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Lagrangian formulation for Mathisson-Papapetrou-Tulczyjew-Dixon equations

Walberto Guzmán Ramírez[1,*] and Alexei A. Deriglazov[1,2,†]
[1]Departamento de Matemática, ICE, Universidade Federal de Juiz de Fora, Juiz de Fora, MG, Brazil
[2]Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Lenin Avenue 30, Russian Federation
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We obtain Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations of a rotating body with given values of spin and momentum starting from Lagrangian action without auxiliary variables. MPTD equations correspond to the minimal interaction of our spinning particle with gravity. We briefly discuss some novel properties deduced from the Lagrangian form of MPTD equations: the emergence of an effective metric instead of the original one, the noncommutativity of coordinates of the representative point of the body, spin corrections to the Newton potential due to the effective metric, as well as spin corrections to the expression for integrals of motion of a given isometry.

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I. INTRODUCTION

The description of spinning bodies in general relativity is an old subject, which has been under intensive study for the past 70 years. The first results concerning equations of motion of a test body in a given background were reported by Mathisson [1] and Papapetrou [2]. They assumed that the structure of the test body could be described by a set of multipoles and have taken the approximation that involves only the first two terms (the pole-dipole approximation). The equations are then derived by the integration of the conservation law for the energy-momentum tensor, $T^{\mu\nu}{}_{;\mu} = 0$. A manifestly covariant formulation was given by Tulczyjew [3] and Dixon [4] and is under detailed investigation by many groups. In this work we will refer Eqs. (6.31)–(6.33) in [4] as Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations. Detailed analysis and interpretation of these equations and their generalizations [5–19] are necessary tasks since they are now widely used to account for spin effects in compact binaries and rotating black holes; see [20–25] and references therein.

It should be interesting to obtain these equations starting from an appropriate Lagrangian action. The vector models of spin yield one possible way to attack the problem. In these models, the basic variables in the spin sector are non-Grassmann vector ω^μ and its conjugated momentum π_μ. The spin tensor is a composed quantity constructed from these variables, $S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu)$. To have a theory with the right number of physical degrees of freedom for the spin (two for an elementary particle with spin one-half, and three for a rotating body in the pole-dipole approximation), some constraints on the eight basic variables must be imposed. This is the main difficulty: besides the equations of motion, the variational problem should produce these constraints. Even for the free theory in flat space, this turns out to be an extremely nontrivial problem [26–30]. We propose the Lagrangian action without auxiliary variables, which, besides the equations of motion, yields all the desired constraints. To point out some advantages of the vector model, let us compare it with the approach developed in [31] for the description of the relativistic top [26] in the curved background. First, in the vector model we have four basic variables in the spin sector instead of six (called ϕ_a in [31]) for the top. Taking into account that we present the Lagrangian without auxiliary variables, the vector model yields more economic formulation. Second, our primary constraints (see T_6 and T_7 below) follow from the variational problem and yield the spin supplementary condition (28). In the work [31] the condition has been added by hand and then considered as a first-class constraint of the formulation. Third, the vector model yields two physical degrees of freedom in the spin sector. Hence, it can be used for the descriptions of both a rotating body (see below) and an elementary particle with spin. In particular, the canonical quantization of the vector model has been considered in [32].

The work is organized as follows. In Sec. II we present Lagrangian action without auxiliary variables[1] for our spinning particle in an arbitrary curved background and obtain its Hamiltonian formulation. Section III contains the detailed derivation and analysis of both Lagrangian and Hamiltonian equations. The particle has a fixed value of spin and two physical degrees of freedom in the spin sector. We also present a modification that yields the model of Hanson-Regge type [26], with an unfixed value of the spin

*wguzman@cbpf.br
†alexei.deriglazov@ufjf.edu.br

[1]The variational problem with four auxiliary variables has been constructed in [33].

and four physical degrees of freedom. In Sec. IV we present the MPTD equations in the form convenient for our analysis. Here we follow the ideas of Dixon [4] and add the mass-shell condition to MPTD equations, transforming them into the Hamiltonian system. This allows us to compare MPTD equations with those of Sec. III. We show that the class of trajectories of MPTD equations with any given values of integration constants (squares of spin and of momentum) is described by our spinning particle with properly chosen mass and spin. In Sec. V we discuss some novel properties that can be immediately deduced from the Lagrangian form of MPTD equations. *Notation*.—The dynamical variables are taken in arbitrary parametrization τ, and then $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$, μ, $\nu = 0, 1, 2, 3$. The covariant derivative is $\nabla P^\mu = \frac{dP^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha P^\beta$ and the curvature is $R^\sigma{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\lambda\nu} - \partial_\nu\Gamma^\sigma{}_{\lambda\mu} + \Gamma^\sigma{}_{\beta\mu}\Gamma^\beta{}_{\lambda\nu} - \Gamma^\sigma{}_{\beta\nu}\Gamma^\beta{}_{\lambda\mu}$. The square brackets mean antisymmetrization, $\omega^{[\mu}\pi^{\nu]} = \omega^\mu\pi^\nu - \omega^\nu\pi^\mu$. We use the condensed notation $\dot{x}^\mu G_{\mu\nu}\dot{x}^\nu = \dot{x}G\dot{x}$, $N^\mu{}_\nu\dot{x}^\nu = (N\dot{x})^\mu$, $\omega^2 = g_{\mu\nu}\omega^\mu\omega^\nu$, and so on. The notation for the scalar functions constructed from second-rank tensors is $\theta S = \theta^{\mu\nu}S_{\mu\nu}$, $S^2 = S^{\mu\nu}S_{\mu\nu}$.

