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Black hole entropy and Lorentz-diffeomorphism Noether charge

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We show that, in the first or second order orthonormal frame formalism, black hole entropy is the horizon Noether charge for a combination of diffeomorphism and local Lorentz symmetry involving the Lie derivative of the frame. The Noether charge for diffeomorphisms alone is unsuitable, since a regular frame cannot be invariant under the flow of the Killing field at the bifurcation surface. We apply this formalism to Lagrangians polynomial in wedge products of the frame field 1-form and curvature 2-form, including general relativity, Lovelock gravity, and "topological" terms in four dimensions.

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I. INTRODUCTION

The entropy of black holes in any diffeomorphism invariant gravity theory can be identified via a variational identity known as the first law of black hole mechanics. In the approach of Wald [1], this identity arises from considering the Hamiltonian H_ξ that generates evolution with respect to the flow of the horizon-generating Killing vector ξ of a black hole solution. The variation δH_ξ at a solution is equal to a variation of boundary terms, and vanishes because ξ generates a symmetry of the dynamical fields. When the boundaries lie at the horizon bifurcation surface and at spatial infinity, the implied relation between the boundary term variations is the first law, from which Wald's formula for the black hole entropy as Noether charge can be inferred.

This method is usually applied in a context where the spacetime geometry is characterized by the metric tensor alone, however in some settings it is necessary or desirable to use instead a formalism with geometry determined by an orthonormal frame and either the associated spin connection (second order formalism) or an independent spin connection (first order formalism). Application of Wald's method in this setting appears at first to yield a vanishing Noether charge at the bifurcation surface where ξ vanishes—and therefore vanishing black hole entropy—because the Noether charge form involves ξ without derivatives. The puzzle this raises has not to our knowledge been discussed explicitly in the literature.

We trace the trouble to the requirement that the frame (hereafter the "orthonormal" qualifier is implicit) has vanishing Lie derivative with respect to ξ. This requirement cannot be met at the bifurcation surface, and implies that the derivative of the frame diverges at the bifurcation surface, so that the spin connection diverges. On the other hand, the diffeomorphism Noether charge form involves the contraction of the vanishing Killing vector with the diverging spin connection. We first show how one can evaluate a finite, nonzero entropy by taking the limit as the bifurcation surface is approached.

Next, in a second approach, we modify the derivation so that the singular behavior does not arise in the first place. In a frame formalism the theory is symmetric under both diffeomorphisms and local Lorentz transformations of the frame. We show in this paper how the black hole entropy can be derived as the Noether charge for a particular combination of these symmetries. The frame can be invariant under the combined symmetry associated with ξ, without having singular derivative at the horizon, so that the extraction of the black hole entropy requires no limit. The variation corresponding to this symmetry is defined by a "Lorentz-Lie" derivative which is covariant under local Lorentz transformations of the frame field. It is defined by adding to the ordinary Lie derivative a connection term built from the frame field and its partial derivatives. Besides allowing for nonsingular invariant frames at the bifurcation surface, this notion of combined Lorentz-diffeomorphism symmetry should allow the symmetry to be implemented on nonparallelizable manifolds, where no global frame field exists. More generally, for theories containing fields charged under a gauge group G, the Noether charge formalism for symmetry under combined diffeomorphisms and local gauge transformations has been formulated recently in terms of fields living on a principal G-bundle over spacetime [2].

This paper is organized as follows. Section II reviews the derivation showing that black hole entropy is the horizon Noether charge associated with the diffeomorphism generated by the horizon-generating Killing vector field. In Sec. III we examine this Noether charge for general relativity in the frame formalism, diagnose the pathology,

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and treat it with a limit. In Sec. IV we introduce the Lorentz-covariant Lie derivative, and in Sec. V we show how black hole entropy is the horizon Noether charge associated with the combined Lorentz and diffeomorphism symmetry it generates. In Sec. V A we use this formalism to evaluate the black hole entropy in Lovelock theory in arbitrary dimensions, and in Sec. V B we apply it in four dimensions to evaluate the contributions of the Holst [3], Euler and Pontryagin terms. We conclude in Sec. VI with a brief discussion.

We work in the units such that $16\pi G = c = 1$. Lower case Greek letters are used for the spacetime indices, and internal Lorentz indices are denoted by lower case Latin letters. The metric signature is $(-+++)$.

II. BLACK HOLE ENTROPY AS DIFFEOMORPHISM NOETHER CHARGE

In this section we sketch Wald's derivation [1] establishing that black hole entropy is the diffeomorphism Noether charge for the horizon-generating Killing field, evaluated at the bifurcation surface. This will set the stage for the application to the frame formalism, and our modified derivation using a Lorentz-diffeomorphism Noether charge.

Wald's derivation applies to any diffeomorphism invariant theory defined by a Lagrangian n-form L, where n is the spacetime dimension. Denoting the dynamical fields collectively by ϕ, the variation δL induced by a field variation $\delta\phi$ can be written as

$$\delta L = E\delta\phi + \mathrm{d}\theta(\phi, \delta\phi). \tag{1}$$

The quantity E defines the field equations, $E = 0$. The $(n-1)$-form θ is constructed locally out of the dynamical fields and their first variation, and is called the "symplectic potential." The antisymmetrized field variation of θ defines an $(n-1)$-form, called the "symplectic current," via

$$\Omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\theta(\phi, \delta_2\phi) - \delta_2\theta(\phi, \delta_1\phi). \tag{2}$$

When integrated over a spatial initial value surface, Ω defines the symplectic form on the phase space of solutions.

