(G)$ gravity near the bouncing point $t = t_s$, which is the point where the type IV singularity occurs too. Consequently, at some point, we shall specify our analysis around the singularity.

For the Hubble rate of Eq. (5), the differential equation (15) that yields the function $P(t)$ becomes

$$\frac{2\beta}{\alpha}(t - t_s)^{1+\alpha}\frac{\mathrm{d}^2P}{\mathrm{d}t^2} + 4(t - t_s)^{\alpha}\beta + 1 = 0, \tag{16}$$

which can be solved analytically to yield

$$P(t) = -\frac{(t - t_s)^{1-2\alpha}((t - t_s)^{\alpha} - 2(t - t_s)^{\alpha}\alpha - 2\beta C_1 + 2\alpha\beta C_1)}{2(-1 + \alpha)(-1 + 2\alpha)\beta} + C_2. \tag{17}$$

Thus, by substituting Eq. (17) into Eq. (14), we obtain the function $Q(t)$ which appears in Appendix A since it is too complicated to include here. By using the resulting expressions for the functions $Q(t)$ and $P(t)$, the final form of Eq. (12) becomes

$$\frac{x^{-1-2\alpha}(4x^{3\alpha}\alpha\beta^3(11x^{\alpha} - 12C_1\beta) - Gx(x^{\alpha} - 2C_1\beta))}{2\beta} = 0, \tag{18}$$

where we have set $x = t - t_s$ for simplicity. It is obvious that the above equation is rather difficult to solve analytically, so we shall simplify it by keeping the dominant terms in the limit $x \to 0$, which corresponds to the limit $t \to t_s$. Therefore, by taking the limit $x \to 0$, Eq. (18) becomes

$$C_1Gx^{-2\alpha} - 24C_1x^{-1+\alpha}\alpha\beta^3 = 0, \tag{19}$$

which yields

$$x = \frac{G^{\frac{1}{(3\alpha-1)}}}{(24\alpha\beta^3)^{\frac{1}{(3\alpha-1)}}}. \tag{20}$$

Then by substituting (20) in $P(t)$ and $Q(t)$, and by using Eq. (13), we obtain the resulting expression for the $F(G)$ gravity near the singular point, which is

$$F(G) = C_2G + AG^{\frac{2\alpha}{-1+3\alpha}} + BG^{\frac{\alpha}{-1+3\alpha}}, \tag{21}$$

where the coefficients A and B are given in Appendix A. We can give, however, a simpler form by exploiting the fact that we are interested in the limit $t \to t_s$. The Gauss-Bonnet invariant of Eq. (8) is written in terms of the variable $x = t - t_s$ which we introduced earlier, as follows,

$$G = 24x^{-1+3\alpha}\alpha\beta^3 + 24x^{4\alpha}\beta^4, \tag{22}$$

from which it is obvious that as $x \to 0$ (or equivalently $t \to t_s$), the Gauss-Bonnet invariant tends also to zero. Hence, by keeping the most dominant terms from the $F(G)$ gravity of Eq. (21), we get the small G limit of it,

$$F(G) \simeq C_2G + BG^{\frac{\alpha}{-1+3\alpha}}. \tag{23}$$

Therefore, in the small G limit or, equivalently, near the type IV singularity, which we chose to be the bouncing point, the $F(G)$ gravity that can generate the Hubble rate (5) is approximately described by the expression of Eq. (23). Since we are discussing cosmological times near the bouncing point, it is worth examining how the evolution of perturbations behave for this $F(G)$ model. This will be done in detail in a later section.

B. Connection with other viable $F(G)$ gravities and possible late-time behavior

In the previous section we investigated which $F(G)$ gravity theory can successfully describe the singular bounce cosmology of Eq. (5), near the singular bounce. However, the complexity of the resulting differential equations forced us to find an approximate solution, with the final $F(G)$ being that of Eq. (23), or a further simplification near the bouncing point,

$$F(G) \simeq BG^{\frac{\alpha}{-1+3\alpha}}. \tag{24}$$

Now, it would be interesting to ask how the complete classical $F(G)$ gravity looks. A possible answer is that the complete classical $F(G)$ gravity description could be one of the four possible forms of $F(G)$ gravity that lead to finite-time singularities, first studied in [10] and further investigated in [38,39]. In particular, the $F(G)$ gravities that lead to singularities are of the following form [10,38,39],

$$F(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2}, \tag{25}$$

$$F(G) = \frac{a_1 G^{n+N} + b_1}{a_2 G^n + b_2}, \tag{26}$$

$$F(G) = a_3 G^n (b_3 G^m + 1), \tag{27}$$

$$F(G) = G^m \frac{a_1 G^n + b_1}{a_2 G^n + b_2}, \tag{28}$$

where the parameters a_i and b_i, with $i = 1,\ 2,\ 3$, are arbitrary real constants. Obviously, the $F(G)$ gravity we found, which for small G is given by Eq. (24), can be the limiting case of the above $F(G)$ gravities, which lead to finite singularities for some values of the parameters. For example, the $F(G)$ gravity of Eq. (28) in the small G limit behaves as $\sim \frac{b_1}{b_2} G^m$, which is clearly similar to the one we obtained in Eq. (24). Note that in the large G limit, the $F(G)$ gravity (28) behaves as $\sim \sim \frac{a_1}{a_2} G^m$, so the late-time behavior of this $F(G)$ gravity is described by a power-law $F(G)$ function and, as was shown in [5–8], such power-law modified gravity theories can serve as models for dark energy.

Before we close this section, we need to stress that the absence of an analytic solution in the case of a singular bounce is exactly due to the existence of the singularity. In other bouncing cosmologies, for which no singularity occurs, this lack of analyticity no longer persists as a problem, and the exact behavior of the $F(G)$ gravity can be found, as in Ref. [37], for example.

IV. STABILITY OF THE $F(G)$ GRAVITY SOLUTIONS NEAR THE BOUNCING POINT

The FRW equations of Eq. (10) for the $F(G)$ gravity constitute a dynamical system which determines the behavior of solutions which satisfy these equations. The stability of a solution of this dynamical system against linear perturbations would mean that this solution is one of the final attractors of the theory. On the contrary, if a solution of the system is unstable against linear perturbations, then it is obvious that this solution is not a final attractor of the theory. The focus in this section is on the stability of the solutions we found for the $F(G)$, near the bouncing point, against linear perturbations of the solutions. Obviously, it is expected that the solutions we found near the bouncing point are unstable since the cosmological evolution does not stop at the bouncing point but continues and the Universe starts to expand. Therefore, the local solutions we found that describe the $F(G)$ gravity near the bouncing point should be unstable against linear perturbations since they do not describe the whole evolution, but a small part of it, near the type IV singularity. In the rest of this section, we shall use some convenient variables, and we shall study the stability of the dynamical system of Eq. (10) against linear perturbations.

We adopt the techniques and notation used in [12], so we consider linear perturbations of the solution $g(N) = H^2$, of the following form,

$$g(N) \to g(N) + \delta g(N), \tag{29}$$

with the function $g(N)$ satisfying the FRW equations (10). Expressing the dynamical system of Eq. (10) in terms of the function $g(N)$, we get

$$\begin{aligned} &288 g^2(N) F''(G)[((g'(N))^2 + g(N))g''(N) + 4g(N)g'(N) + 4g(N)g'(N)] \\ &6g(N) + F(G) - 12g(N)(g'(N) + 2g(N))F'(G) = 0. \end{aligned} \tag{30}$$

The conditions that ensure the stability of the dynamical system (30) against linear perturbations are the following,

$$\frac{J_2}{J_1} > 0, \qquad \frac{J_3}{J_1} > 0, \tag{31}$$

where we introduced the variable J_1 which is equal to

$$J_1 = 288 g(N)^3 F''(G), \tag{32}$$

while the variable J_2 stands for

$$J_2 = 432g(N)^2((2g(N) + g'(N))F''(G) + 8g(N)(g'(N)^2 + g(N)(4g'(N) + g''(N)))F''(G)), \tag{33}$$

and the parameter J_3 is equal to

$$J_3 = 6(1 + 24g(N)(-8g(N)^2 + 3g'(N)^2 + 6g(N)(3g'(N) + g''(N)))F''(G) \\ + 24g(N)(4g(N) + g'(N))(g'(N)^2 + g(N)(4g(N) + g''(N)))F''(G)). \tag{34}$$