II. LAGRANGIAN AND HAMILTONIAN FORMULATIONS

The variational problem for the vector model of the spin interacting with electromagnetic and gravitational fields can be formulated with various sets of auxiliary variables [32–35]. For the free theory in flat space there is the Lagrangian action without auxiliary variables. The configuration space consists of the position $x^\mu(\tau)$ and the vector $\omega^\mu(\tau)$ attached to the point x^μ. The action reads

$$S = -\frac{1}{\sqrt{2}}\int d\tau\sqrt{m^2c^2 - \frac{\alpha}{\omega^2}} \times\sqrt{-\dot{x}N\dot{x} - \dot{\omega}N\dot{\omega} + \sqrt{[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}]^2 - 4(\dot{x}N\dot{\omega})^2}}. \quad (1)$$

The matrix $N_{\mu\nu}$ is the projector on the plane orthogonal to ω^ν,

$$N_{\mu\nu} = \eta_{\mu\nu} - \frac{\omega_\mu\omega_\nu}{\omega^2}, \quad \text{and then } N_{\mu\nu}\omega^\nu = 0. \quad (2)$$

Below we use the notation

$$T \equiv [\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}]^2 - 4(\dot{x}N\dot{\omega})^2. \quad (3)$$

The double square-root structure in the expression (1) seem to be typical for the vector models of spin [26]. The Lagrangian depends on one free parameter α that determines the value of the spin. The value $\alpha = \frac{3\hbar^2}{4}$ corresponds to a spin one-half particle. In the spinless limit, $\alpha = 0$ and $\omega^\mu = 0$, Eq. (1) reduces to the standard expression, $-mc\sqrt{-\dot{x}^\mu\dot{x}_\mu}$. The equivalent Lagrangian with one auxiliary variable $\lambda(\tau)$ is

$$L = \frac{1}{4\lambda}[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega} - T^{\frac{1}{2}}] - \frac{\lambda}{2}\left(m^2c^2 - \frac{\alpha}{\omega^2}\right). \quad (4)$$

Switching off the spin variables ω^μ from Eq. (4), we arrive at the familiar Lagrangian of spinless particle $L = \frac{1}{2\lambda}\dot{x}^2 - \frac{\lambda}{2}m^2c^2$. In this formulation the model admits interaction with an arbitrary electromagnetic field. The interacting theory is obtained [35] adding the minimal interaction term, $\frac{e}{c}A_\mu\dot{x}^\mu$, and replacing $\dot{\omega}^\mu$ by $D\omega^\mu \equiv \dot{\omega}^\mu - \lambda\frac{e\mu}{c}F^{\mu\nu}\omega_\nu$, where μ is the magnetic moment.

The Frenkel spin tensor [36] in our model is a composite quantity constructed from ω^μ, and its conjugated momentum $\pi^\mu = \frac{\partial L}{\partial\dot{\omega}_\mu}$ as follows:

$$S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu) = (S^{i0} = D^i, S_{ij} = 2\epsilon_{ijk}S_k). \quad (5)$$

Here S_i is a three-dimensional spin vector and D_i is a dipole electric moment [37]. The model is invariant under reparametrizations and local spin-plane symmetries [38]. The latter symmetry acts on ω^μ and π^μ but leaves $S^{\mu\nu}$ invariant. So only $S^{\mu\nu}$ is an observable quantity. In their work [26], Hanson and Regge analyzed whether the spin tensor interacts directly with an electromagnetic field and concluded with the impossibility of constructing the interaction in closed form. In our model an electromagnetic field interacts with ω^μ from which the spin tensor is constructed.

The minimal interaction with gravitational field can be achieved by covariantization of the formulation (1). In the expressions (1)–(3) we replace $\eta_{\mu\nu} \to g_{\mu\nu}$ and the usual derivative by the covariant one, $\dot{\omega}^\mu \to \nabla\omega^\mu = \frac{d\omega^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\omega^\beta$. Thus our Lagrangian in a curved background reads

$$L = -\frac{1}{\sqrt{2}}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{\frac{1}{2}}L_0,$$
$$L_0 \equiv \sqrt{-\dot{x}N\dot{x} - \nabla\omega N\nabla\omega + T^{\frac{1}{2}}}. \quad (6)$$

Velocities $\dot{x}^\mu$, $\nabla\omega^\mu$ and projector $N_{\mu\nu}$ transform like contravariant vectors and the covariant tensor, so the action is manifestly invariant under general-coordinate transformations.

Let us construct the Hamiltonian formulation of the model (6). Conjugate momenta for x^μ and ω^μ are $p_\mu = \frac{\partial L}{\partial\dot{x}^\mu}$ and $\pi_\mu = \frac{\partial L}{\partial\dot{\omega}^\mu}$, respectively. Because of the presence of Christoffel symbols in $\nabla\omega^\mu$, the conjugated momentum p_μ does not transform as a vector, so it is convenient to introduce the canonical momentum

$$P_\mu \equiv p_\mu - \Gamma^\beta_{\alpha\mu}\omega^\alpha\pi_\beta, \quad (7)$$

and the latter transforms as a vector under the general transformations of the coordinates. The manifest form of the momenta is as follows:

$$P_\mu = \frac{1}{\sqrt{2}L_0}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{\frac{1}{2}}[N_{\mu\nu}\dot{x}^\nu - K_\mu], \tag{8}$$

$$\pi_\mu = \frac{1}{\sqrt{2}L_0}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{\frac{1}{2}}[N_{\mu\nu}\nabla\omega^\nu - R_\mu], \tag{9}$$

with

$$K_\mu = T^{-1/2}[(\dot{x}N\dot{x} + \nabla\omega N\nabla\omega)(N\dot{x})_\mu - 2(\dot{x}N\nabla\omega)(N\nabla\omega)_\mu],$$
$$R_\mu = T^{-1/2}[(\dot{x}N\dot{x} + \nabla\omega N\nabla\omega)(N\nabla\omega)_\mu - 2(\dot{x}N\nabla\omega)(N\dot{x})_\mu].$$

These vectors obey the following algebraic identities:

$$K^2 = \dot{x}N\dot{x}, \quad R^2 = \nabla\omega N\nabla\omega, \quad KR = -\dot{x}N\nabla\omega,$$
$$\dot{x}R + \nabla\omega K = 0, \quad K\dot{x} + R\nabla\omega = T^{\frac{1}{2}}. \tag{10}$$