Now consider the variation induced by a diffeomorphism generated by a vector field ξ,

$$\delta_\xi\phi = \mathcal{L}_\xi\phi. \tag{3}$$

Diffeomorphism invariance of the theory means that the Lagrangian is constructed only from the dynamical fields, without any background structure. In this case, the variation of L induced by the field variation $\delta_\xi\phi$ is equal to the Lie derivative of the Lagrangian itself,

$$\delta_\xi L = \mathcal{L}_\xi L = \mathrm{d}\mathrm{i}_\xi L. \tag{4}$$

Since this is a total derivative we learn that the vector fields on the spacetime generate symmetries of the dynamics. With each ξ is associated an $(n-1)$-form called the Noether current form, defined as

$$j_\xi = \theta(\phi, \mathcal{L}_\xi\phi) - \mathrm{i}_\xi L, \tag{5}$$

whose exterior derivative is given [according to (1), (3), and (4)] by

$$\mathrm{d}j_\xi = -E\mathcal{L}_\xi\phi. \tag{6}$$

For all vector fields ξ, the current j_ξ is therefore closed "on shell," i.e. when $E = 0$. This implies [4] that, on shell, j_ξ is an exact form,

$$j_\xi = \mathrm{d}Q_\xi, \tag{7}$$

where Q_ξ is some $(n-2)$-form that is constructed locally from the fields and their derivatives. The integral of Q_ξ over a closed $(n-2)$-surface S is called the "Noether charge" of S relative to ξ.

In the covariant framework used by Wald, the space of solutions to the field equations is the phase space of the theory, and the on shell variation $\delta_\xi\phi$ is the phase space flow vector corresponding to the 1-parameter family of diffeomorphisms generated by ξ. The Hamiltonian H_ξ generating this flow is related to the symplectic form via Hamilton's equations, $\delta H_\xi = \int_\Sigma \Omega(\phi, \delta\phi, \mathcal{L}_\xi\phi)$, where Σ is a Cauchy surface. On shell this variation is a boundary term:

$$\delta H_\xi = \int_\Sigma \Omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) \tag{8}$$

$$= \int_\Sigma \delta\theta(\phi, \mathcal{L}_\xi\phi) - \mathcal{L}_\xi\theta(\phi, \delta\phi) \tag{9}$$

$$= \int_\Sigma \delta j_\xi + \delta(i_\xi L) - \mathrm{i}_\xi \mathrm{d}\theta - \mathrm{d}\mathrm{i}_\xi\theta \tag{10}$$

$$= \oint_{\partial\Sigma} \delta Q_\xi - \mathrm{i}_\xi\theta. \tag{11}$$

In the second line we used (2), in the third line (5), and in the fourth line (7) and (1). If ξ generates a symmetry of the fields in a solution ϕ, then $\mathcal{L}_\xi\phi = 0$, and thus (8) implies $\delta H_\xi = 0$, so that (11) yields an identity relating the surface term variations away from that solution, $\oint_{\partial\Sigma} \delta Q_\xi - \mathrm{i}_\xi\theta = 0$.

Now consider a stationary, axisymmetric black hole with a Killing field ξ that generates a Killing horizon with nonzero, constant surface gravity κ, and vanishes on a bifurcation surface $\mathcal{B}$. If we choose the hypersurface Σ to have its only boundaries at spatial infinity and at $\mathcal{B}$, then the variational identity takes the form

$$\oint_{\mathcal{B}} \delta Q_\xi = \oint_\infty \delta Q_\xi - \mathrm{i}_\xi\theta, \tag{12}$$

where the orientations of both surfaces are induced by a vector pointing toward infinity. The right-hand side can be shown to be equal to $\delta\mathcal{E} - \Omega_H \delta\mathcal{J}$ where $\mathcal{E}$ and $\mathcal{J}$ are the asymptotically defined total energy and angular momentum, respectively, and Ω_H is the angular velocity of the horizon. To evaluate the left-hand side, note that since ξ is a Killing vector, its second and higher derivatives can be written in terms of ξ and its first derivative, together with the Riemann tensor and its derivatives, so Q_ξ depends on ξ only algebraically via ξ and $\nabla\xi$. At $\mathcal{B}$ the vector ξ vanishes, and

$$\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu = \kappa n_\mu{}^\nu, \tag{13}$$

where $n_{\mu\nu}$ is the binormal to $\mathcal{B}$ (i.e. the normal 2-form, normalized to -2), oriented as determined by the derivative of the Killing vector in (13). Hence all the ξ dependence of Q for the background solution is contained in the specification of the bifurcation surface and the (constant) surface gravity κ. Moreover, the replacement $\nabla_\mu \xi^\nu \to \kappa n_\mu{}^\nu$ may be made before the variation is taken: the quantity $a^\mu b_\nu \delta n_\mu{}^\nu$ vanishes unless a^μ is normal and b^ν is tangent to $\mathcal{B}$, yet there are no normal-tangential components in the background tensor because they would not be invariant under the Killing flow of ξ at $\mathcal{B}$ (which acts as a boost normal to $\mathcal{B}$). The identity (12) therefore takes the form of the so-called first law of black hole thermodynamics,