Having these at hand, and also the stability conditions, lets us investigate whether the solution (23) is stable towards linear perturbations. The $F(G)$ gravity of Eq. (23) can be further simplified since α satisfies the condition $1 < \alpha < 2$, so the most dominant term near the bouncing point (or equivalently at the small G limit as we showed in the previous section) is the second term in Eq. (23) and, hence, the $F(G)$ function can be approximated by

$$F(G) \simeq BG^{\frac{\alpha}{-1+3\alpha}}. \tag{35}$$

By using the latter form of the $F(G)$ gravity, the variables J_1, J_2, and J_3 can easily be computed, and we give their detailed form in Appendix B. Note that the function $g(N)$ is expressed in terms of the e-folding number N which is equal to $N = \ln a$, since we have set $a_0 = 1$, so for the Hubble rate of Eq. (5), the function $g(N)$ reads

$$g(N) = \frac{\beta^2 N^\gamma}{f_0}, \tag{36}$$

where $\gamma = \frac{2\epsilon+1}{1+\epsilon}$. So, finally, the stability conditions for the $F(G)$ gravity of Eq. (35) read

$$\frac{J_2}{J_1} = \frac{3(f_0^2 N(2N+\gamma) + 8N^{2\gamma}\beta^4\gamma(-1+4N+2\gamma))}{2f_0^2 N^2} \tag{37}$$

and, moreover, J_3/J_1 is equal to

$$\frac{J_3}{J_1} = -\frac{1}{Bf_0\alpha(-1+2\alpha)} 3^{1+\frac{\alpha}{1-3\alpha}} 4^{\frac{\alpha}{1-3\alpha}} N^{-2+\gamma}(1-3\alpha)^2\beta^2(\gamma)^2 \left(\frac{N^{-1+2\gamma}\beta^4(\gamma)}{f_0^2}\right)^{\frac{\alpha}{1-3\alpha}} \\ \times \left(1 + \frac{2^{\frac{1-\alpha}{-1+3\alpha}} 3^{\frac{1-2\alpha}{-1+3\alpha}} Bf_0 N^{-\gamma}\alpha(-1+2\alpha)(\frac{N^{-1+2\gamma}\beta^4(2N+\gamma)}{f_0^2})^{\frac{\alpha}{-1+3\alpha}}(8N^2 - 18N\gamma + 3(2-3\gamma)\gamma)}{(1-3\alpha)^2\beta^2(2N+\gamma)^2} \right. \\ \left. - \frac{2^{\frac{1-\alpha}{-1+3\alpha}} 3^{\frac{1-2\alpha}{-1+3\alpha}} B\alpha(-1+2\alpha)(\frac{N^{-1+2\gamma}\beta^4(\gamma)}{f_0^2})^{\frac{\alpha}{-1+3\alpha}}(\gamma)(\gamma(-1+2\gamma))}{N(1-3\alpha)^2(\gamma)^2}\right). \tag{38}$$

Since we are interested in the behavior of J_2/J_1 and J_3/J_1 near the bouncing point, we should know what the limit $t \to t_s$ means in terms of the e-folding number N. Actually, since when $t \to t_s$, the scale factor tends to unity, then, in the limit $t \to t_s$, the e-folding number tends to zero. Therefore, we shall find how J_2/J_1 and J_3/J_1 behave for $N \to 0$ and, taking the limit of the expressions appearing in Eqs. (37) and (38), we obtain

$$\frac{J_2}{J_1} = 3 + \frac{3\gamma}{2N}, \\ \frac{J_3}{J_1} = -\mathcal{A}N^{-2+\gamma+\frac{\alpha-2\alpha\gamma}{-1+3\alpha}}, \tag{39}$$

where the parameter $\mathcal{A}$ is positive and can be found in Appendix B. As is obvious from Eq. (39), the parameter J_3/J_1 is negative and, therefore, the system is unstable, as we anticipated. This means that the solution (35) is not a final attractor and, therefore, it can describe the system near the bouncing point, but only for a short time, since the system continues its evolution after that point. What now remains is to study the evolution of scalar perturbations of the cosmological evolution, with an emphasizing on the description near the type IV singularity. This issue is addressed in the next section.

V. EVOLUTION OF SCALAR PERTURBATIONS

Now that we have the qualitative behavior of the $F(G)$ gravity that generates the singular bounce, it is worth examining how the scalar perturbations evolve, assuming the flat FRW background of Eq. (9). So we consider scalar linear perturbations of the flat FRW background of Eq. (9) of the form

$$\mathrm{d}s^2 = -(1+\psi)\mathrm{d}t^2 - 2a(t)\partial_i\beta \mathrm{d}t\mathrm{d}x^i \\ + a(t)^2(\delta_{ij} + 2\phi\delta_{ij} + 2\partial_i\partial_j\gamma)\mathrm{d}x^i\mathrm{d}x^j, \tag{40}$$

with ψ, ϕ, γ, and β being the smooth scalar perturbations. For the perturbation study, we follow the approach and master equation given in Ref. [40], but we specify everything for the $F(G)$ case, which is a special case of $F(R,G)$ gravity studied in [40]. For convenience, perturbations are usually analyzed in terms of gauge-invariant quantities; therefore, we shall be interested in the following gauge-invariant quantity (the corresponding comoving curvature perturbation), the evolution of which we study in this section:

$$\Phi = \phi - \frac{H\delta\xi}{\dot{\xi}} \tag{41}$$

where $\xi = \frac{dF}{dG}$. In the $F(G)$ gravity case, the propagating scalar modes contain no k^4 terms, so no superluminal propagation occurs, and only the usual k^2 terms appear [40]. The perturbation equation that governs the scalar modes in $F(G)$ gravity is the following,

$$\frac{1}{a(t)^3 Q(t)}\frac{d}{dt}(a(t)^3 Q(t)\dot{\Phi}) + B_1(t)\frac{k^2}{a(t)^2}\Phi = 0, \tag{42}$$

where we can see the above equation has the usual form for scalar perturbations, in which k^2 terms dominate in the evolution. It is conceivable that the speed of propagation is determined by the term $B_1(t)$, which for $F(G)$ theories of gravity is defined as

$$B_1(t) = 1 + \frac{2\dot{H}}{H^2}. \tag{43}$$

Moreover, the term $Q(t)$, appearing in Eq. (42) for the $F(G)$ case, is equal to

$$Q(t) = \frac{6(\frac{d^2F}{dG^2})^2\dot{G}^2(1+4F''(G)\dot{G}H)}{(1+6HF''(G)\dot{G})^2}, \tag{44}$$

where the prime denotes differentiation with respect to G, while the dot as usual denotes differentiation with respect to the cosmic time. Note additionally that we used the fact that $\dot{\xi} = \frac{dF^2}{dG^2}\dot{G}$.

It is conceivable that finding an analytic solution of Eq. (42) is rather difficult, so either a numerical study or an approximate solution is required. We shall choose the latter approach and seek an approximate solution near the bouncing point. Before continuing, we rewrite the differential equation (42) as follows:

$$a(t)^3 Q(t)\ddot{\Phi} + (3a(t)^2\dot{a}Q(t) + a(t)^3\dot{Q}(t))\dot{\Phi} + B_1(t)Q(t)a(t)k^2\Phi = 0. \tag{45}$$

After some tedious calculations, by using the resulting $F(G)$ gravity near the bouncing point—namely, the one appearing in Eq. (23)—and by keeping the most dominant terms near the bouncing point, the differential equation that governs the evolution of perturbations near $t = t_s$ reads

$$(t-t_s)^{1+\alpha}\Omega_4\ddot{\Phi} - (t-t_s)^{\alpha}\Omega_2\dot{\Phi} + \Omega_1\Phi = 0, \tag{46}$$

where the parameters Ω_i $(i = 1, 2, 4)$ are given in Appendix D. Note that we omitted a term $\sim\Omega_3\dot{\Phi}(t-t_s)^{2\alpha}$, which is subdominant compared to the term $(t-t_s)^{\alpha}\Omega_2\dot{\Phi}$. The parameter Ω_3 can also be found in Appendix D. Then, by solving the differential equation (45), we obtain the following analytic solution which describes the evolution of scalar perturbations near the singular point $t = t_s$,

$$\Phi(t) = \Delta_1 x^{\frac{\mu}{2(-1+\alpha)}} J_{\mu}(\zeta x^{\frac{1-\alpha}{2}}) + \Delta_2 x^{\frac{\mu}{2(-1+\alpha)}} J_{-\mu}(\zeta x^{\frac{1-\alpha}{2}}), \tag{47}$$

where $x = t - t_s$ and the constants μ and ζ depend on the parameters Ω_i, with their full detailed form appearing in Appendix D. The function $J_\mu(y)$ is the Bessel function of the first kind, and, in addition, the parameters Δ_i, $i = 1, 2$ are given as follows:

$$\Delta_1 = \left(-1+\frac{1}{\alpha}\right)^{\mu}\alpha^{\mu}\Omega_1^{-\frac{\mu}{2}}\Omega_4^{\frac{\mu}{2}}C_3\Gamma\left[\frac{\alpha}{-1+\alpha} + \frac{\Omega_2}{(-1+\alpha)\Omega_4}\right],$$
$$\Delta_2 = \left(-1+\frac{1}{\alpha}\right)^{3\mu}\alpha^{\mu}\Omega_1^{-\frac{\mu}{2}}\Omega_4^{\frac{\mu}{2}}C_4\Gamma\left[\frac{\Omega_2}{\Omega_4-\alpha\Omega_4} + \frac{2\Omega_4}{\Omega_4-\alpha\Omega_4} - \frac{\alpha\Omega_4}{\Omega_4-\alpha\Omega_4}\right]. \tag{48}$$