Using (2) we conclude that $\omega\pi = 0$ and $P\omega = 0$; that is, we found two primary constraints. Using the relations (10) we find one more primary constraint $P\pi = 0$. Besides, computing $P^2 + \pi^2$ given by (8) and (9) we see that all the terms with derivatives vanish, and we obtain the last primary constraint

$$T_1 \equiv P^2 + m^2c^2 + \pi^2 - \frac{\alpha}{\omega^2} = 0. \tag{11}$$

In the result, the action (6) implies four primary constraints, T_1 and

$$T_5 \equiv \omega\pi = 0, \quad T_6 \equiv P\omega = 0, \quad T_7 \equiv P\pi = 0. \tag{12}$$

The Hamiltonian is constructed excluding velocities from the expression

$$H = p_\mu\dot{x} + \pi\dot{\omega} - L + \lambda_i T_i \equiv P\dot{x} + \pi\nabla\omega - L + \lambda_i T_i, \tag{13}$$

where λ_i ($i = 1,\ 5,\ 6,\ 7$) are the Lagrangian multipliers associated with the primary constraints. From (8) and (9), we observe the equalities $P\dot{x} = (\sqrt{2}L_0)^{-1}(m^2c^2 - \frac{\alpha}{\omega^2})^{\frac{1}{2}}[\dot{x}N\dot{x} - \dot{x}K]$ and $\pi\nabla\omega = (\sqrt{2}L_0)^{-1}(m^2c^2 - \frac{\alpha}{\omega^2})^{\frac{1}{2}}[\nabla\omega N\nabla\omega - \nabla\omega R]$. Together with (10) they imply $P\dot{x} + \pi\nabla\omega = L$. Using this in (13), we conclude that the Hamiltonian is composed from the primary constraints

$$H = \frac{\lambda_1}{2}\left(P^2 + m^2c^2 + \pi^2 - \frac{\alpha}{\omega^2}\right) + \lambda_5(\omega\pi) + \lambda_6(P\omega)$$
$$+ \lambda_7(P\pi). \tag{14}$$

The full set of phase-space coordinates consists of the pairs x^μ, p_μ and ω^μ, π_μ. They fulfill the fundamental Poisson brackets $\{x^\mu, p_\nu\} = \delta^\mu_\nu$ and $\{\omega^\mu, \pi_\nu\} = \delta^\mu_\nu$, and then $\{P_\mu, P_\nu\} = R^\sigma{}_{\lambda\mu\nu}\pi_\sigma\omega^\lambda$, $\{P_\mu, \omega^\nu\} = \Gamma^\nu_{\mu\alpha}\omega^\alpha$, $\{P_\mu, \pi_\nu\} = -\Gamma^\alpha_{\mu\nu}\pi_\alpha$. For the quantities x^μ, P^μ, and $S^{\mu\nu}$ these brackets imply the typical relations used by people for spinning particles in the Hamiltonian formalism

$$\{x^\mu, P_\nu\} = \delta^\mu_\nu, \qquad \{P_\mu, P_\nu\} = -\frac{1}{4}R_{\mu\nu\alpha\beta}S^{\alpha\beta},$$
$$\{P_\mu, S^{\alpha\beta}\} = \Gamma^\alpha_{\mu\sigma}S^{\sigma\beta} - \Gamma^\beta_{\mu\sigma}S^{\sigma\alpha},$$
$$\{S^{\mu\nu}, S^{\alpha\beta}\} = 2(g^{\mu\alpha}S^{\nu\beta} - g^{\mu\beta}S^{\nu\alpha} - (\alpha\leftrightarrow\beta)). \tag{15}$$

To reveal the higher-stage constraints and the Lagrangian multipliers, we study the equations $\dot{T}_i = \{T_i, H\} = 0$. T_5 implies the secondary constraint

$$\dot{T}_5 = 0 \Rightarrow T_3 \equiv \pi^2 - \frac{\alpha}{\omega^2} \approx 0, \tag{16}$$

and then T_1 can be replaced on $P^2 + m^2c^2 \approx 0$. The preservation in time of T_7 and T_6 gives the Lagrangian multipliers λ_6 and λ_7

$$\lambda_6 = \frac{\lambda_1 R_{(\pi)}}{2M^2c^2}, \qquad \lambda_7 = -\frac{\lambda_1 R_{(\omega)}}{2M^2c^2}, \tag{17}$$

where we have denoted

$$R_{(\pi)} = 2R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\pi^\mu P^\nu,$$
$$R_{(\omega)} = 2R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\omega^\mu P^\nu; \tag{18}$$

$$M^2 = m^2 + \frac{1}{c^2}R_{\alpha\mu\beta\nu}\omega^\alpha\pi^\mu\omega^\beta\pi^\nu \equiv m^2 + \frac{1}{16c^2}\theta S, \tag{19}$$

$$\theta_{\mu\nu} \equiv R_{\alpha\beta\mu\nu}S^{\alpha\beta}. \tag{20}$$

The preservation in time of T_1 gives the equation $\lambda_6 R_{(\omega)} + \lambda_7 R_{(\pi)} = 0$, which is identically satisfied by virtue of (17). No more constraints are generated after this step. We summarize the algebra of Poisson between the constraints in Table I. T_6 and T_7 represent a pair of second-class constraints, while T_3, T_5, and the combination

$$T_0 = T_1 + \frac{R_{(\pi)}}{M^2c^2}T_6 - \frac{R_{(\omega)}}{M^2c^2}T_7 \tag{21}$$

are the first-class constraints. The presence of two primary first-class constraints T_5 and T_0 is in correspondence with the fact that two Lagrangian multipliers remain undetermined. This also is in agreement with the invariance of our

TABLE I. Algebra of constraints.