$$T_H \delta S = \delta\mathcal{E} - \Omega_H \delta\mathcal{J}, \tag{14}$$

where $T_H = \hbar\kappa/2\pi$ is the Hawking temperature, and

$$S = \frac{2\pi}{\hbar} \oint_{\mathcal{B}} \hat{Q}_\xi, \tag{15}$$

where $\hat{Q}_\xi$ (for the background as well as for the varied solution) is obtained from Q_ξ by the replacement $\nabla_\mu \xi_\nu \to n_{\mu\nu}$. The black hole entropy S is thus proportional to the horizon Noether charge corresponding to the horizon-generating diffeomorphism. (For a more complete discussion see [5].)

In order for the entropy to be nonzero, it would seem that Q_ξ must depend on $\nabla\xi$, so j_ξ, and therefore $\theta(\phi, \mathcal{L}_\xi \phi)$, must depend on $\nabla\nabla\xi$. Since the Lie derivative of a tensor field depends on $\nabla\xi$, this requires that $\theta(\phi, \delta\phi)$ depends on at least one derivative of $\delta\phi$, and therefore that L involves at least second derivatives. Since the first order orthonormal frame formalism involves only one derivative, it thus appears that the black hole entropy would *vanish* in that formalism, but that conclusion is obviously erroneous. The right-hand side of the first law (14) is of course independent of which formalism is used. In the next section we compute the horizon Noether charge for general relativity using the frame formalism, diagnose the flaw in the above reasoning, and show how to evade the problem.

III. DIFFEOMORPHISM NOETHER CHARGE FOR GENERAL RELATIVITY WITH ORTHONORMAL FRAMES

In the first order orthonormal frame formalism, the Lagrangian for general relativity in n dimensions is written in terms of the frame field 1-form e^a, which is $SO(n-1,1)$ vector valued, and the $SO(n-1,1)$ connection 1-form $\omega^a{}_b$. These are the independent dynamical variables of the theory. The spacetime metric is given by $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$, where η_{ab} is the Minkowski metric, and the curvature 2-form is defined by $R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$. We raise and lower Lorentz indices with η_{ab} and its inverse, η^{ab}. We sometimes omit the Lorentz indices when that will not cause confusion.

The Lagrangian n-form for general relativity in n-dimensions is a function of the frame and the spin connection via the curvature 2-form,

$$L(e,\omega) = \epsilon_{a\cdots bcd} e^a \wedge \cdots \wedge e^b \wedge R^{cd}. \tag{16}$$

This is manifestly gauge invariant and diffeomorphism covariant. The variation is given by

$$\delta L = \delta e^a \wedge \frac{\partial L}{\partial e^a} + \mathrm{D}\delta\omega^{ab} \wedge \frac{\partial L}{\partial R^{ab}} \tag{17}$$

$$= \delta e^a \wedge \frac{\partial L}{\partial e^a} + \delta\omega^{ab} \wedge \mathrm{D}\frac{\partial L}{\partial R^{ab}} + \mathrm{d}\left(\delta\omega^{ab} \wedge \frac{\partial L}{\partial R^{ab}}\right), \tag{18}$$

where D is the Lorentz covariant exterior derivative [6], and we have used the identity $\delta R^{ab} = \mathrm{D}\delta\omega^{ab}$. (The variation forms are placed in the first position in order to avoid the need for a dimension-dependent minus sign that would arise when integrating by parts on the D.) The equations of motion are given by

$$\epsilon_{abc\cdots df} e^c \wedge \cdots \wedge e^d \wedge \mathrm{D}e^f = 0, \tag{19}$$

$$\epsilon_{ab\cdots cde} e^b \wedge \cdots \wedge e^c \wedge R^{de} = 0. \tag{20}$$

The first of these equations implies (assuming e^a is nondegenerate) the "torsion-free" condition $\mathrm{D}e^a = 0$, which can be solved for the connection $\omega = \omega^e$. When this is substituted in the second equation of motion, that becomes equivalent to the vanishing of the Ricci tensor of $g_{\mu\nu}$, so one recovers the (vacuum) Einstein equation. If one puts $\omega = \omega^e$ in the Lagrangian at the beginning, one has the second order frame formalism, and (19) is true as an identity. The diffeomorphism Noether current (5) involves the Lie derivative of the connection, $\mathcal{L}_\xi\omega$, which is given by

$$\mathcal{L}_\xi \omega = \mathrm{i}_\xi \mathrm{d}\omega + \mathrm{d}(\mathrm{i}_\xi \omega) = \mathrm{i}_\xi R + \mathrm{D}(\mathrm{i}_\xi \omega). \tag{21}$$

Here we are treating the connection components as a collection of 1-forms, and we shall do the same with the frame components. If the relevant manifold cannot be covered by a single frame field—i.e. is not parallelizable—this strategy would not be available, because under a change of local Lorentz gauge the Lie derivative would not transform properly so as to determine a well-defined symmetry operation. In that case, something like the Lorentz-Lie derivative discussed below would be required.