Note that in the parameters Δ_i, $i = 1, 2$ appear the constants C_3 and C_4, which are arbitrary constants of integration that result after solving the differential equation (45) without any initial conditions. Therefore, the solution of Eq. (47) is a general solution, and the constants of integration of this general solution are the parameters C_3 and C_4, appearing in the parameters Δ_i, $i = 1, 2$ of Eq. (48). Since we are considering the limit $x \to 0$, we can further approximate the solution by using the limit of the Bessel function for small arguments, which is

$$J_\mu(y) \simeq \frac{y^\mu 2^{-\mu}}{\Gamma[1+\mu]}, \tag{49}$$

so the approximate evolution of the scalar perturbations (47) near the type IV singularity behaves as follows:

$$\Phi(t) \simeq \Delta_2 \frac{2^{-\mu}\zeta^\mu}{\Gamma[1+\mu]} x^{\frac{(2-2\alpha+\alpha^2)\mu}{2(-1+\alpha)}}. \tag{50}$$

The resulting expression for the evolution of perturbations in the absence of matter fluids is described by a power-law function of the variable $x = t - t_s$. It is worthwhile to further investigate the power spectrum and check whether it is scale invariant. Note, however, that we already are within an approximation and, therefore, we should stress that our results should be considered only approximate and also that the full $F(G)$ solution is needed in order to answer the problem in a consistent way. Nevertheless, near the bouncing point, it is still interesting to find out how the power spectrum behaves within the context of the classical $F(G)$ theory. This may indicate how the full quantum description of the bounce theory will remedy any problematic issues.

We start from the gauge-invariant variable Φ given in Eq. (41), which was shown in [40] to satisfy the following second-order perturbed action,

$$\mathcal{S}_p = \int \mathrm{d}x^4 a(t)^3 Q_s \left(\frac{1}{2}\dot{\Phi} - \frac{1}{2}\frac{c_s^2}{a(t)^2}(\nabla\Phi)^2\right), \tag{51}$$

where $Q_s = \frac{4}{\kappa^2} Q(t)$, and $Q(t)$ appears in Eq. (44). Following the standard approach in perturbation theory [41–43], the power spectrum of curvature perturbations for the scalar field Φ is

$$\mathcal{P}_R = \frac{4\pi k^3}{(2\pi)^3}|\Phi|^2_{k=aH}. \tag{52}$$

It is straightforward to show that the power spectrum is not scale invariant by simply looking at the forms of Ω_i, Δ_2, and ζ from Appendix D. As it can be seen, the wave number k is contained only in the parameters Ω_1, Δ_2 and in the parameter ζ, since the latter depends explicitly on the parameter Ω_1 implicitly via Ω_1. These have the following functional dependence with respect to k,

$$\Omega_1 \sim k^2, \qquad \zeta \sim \sqrt{\Omega_1}, \qquad \Delta_2 \sim \Omega_1^{-\frac{\mu}{2}}, \tag{53}$$

and since the power spectrum depends on the combination $\Delta\zeta^\mu$, it follows from Eq. (53) that

$$\mathcal{P}_R \sim k^3 |C_4 (t-t_s)^{\frac{(2-2\alpha+\alpha^2)\mu}{2(-1+\alpha)}}|^2_{k=aH}. \tag{54}$$

However, we cannot determine at this point if the spectrum is scale invariant since the parameter C_4, which is the constant of integration appearing in Δ_2 in Eq. (48), depends on k, and its exact form will depend on the initial conditions of the vacuum form of $\Phi(t)$. In addition, the term $(t-t_s)^{\frac{(2-2\alpha+\alpha^2)\mu}{2(-1+\alpha)}}$ also has a dependence on k since the power spectrum is computed at $k = aH$, that is, at the horizon crossing. So let us calculate these in detail.

Before we start, it is worth recalling that we are working in cosmological times for which $t - t_s \to 0$, and, hence, the conformal time τ, defined as $\mathrm{d}\tau = a^{-1}(t)\mathrm{d}t$, is approximately equal to t since for $t - t_s \to 0$, the scale factor appearing in Eq. (1) behaves as $a \simeq 1$. Also, since $a \sim 1$ for $t - t_s \to 0$, we have that $k \simeq H$ at the horizon crossing, and this means that

$$\beta(t-t_s)^\alpha \simeq k, \tag{55}$$

which, by solving with respect to $(t - t_s)$, yields

$$t - t_s \simeq \left(\frac{k}{\beta}\right)^{\frac{1}{\alpha}}. \tag{56}$$

Hence, Eq. (56) shows how $t - t_s$ behaves as a function of the wave number k near the horizon crossing. We now proceed to find the k dependence of C_4. To do so, we shall introduce the canonical variable $u = z_s\Phi$, which is very frequently used in the literature [41,42], with $z_s = Q(t)a(t)$, and since $a(t) \simeq 1$ for $(t - t_s) \to 0$, we have that $z_s \simeq Q(t)$ and, therefore,

$$u \sim \Phi Q(t), \tag{57}$$

with $Q(t)$ as defined in Eq. (44). In terms of u, the action of Eq. (51) near the bounce becomes

$$\mathcal{S}_u \simeq \int \mathrm{d}^3\mathrm{d}\tau\left[\frac{u'}{2} - \frac{1}{2}(\nabla u)^2 + \frac{z_z''}{z_s}u^2\right], \tag{58}$$

where the prime indicates differentiation with respect to the conformal time, which as we demonstrated earlier is approximately identical to the cosmological time t, for $(t - t_s) \to 0$. In addition, the action of Eq. (58) is defined modulo a factor of a^{-1}, which is approximately equal to one near the time instance $t \simeq t_s$. The vacuum state of the canonical field u is the Bunch-Davies quantum fluctuating vacuum [42] at exactly $t = t_s$, hence, $u \sim \frac{e^{-ik\tau}}{\sqrt{k}}$. Note that the imaginary phase will disappear when we compute the norm of the comoving curvature $|\Phi(t = t_s)|^2$, so the relevant form of u for the k dependence issue of C_4 is $u \sim \frac{1}{\sqrt{k}}$. As a result of Eq. (57), we obtain

$$\Phi(t = t_s) \sim C_4 \sim \frac{1}{\sqrt{k}Q(t)}. \tag{59}$$

By using the fact that $F(G)$ for $(t - t_s) \to 0$ is approximated by Eq. (24), we get that the function $Q(t)$ is approximately equal to

$$Q(t) \simeq \frac{2^{-5+\frac{6\alpha}{-1+3\alpha}}3^{-1+\frac{2\alpha}{-1+3\alpha}}B^2(t-t_s)^{-4\alpha}(1-2\alpha)^2(\alpha\beta^3)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4\beta^6}, \tag{60}$$

and since $(t-t_s)$ is given by Eq. (56), we get

$$Q(t(k)) \simeq \mathcal{Z}_1 k^{\frac{-4\alpha}{\alpha}} = \mathcal{Z}_1 k^{-4}, \tag{61}$$

where $\mathcal{Z}_1$ stands for

$$\mathcal{Z}_1 = \frac{2^{-5+\frac{6\alpha}{-1+3\alpha}}3^{-1+\frac{2\alpha}{-1+3\alpha}}B^2(1-2\alpha)^2(\alpha\beta^3)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4\beta^6\beta^{1/\alpha}}. \tag{62}$$

By substituting $Q(t)$ from Eq. (61) into Eq. (59), we finally obtain that C_4 behaves as

$$C_4 \simeq \frac{1}{\mathcal{Z}_1}\frac{1}{\sqrt{k}k^{-4}} \sim k^{\frac{7}{2}}. \tag{63}$$

So combining Eqs. (63), (56), and (54), we finally get that the k dependence of the power spectrum $\mathcal{P}_R$ is of the form

$$\mathcal{P}_R \sim k^{\frac{7}{2}+3+\frac{(2-2\alpha+\alpha^2)\mu}{2(-1+\alpha)}}, \tag{64}$$

and hence we conclude that the power spectrum is not scale invariant. We can use the approximate value for the power spectrum of primordial curvature perturbations given in Eq. (64) in order to calculate the spectral index of primordial curvature perturbations n_s as follows: Combining the expression for μ given in Eq. (D2), and also the values of Ω_2 and Ω_4 given in Eq. (D1), we get that μ becomes $\mu = 11/(1-\alpha)$. Therefore, the spectral index of primordial curvature perturbations is equal to

$$\begin{aligned} n_s - 1 &\equiv \frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k} = \frac{7}{2} + 3 + \frac{2-2\alpha+\alpha^2}{2(\alpha-1)}\mu \\ &= 1 - \frac{11}{2(\alpha-1)^2}. \end{aligned} \tag{65}$$

Thus, the spectral index can be in agreement with the latest observational constraints on n_s [2] for two values of α: one which satisfies $\alpha < -1$ and one which satisfies $\alpha > 1$. The $\alpha < -1$ case leads to a big rip singularity, while the $\alpha > 1$ case leads to a type IV singularity, which is the case we studied in this paper. However, the value of α would need to be quite large, and so it would be a large deformation of a symmetric bounce, which we assumed in this paper. Therefore, this result does not appear to be very physical.