	T_1	T_3	T_5	T_6	T_7
$T_1 = P^2 + m^2c^2$	0	0	0	$R_{(\omega)}$	$R_{(\pi)}$
$T_3 = \pi^2 - \frac{\alpha}{\omega^2}$	0	0	$-2T_3$	$-2T_7$	$-2T_6/\omega^2$
$T_5 = \omega\pi$	0	0	0	$-T_6$	T_7
$T_6 = P\omega$	$-R_{(\omega)}$	$2T_7$	T_6	0	$-M^2c^2$
$T_7 = P\pi$	$-R_{(\pi)}$	$2T_6/\omega^2$	$-T_7$	M^2c^2	0

action with respect to two local symmetries mentioned above. Taking into account that each second-class constraint rules out one phase-space variable, whereas each first-class constraint rules out two variables, we have the right number of spin degrees of freedom, $8-(2+4)=2$.

We point out that the first-class constraint $T_3=\pi^2-\frac{\alpha}{\omega^2}\approx 0$ can be replaced on the pair

$$\pi^2=\mathrm{const},\qquad \omega^2=\mathrm{const}, \tag{22}$$

and this gives an equivalent formulation of the model. The Lagrangian that implies the constraints (12) and (22) has been studied in [32–34,39]. Hamiltonian and Lagrangian equations for physical variables of the two formulations coincide [35], which proves their equivalence.

Using (17), we can present the Hamiltonian (14) in the form

$$H=\frac{\lambda_1}{2}\left(P^2+m^2c^2+\frac{R_{(\pi)}(P\omega)-R_{(\omega)}(P\pi)}{M^2c^2}\right)+\frac{\lambda_1}{2}\left(\pi^2-\frac{\alpha}{\omega^2}\right)+\lambda_5(\omega\pi). \tag{23}$$

III. EQUATIONS OF MOTION

The dynamics of basic variables is governed by Hamiltonian equations $\dot{z}=\{z,H\}$, where $z=(x,p,\omega,\pi)$, and the Hamiltonian is given in (23). The equations can be written in a manifestly covariant form as follows:

$$\dot{x}^\mu=\lambda_1[P^\mu+(2M^2c^2)^{-1}(R_{(\pi)}\omega^\mu-R_{(\omega)}\pi^\mu)], \tag{24}$$

$$\nabla P_\mu=R^\alpha{}_{\beta\mu\nu}\pi_\alpha\omega^\beta\dot{x}^\nu, \tag{25}$$

$$\nabla\omega^\mu=-\lambda_1\frac{R_{(\omega)}}{2M^2c^2}P^\mu+\lambda_5\omega^\mu+\lambda_1\pi^\mu, \tag{26}$$

$$\nabla\pi_\mu=-\lambda_1\frac{R_{(\pi)}}{2M^2c^2}P_\mu-\lambda_5\pi_\mu-\lambda_1\frac{\omega_\mu}{\omega^2}. \tag{27}$$

Neither constraints nor equations of motion determine the functions λ_1 and λ_5. Their presence in the equations of motion implies that evolution of our basic variables is ambiguous. This is in correspondence with two local symmetries presented in the model. According to general theory [40–42], variables with ambiguous dynamics do not represent observable quantities, so we need to search for variables that can be candidates for observables. Consider the antisymmetric tensor (5). As a consequence of $T_6=0$ and $T_7=0$, this obeys the Tulczyjew supplementary condition

$$S^{\mu\nu}P_\nu=0. \tag{28}$$

Besides, the constraints T_3 and T_5 fix the value of square

$$S^{\mu\nu}S_{\mu\nu}=8\alpha, \tag{29}$$

so we identify $S^{\mu\nu}$ with the Frenkel spin tensor [36]. Equations (28) and (29) imply that only two components of the spin tensor are independent, as it should be for a spin one-half particle. Equations of motion for $S^{\mu\nu}$ follow from (26) and (27). Besides, using (18) we express Eqs. (24) and (25) in terms of the spin tensor. This gives the system

$$\dot{x}^\mu=\lambda_1\left[P^\mu+\frac{1}{8M^2c^2}S^{\mu\beta}\theta_{\beta\alpha}P^\alpha\right], \tag{30}$$

$$\nabla P_\mu=-\frac{1}{4}R_{\mu\nu\alpha\beta}S^{\alpha\beta}\dot{x}^\nu\equiv-\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu, \tag{31}$$

$$\nabla S^{\mu\nu}=2(P^\mu\dot{x}^\nu-P^\nu\dot{x}^\mu), \tag{32}$$

where θ has been defined in (20). Equation (32), contrary to Eqs. (26) and (27) for ω and π, does not depend on λ_5. This proves that the spin tensor is invariant under local spin-plane symmetry. The remaining ambiguity due to λ_1 is related with reparametrization invariance and disappears when we work with physical dynamical variables $x^i(t)$. Equations (30)–(32) together with (28) and (29) form a closed system that determines evolution of a spinning particle.

To obtain the Hamiltonian equations we can equally use the Dirac bracket constructed with the help of second-class constraints

$$\{A,B\}_D=\{A,B\}-\frac{1}{M^2c^2}[\{A,T_6\}\{T_7,B\}-\{A,T_7\}\{T_6,B\}]. \tag{33}$$

Since the Dirac bracket of a second-class constraint with any quantity vanishes, we can now omit T_6 and T_7 from (23); this yields the Hamiltonian

$$H_1=\frac{\lambda_1}{2}(P^2+m^2c^2)+\frac{\lambda_1}{2}\left(\pi^2-\frac{\alpha}{\omega^2}\right)+\lambda_5(\omega\pi). \tag{34}$$

Then Eqs. (24)–(27) can be obtained according to the rule $\dot{z}=\{z,H_1\}_D$. The quantities x^μ, P^μ, and $S^{\mu\nu}$, being invariant under spin-plane symmetry, have vanishing brackets with the corresponding first-class constraints T_3 and T_5. So, obtaining equations for these quantities, we can omit the last two terms in H_1, arriving at the familiar relativistic Hamiltonian

$$H_2=\frac{\lambda_1}{2}(P^2+m^2c^2). \tag{35}$$

Equations (30)–(32) can be obtained according to the rule $\dot{z}=\{z,H_2\}_D$. From (35) we conclude that our model describes the spinning particle without a gravimagnetic moment. In the Hamiltonian formulation, equations of motion with a gravimagnetic moment κ have been proposed by Khriplovich [8,20] adding nonminimal interaction $\frac{\lambda_1}{2}\frac{\kappa}{16}R_{\mu\nu\alpha\beta}S^{\mu\nu}S^{\alpha\beta}$ to the expression for H_2. It would be interesting to find the corresponding Lagrangian formulation of the model.