From (18) we can read off the symplectic potential defined in (1),

$$\theta = \delta\omega^{ab} \wedge \frac{\partial L}{\partial R^{ab}}. \tag{22}$$

Using (21), the diffeomorphism Noether current (23) can thus be written as

$$j_\xi = \mathrm{d}\left(\mathrm{i}_\xi\omega^{ab} \wedge \frac{\partial L}{\partial R^{ab}}\right) - (\mathrm{i}_\xi\omega^{ab}) \wedge \mathrm{D}\frac{\partial L}{\partial R^{ab}} + (\mathrm{i}_\xi R^{ab}) \wedge \frac{\partial L}{\partial R^{ab}} - \mathrm{i}_\xi L. \tag{23}$$

(In the first and second terms, the first factor is a 0-form, so the wedge product is just ordinary multiplication. Throughout this paper we sometimes include such unnecessary wedge notations since they seem helpful in organizing the structure of the expressions.)

The second term in the Noether current (23) vanishes by the ω equation of motion. Moreover, the Lagrangian (16) has the nice property

$$\mathrm{i}_\xi L = (\mathrm{i}_\xi e^a) \wedge \frac{\partial L}{\partial e^a} + (\mathrm{i}_\xi R^{ab}) \wedge \frac{\partial L}{\partial R^{ab}}, \tag{24}$$

from which it follows that, taken together, the third and fourth terms of (23) vanish by the e equation of motion. Thus we may simply read off the Noether charge $(n-2)$-form,

$$Q_\xi = \mathrm{i}_\xi\omega^{ab} \wedge \frac{\partial L}{\partial R^{ab}}. \tag{25}$$

Notice that this is linear in ξ, with no derivative on ξ. If ξ is a horizon generating Killing field, Q_ξ therefore appears to *vanish* when evaluated at the bifurcation surface $\mathcal{B}$ of the Killing horizon. This would imply that the entropy (15) vanishes, but obviously something is wrong with this argument.

The problem arises because, in showing that the entropy is proportional to the horizon Noether charge, we assumed that the dynamical fields have vanishing Lie derivative with respect to ξ. Because of this, the connection ω^e diverges as $\mathcal{B}$ is approached. We shall explain shortly from a geometric viewpoint why the connection diverges, but first let us show that $i_\xi\omega^e$ has a finite, nonzero limit at $\mathcal{B}$, and use this to find the entropy.

The Lie derivative of the frame is given by

$$\begin{aligned}\mathcal{L}_\xi e^a &= \mathrm{i}_\xi \mathrm{d}e^a + \mathrm{d}\mathrm{i}_\xi e^a \\ &= \mathrm{i}_\xi \mathrm{D}e^a + \mathrm{D}\mathrm{i}_\xi e^a - \mathrm{i}_\xi\omega^a{}_b \wedge e^b.\end{aligned} \tag{26}$$

Setting this equal to zero, and using the field equation $\mathrm{D}e^a = 0$ (or the definition of ω^e in the second order formulation), we obtain

$$\mathrm{i}_\xi(\omega^e)^a{}_b = e^\mu_b \mathrm{D}_\mu(\mathrm{i}_\xi e^a), \tag{27}$$

where e^μ_b is the inverse frame. To evaluate the right-hand side note that the action of D_μ on tensors has not so far been specified (other than being torsion-free) hence we may choose it to act on tensor indices as the torsion-free covariant derivative ∇_μ determined by the metric. With this choice we have $\mathcal{D}_\mu e^a_\nu = 0$, where $\mathcal{D}_\mu$ denotes the *full* derivative including both the spacetime and spin connections. Then, using the Leibniz rule, (27) becomes

$$\mathrm{i}_\xi(\omega^e)^a{}_b = e^\mu_b e^a_\nu \nabla_\mu \xi^\nu. \tag{28}$$

The limiting value at $\mathcal{B}$ is given by

$$\lim_{\to\mathcal{B}} \mathrm{i}_\xi(\omega^e)^{ab} = -\kappa n^{ab}, \tag{29}$$

where again κ (13) is the surface gravity, and $n^{ab} = n^{\mu\nu}e^a_\mu e^b_\nu$ is the binormal to $\mathcal{B}$, converted to a Lorentz tensor. Thus, despite appearances, $\mathrm{i}_\xi\omega^e$ does *not* vanish at the bifurcation surface. This can only happen because ω^e blows up there.

Using (29), we find the Noether charge form (25) is given by

$$\lim_{\to\mathcal{B}}(Q_\xi) = -\kappa n^{ab}\epsilon_{abc\cdots d}e^c \wedge \cdots \wedge e^d. \tag{30}$$

This is just 2κ times the "area" element on $\mathcal{B}$, hence $\oint_{\mathcal{B}} Q_\xi = \kappa A/8\pi G$ (restoring the $16\pi G$), so the entropy (15) is $S_{\mathrm{BH}} = A/4\hbar G$, the Bekenstein-Hawking entropy.