Still we need to correctly interpret this result since what we have at hand is that the classical approximation of the $F(G)$ theory that describes the singular bounce (5) near the bouncing point fails to produce a scale-invariant spectrum. This could be a strong motivation to use a Loop Quantum Cosmology-corrected $F(G)$ gravity theory [44] and investigate whether the same picture persists with quantum corrections added. If the answer lies in the affirmative, then this would probably mean that this effect is a feature of the type IV singularity, which needs to be further investigated.

However, if we use more physical values of α in the range $1 < \alpha < 2$, this would yield a very red spectrum, and one would strongly doubt that quantum corrections, e.g., from LQC corrected $F(G)$ gravity, could adjust this to a nearly scale-invariant spectrum. This argument is very rigid and solid and, therefore, should be investigated to explicitly demonstrate its validity, a task that we defer for a future work.

A. Discussion

Before we close this section, we need to discuss in detail the physical results we obtained in this section. As we demonstrated, the power spectrum of primordial curvature perturbations is not scale invariant when evaluated for cosmic times near the bouncing point at $t = t_s$, which is also the point at which the type IV singularity occurs. However, we need to discuss the physical implications of this result. To this end, let us recall some fundamental issues for the dynamics of perturbations and related issues, but in the context of bouncing cosmology. Recall that the problem of initial conditions in the standard big bang cosmology was due to the fact that the Universe appears to be nearly flat and homogeneous in large scales, which cannot have causally communicated in the past. Therefore, a successful bouncing cosmology should, in some way, provide an elegant solution to these problems.

Before we see what happens in the bouncing cosmology we studied in this paper, it is worth presenting what happens in inflationary theories, so that we can compare the inflationary picture to the bouncing cosmology picture. In the inflationary cosmology picture, the primordial curvature perturbations[2] which are of interest at present time—during inflation at subhorizon scales, with "horizon" usually referring to the Hubble radius $1/a(t)H(t)$—freeze once they exit the horizon, which happens when the contracting horizon becomes comparable to their wavelength. Subsequently, these freezed perturbations become classical superhorizon perturbations which re-enter the horizon as the horizon expands again during the Hubble evolution of the Universe, after the reheating. Eventually, the gravitational collapse of the frozen superhorizon perturbations leads to the formation of the large-scale structure of the Universe, and the cosmic microwave background anisotropies correspond to superhorizon modes which were initially subhorizon during inflation but froze after the horizon crossing. Moreover, anisotropies are due to modes which have reentered the horizon, but still

[2]We are referring to the comoving curvature of Eq. (41), so these are fluctuating vacuum scalar perturbations.

these modes have the same origin on subhorizon scales during inflation.

At this point, a more concise presentation is needed so that we gain deeper insights into the full picture. The novel feature of the inflationary description is that, during inflation, the Hubble radius, to which we refer as the horizon, actually shrinks dramatically. So at the initial singularity, the horizon was very large, and the primordial perturbations were actually subhorizon scales since their comoving wave number was at subhorizon scales, that is,

$$k \gg H(t)a(t). \tag{66}$$

Note that, in principle, perturbations are created at all length scales, but the most relevant are those whose wave number is at subhorizon scales. Also note that we switched our description by using the wave number, but it is equivalent to using the wavelength, in which case Eq. (66) would be $\lambda \ll (H(t)a(t))^{-1}$. During the inflationary era, the Hubble radius (the horizon) shrinks, so at some point the cosmologically relevant perturbations of wave number k satisfying Eq. (66) exit the horizon and freeze, meaning that these become classical. So these become superhorizon perturbations, in which case, their wave number satisfies

$$k \ll a(t)H(t). \tag{67}$$

Once the horizon crossing occurs, the comoving curvature perturbations corresponding to the wave number k cease to be of quantum nature, and the corresponding quantum expectation value of the comoving curvature perturbation is practically the classical ensemble stochastic average of a classical stochastic field. This is what we mean by the "freezing" of these modes. The conservation of the average value of the comoving curvature perturbation in superhorizon scales is what actually enables us to relate the predictions corresponding to the time that the horizon is crossed, which actually corresponds to high energies, to the observable quantities corresponding to the horizon reentry of the modes after reheating, which in turn corresponds to low energies. Note that the era between the horizon exit and reentry is an era that is not fully understood, even now. In addition, this issue may appear also in bouncing cosmology, since in some cases what is needed is a quantum bounce description followed by some other model; see, for example, [45]. We shall discuss this issue in more detail later on in this section. Coming back to the inflationary picture, at present time we are able to compute the inflationary observable quantities because the subhorizon wavelengths froze out at the horizon exit and evolved in a classical way until nearly the present time era, after reheating and after reentering the horizon. Note that, in principle, the primordial perturbations may freeze out well before the inflationary era ends, so these correspond to an expanding quantum era. Hence, the primordial curvature perturbations corresponding to the expanding quantum era may be directly related to the cosmic microwave background observables and also to other present-time observables since the quantum fluctuations freeze after the horizon exit.

Let us now turn our focus to the singular bouncing cosmology of Eq. (5). In this case, the Hubble radius as a function of the cosmological time is equal to

$$r_H(t) = \frac{e^{-\frac{(t-t_s)^{1+\alpha}\beta}{1+\alpha}}(t-t_s)^{-\alpha}}{\beta}, \tag{68}$$

where for notational simplicity we denoted the Hubble radius as $r_H(t) = \frac{1}{a(t)H(t)}$. As is obvious from Eq. (68), in the case of a type IV singularity, since $\alpha > 1$, the Hubble radius at the bouncing point, which is also chosen to be the singularity point, diverges due to the existence of the term $\sim(t-t_s)^{-\alpha}$. Hence, all the cosmologically relevant modes are in subhorizon scales, since $k \gg H(t_s)a(t_s) = 0$ at that point. Equivalently, at the singularity point, the cosmologically relevant modes are inside the Hubble radius which is infinite at that point, so the wavelength of these modes satisfies $\lambda \ll r_H$. Immediately after the bouncing point, the Hubble radius drops and starts to shrink as the time evolves. This can also be seen in Fig. 1, where we can see that the Hubble radius drops after the bouncing point in a radical way. In the left plot, we plotted until $t \simeq 10^{-20}$ sec, and in the right plot, until $t \simeq 10^{-10}$ sec, and as can be seen, the Hubble radius r_H fraction corresponding to the two cases is of the order $\frac{r_H(10^{-20})}{r_H(10^{-10})} \simeq 10^{12}$. Subsequently, as the Hubble radius shrinks, the cosmologically relevant modes will eventually exit the horizon, when the Hubble radius becomes of the order of their wavelength $\lambda \sim r_H$. But which modes can be cosmologically relevant in the singular bounce at hand? In principle, the time era near the bouncing point is governed by the quantum theory of gravity, so after the bouncing point we may still have the quantum era primordial modes. But to which cosmological time does the "near the bounce" expression refer? Since we assume that $t_s \simeq 10^{-35}$ sec, then near the bouncing point, from a mathematical point of view, corresponds to cosmological times of the order $t \simeq 10^{-10}$ sec, which can also be considered as being near the bounce since what we assumed in the calculation of the spectrum of the primordial curvature perturbations is that $t - t_s \to 0$. Thereby, in inflationary terms these cosmic times correspond to times after the exit from the inflationary era, so the relevant modes today have already exited the horizon well before $t \simeq 10^{-10}$ sec. Hence, for the singular bounce, the modes we took into account in the calculation of the spectrum of the primordial curvature perturbations are actually the cosmologically relevant ones today—the quantum modes. Therefore, in the singular bounce case, the modes with wavelengths of the order of the horizon corresponding to