Similar to the spinless particle, we can exclude momenta P^μ from the Hamiltonian equations by using the mass-shell condition. This yields a second-order equation for the particle's position $x^\mu(\tau)$ (so we refer to the resulting equations as the Lagrangian form of MPTD equations). To achieve this, we observe that Eq. (30) is linear on P,

$$\dot{x}^\mu = \lambda_1 T^\mu{}_\nu P^\nu, \quad \text{with} \quad T^\mu{}_\nu = \delta^\mu_\nu + \frac{1}{8M^2c^2}S^{\mu\alpha}\theta_{\alpha\nu}. \tag{36}$$

Using the identity

$$(S\theta S)^{\mu\nu} = -\frac{1}{2}(S\theta)S^{\mu\nu}, \quad \text{where } S\theta = S^{\alpha\beta}\theta_{\alpha\beta}, \tag{37}$$

we find the inverse of the matrix T

$$\tilde{T}^\mu{}_\nu = \delta^\mu{}_\nu - \frac{1}{8m^2c^2}S^{\mu\sigma}\theta_{\sigma\nu}, \qquad T\tilde{T} = 1, \tag{38}$$

so (36) can be solved with respect to P^μ, $P^\mu = \frac{1}{\lambda_1}\tilde{T}^\mu{}_\nu\dot{x}^\nu$. We substitute P^μ into the constraint $P^2 + m^2c^2 = 0$, and this gives the expression for λ_1,

$$\lambda_1 = \frac{\sqrt{-G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}{mc} \equiv \frac{\sqrt{-\dot{x}G\dot{x}}}{mc}. \tag{39}$$

We have introduced the effective metric [43]

$$G_{\mu\nu} \equiv \tilde{T}^\alpha{}_\mu g_{\alpha\beta}\tilde{T}^\beta{}_\nu. \tag{40}$$

From (36) and (39) we obtain the expression for P_μ,

$$P^\mu = \frac{mc}{\sqrt{-\dot{x}G\dot{x}}}\left[\dot{x}^\mu - \frac{1}{8m^2c^2}S^{\mu\nu}\theta_{\nu\sigma}\dot{x}^\sigma\right], \tag{41}$$

and the Lagrangian form of the Tulczyjew condition

$$S^{\mu\nu}P_\nu = S^{\mu\nu}\tilde{T}_{\nu\sigma}\dot{x}^\sigma = 0. \tag{42}$$

Using Eqs. (41) and (42) in (31) and (32) we finally obtain

$$\nabla\left[\frac{\tilde{T}^\mu{}_\nu\dot{x}^\nu}{\sqrt{-\dot{x}G\dot{x}}}\right] = -\frac{1}{4mc}R^\mu{}_{\nu\alpha\beta}S^{\alpha\beta}\dot{x}^\nu, \tag{43}$$

$$\nabla S^{\mu\nu} = \frac{1}{4mc\sqrt{-\dot{x}G\dot{x}}}\dot{x}^{[\mu}S^{\nu]\sigma}\theta_{\sigma\alpha}\dot{x}^\alpha. \tag{44}$$

These equations, together with the conditions (42) and (29), form a closed system for the set $(x^\mu, S^{\mu\nu})$. The consistency of the constraints (42) and (29) with the dynamical equations is guaranteed by the Dirac procedure for singular systems.

The Lagrangian considered above yields the fixed value of spin; that is, this corresponds to an elementary particle. Let us present the modification that leads to the theory with an unfixed spin, and, similar to the Hanson-Regge approach [26], with a mass-spin trajectory constraint. Consider the following Lagrangian:

$$L = -\frac{mc}{\sqrt{2}}\sqrt{-\dot{x}N\dot{x} - l^2\frac{\nabla\omega N\nabla\omega}{\omega^2} + T^{\frac{1}{2}}},$$
$$T \equiv \left[\dot{x}N\dot{x} + l^2\frac{\nabla\omega N\nabla\omega}{\omega^2}\right]^2 - 4l^2\frac{(\dot{x}N\nabla\omega)^2}{\omega^2}, \tag{45}$$

where l is a parameter with the dimension of length. Applying the Dirac procedure as in Sec. II, we obtain the Hamiltonian

$$H = \frac{\lambda_1}{2}\left(P^2 + m^2c^2 + \frac{\pi^2\omega^2}{l^2}\right) + \lambda_5(\omega\pi)$$
$$+ \lambda_6(P\omega) + \lambda_7(P\pi), \tag{46}$$

which turns out to be a combination of the first-class constraints $P^2 + m^2c^2 + \frac{\pi^2\omega^2}{l^2} = 0$, $\omega\pi = 0$ and the second-class constraints $P\omega = 0$, $P\pi = 0$. The Dirac procedure stops on the first stage; that is, there are no secondary constraints. As compared with (6), the first-class constraint $\pi^2 - \frac{\alpha}{\omega^2} = 0$ does not appear in the present model. Because of this, the square of the spin is not fixed, $S^2 = 8(\omega^2\pi^2 - \omega\pi) \approx 8\omega^2\pi^2$. Using this equality, the mass-shell constraint acquires the stringlike form

$$P^2 + m^2c^2 + \frac{1}{8l^2}S^2 = 0. \tag{47}$$

The model has four physical degrees of freedom in the spin sector. As the independent gauge-invariant degrees of freedom, we can take three components S^{ij} of the spin tensor together with any one product of conjugate coordinates, for instance, $\omega^0\pi^0$.