To explain why and how the connection diverges at the bifurcation surface, we employ a simple analogy with a two-dimensional Euclidean space. The Killing vector field that generates the rotation around the origin is given by $\xi = \partial_\theta$ in polar coordinates (r, θ). The origin is a fixed point of the rotational isometry, i.e. ξ vanishes there, so it is analogous to the bifurcation surface. A frame that has zero Lie derivative with respect to this rotation Killing field rotates by 2π when traversing a circle around the origin. For a circle closer to the origin, the frame rotates faster, because the circumference shrinks. At the origin the frame has to rotate infinitely fast, which implies that the connection diverges. Explicitly, let the frame be given by $e^1 = \mathrm{d}r$ and $e^2 = r\mathrm{d}\theta$, so that $\mathcal{L}_\xi e^a = 0$. The nonzero connection components are given by $\omega^2{}_1 = -\omega^1{}_2 = \mathrm{d}\theta$. The norm of $\mathrm{d}\theta$ is $(g^{\theta\theta})^{1/2} = 1/r$, so $\mathrm{d}\theta$, and therefore the connection, diverges at the origin, although the contraction $\mathrm{i}_\xi\omega^2{}_1 = 1$ is finite and nonzero. At the bifurcation surface of a black hole space-time one has a hyperbolic version of this phenomenon. For instance, for a Schwarzschild black hole we have $e^0 = N\mathrm{d}t$ and $e^1 = N^{-1}\mathrm{d}r$, with $N = (1-2M/r)^{1/2}$ the norm of the Killing vector ∂_t. Then $\omega^0{}_1 = \kappa\mathrm{d}t$, where $\kappa = 1/4M$ is the surface gravity. The connection diverges since the norm of $\mathrm{d}t$ is N^{-1}, although $\mathrm{i}_{\partial_t}\omega^0{}_1 = \kappa$ is finite.

If we are to avoid the occurrence of a singular spin connection in the Noether charge computation of black hole entropy, we must modify the realization of the diffeomorphism symmetry, so that a frame can be invariant under the symmetry and yet nonsingular at the bifurcation surface. The next section introduces this realization.

IV. LORENTZ-LIE DERIVATIVE

The Lie derivative of tensor fields with respect to a vector field ξ is defined, with no additional structure, as the rate of change of the pull-back along the flow of ξ. A frame consists of covectors which are carried by the flow in a unique way. The covectors remain orthonormal under the flow of a Killing vector, but they undergo a Lorentz transformation. Therefore the Lie derivative of a frame with respect to a Killing vector is generally nonzero. However, given a frame, one can define a modified derivative which includes a compensating local Lorentz transformation, so that the modified derivative of the frame with respect to a Killing vector is always zero. We call this the *Lorentz-Lie (LL) derivative*. The LL derivative we employ has been introduced several times, using various formalisms (see [7–12] and references therein). Acting on a spinor field, the LL derivative agrees with the definition given by Kosmann [7]. It was called the Yano derivative in [11], where other notions of generalized Lie derivative are also discussed.

We denote the Lorentz-Lie derivative by $\mathcal{K}^e_\xi$ (the notation is chosen in honor of Kosmann). It is the Lie derivative supplemented with a local $SO(n-1,1)$ gauge transformation generated by a particular λ^e_ξ which is determined by a frame e^a as follows. Note first that metric compatibility, i.e. the vanishing of $\mathcal{K}^e_\xi\eta^{ab}$, implies antisymmetry of λ^e_ξ, that is, $(\lambda^e_\xi)^{(ac)} = (\lambda^e_\xi)^{(a}{}_b\eta^{c)b} = 0$. Now consider the action of $\mathcal{K}^e_\xi$ on e^a,

$$\mathcal{K}^e_\xi e^a = \mathcal{L}_\xi e^a + (\lambda^e_\xi)^a{}_b e^b. \tag{31}$$

The spacetime tensor $e_a\mathcal{K}^e_\xi e^a$ can be decomposed into its symmetric and antisymmetric parts,

$$e_{a\mu}\mathcal{K}^e_\xi e^a_\nu = e_{a(\mu}\mathcal{K}^e_\xi e^a_{\nu)} + e_{a[\mu}\mathcal{K}^e_\xi e^a_{\nu]}. \tag{32}$$

Owing to the antisymmetry of $(\lambda^e_\xi)^{ab}$, the symmetric part is independent of λ^e_ξ, and is given by

$$e_{a(\mu}\mathcal{K}^e_\xi e^a_{\nu)} = \frac{1}{2}\mathcal{L}_\xi g_{\mu\nu}. \tag{33}$$

The LL derivative $\mathcal{K}^e_\xi e^a$ will therefore vanish when ξ is a Killing vector if and only if the antisymmetric part vanishes. The antisymmetric part,

$$e_{a[\mu}\mathcal{K}^e_\xi e^a_{\nu]} = e_{a[\mu}\mathcal{L}_\xi e^a_{\nu]} + e_{a\mu}e_{b\nu}(\lambda^e_\xi)^{ab}, \tag{34}$$

can be set to zero by choosing

$$(\lambda^e_\xi)^{ab} = e^{\sigma[a}\mathcal{L}_\xi e^{b]}_\sigma. \tag{35}$$

This choice of λ^e_ξ defines the LL derivative associated with e^a. The LL derivative of e^a with respect to an arbitrary vector field is thus given by

$$\mathcal{K}^e_\xi e^a_\mu = \frac{1}{2}e^{a\nu}\mathcal{L}_\xi g_{\mu\nu}. \tag{36}$$

In particular, when ξ^a is a Killing vector field we have $\mathcal{K}_\xi e^a = 0$.