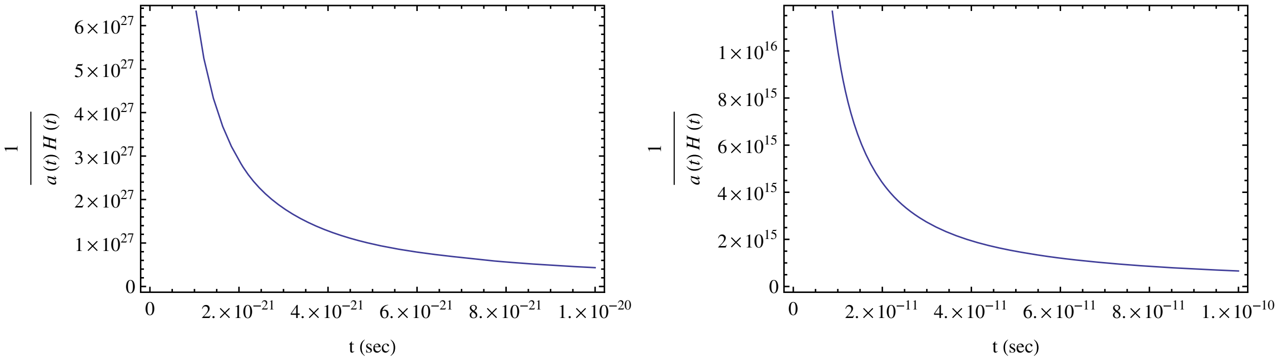

FIG. 1 (color online). The Hubble radius $r_H = \frac{1}{a(t)H(t)}$ as a function of time, for $t_s = 10^{-35}$ sec, $\alpha = 13/11$, $\beta = 0.001$ $(\mathrm{sec})^{-\alpha-1}$. As can be seen from the left and right plot, the Hubble radius r_H fraction corresponding to $t \simeq 10^{-20}$ sec and $t \simeq 10^{-10}$ sec is of the order $\frac{r_H(10^{-20})}{r_H(10^{-10})} \simeq 10^{12}$.

cosmic times near the bouncing point are the cosmologically relevant ones since these can reveal the quantum era of primordial expansion. After these modes exit the Hubble radius, freeze out and the quantum expectation value of the comoving curvature perturbation are described by the classical ensemble stochastic average of a classical stochastic field. Practically, the conservation of the average value of the comoving curvature perturbation at super-horizon scales will eventually enable us to relate the horizon crossing predictions (high-energy ones) to the late-time ones, which correspond to the horizon reentry era (low-energy).

However, we need to explicitly demonstrate that in the case of the singular bounce we studied in this paper, the comoving curvature perturbations are conserved after the modes exit the horizon. This is important since it is not granted that the comoving curvature perturbations will be conserved, as in the inflationary cosmology, for example. The matter bounce cosmology case [41] is an example of when the comoving curvature perturbations grow after the modes exit the horizon. We will now study the evolution of the comoving curvature perturbations in the context of the singular bounce. We are interested in the cosmological times for which the Hubble rate and scale factor satisfy $k \ll a(t)H(t)$, so times much later than the horizon crossing. Note that this does not mean that t corresponds to late times, or $t \gg 1$, but the cosmic time is $t \gg t_H$, with t_H the time at which the horizon crossing occurs. Since $k \ll a(t)H(t)$, this means that the last term in the differential equation of Eq. (42) can be neglected, so the differential equation becomes

$$\frac{1}{a(t)^3 Q(t)} \frac{\mathrm{d}}{\mathrm{d}t} (a(t)^3 Q(t) \dot{\Phi}) = 0, \tag{69}$$

which can be easily solved and the solution is

$$\Phi(t) = \mathcal{C}_1 + \mathcal{C}_2 \int \frac{1}{a(t)^3 Q(t)} \mathrm{d}t, \tag{70}$$

with $Q(t)$ being defined in Eq. (44). In order to see if the comoving curvature perturbations are conserved after the horizon crossing, we need to examine the behavior of the integral term in Eq. (70). Obviously, the key point to determine the behavior of the comoving curvature perturbations is to determine $Q(t)$, and therefore to find which $F(G)$ gravity describes the cosmological evolution at the cosmological times for which $k \ll a(t)H(t)$. In order to proceed, we need to classify the problem into two subcases or scenarios, more preferably, which we list below:

(1) In the first scenario, which we call scenario I, the cosmic times for which the relation $k \ll a(t)H(t)$, satisfy $t \gg t_s$, and also $t \ll 1$, where t_s the time at which the type IV singularity occurs. This is the most plausible scenario, from a physical point of view, and it could be realized like this: Suppose that $t_s = 10^{-35}$ sec and the time long after the horizon crossing is at $t = 10^{-15}$ sec. Hence, in this case, the time t is 10^{20} times larger than t_s and also satisfies $t \ll 1$. Note that the time at which horizon crossing occurs, is somewhere in between t_s and t, that is, $t_s < t_H < t$.

(2) In the second scenario, which we call scenario II, we have again $t \gg t_s$, but $t > 1$.

Scenario I is much more likely to occur since the time for which $k \ll a(t)H(t)$ holds true is after the horizon crossing, which for inflationary cosmology is (possibly) of the order 10^{-30} sec, so this is the most appealing case and we start our analysis with scenario I. For completeness, we also deal with scenario II later on.

So in the context of scenario I, the cosmic time satisfies $t \gg t_s$, but still $t \ll 1$ and, consequently, in this case $t - t_s \to 0$. Therefore, the $F(G)$ gravity that is responsible for the cosmological evolution at the time t, is still given by Eq. (24) and, therefore, the function $Q(t)$ can easily be calculated to yield

$$Q(t) \simeq \mathcal{Z}_2 t^{-4\alpha}, \tag{71}$$

where in Eq. (71) we kept the most dominant term, and also we used the fact that $t-t_s \simeq t \ll 1$. Therefore, since $a(t) \simeq e^{f_0 t^{\alpha+1}}$ for the singular bounce [see Eq. (1)], the term $\frac{1}{a(t)^3 Q(t)}$, behaves as follows:

$$\frac{1}{a(t)^3 Q(t)} \sim \frac{t^{4\alpha}}{e^{f_0 t^{\alpha+1}}}. \tag{72}$$

Clearly, the exponential dominates, so the integral term decays as t increases, that is,

$$\int \frac{1}{a(t)^3 Q(t)} \to 0, \tag{73}$$

and, hence, the comoving curvature perturbation is approximately equal to

$$\Phi(t) = \mathcal{C}_1, \tag{74}$$

which means that the comoving curvature perturbation for scenario I is conserved after the horizon crossing.

Now we turn our focus to scenario II, for which $t \gg t_s$ and $t > 1$. As we already mentioned, this scenario probably corresponds to an era much later than the bounce, and possibly much later than the horizon crossing, so it is rather less physically appealing. Regardless, we shall study this scenario for completeness. It is conceivable that the $F(G)$ gravity which describes the cosmological evolution is no longer given by the one appearing in Eq. (24) since $(t-t_s)$ is not small anymore. Therefore, we need to find the $F(G)$ gravity that describes the cosmological evolution. This is, however, a formidable task since by employing the reconstruction method we used in the previous sections, we end up at the following differential equation,

$$2t^{\alpha+1}\ddot{P}(t) - 2t^{1+2\alpha}\alpha\beta^4\dot{P}(t) + 1 = 0, \tag{75}$$

which is very difficult to solve analytically. So, in order to proceed, we speculate about the possible behavior of the $F(G)$ gravity. If, for example, the $F(G)$ gravity has polynomial form, that is $F(G) \sim BG^\gamma$, then, since for large t the Gauss-Bonnet invariant becomes approximately equal to $G \sim t^{4\alpha}$, the resulting expression for $Q(t)$ is

$$Q(t) \simeq \mathcal{Z}_3 t^{-2+8\alpha+8\alpha\gamma}, \tag{76}$$

with $\mathcal{Z}_3$ being equal to

$$\mathcal{Z}_3 = 2^{12+6\gamma}3^{2+2\gamma}B^2\alpha^2\beta^{8+8\gamma}. \tag{77}$$

Therefore, the term $\sim 1/(a(t)^3 Q(t))$ is equal to

$$\frac{1}{a(t)^3 Q(t)} \sim \frac{1}{e^{f_0 t^{\alpha+1}} t^{-2+8\alpha+8\alpha\gamma}}. \tag{78}$$

Hence, even in the case that γ is a large negative real number, the exponential in the expression appearing in Eq. (78) dominates and, thereby, the integral in Eq. (70) decays and becomes subdominant. Hence, in this case too, the comoving curvature perturbations after the horizon crossing are conserved since $\Phi(t) = \mathcal{C}_1$.

In the case that the $F(G)$ is such that $Q(t)$ dominates over the exponential scale factor, then the integral in Eq. (70) dominates the evolution and possibly the comoving curvature perturbations grow as time passes. For example, if $F(G) \sim e^{-G^\gamma}$, then the function $Q(t)$ becomes approximately equal to

$$Q(t) \sim e^{-\mathcal{A}_1 t^{4\alpha\gamma}} t^{-2+8\alpha}, \tag{79}$$

with $\mathcal{A}_1 = 2^{1+3\gamma}3^\gamma\beta^{4\gamma}$, and therefore the term $\sim 1/(a(t)^3 Q(t))$ becomes approximately equal to

$$\frac{1}{a(t)^3 Q(t)} \sim \frac{e^{\mathcal{A}_1 t^{4\alpha\gamma}}}{e^{f_0 t^{\alpha+1}} t^{-2+8\alpha}}, \tag{80}$$

which clearly does not decay. So the integral in the expression (70) dominates the evolution of the comoving curvature perturbations after the horizon crossing, and the perturbations grow as the time grows larger. Therefore, in the case of scenario II, the curvature perturbations depend strongly on the form of the $F(G)$ gravity. However, scenario II is rather unlikely to occur, since the requirement $t > 1$ sec means that the Universe is at the lepton epoch, which is much later than the time when the horizon crossing occurred. At the time $t > 1$, it is possible that the singular bounce does not describe the Universe anymore since another description must be found to generate the Hubble radius expansion, because the Hubble radius decreases in the context of the singular bounce. It is highly likely that a scenario like the one used in [45] will take place and describe the Universe's evolution. Work is in progress for realizing such a scenario, but it is worth analyzing this a bit more.