IV. MPTD EQUATIONS AND DYNAMICS OF REPRESENTATIVE POINT OF A ROTATING BODY

In this section we discuss the MPTD equations of a rotating body in the form studied by Dixon (for the relation of the Dixon equations with those of Papapetrou and Tulczyjew see p. 335 in [4]),

$$\nabla P^\mu = -\frac{1}{4}R^\mu{}_{\nu\alpha\beta}S^{\alpha\beta}\dot{x}^\nu \equiv -\frac{1}{4}(\theta\dot{x})^\mu, \tag{48}$$

$$\nabla S^{\mu\nu} = 2(P^\mu \dot{x}^\nu - P^\nu \dot{x}^\mu), \tag{49}$$

$$S^{\mu\nu} P_\nu = 0, \tag{50}$$

and compare them with equations of motion of our spinning particle. In particular, we show that the effective metric $G_{\mu\nu}$ also emerges in this formalism. MPTD equations appeared in the multipole approach to the description of a body [1–7,44], where the energy momentum of the body is modeled by a set of multipoles. In this approach $x^\mu(\tau)$ is called the representative point of the body, and we take it in arbitrary parametrization τ (contrary to Dixon, we do not assume the proper-time parametrization; that is, we do not add the equation $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2$ to the system above). $S^{\mu\nu}(\tau)$ is associated with the inner angular momentum, and $P^\mu(\tau)$ is called momentum. The first-order equations (48) and (49) appear in the pole-dipole approximation, while the algebraic equation (50) has been added by hand.[2] After that, the number of equations coincides with the number of variables.

To compare MPTD equations with those of the previous section, we first observe some useful consequences of the system (48)–(50).

Take the derivative of the constraint, $\nabla(S^{\mu\nu}P_\nu) = 0$, and use (48) and (49); this gives the expression

$$(P\dot{x})P^\mu = P^2 \dot{x}^\mu + \frac{1}{8}(S\theta\dot{x})^\mu, \tag{51}$$

which can be written in the form

$$P^\mu = \frac{P^2}{(P\dot{x})}\left(\delta^\mu{}_\nu + \frac{1}{8P^2}(S\theta)^\mu{}_\nu\right)\dot{x}^\nu \equiv \frac{P^2}{(P\dot{x})}\tilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu. \tag{52}$$

Contract (51) with $\dot{x}_\mu$. Taking into account that $(P\dot{x}) < 0$, this gives $(P\dot{x}) = -\sqrt{-P^2}\sqrt{-\dot{x}\tilde{\mathcal{T}}\dot{x}}$. Using this in Eq. (52) we obtain

$$P^\mu = \frac{\sqrt{-P^2}}{\sqrt{-\dot{x}\tilde{\mathcal{T}}\dot{x}}}(\tilde{\mathcal{T}}\dot{x})^\mu, \quad \tilde{\mathcal{T}}^\mu{}_\nu = \delta^\mu{}_\nu + \frac{1}{8P^2}(S\theta)^\mu{}_\nu. \tag{53}$$

For the latter use we observe that in our model with composite $S^{\mu\nu}$ we used the identity (37) to invert T, then the Hamiltonian equation (30) has been written in the form (41), and the latter can be compared with (53).

Contracting (49) with $S_{\mu\nu}$ and using (50) we obtain $\frac{d}{d\tau}(S^{\mu\nu}S_{\mu\nu}) = 0$; that is, the square of the spin is a constant of motion. The contraction of (51) with P_μ gives $(PS\theta\dot{x}) = 0$. The contraction of (51) with $(\dot{x}\theta)_\mu$ gives $(P\theta\dot{x}) = 0$. The contraction of (48) with P_μ gives $\frac{d}{d\tau}(P^2) = -\frac{1}{2}(P\theta\dot{x}) = 0$; that is, P^2 is one more constant of motion, say k, $\sqrt{-P^2} = k = \text{const}$ (in our model this is fixed as $k = mc$). Substituting (53) into Eqs. (48)–(50) we now can exclude P^μ from these equations, modulo to the constant of motion $k = \sqrt{-P^2}$.

Thus, the square of momentum cannot be excluded from the system (48)–(51); that is, MPTD equations in this form do not represent a Hamiltonian system for the pair x^μ, P^μ. To improve this point, we note that Eq. (53) acquires a conventional form (as the expression for conjugate momenta of x^μ in the Hamiltonian formalism), if we add to the system (48)–(50) one more equation, which fixes the remaining quantity P^2 (Dixon noticed this for the body in the electromagnetic field; see his Eq. (4.5) in [44]). To see how the equation could look, we note that for the nonrotating body (pole approximation) we expect equations of motion of the spinless particle, $\nabla p^\mu = 0$, $p^\mu = \frac{mc}{\sqrt{-\dot{x}g\dot{x}}}\dot{x}^\mu$, $p^2 + (mc)^2 = 0$. Independent equations of the system (48)–(51) in this limit read $\nabla P^\mu = 0$, $P^\mu = \frac{\sqrt{-P^2}}{\sqrt{-\dot{x}g\dot{x}}}\dot{x}^\mu$. Comparing the two systems, we see that the missing equation is the mass-shell condition $P^2 + (mc)^2 = 0$. Returning to the pole-dipole approximation, an admissible equation should be $P^2 + (mc)^2 + f(S, \ldots) = 0$, where f must be a constant of motion. Since the only constant of motion in the arbitrary background is S^2, we have finally

$$P^2 = -(mc)^2 - f(S^2). \tag{54}$$

With this value of P^2, we can exclude P^μ from MPTD equations, obtaining a closed system with the second-order equation for x^μ. We substitute (53) into (48)–(50), and this gives

$$\nabla\frac{(\tilde{\mathcal{T}}\dot{x})^\mu}{\sqrt{-\dot{x}\tilde{\mathcal{T}}\dot{x}}} = -\frac{1}{4\sqrt{-P^2}}(\theta\dot{x})^\mu, \tag{55}$$

$$\nabla S^{\mu\nu} = -\frac{1}{4\sqrt{-P^2}\sqrt{-\dot{x}\tilde{\mathcal{T}}\dot{x}}}\dot{x}^{[\mu}(S\theta\dot{x})^{\nu]}, \tag{56}$$

$$(SS\theta\dot{x})^\mu = -8P^2(S\dot{x})^\mu, \tag{57}$$

where (54) is implied. They determine the evolution of x^μ and $S^{\mu\nu}$ for each given function $f(S^2)$.