It will be useful to find an explicit expression for λ^e_ξ (35) in terms of $\nabla\xi$. We have

$$(\lambda^e_\xi)^{ab} = e^{\mu[a}\mathcal{L}_\xi e^{b]}_\mu \tag{37}$$

$$= e^{\mu[a}\xi^\nu\nabla_\nu e^{b]}_\mu + e^{\mu[a}(\nabla_\mu\xi^\nu)e^{b]}_\nu \tag{38}$$

$$= \mathrm{i}_\xi(\omega^e)^{ab} + e^{\mu[a}e^{b]}_\nu\nabla_\mu\xi^\nu. \tag{39}$$

In the second line we expressed the Lie derivative using the torsion-free metric compatible derivative ∇, and in the third line we used $\nabla e^b = \mathcal{D}e^b - (\omega^e)^b{}_c e^c = -(\omega^e)^b{}_c e^c$.

Under a Lorentz transformation of the frame, $e^a \to L^a{}_b e^b$, the quantity λ^e_ξ transforms like a connection for the Lie derivative,

$$\lambda^{Le}_\xi = L\lambda^e_\xi L^{-1} + L\mathcal{L}_\xi L^{-1}. \tag{40}$$

This makes the LL derivative covariant under $SO(n-1,1)$ gauge transformations. The action of the LL derivative is extended to any Lorentz tensor by requiring that it be a derivation, i.e. by stipulating that the Leibniz product rule applies. Its action on any $SO(n-1,1)$ connection is defined so that the λ_ξ term implements the infinitesimal gauge transformation of a connection,

$$\mathcal{K}^e_\xi\omega^{ab} = \mathcal{L}_\xi\omega^{ab} - \mathrm{D}(\lambda^e_\xi)^{ab} \tag{41}$$

$$= \mathrm{i}_\xi R^{ab} + \mathrm{D}(\mathrm{i}_\xi\omega - \lambda^e_\xi)^{ab}. \tag{42}$$

This result will be key when evaluating the entropy using the Lorentz-diffeomorphism Noether charge.

Let us illustrate the action of the LL derivative in a two-dimensional flat Euclidean space. The frame we considered above has zero Lie derivative along the rotation Killing vector field $\xi = \partial_\theta$. Hence for that frame and that vector field we have $\lambda_\xi = 0$, so the LL derivative is just the Lie derivative, which vanishes on the frame. The problem with such a frame, as explained above, is that it is singular at the fixed point of the Killing flow. Next we consider a Cartesian frame, $e^1 = dx$ and $e^2 = dy$. Writing the same Killing vector as $\xi = x\partial_y - y\partial_x$, it is simple to see that $(\mathcal{L}_\xi e)^1 = -e^2$ and $(\mathcal{L}_\xi e)^2 = e^1$. Although this frame is not rotationally invariant, its LL derivative must vanish since ξ is a Killing field. Indeed we have $(\lambda_\xi^e)^1{}_2 = -(\lambda_\xi^e)^2{}_1 = 1$, so $(\mathcal{K}^e e)^1 = (\mathcal{L}_\xi e)^1 + (\lambda_\xi^e)^1{}_2 e^2 = -e^2 + e^2 = 0$, and similarly $(\mathcal{K}^e e)^2 = (\mathcal{L}_\xi e)^2 + (\lambda_\xi^e)^2{}_1 e^1 = e^1 - e^1 = 0$. In effect, the gauge transformation cancels the nonzero Lie derivative with respect to a Killing vector. (If we consider instead the shear vector field $x\partial_y$, which is *not* a Killing vector, then both the Lie and LL derivatives of the Cartesian frame are nonvanishing, and they differ from each other.) Similarly, the rotation invariant frame has a nonvanishing Lie derivative with respect to the translation Killing vector ∂_x, but its LL derivative with respect to ∂_x vanishes.

V. BLACK HOLE ENTROPY AS LORENTZ-DIFFEOMORPHISM NOETHER CHARGE

We may now repeat the steps in the Noether charge construction of Sec. II, replacing the Lie derivative variation by the LL derivative,

$$\delta\phi = \mathcal{K}_\xi^e \phi. \tag{43}$$

Assuming the diffeomorphism-covariant Lagrangian is a Lorentz scalar, its variation is the same whether the fields of which it is built vary by the Lie derivative, or the LL derivative, hence it satisfies $\mathcal{K}_\xi^e L = \mathcal{L}_\xi L = \mathrm{d}\mathrm{i}_\xi L$.