The problem with the singular bouncing cosmology occurs because there exists no mechanism to make the Hubble radius eventually increase unless we assume that the singular bounce governs early times and, after some time, the evolution is described by another scale factor, as in Ref. [45] for example; see also [46] for an $F(R)$ description of the model of Ref. [45]. In the case of the model studied in Ref. [45], the quantum radiation era was followed by a perfect fluid evolution; see [45]. So in the singular bouncing case, it is necessary to find another scenario that will describe the evolution of the Universe after the quantum bouncing era, for which the new scenario, the reentering of the modes in the expanding horizon, will be possible. However, the time in between the two horizons era leaves a gap in our description, and, as we noted, this also occurs in the standard inflationary cosmology. This task exceeds the purposes of this paper, but we hope in a future work to provide a model with two or more phases, for which a successful cosmological description may be achieved. However, this will not be in the context of $F(G)$ gravity since, as we demonstrated, the spectrum of primordial curvature perturbations is not scale invariant, and this is a rather discouraging feature, but perhaps the LQC-corrected $F(G)$ gravity may have more appealing features.

VI. SINGULAR BOUNCE FROM MIMETIC $F(G)$ GRAVITY

As a final study, we shall investigate which $F(G)$ gravity can generate the singular bounce of Eq. (5), but in the context of mimetic $F(G)$ gravity. For a detailed account on this issue, see [36]. The mimetic $F(G)$ gravity approach uses the same action as the one appearing in Eq. (6), but the Jordan frame metric is parametrized as follows [47–50]:

$$g_{\mu\nu} = -\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi. \tag{81}$$

Upon varying the metric tensor, we get

$$\begin{aligned}\delta g_{\mu\nu} = {}& \hat{g}^{\rho\tau}\delta\hat{g}_{\tau\omega}\hat{g}^{\omega\sigma}\partial_\rho\phi\partial_\sigma\phi\hat{g}_{\mu\nu} - \hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\delta\hat{g}_{\mu\nu} \\ & - 2\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\delta\phi\hat{g}_{\mu\nu},\end{aligned}$$

and upon variation of the Jordan frame action (6), with respect to the redefined metric $\hat{g}_{\mu\nu}$, and with respect to the mimetic scalar ϕ, we obtain the following equations of motion:

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 8\left[R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho} + \frac{R}{2}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})\right]\nabla^\rho\nabla^\sigma F_{\mathcal{G}} + (F_G G - F(G))g_{\mu\nu} \\ & + \partial_\mu\phi\partial_\nu\phi\left(-R + 8\left(-R_{\rho\sigma} + \frac{1}{2}Rg_{\rho\sigma}\right)\nabla^\rho\nabla^\sigma F_{\mathcal{G}} + 4(F_G G - F(G))\right) = T_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi T, \end{aligned} \tag{82}$$

with F_G standing for $F_G = dF(G)/dG$. Moreover, upon variation of the action (6) with respect to the mimetic scalar field ϕ, we get

$$\nabla^\mu\left(\partial_\mu\phi\left(-R + 8\left(-R_{\rho\sigma} + \frac{1}{2}Rg_{\rho\sigma}\right)\nabla^\rho\nabla^\sigma F_{\mathcal{G}} + 4(F_G G - F(G)) - T\right)\right) = 0. \tag{83}$$

Since the following relation holds true [36],

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -1, \tag{84}$$

and owing to the fact that the mimetic scalar ϕ depends only on the cosmic time, we get the constraint $\phi = t$. Hence, the (t, t) component of the expression appearing in Eq. (82) becomes

$$\begin{aligned} & 2\dot{H} + 3H^2 + 16H(\dot{H} + H^2)\frac{dF_{\mathcal{G}}}{dt} + 8H^2\frac{d^2F_G}{dt^2} \\ & - (F_G\mathcal{G} - F(G)) = -p. \end{aligned} \tag{85}$$

Upon integration of Eq. (83), we get

$$\begin{aligned} & -R + 8\left(-R_{\rho\sigma} + \frac{1}{2}Rg_{\rho\sigma}\right)\nabla^\rho\nabla^\sigma F_{\mathcal{G}} \\ & + 4(F_G G - F(G)) + \rho - 3p = -\frac{C}{a^3}, \end{aligned} \tag{86}$$

which can be rewritten as follows:

$$\begin{aligned} & \dot{H} + 2H^2 + 4H^2\frac{d^2F_G}{dt^2} + 4H(2\dot{H} + 3H^2)\frac{dF_{\mathcal{G}}}{dt} \\ & + \frac{2}{3}(F_G G - F(G)) + \frac{\rho}{6} - \frac{p}{2} = -\frac{C}{a^3}, \end{aligned} \tag{87}$$

where C is an arbitrary constant.

The combination of Eqs. (85) and (87) results in

$$\dot{H} + 4H^2\frac{d^2F_{\mathcal{G}}}{dt^2} + 4H(2\dot{H} - H^2)\frac{dF_{\mathcal{G}}}{dt} = -\frac{1}{2}(\rho + p) - \frac{C}{a^3}. \tag{88}$$

For convenience, we introduce the function $g(t)$, which is defined as

$$g(t) = \frac{dF_G}{dt}$$

and satisfies the following equation:

$$4H^2\frac{dg(t)}{dt} + 4H(2\dot{H} - H^2)g(t) = -\dot{H} - \frac{1}{2}(\rho + p) - \frac{C}{a^3}. \tag{89}$$

The general form of the solution of the differential equation (89) is

$$g(t) = g_0\left(\frac{H_0}{H}\right)^2 \exp\left(\int_0^t H dt\right) + \frac{1}{4H^2}\int_0^t dt_1 B(t_1)\exp\left(\int_{t_1}^t H(\tau)d\tau\right), \tag{90}$$

where $B(t) = -\dot{H} - \frac{1}{2}(\rho + p) - \frac{C}{a^3}$ and, in addition, g_0 is an integration constant, while $H_0 = H(0)$. Having Eq. (90) for a given cosmological evolution in terms of the Hubble rate, we can easily obtain the function $g(t)$. Consequently, the function $F(t)$ reads

$$F_G(t) = \int g(t)dt.$$

Then by exploiting the expression (8), and solving with respect to t, we have the function $t = t(G)$. By substituting this into $F_G(t)$, and by integrating with respect to G, we easily obtain the $F(G)$ function. Let us apply this method for the case of the singular bounce of Eq. (5). By substituting the Hubble rate of Eq. (5) in Eq. (90), and by keeping the most dominant terms, we obtain

$$g(t) = \frac{e^{f_0(t-t_s)^{1+\alpha}} g_0 H_0^2 (t-t_s)^{-2\alpha}}{f_0^2(1+\alpha)^2} + \frac{(t-t_s)^{-2\alpha}(-\frac{Ct^2}{2} - \frac{pt^2}{4} - \frac{t^2\rho}{4})}{4f_0^2(1+\alpha)^2}. \tag{91}$$

Integrating this with respect to t, we obtain the approximate form of the function $F_G(t)$ near the bouncing point which is

$$F_G(t) = \frac{(t-t_s)^{1-2\alpha}(-16g_0H_0^2(3-5\alpha+2\alpha^2) + (t_s^2 + t(t_s - 2t_s\alpha) + t^2(1-3\alpha+2\alpha^2))(2C+p+\rho))}{16f_0^2(-1+\alpha)(1+\alpha)^2(-3+2\alpha)(-1+2\alpha)} + \frac{f_0^{-\frac{3}{1+\alpha}} g_0 H_0^2 \Gamma[\frac{3}{1+\alpha}]}{(1+\alpha)(2-5\alpha+2\alpha^2)}. \tag{92}$$