It is convenient to introduce the effective metric $\mathcal{G}$ composed from the "tetrad field" $\tilde{\mathcal{T}}$,

$$\mathcal{G}_{\mu\nu} \equiv g_{\alpha\beta}\tilde{\mathcal{T}}^\alpha{}_\mu \tilde{\mathcal{T}}^\beta{}_\nu. \tag{58}$$

Equation (57) implies the identity

[2] For geometric interpretation of the spin supplementary condition in the multipole approach see [7].

$$\dot{x}\tilde{\mathcal{T}}\dot{x} = \dot{x}\mathcal{G}\dot{x}, \tag{59}$$

so we can replace $\sqrt{-\dot{x}\tilde{\mathcal{T}}\dot{x}}$ in (55)–(57) by $\sqrt{-\dot{x}\mathcal{G}\dot{x}}$.

In summary, we have presented MPTD equations in the form

$$P^\mu = \frac{\sqrt{-P^2}}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}(\tilde{\mathcal{T}}\dot{x})^\mu, \qquad \nabla P^\mu = -\frac{1}{4}(\theta\dot{x})^\mu,$$
$$\nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]}, \qquad S^{\mu\nu}P_\nu = 0, \tag{60}$$

$$P^2 + (mc)^2 + f(S^2) = 0, \tag{61}$$

$$S^2 \text{is a constant of motion}, \tag{62}$$

with $\tilde{\mathcal{T}}$ given in (53). Now we are ready to compare them with Hamiltonian equations of our spinning particle, which we write here in the form

$$P^\mu = \frac{mc}{\sqrt{-\dot{x}G\dot{x}}}(\tilde{T}\dot{x})^\mu, \qquad \nabla P^\mu = -\frac{1}{4}(\theta\dot{x})^\mu,$$
$$\nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]}, \qquad S^{\mu\nu}P_\nu = 0, \tag{63}$$

$$P^2 + (mc)^2 = 0, \tag{64}$$

$$S^2 = 8\alpha, \tag{65}$$

with $\tilde{T}$ given in (38). Comparing the systems, we see that our spinning particle has fixed values of spin and canonical momentum, while for the MPTD particle the spin is a constant of motion and the momentum is a function of spin. We conclude that all the trajectories of a body with given m and $S^2 = \beta$ are described by our spinning particle with spin $\alpha = \frac{\beta}{8}$ and with the mass equal to $\sqrt{m^2 - \frac{f^2(\beta)}{c^2}}$. In this sense our spinning particle is equivalent to the MPTD particle.[3]

MPTD equations in the Lagrangian form (55)–(57) can be compared with (42)–(44).

V. LAGRANGIAN FORM OF MPTD EQUATIONS

Here we briefly discuss some immediate consequences that can be obtained from the Lagrangian form (42)–(44), (29) of MPTD equations.

In the spinless limit Eq. (43) turns into the geodesic equation. Spin causes deviations from the geodesic motion due to the right-hand side of this equation, as well as due to the presence of the tetrad field $\tilde{T}$ and the effective metric G in the left-hand side. In the Newtonian limit the original metric $g_{\mu\nu}(x)$ can be presented through the Newton potential in which a test body is immersed. The presence of $G_{\mu\nu}$ could be thought of as a contribution to this potential when the spin of the body is taken into account. Let us compute the manifest form of G in the field with nearly flat metric

$$g_{\mu\nu} = \eta_{\nu\mu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1. \tag{66}$$

To linear order in $h_{\mu\nu}$ the curvature tensor is $R^{(1)}_{\mu\nu\alpha\beta} = \frac{1}{2}(h_{\mu\beta,\nu\alpha} + h_{\nu\alpha,\mu\beta} - h_{\nu\beta,\mu\alpha} - h_{\mu\alpha,\nu\beta})$; hence, $\theta^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu\alpha\beta}S^{\alpha\beta} = (h_{\mu\alpha,\beta\nu} - h_{\nu\alpha,\beta\mu})S^{\beta\alpha}$, where the comma denotes the partial derivative. The effective metric in the weak field approximation reads

$$G^{(1)}_{\mu\nu} = g_{\mu\nu} - \frac{1}{8m^2c^2}(\eta_{\mu\alpha}S^{\alpha\beta}\theta^{(1)}_{\beta\nu} + \eta_{\nu\alpha}S^{\alpha\beta}\theta^{(1)}_{\beta\mu}). \tag{67}$$

Let us consider the Newtonian solution to the linearized Einstein equations

$$h_{00} = -2\phi, \qquad h_{ij} = -2\delta_{ij}\phi, \qquad h_{\mu 0} = 0, \tag{68}$$

with $\phi = -\frac{k}{r}$. Using the three-dimensional spin vector and the dipole electric moment (5), the time-time component of the effective metric is

$$G_{00} = -1 + \frac{2k}{r} + \frac{k}{2m^2c^2r^3}[3(\mathbf{D}\cdot\mathbf{n})^2 - \mathbf{D}^2], \tag{69}$$

where $\mathbf{n} = \mathbf{r}/r$. Contrary to the Newtonian solution (68), the space-time components of $G_{\mu\nu}$ are different from zero,

$$G_{i0} = \frac{3k}{4m^2c^2r^3}[(\mathbf{D}\times\mathbf{s})_i - 2(\mathbf{D}\cdot\mathbf{n})(\mathbf{n}\times\mathbf{s})_i - n_i(\mathbf{D}\times\mathbf{s})\cdot\mathbf{n}]. \tag{70}$$

For the space-space components we found

$$G_{ij} = \delta_{ij} + \frac{2k}{r}\delta_{ij} + \frac{k}{2m^2c^2r^3}\Big\{[3\hat{n}_i\hat{n}_j - 5\delta_{ij}]\mathbf{s}^2 - 5s_is_j + D_iD_j - \frac{3}{2}[(\mathbf{s}\cdot\mathbf{n})s_{(i}n_{j)} + (\mathbf{D}\cdot\mathbf{n})D_{(i}n_{j)}] - 12(\mathbf{n}\times\mathbf{s})_i(\mathbf{n}\times\mathbf{s})_j\Big\}. \tag{71}$$

We point out that the expressions (67)–(71) are written without any approximation with respect to the spin. The contributions due to spin over long distances will be very small, and then in the Newtonian limit a spinning particle behaves almost as a spinless one. Probably at short distances the contributions may be important; to verify this, other geometries should be considered.

