Cardy formula for charged black holes with anisotropic scaling

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We first observe that for Lifshitz black holes of which the only charge is the mass, the resulting Smarr relation is a direct consequence of the Lifshitz Cardy formula. From this observation, we propose to extend the Cardy formula to the case of electrically charged Lifshitz black holes satisfying as well a Smarr relation. The expression of our formula depends on the dynamical exponent, the energy and the charge of the ground state which is played by a magnetically charged soliton obtained through a double Wick rotation. The expression also involves a factor multiplying the chemical potentials which varies in function of the electromagnetic theory considered. This factor is precisely the one that appears in the Smarr formula for charged Lifshitz black holes. We test the validity of this Cardy formula in different situations where electrically Lifshitz charged black holes satisfying a Smarr relation are known. We then extend these results to electrically charged black holes with hyperscaling violation. Finally, an example in the charged anti-de Sitter case is also provided.

I. INTRODUCTION

Recently, there has been important interest in extending the ideas underlying the standard relativistic AdS/CFT correspondence [1] to physical systems that exhibit a dynamical scaling near fixed points. The latter are characterized by an anisotropic invariance encoded by the fact that the space and the time scale with different weights,

$$t \rightarrow \lambda^z t, \qquad \vec{x} \rightarrow \lambda \vec{x}. \qquad (1)$$

The constant z which is called the dynamical exponent precisely reflects this anisotropic symmetry. In analogy with the anti-de Sitter (AdS) case $z = 1$, the gravity dual metric in D dimensions refereed as the Lifshitz metric was given in Ref. [2],

$$ds^2 = -\left(\frac{r}{l}\right)^{2z} dt^2 + \frac{l^2}{r^2} dr^2 + \frac{r^2}{l^2} \sum_{i=1}^{D-2} dx_i^2, \qquad (2)$$

and it is easy to see that the anisotropic transformations (1) together with the rule $r \rightarrow \lambda^{-1} r$ act as an isometry for this metric. Nevertheless, in contrast with the AdS case, Lifshitz spacetimes or their black hole extensions are not solutions of standard General Relativity and instead require the introduction of some source that may be materialized by some extra fields [3–6] or/and by considering higher-order gravity theories [7–11]. The thermodynamical properties of the Lifshitz black holes, in spite of their rather unconventional asymptotic behaviors, have been intensively studied; see e.g. Refs. [12–14]. One of the most appealing properties

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of the Lifshitz black holes of which the only charge is the mass Δ concerns their entropy $\mathcal{S}$ which scales with respect to the temperature T as

$$\mathcal{S} \propto T^{\frac{D-2}{z}}. \qquad (3)$$

As a direct consequence, the Smarr formula [15] takes the following form [16]:

$$\Delta = \frac{D-2}{D+z-2} T\mathcal{S}. \qquad (4)$$

In three dimensions, this last relation (4) can be obtained by exploiting the fact that the Lifshitz algebras in two dimensions with dynamical exponents z and z^{-1} are isomorphic [17]. As shown precisely in this last Ref. [17], this isomorphism is translated into a duality between the low and high temperature regimes and allows one to derive a formula for the asymptotic growth number of states in three dimensions where the ground state is played by the soliton obtained through a double Wick rotation,

$$\mathcal{S} = 2\pi l(z+1)\left[\left(\frac{\Delta_0}{z}\right)^z \Delta\right]^{\frac{1}{z+1}}, \qquad (5)$$

where $-\Delta_0$ corresponds to the mass of the soliton. In the isotropic case $z = 1$, this expression becomes the standard Cardy formula. Note also that the validity of Eq. (5) has been checked in the case of the Lifshitz black hole solution with $z = 3$ of new massive gravity [7] (see Ref. [17]) and also in the presence of a source given by a nonminimal scalar field for the same gravity theory [5]. The first law $d\Delta = Td\mathcal{S}$ applied to the relation (5) will then imply that the mass can be expressed as

$$\Delta = \frac{\Delta_0}{z}(2\pi lT)^{1+\frac{1}{z}}, \tag{6}$$

and combining together the two expressions (5)–(6), one easily obtains the Smarr formula (4) for $D = 3$. Hence from this simple exercise, we have highlighted a certain correlation between the Smarr formula and the generalized Cardy formula in three dimensions.