Using (8), and also Eq. (20), we obtain the function $F_G(G)$, so upon integration with respect to G we finally get

$$F(G) \simeq -24f_0^3\alpha(1+\alpha)^3(a_7 G^{-\frac{3\alpha}{1-3\alpha}}) + \frac{-24f_0^3\alpha(1+\alpha)^3 a_1}{a_6} G^{\frac{1+\alpha}{-1+3\alpha}}\left(a_2 + 3 + a_4\left(t_s + 24^{\frac{1}{1-3\alpha}} G^{\frac{1}{-1+3\alpha}}\left(-\frac{1}{f_0^3\alpha(1+\alpha)^3}\right)^{\frac{1}{-1+3\alpha}}\right)\right) + \frac{-24f_0^3\alpha(1+\alpha)^3(2C+p+\rho)a_5 a_1 G^{\frac{1+\alpha}{-1+3\alpha}}}{a_6}\left(t_s + 24^{\frac{1}{1-3\alpha}} G^{\frac{1}{-1+3\alpha}}\left(-\frac{1}{f_0^3\alpha(1+\alpha)^3}\right)^{\frac{1}{-1+3\alpha}}\right)^2, \tag{93}$$

where the parameters $a_i, i = 1, 2, \ldots 7$ are given in Appendix C. Since we are considering the limit $t \to t_s$, which means $G \to 0$ as we explained earlier, by keeping the most dominant term in Eq. (93), we get

$$F(G) = -\frac{24f_0^3\alpha(1+\alpha)^3(2C+p+\rho)t_s^2 a_5 a_1}{a_6} G^{\frac{1+\alpha}{-1+3\alpha}}. \tag{94}$$

It is conceivable that the resulting mimetic $F(G)$ gravity of Eq. (94) is different for the vacuum $F(G)$ gravity of Eq. (23), but we need to further analyze this issue. In the context of the mimetic $F(G)$ gravity, some extra conformal degrees of freedom arise in the FRW equations of motion. Therefore, in the context of mimetic $F(G)$ gravity, we have a new reconstruction method in which we can choose the internal degrees of freedom and a specific $F(G)$ gravity so that some fixed cosmological evolution is generated. This is different from the ordinary vacuum $F(G)$ gravity case since in this case no internal degrees of freedom are taken into account. The $F(G)$ gravity that can generate the same Hubble rate as the mimetic $F(G)$ is, in principle, different from the resulting expression of the mimetic $F(G)$ gravity. Of course, in both cases we are using approximations, so one should be cautious when dealing with both theories. Finally, let us note that in the mimetic $F(G)$ gravity, much more freedom is offered for successfully generating various cosmological scenarios; see for example [50]. This is because of the presence of these internal conformal degrees of freedom. It is then easy to reconstruct any cosmology by suitably adjusting these degrees of freedom and, in some approaches, the potential and the Lagrange multiplier. It is questionable, however, if these results can be trusted because the resulting picture is complicated. If simplicity is to be an important feature of a physical theory, then probably these theories are of mathematical importance only. However, concordance with observations is an appealing feature, and therefore these theories can be valuable from a physical point of view. Since this discussion should be addressed in more detail, we defer it to a future work. Regardless of the differences, in both cases, the $F(G)$ gravity that can realize the singular bounce of Eq. (5) is described by a power-law function. In our opinion, however, the vacuum $F(G)$ is conceptually a simpler theory, so from this aspect it offers a more appealing physical description of the singular bounce.

VII. CONCLUSIONS

In this paper, we studied a bounce cosmology with a type IV singularity at the bouncing point in the context of classical $F(G)$ gravity. In particular, we investigated which classical pure (vacuum) $F(G)$ gravity can generate the type IV singular bounce cosmology, emphasizing cosmic times near the bouncing point. As we explicitly demonstrated, the resulting $F(G)$ gravity has the form $F(G) \sim C_2 G + B G^{\frac{\alpha}{-1+3\alpha}}$, so it is a power-law modified gravity theory. Since this result holds true only near the singularity point, we discussed the possibility that this $F(G)$ gravity is the limiting case of some viable $F(G)$ gravity, in which case the full solution would also be interesting since the late-time and early-time acceleration could be simultaneously described by the same theory. We also discussed the stability of the resulting $F(G)$ theory, from a dynamical point of view, examining if it can be the final attractor of the theory. As we anticipated, the answer to this question does not lie in the affirmative and, hence, instability of the solution cannot be avoided. This feature is welcome since the cosmological evolution does not stop at the bouncing point and, therefore, the resulting $F(G)$ gravity is not anticipated to be a stable solution of the cosmological system. Moreover, we investigated how the scalar cosmological perturbations of the background flat FRW metric behave near the bouncing point, and we explicitly calculated the spectrum of primordial curvature perturbations. As we showed, the spectrum is not scale invariant, and as we claimed in the main text, this result should be further investigated. This is due to the fact that we cannot be sure if it is a universal feature of the theory that owes its existence to the type IV singularity or an artifact of the approximations we made to obtain the resulting $F(G)$ gravity. The latter seems more plausible; however, this feature has to be thoroughly addressed. Another important point that we need to stress, with regards to the nonscale invariance of the spectrum of primordial curvature perturbations, is that since we are studying a classical theory near the bouncing point, it might be possible that at these cosmic time scales, quantum effects take place. So, effectively, the lack of scale invariance in the power spectrum might be an effect of the classical approach to the problem, so the same problem should be addressed in the context of loop quantum cosmology [51] and, especially, in the context of $F(G)$ LQC, which was developed in [44]. We also studied which mimetic $F(G)$ gravity can describe the singular bounce near the bouncing point by adopting the formalism of [36] and the resulting $F(G)$ gravity that has a power-law functional form.

Finally, with regards to the classical $F(G)$ gravity approach, since the $F(G)$ gravity is a special case of the most general class of $F(R, G)$ theory [10,11,40,52,53], the same problem we investigated in this problem should be addressed in the context of $F(R, G)$ theory. Actually, this problem should also be compared with the $F(R, G)$ gravity inflation properties, as was done in [53], but this time by using a type IV singular bounce. In addition, a compelling task is to include matter fluids in the theory and investigate how the physical picture is affected by the presence of matter. Moreover, as was demonstrated in Ref. [34], a type IV singular bounce may play a crucial role in the graceful exit from inflation, but the study was focused on scalar field models. It is worth examining the effect of the type IV singularity on the Jordan frame $F(G)$ theories but also on the $F(R)$ and $F(T)$ theories. With regards to the latter, see [54]. We hope to address these projects in a future work.

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APPENDIX A: ANALYTIC FORM OF $Q(t)$ AND OF A, B

In this appendix we quote the exact form of the function $Q(t)$ and of A, B appearing in Eqs. (17) and (21). In particular, the function $Q(t)$ reads

$$
\begin{aligned}
Q(t) = & -2(t-t_s)^{-1+2\alpha}\alpha\beta^2 - 72(t-t_s)^{-1+3\alpha}\alpha\beta^3\Bigg(-\frac{(t-t_s)^{1-2\alpha}((t-t_s)^{-1+\alpha}\alpha - 2(t-t_s)^{-1+\alpha}\alpha^2)}{2(-1+\alpha)(-1+2\alpha)\beta} \\
& - \frac{(t-t_s)^{-2\alpha}(1-2\alpha)((t-t_s)^{\alpha} - 2(t-t_s)^{\alpha}\alpha - 2C_1\beta + 2C_1\alpha\beta)}{2(-1+\alpha)(-1+2\alpha)\beta}\Bigg) \\
& \times 24(t-t_s)^{3\alpha}\beta^3\Bigg(-\frac{(t-t_s)^{-2\alpha}(1-2\alpha)((t-t_s)^{-1+\alpha}\alpha - 2(t-t_s)^{-1+\alpha}\alpha^2)}{(-1+\alpha)(-1+2\alpha)\beta} \\
& - \frac{(t-t_s)^{1-2\alpha}((t-t_s)^{-2+\alpha}(-1+\alpha)\alpha - 2(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha^2)}{2(-1+\alpha)(-1+2\alpha)\beta} \\
& + \frac{(t-t_s)^{-1-2\alpha}(1-2\alpha)\alpha((t-t_s)^{\alpha} - 2(t-t_s)^{\alpha}\alpha - 2C_1\beta + 2C_1\alpha\beta)}{(-1+\alpha)(-1+2\alpha)\beta}\Bigg).
\end{aligned} \tag{A1}
$$

Also, the exact analytic form of the coefficients A and B appearing in Eq. (21) is the following:

$$
\begin{aligned}
A &= 1124^{\frac{2\alpha}{1-3\alpha}}\beta^2(\alpha\beta^3)^{\frac{2\alpha}{1-3\alpha}} \\
B &= \frac{(24^{\frac{1-2\alpha}{1-3\alpha}}C_1 - 24^{\frac{1-2\alpha}{1-3\alpha}}C_1\alpha - \frac{2^{-1+\frac{3(1-2\alpha)}{1-3\alpha}}3^{\frac{1-2\alpha}{1-3\alpha}}}{\beta} + \frac{24^{\frac{1-2\alpha}{1-3\alpha}}\alpha}{\beta})(\alpha\beta^3)^{\frac{1-2\alpha}{1-3\alpha}}}{1-3\alpha+2\alpha^2} - 24^{1+\frac{\alpha}{1-3\alpha}}C_1 G^{\frac{\alpha}{-1+3\alpha}}\beta^3(\alpha\beta^3)^{\frac{\alpha}{1-3\alpha}}.
\end{aligned} \tag{A2}
$$