[3] We point out that our final conclusion remains true even when we do not add (54) to MPTD equations: to study the class of trajectories of a body with $\sqrt{-P^2} = k$ and $S^2 = \beta$ we take our spinning particle with $m = \frac{k}{c}$ and $\alpha = \frac{\beta}{8}$.

Our formulation reveals one more novel property of MPTD equations: the mean position of a rotating body will be represented by noncommutative operators in quantum theory. Indeed, to construct the quantum theory of a system with second-class constraints, one should pass from the Poisson to the Dirac bracket [40–42]. Then one looks for operators of basic variables with commutators resembling the Dirac bracket. For our case the Dirac bracket is given by (33). This yields highly noncommutative algebra for the position variables

$$\{x^\mu, x^\nu\}_D = \frac{2\omega^{[\mu}\pi^{\nu]}}{M^2c^2} \equiv \frac{S^{\mu\nu}}{M^2c^2}. \qquad (72)$$

In the result, the position space is endowed with a noncommutative structure that originates from the accounting of the spin degrees of freedom. We point out that a nonrelativistic spinning particle implies canonical algebra of position operators; see [38,45]. So the deformation (72) arises as a relativistic correction induced by spin. It is known that formalism of dynamical systems with second-class constraints implies a natural possibility to incorporate noncommutative geometry into the framework of classical and quantum theory [26,46–49]. Our model represents an example where a physically interesting noncommutative particle (72) emerges in this way. For the case, the "parameter of noncommutativity" is proportional to the spin tensor. This allowed us [33] to explain contradictory results concerning the first relativistic corrections due to the spin obtained by different authors.

Consider the background metric that admits the Killing vector ξ_μ, $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$ (the semicolon means the covariant derivative). Then the infinitesimal transformation

$$x'^\mu = x^\mu + \varepsilon\xi^\mu(x), \qquad \varepsilon \ll 1, \qquad (73)$$

generates the isometry of the metric, that is, leaves it form invariant, $g'_{\mu\nu}(y) = g_{\mu\nu}(y)$. For the spinless particle the isometry generates the conserved quantity $\frac{\partial L}{\partial \dot{x}^\mu}\xi^\mu$. A natural question is, does this remain true for a vector model of spin, where the particle does not follow a geodesic trajectory? From the transformation law of ω^μ,

$$\omega'^\mu(\tau) = \frac{\partial x'^\mu}{\partial x^\alpha}\omega^\alpha(\tau) = (\delta^\mu_\alpha + \varepsilon\xi^\mu_{,\alpha})\omega^\alpha(\tau), \qquad (74)$$

we deduce that $\delta\omega^\mu = \omega^\mu(\tau) - \omega^\mu(\tau) = \varepsilon\omega^\nu\xi^\mu{}_{,\nu}$, which corresponds to the transformation law of a form-invariant vector field. By Noether's theorem the quantity

$$J^{(\xi)} = \frac{\partial L}{\partial \dot{x}^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{\omega}^\mu}\delta\omega^\mu = p_\mu\xi^\mu + \xi^\mu{}_{,\nu}\pi_\mu\omega^\nu \qquad (75)$$

is conserved. In terms of $S^{\mu\nu}$ and P_μ this coincides with that of [11], $J^{(\xi)} = P^\mu\xi_\mu - \frac{1}{4}S^{\mu\nu}\xi_{\mu;\nu}$. Using Eqs. (31) and (32), it is easy to confirm that $J^{(\xi)}$ is conserved. We conclude that an isometry of the spinless particle remains the isometry for the vector models of spin. However, the conserved quantity acquires the spin-dependent term $-\frac{1}{4}S^{\mu\nu}\xi_{\mu;\nu}$.

VI. CONCLUSIONS

In this work we have presented the Lagrangian action without auxiliary variables (6) for a description of the spinning particle in an arbitrary curved background. The supplementary spin conditions (28) and (29) are guaranteed by the set of constraints (12) and (16) arising from our singular Lagrangian in the Hamiltonian formalism. Because of this, the spin has two physical degrees of freedom, as it should for a spin one-half particle. Besides, the reparametrization invariance of the action generates the mass-shell constraint $P^2 + (mc)^2 = 0$. The description of the spin on the base of a vectorlike variable allows us to construct also the Lagrangian (45) with an unfixed value of spin and stringlike mass-shell constraint (47), as in the Hanson-Regge model of a relativistic top. In the model (45) appeared the fundamental length scale and the spin has four physical degrees of freedom.

We showed that our spinning particle can be used to study dynamics of a rotating body in curved background: all the trajectories of MPTD equations with given values of integration constants, $\sqrt{-P^2} = k$ and $S^2 = \beta$, are described by our spinning particle with $m = \frac{k}{c}$ and $\alpha = \frac{\beta}{8}$. In this sense the expression (6) yields the Lagrangian formulation of MPTD equations, and the latter corresponds to minimal interaction of the particle with gravity. This demonstrates the effectiveness of the classical description of spin on the base of a vectorlike non-Grassmann variable. We have explored our formulation to obtain, in an unambiguous way, the closed system of Eqs. (42)–(44), (29) for the set x^μ, $S^{\mu\nu}$. Some immediate consequences of this form of MPTD equations have been discussed in Sec. V. In particular, in the Lagrangian form of MPTD equations, instead of the original metric $g_{\mu\nu}$ emerges the effective metric $G_{\mu\nu} = g_{\mu\nu} + H_{\mu\nu}$ with spin and field-dependent contribution $H_{\mu\nu}$. According to (40), the matrix (38), which links canonical momentum and velocity, plays the role of a tetrad field to compose the effective metric.

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