The Noether current associated with the LL symmetry is defined by

$$j_\xi^{\mathcal{K}} = \theta(\phi, \mathcal{K}_\xi^e \phi) - \mathrm{i}_\xi L, \tag{44}$$

which is closed on shell for all ξ, and hence is the exterior derivative of a Noether charge $(n-2)$-form,

$$j_\xi^{\mathcal{K}} = \mathrm{d}Q_\xi^{\mathcal{K}}. \tag{45}$$

The derivation of the first law of black hole mechanics proceeds as in the case of the diffeomorphism Noether current, but the role of the Lie derivative is played by the LL derivative. In particular, to make use of the correspondingly modified variational identity (11), the background fields must now satisfy $\mathcal{K}_\xi^e \phi = 0$, so that the variation of the Hamiltonian generating the combined Lorentz-diffeomorphism symmetry will vanish. This leads to a new expression for the black hole entropy,

$$S = \frac{2\pi}{\hbar} \oint_{\mathcal{B}} \hat{Q}_\xi^{\mathcal{K}}, \tag{46}$$

where again the hat on Q indicates the replacement $\nabla_\mu \xi_\nu \to n_{\mu\nu}$. In order to evaluate this for a particular theory one needs first to find the Noether current and then the corresponding Noether charge form. Let us see how it works out for general relativity and some closely related theories.

A. Lovelock gravity

The analysis for general relativity in Sec. III actually applies more generally to any Lagrangian $L(e, \omega)$ that is constructed from wedge products of frames and curvature 2-forms, since the nice property (24) continues to hold, and the rest of the derivation is generic. In particular, the expression for the Noether charge form (25) applies to all such Lagrangians. These Lagrangians correspond to Lovelock gravity theories, together with various "topological" terms that do not affect the equations of motion.

Comparison of the expressions (42) and (21) for the LL and Lie derivatives of the connection reveals that, to obtain the Noether charge form, we merely need to replace $\mathrm{i}_\xi\omega$ by $\mathrm{i}_\xi\omega - \lambda_\xi^e$ in (25). This yields

$$Q_\xi^{\mathcal{K}} = (\mathrm{i}_\xi\omega - \lambda_\xi^e)^{ab} \wedge \frac{\partial L}{\partial R^{ab}}. \tag{47}$$

The key point now is that since the frame is LL invariant and not Lie invariant, it can be assumed to be regular at $\mathcal{B}$. Therefore the quantity $\mathrm{i}_\xi\omega$ vanishes at $\mathcal{B}$, and from (39) and (13) we have there

$$(\lambda_\xi^e)^{ab} = \kappa n^{ab}. \tag{48}$$

When this is substituted in (47), the result is identical to what we obtained using the limiting expression (29) with a singular, Lie invariant frame. That is,

$$\hat{Q}_\xi^{\mathcal{K}} = -\kappa n^{ab} \wedge \frac{\partial L}{\partial R^{ab}}, \tag{49}$$

and integrating this gives the entropy (46).

The Lagrangian for Lovelock gravity is

$$L(e,\omega) = \epsilon_{a\cdots bcd}(c_0 e^a \wedge \cdots \wedge e^b \wedge e^c \wedge e^d + c_1 e^a \wedge \cdots \wedge e^b \wedge R^{cd} + \cdots), \tag{50}$$

where c_i is a coupling constant for the term with i factors of the curvature, and the terms indicated by the ellipses each contain one more factor of R than the previous term. The c_0 term is a cosmological constant, and the c_1 term is the Einstein-Hilbert Lagrangian. The form $Q_\xi^{\mathcal{K}}$ is obtained from L by moving, in turn, each factor of R all the way to the first position and replacing it by $-\kappa n^{ab}$. Contracting n^{ab} with the rank-n Lorentz ϵ in L produces twice the rank-$(n-2)$ Lorentz ϵ associated via the frame with the $SO(n-2)$ group of the tangent space of $\mathcal{B}$. The remaining Lorentz indices are thus all projected into this subspace. The coefficient of the term in $Q_\xi^{\mathcal{K}}$ with $m-1$ factors of R is thus $2\kappa m c_m$.

The curvatures in the entropy integrand are those of the connection ω, whose equation of motion is $\mathrm{D}\partial L/\partial R^{ab}=0$. One way—and generically the only way—to satisfy this is to have $\mathrm{D}e^a=0$, i.e. for ω to be the spin connection ω^e determined by e. For such solutions the curvature appearing in the entropy is the one determined by e. These curvature 2-forms are all pulled back to $\mathcal{B}$ and, as explained above, their Lorentz indices are all projected into the $\mathcal{B}$-subspace. Moreover, the extrinsic curvature of the bifurcation surface vanishes, so these curvatures all reduce to intrinsic curvatures of $\mathcal{B}$. The entropy is therefore determined by the *intrinsic* geometry of the horizon [13].

The first and second order formalisms for Lovelock gravity are not strictly identical in more than four dimensions, since there exist solutions in the first order formalism for which $\omega\neq\omega^e$. That is, the connection may have torsion. In fact, black hole solutions with this property exist, and their entropy might involve this torsion via the curvature (see for example [14] and references therein). In [15] black hole solutions in Born-Infeld gravity (which is a special case of Lovelock gravity in even dimensions) supporting nonzero torsion were constructed. However, by construction all the Noether charges for these solutions vanish, including the entropy. It would be interesting to find solutions with nontrivial torsion contributing to the black hole entropy.