APPENDIX B: EXACT FORM OF THE PARAMETERS J_1,J_2,J_3 AND OF $\mathcal{A}$

Here we quote the exact form of the parameters J_1, J_2, J_3 and of $\mathcal{A}$. In particular, the parameter J_1 is

$$
J_1 = \frac{2^{1+\frac{2\alpha}{-1+3\alpha}}3^{\frac{\alpha}{-1+3\alpha}}BN^{\gamma}\alpha^2\beta^2(\frac{N^{\gamma}\beta^2(\frac{2N^{\gamma}\beta^2}{f_0}+\frac{N^{-1+\gamma}\beta^2\gamma}{f_0})}{f_0})^{\frac{\alpha}{-1+3\alpha}}}{f_0(-1+3\alpha)^2(\frac{2N^{\gamma}\beta^2}{f_0}+\frac{N^{-1+\gamma}\beta^2\gamma}{f_0})^2} - \frac{2^{1+\frac{2\alpha}{-1+3\alpha}}3^{\frac{\alpha}{-1+3\alpha}}BN^{\gamma}\alpha\beta^2(\frac{N^{\gamma}\beta^2(\frac{2N^{\gamma}\beta^2}{f_0}+\frac{N^{-1+\gamma}\beta^2\gamma}{f_0})}{f_0})^{\frac{\alpha}{-1+3\alpha}}}{f_0(-1+3\alpha)(\frac{2N^{\gamma}\beta^2}{f_0}+\frac{N^{-1+\gamma}\beta^2\gamma}{f_0})^2}, \tag{B1}
$$

while J_2 is

$$
\begin{aligned}
J_2 = & \frac{1}{f_0^2}432N^{2\gamma}\beta^4\left(\frac{N^{\gamma}\beta^2(\frac{2N^{\gamma}\beta^2}{f_0}+\frac{N^{-1+\gamma}\beta^2\gamma}{f_0})}{f_0}\right)^{-2+\frac{\alpha}{-1+3\alpha}}\left(\frac{12^{-2+\frac{\alpha}{-1+3\alpha}}B\alpha(-1+\frac{\alpha}{-1+3\alpha})(\frac{2N^{\gamma}\beta^2}{f_0}+\frac{N^{-1+\gamma}\beta^2\gamma}{f_0})}{-1+3\alpha}\right. \\
& \left. + \frac{2^{-1+\frac{2\alpha}{-1+3\alpha}}3^{-2+\frac{\alpha}{-1+3\alpha}}BN^{\gamma}\alpha(-1+\frac{\alpha}{-1+3\alpha})\beta^2(\frac{N^{-2+2\gamma}\beta^4\gamma^2}{f_0^2}+\frac{N^{\gamma}\beta^2(\frac{4N^{-1+\gamma}\beta^2\gamma}{f_0}+\frac{N^{-2+\gamma}\beta^2(-1+\gamma)\gamma}{f_0})}{f_0})}{f_0(-1+3\alpha)}\right),
\end{aligned} \tag{B2}
$$

and finally J_3 is

$$
\begin{aligned}
J_3 = & \, 62^{-1+\frac{2\alpha}{-1+3\alpha}}3^{-1+\frac{\alpha}{-1+3\alpha}}BN^{\gamma}\alpha\left(-1+\frac{\alpha}{-1+3\alpha}\right)\beta^2\left(\frac{N^{\gamma}\beta^2(\frac{2N^{\gamma}\beta^2}{f_0}+\frac{N^{-1+\gamma}\beta^2\gamma}{f_0})}{f_0}\right)^{-2+\frac{\alpha}{-1+3\alpha}} \\
& \times\left(1+\frac{\left(\frac{4N^{\gamma}\beta^2}{f_0}+\frac{N^{-1+\gamma}\beta^2\gamma}{f_0}\right)\left(\frac{N^{-2+2\gamma}\beta^4\gamma^2}{f_0^2}+\frac{N^{\gamma}\beta^2(\frac{4N^{\gamma}\beta^2}{f_0}+\frac{N^{-2+\gamma}\beta^2(-1+\gamma)\gamma}{f_0})}{f_0}\right)}{f_0(-1+3\alpha)}\right. \\
& \left. + \frac{\left(-\frac{8N^{2\gamma}\beta^4}{f_0^2}+\frac{3N^{-2+2\gamma}\beta^4\gamma^2}{f_0^2}+\frac{6N^{\gamma}\beta^2(\frac{3N^{-1+\gamma}\beta^2\gamma}{f_0}+\frac{N^{-2+\gamma}\beta^2(-1+\gamma)\gamma}{f_0})}{f_0}\right)}{f_0(-1+3\alpha)}\right).
\end{aligned} \tag{B3}
$$

Also the parameter $\mathcal{A}$ which appears in Eq. (39) is equal to

$$\mathcal{A} = \frac{3^{1+\frac{\alpha}{1-3\alpha}} 4^{\frac{\alpha}{1-3\alpha}} f_0 (1-3\alpha)^2 \gamma \left(\frac{\beta^4 \gamma}{f_0^2}\right)^{1+\frac{\alpha}{1-3\alpha}}}{B\alpha(-1+2\alpha)\beta^2}. \tag{B4}$$

APPENDIX C: DETAILED FORM OF THE PARAMETERS $a_1, a_2, a_3, a_4, a_5, a_6, a_7$

The detailed form of the parameters a_i, $i = 1, 2, \ldots 7$ that appear in Eq. (93) are

$$\begin{aligned}
a_1 &= \left(24^{\frac{1}{1-3\alpha}} \left(-\frac{1}{f_0^3 \alpha (1+\alpha)^3}\right)^{-\frac{1}{-1+3\alpha}}\right)^{1+\alpha}, \\
a_2 &= -16 g_0 H_0^2 (18 - 15\alpha - 10\alpha^2 + 5\alpha^3 + 2\alpha^4) \\
a_3 &= t_s^2 (11 - 6\alpha + 3\alpha^2), \\
a_4 &= t_s (5 - 6\alpha - 9\alpha^2 + 2\alpha^3) \\
a_5 &= (2 - 3\alpha - 4\alpha^2 + 3\alpha^3 + 2\alpha^4), \\
a_6 &= 16 f_0^2 (1+\alpha)^3 (-1+2\alpha)(-2+\alpha+\alpha^2) \\
&\quad \times (-9 + 3\alpha + 2\alpha^2) \\
a_7 &= \frac{f_0^{-\frac{3}{1+\alpha}} g_0 H_0^2 (24^{\frac{1}{1-3\alpha}} (-\frac{1}{f_0^3 \alpha (1+\alpha)^3})^{-\frac{1}{1-3\alpha}})^{3\alpha} \Gamma[\frac{3}{1+\alpha}]}{6\alpha - 9\alpha^2 - 9\alpha^3 + 6\alpha^4}.
\end{aligned} \tag{C1}$$

APPENDIX D: THE PARAMETERS Ω_1, Ω_2, Ω_3, Ω_4, μ AND ζ

Here we quote the detailed form of the parameters Ω_1, Ω_2, Ω_3, Ω_4 and Δ_i, $i = 1, 2$, μ and ζ. In particular, the parameters Ω_1, Ω_2, Ω_3, Ω_4 appearing in Eq. (46) are equal to

$$\begin{aligned}
\Omega_1 &= \frac{2^{-4+\frac{6\alpha}{-1+3\alpha}} 3^{-1+\frac{2\alpha}{-1+3\alpha}} B^2 k^2 (1-2\alpha)^2 \alpha (\alpha\beta^3)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4 \beta^7}, \\
\Omega_2 &= -\frac{2^{-3+\frac{6\alpha}{-1+3\alpha}} 3^{-1+\frac{2\alpha}{-1+3\alpha}} B^2 (1-2\alpha)^2 \alpha (\alpha\beta^3)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4 \beta^6}, \\
\Omega_3 &= \frac{2^{-5+\frac{6\alpha}{-1+3\alpha}} 3^{\frac{2\alpha}{-1+3\alpha}} B^2 (1-2\alpha)^2 \alpha (\alpha\beta^3)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4 (1+\alpha) \beta^5}, \\
\Omega_4 &= \frac{2^{-5+\frac{6\alpha}{-1+3\alpha}} 3^{-1+\frac{2\alpha}{-1+3\alpha}} B^2 (1-2\alpha)^2 (\alpha\beta^3)^{\frac{2\alpha}{-1+3\alpha}}}{(1-3\alpha)^4 \beta^6}.
\end{aligned} \tag{D1}$$

In addition, the parameters μ and ζ appearing in Eq. (47) are equal to

$$\begin{aligned}
\mu &= \frac{\Omega_2 + \Omega_4}{(-1+\alpha)\Omega_4}, \\
\zeta &= \frac{2\sqrt{\Omega_1}}{(-1+\frac{1}{\alpha})\alpha\sqrt{\Omega_4}}.
\end{aligned} \tag{D2}$$

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