The main aim of this paper is to extend the formula (5) to the charged case. In doing so, we will inspire ourselves from the fact that the Smarr formula in the case of charged solutions must be a consequence of the Cardy formula as it occurs in the neutral case. This problem has a certain interest since electrically charged Lifshitz black holes have also been found in the current literature; see e.g. Refs. [18–23]. Such examples occur for example in the case of Einstein gravity with a source given by a Proca-Maxwell action [20] or in the presence of N-Abelians $U(1)$ fields with a dilaton [24] as well as in the case of nonlinear electrodynamics [21,22]. In all these examples, a Smarr formula generalizing the expression (4) can be derived and is generically written as [16]

$$\Delta = \frac{D-2}{D+z-2}T\mathcal{S} + \alpha\Phi_e\mathcal{Q}_e, \tag{7}$$

where Φ_e is the electric potential and $\mathcal{Q}_e$ the electric charge. In this relation, the value of the constant α varies in function of the electromagnetic Lagrangian considered. From now on, it is important to emphasize the nonuniversal character of the Smarr formula in the charged case reflected by the presence of the constant α. In other words, this means that the constant α does not depend only on the dynamical exponent z and the dimension D but also depends on the theory considered as we will see in the different examples listed below.

In this paper, we will show that for electrically charged Lifshitz black holes satisfying a Smarr relation of the form (7) in three dimensions, the Cardy formula (5) becomes

$$\mathcal{S} = 2\pi l(z+1)(|\Delta_0 z^{-1} + \alpha\Phi_m\mathcal{Q}_m|^z|\Delta - \alpha\Phi_e\mathcal{Q}_e|)^{\frac{1}{z+1}}, \tag{8}$$

where Φ_m (respectively, $\mathcal{Q}_m$) denotes the magnetic potential (respectively, the magnetic charge) of the magnetically charged soliton obtained from the electric solution by means of a double Wick rotation. Since the Wick rotation switches the role of the time coordinate t with the angular coordinate $x = \varphi$, the field strengths of the resulting magnetically charged soliton will be complex in general with a magnetic charge and potential both purely imaginary. Nevertheless, this will not be dramatic since in the proposed formula (8), only their product which is always real appears. The Wick rotation is also responsible for the apparent discrepancy of the sign appearing in front of the constant α accompanying the magnetic and electric parts in (8).

In what follows, we will test the validity of the formula (8) in different theories where charged Lifshitz black holes satisfying a Smarr formula of the form (7) are known. In each case, we will derive the corresponding magnetically charged soliton and compute the mass of the soliton through the quasilocal method given in Refs. [25,26] as well as their magnetic charge. We will then extend these results to the case of charged hyperscaling violation black holes. Finally, the last section will be dedicated to some comments regarding the isotropic AdS case $z = 1$.

II. CHARGED LIFSHITZ BLACK HOLE AND SOLITON SOLUTIONS

In all the examples given below, the Lagrangian $\mathcal{L}$ will involve a gravity part encoded by the metric g as well as different Abelian fields denoted generically by $A_{(i)\mu}$ and eventually a scalar field ϕ with its standard kinetic term $\partial_\mu\phi\partial^\mu\phi$,

$$\mathcal{L} = \mathcal{L}(g, A_{(i)\mu}, \phi). \tag{9}$$

The corresponding action will be given by

$$S[g, \phi, A_{(i)\mu}] = \int d^3x\sqrt{-g}\mathcal{L}. \tag{10}$$

The mass of the charged black hole and soliton will be computed through the quasilocal method described in Refs. [25,26] where the charge Δ which corresponds to the mass is given by

$$\Delta(\xi) = \int_{\mathcal{B}} dx_{\mu\nu}\left(\delta K^{\mu\nu}(\xi) - 2\xi^{[\mu}\int_0^1 ds\Theta^{\nu]}(\xi|s)\right). \tag{11}$$

Here $\delta K^{\mu\nu}(\xi) \equiv K^{\mu\nu}_{s=1}(\xi) - K^{\mu\nu}_{s=0}(\xi)$ denotes the difference of the Noether potential between the interpolated solutions, $dx_{\mu\nu}$ represents the integration over the codimension-2 boundary $\mathcal{B}$, $\xi^t = (1, 0, 0)$ is the timelike Killing vector field, and Θ^ν represents the surface term. In the case of a Lagrangian given by (9), the involved quantities are given by

$$\Theta^\mu = 2\sqrt{-g}\left[P^{\mu(\alpha\beta)\gamma}\nabla_\gamma\delta g_{\alpha\beta} - \delta g_{\alpha\beta}\nabla_\gamma P^{\mu(\alpha\beta)\gamma}\right.$$
$$\left. + \frac{1}{2}\sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_{(i)\nu})}\delta A_{(i)\nu}\right) + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right], \tag{12}$$

$$K^{\mu\nu} = \sqrt{-g}\left[2P^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma - 4\xi_\sigma\nabla_\rho P^{\mu\nu\rho\sigma}\right.$$
$$\left. -\sum_i\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_{(i)\nu})}\xi^\sigma A_{(i)\sigma}\right], \tag{13}$$

where $P^{\mu\nu\rho\sigma} = \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}$ with $R_{\mu\nu\rho\sigma}$ being the Riemann tensor.