B. Topological terms

As a further application of the Lorentz-diffeomorphism symmetry discussed here, we now look into the contributions of topological terms to the black hole entropy in four-dimensional general relativity. The contributions of these terms have been studied before, using various formalisms; see for example Refs. [12,16–18].

The Lagrangian 4-form is given by

$$L(e,\omega)=(*(e^a\wedge e^b)+c_{\mathrm{H}}e^a\wedge e^b+c_{\mathrm{E}}*R^{ab}+c_{\mathrm{P}}R^{ab})\wedge R_{ab},\tag{51}$$

where $*$ denotes Lorentz dual, e.g. $*R^{ab}=\frac{1}{2}\epsilon^{abcd}R_{cd}$. The coupling constants are c_{H} for the Holst term [3], c_{E} for the Euler (Gauss-Bonnet) invariant, and c_{P} for the Pontryagin invariant. The Holst term modifies the connection equation of motion, but does not affect its solution ω^e, and it drops out of the frame equation of motion when the connection is on shell. The Euler and Pontryagin terms depend only on the connection. The Euler and Pontryagin terms are exact forms, so do not affect the equations of motion. Were they exterior derivatives of gauge-invariant forms, we could absorb those forms into the symplectic potential θ (1), and from general considerations conclude that the entropy is unaffected by them [19]. However, those forms are not gauge invariant, hence these terms might contribute to the black hole entropy.

The black hole entropy (46) for the Lagrangian (51) is given by

$$S=\frac{2\pi}{\hbar}\oint_{\mathcal{B}}n^{cd}(*e_c\wedge e_d+c_{\mathrm{H}}e_c\wedge e_d+2c_{\mathrm{E}}*R_{cd}+2c_{\mathrm{P}}R_{cd}).\tag{52}$$

The Einstein-Hilbert term is proportional to the area of $\mathcal{B}$, as we saw before. The Holst term vanishes because the binormal is orthogonal to $\mathcal{B}$. The Euler term is one of the terms in the general Lovelock Lagrangian (50). Therefore, as explained above, it involves only the intrinsic curvature of $\mathcal{B}$. In the present case, since $\mathcal{B}$ is two dimensional, that just amounts to the Ricci scalar. The integral of this term in the entropy is a topological invariant, proportional to the Euler characteristic of the horizon [13]. In higher, even dimensions, similar terms exist, involving $(n-2)/2$ curvature tensors. Finally, it turns out that, since the extrinsic curvature vanishes, the Pontryagin term is an exact form on $\mathcal{B}$, so its integral vanishes. To see that the pull-back of $n^{cd}R_{cd}$ to $\mathcal{B}$ is exact, let l^a and n^a be null normals to $\mathcal{B}$ satisfying $l_cn^c=-2$, so $n^{cd}=l^{[c}n^{d]}$. Then we have $n^{cd}R_{cd}=l_cD^2n^c=d(l_cDn^c)-Dl_c\wedge Dn^c$. Since the extrinsic curvature of $\mathcal{B}$ vanishes, the null normals must be parallel transported along $\mathcal{B}$ into multiples of themselves, so pulled back to $\mathcal{B}$ we have $Dl^c=\sigma l^c$ and $Dn^c=-\sigma n^c$ for some 1-form σ. Hence $Dl_c\wedge Dn^c=-2\sigma\wedge\sigma=0$.

VI. DISCUSSION

In this paper we have made use of the Lorentz-Lie derivative $\mathcal{K}_\xi$ to define a particular variation of the frame field (and other Lorentz tensors) under a diffeomorphism generated by a vector field ξ. In words, the LL derivative is defined by combining the usual Lie derivative with a term that subtracts the local Lorentz transformation induced on the frame by the flow. This subtraction term depends on the frame field, and amounts to a connection that covariantizes the Lie derivative with respect to local Lorentz transformations. A key property of this definition is that if ξ is a Killing vector, the LL derivative of the frame vanishes. This property makes it possible for the frame to be LL-invariant at the bifurcation surface of a Killing horizon while remaining regular there. Using this formalism, we showed how the LL Noether charge yields the black hole entropy. We illustrated the computational convenience of this method by evaluating the black hole entropy for Lagrangians that are polynomial in wedge products of the frame field 1-form and curvature 2-form.

The computations in this paper were carried out using a single "local Lorentz gauge," so in effect we assumed that the relevant portion of the spacetime could be covered by a single gauge patch. Further analysis would be required to deal with situations where that is not the case. For example, one could use the frame bundle formalism, which has been discussed in this setting in Refs. [2,12].

We have restricted attention here to Lagrangians that are Lorentz scalar n-forms. It could be interesting to study the Noether charge formalism allowing for Lagrangians having this property only up to the exterior derivative of a nonscalar n-form. This would shed a different light on the contributions of the Euler and Pontryagin terms studied here, and could be useful in further generalizations.

Finally, the combined diffeomorphism-gauge Noether current analysis can also be applied when the local gauge symmetry is internal, as in Yang-Mills theory. A simple example involving the electromagnetic field is discussed in Appendix E1 of [20]. It employs the notion of "gauge covariant Lie derivative" to arrive directly at a gauge-covariant Noether current. A general analysis is provided in Ref. [2].

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