The black hole metric will be parametrized by the following line element,

$$ds^2 = -\frac{r^{2z}}{l^{2z}} f(r)dt^2 + \frac{l^2}{r^2 f(r)} dr^2 + \frac{r^2}{l^2} d\varphi^2, \qquad (14)$$

and the ansatz for the gauge fields and eventually the scalar field read

$$A_{(i)\mu}dx^\mu = A_{(i)t}(r)dt, \qquad \phi = \phi(r). \qquad (15)$$

The Euclidean version of (14) obtained by means of the transformation $t = i\tau$ requires the Euclidean time to be periodic with period $\beta = T^{-1}$ in order to avoid conical singularity while the angle keeps identified as $0 \leq \varphi < 2\pi l$. Under the Euclidean diffeomorphism defined by

$$(\tau, r, \varphi) \mapsto \left(\bar{\tau} = \left(\frac{2\pi l}{\beta}\right)^{\frac{1}{z}} \varphi, \bar{r} = \frac{\beta}{2\pi z} \left(\frac{r}{l}\right)^z, \bar{\varphi} = \frac{2\pi l}{\beta} \tau \right), \qquad (16)$$

the Euclidean Lifshitz black hole is diffeomorphic to another asymptotically Lifshitz solution with dynamical exponent z^{-1}, scale lz^{-1} and inverse temperature

$$\bar{\beta} = (2\pi l)^{1+\frac{1}{z}} \beta^{-\frac{1}{z}}, \qquad (17)$$

and finally the Lorentzian soliton will be obtained from $\bar{\tau} = i\bar{t}$ yielding

$$ds^2 = -\left(\frac{z\bar{r}}{l}\right)^{\frac{2}{z}} d\bar{t}^2 + \frac{l^2}{z^2 \bar{r}^2 h(\bar{r})} d\bar{r}^2 + \frac{z^2 \bar{r}^2}{l^2} h(\bar{r}) d\bar{\varphi}^2. \qquad (18)$$

As mentioned before, this double Wick rotation will be responsible of the fact that the field strengths of the corresponding soliton will be purely imaginary. Note that in the case of the scalar field which depends only on the radial coordinate, this double Wick rotation does not yield to a complex scalar field for the soliton solution [5]. We may also emphasize that the set of parameters as well as the range of admissible values of the dynamical exponent z are the same for the electrically charged black hole and for the magnetically charged soliton.

Also in order to simplify the expressions, the volume of the one-dimensional sphere is denoted by Ω_1 with

$$\Omega_1 = 2\pi l. \qquad$$

We are now in a position to check the validity of the expression (8) in different contexts presented below.

A. Case of Einstein gravity with two Abelian fields and a dilaton

We first analyze the case of Einstein gravity with two Abelian fields and a dilaton for which the Lagrangian reads

$$\mathcal{L} = \frac{1}{2\kappa} \left(R - 2\Lambda - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \sum_{i=1}^{2} e^{\lambda_i \phi} F_{(i)}^2 \right), \qquad (19)$$

with $F_{(i)}^2 = F_{(i)\mu\nu} F_{(i)}^{\mu\nu}$ for $i = 1, 2$.

For an ansatz of the form (14)–(15), the solution given in Ref. [24] reads

$$f(r) = 1 - m\left(\frac{r_h}{r}\right)^{z+1} + (m-1)\left(\frac{r_h}{r}\right)^{2z}, \qquad (20a)$$

$$F_{(1)rt} = \sqrt{2(z^2-1)}\mu^{\sqrt{\frac{1}{2(z-1)}}} \left(\frac{r}{l}\right)^z, \qquad (20b)$$

$$F_{(2)rt} = \sqrt{2(m-1)(z-1)}\mu^{-\sqrt{\frac{z-1}{2}}} \left(\frac{r_h}{l}\right)^z r^{-z}, \qquad (20c)$$

$$e^\phi = \mu r^{\sqrt{2(z-1)}}, \qquad (20d)$$

where m and μ are two integration constants and r_h stands for the location of the horizon. Note that we have opted for this parametrization of the solution for latter convenience, but the expressions (20) are equivalent to those given in Ref. [24] after some redefinitions of the constants. This solution is defined provided that the parameters are fixed as follows,

$$\Lambda = -\frac{z(z+1)}{2l^2}, \quad \lambda_1 = -\sqrt{\frac{2}{z-1}}, \quad \lambda_2 = \sqrt{2(z-1)}, \qquad (21)$$

while the range for the admissible values of the dynamical exponent is $z > 1$.

In this case, the Wald entropy together with the Hawking temperature read

$$S_W = \frac{2\pi \Omega_1}{\kappa} \left(\frac{r_h}{l}\right), \qquad (22a)$$

$$T = \frac{1}{4\pi l} [2z + (1-z)m] \left(\frac{r_h}{l}\right)^z = \frac{\sigma}{l} \left(\frac{r_h}{l}\right)^z, \qquad (22b)$$

where we have defined

$$\sigma = \frac{1}{4\pi} [2z + (1-z)m]. \qquad (23)$$

On the other hand, the electric charge and electric potential read, respectively,

$$Q_e = \frac{\sqrt{2(m-1)(z-1)}\mu^{\frac{1}{2}\sqrt{2(z-1)}}\Omega_1}{2\kappa l^2} r_h^z \qquad (24)$$

and

$$\Phi_e = -A_{(2)t}(r_h) = \frac{\sqrt{2(m-1)}\mu^{-\frac{1}{2}\sqrt{2(z-1)}}}{\sqrt{z-1}\,l^z}\,r_h. \quad (25)$$

Introducing a one-parameter family of locally equivalent solutions, the variation of the Noether potential and the surface term (12)–(13) are given by

$$\delta K^{rt} = -\frac{(z-1)m}{2\kappa l}\left(\frac{r_h}{l}\right)^{z+1} + \frac{(m-1)r_h^{2z}}{\kappa l^{z+2}}\,r^{-z+1},$$

$$\int_0^1 ds\Theta^r = \frac{mz}{2\kappa l}\left(\frac{r_h}{l}\right)^{z+1} - \frac{(m-1)r_h^{2z}}{\kappa l^{z+2}}\,r^{-z+1}.$$

From these expressions, we obtain the mass of the Lifshitz black hole to be

$$\Delta = \frac{m\Omega_1}{2\kappa l}\left(\frac{r_h}{l}\right)^{z+1}, \quad (26)$$

and we easily check that the first law holds:

$$d\Delta = Td\mathcal{S}_{\mathrm{W}} + \Phi_e d\mathcal{Q}_e. \quad (27)$$

The Smarr formula turns to be

$$\Delta = \frac{1}{z+1}\left(T\mathcal{S}_{\mathrm{W}} + z\Phi_e\mathcal{Q}_e\right) \quad (28)$$

and corresponds to the expression (7) with $\alpha = \frac{z}{z+1}$.

The metric function of the corresponding solitonic spacetime (18) is given by

$$h(\bar{r}) = 1 - \frac{m}{(2\pi\sigma)^{\frac{z+1}{z}}}\left(\frac{l}{z\bar{r}}\right)^{\frac{z+1}{z}} + \frac{(m-1)}{(2\pi\sigma)^2}\left(\frac{l}{z\bar{r}}\right)^2, \quad (29)$$

where σ is defined in (23), and the Abelian gauge fields and the dilaton read

$$F_{(1)\bar{r}\bar{\varphi}} = i\frac{\sqrt{2(z^2-1)}\mu^{\sqrt{\frac{1}{2(z-1)}}}}{l}\,\bar{r}^{\frac{1}{z}}, \quad (30a)$$

$$F_{(2)\bar{r}\bar{\varphi}} = i\frac{\sqrt{2(m-1)(z-1)}\mu^{-\sqrt{\frac{z-1}{2}}}}{2\pi\sigma z}\,\bar{r}^{\frac{1-2z}{z}}, \quad (30b)$$

$$e^{\phi} = \mu\bar{r}^{\frac{\sqrt{2(z-1)}}{z}}. \quad (30c)$$

As before, the variation of the Noether potential and the surface term read

$$\delta K^{\bar{r}\bar{t}} = \frac{1}{2\kappa}\left[\frac{(m-1)}{2\pi^2\sigma^2 z\bar{r}}\left(\frac{zr}{l}\right)^{\frac{1}{z}} - \frac{2m}{(2\pi\sigma)^{\frac{z+1}{z}}l}\right],$$

$$\int_0^1 ds\Theta^{\bar{r}} = \frac{1}{2\kappa}\left[-\frac{(m-1)}{2\pi^2\sigma^2 z\bar{r}}\left(\frac{z\bar{r}}{l}\right)^{\frac{1}{z}} + \frac{m(z+2)}{(2\pi\sigma)^{\frac{z+1}{z}}l}\right],$$

yielding to

$$\Delta_0 = \frac{zm\Omega_1}{2\kappa l(2\pi\sigma)^{\frac{z+1}{z}}}. \quad (31)$$

Finally, the magnetic charge and potential are, respectively, expressed as

$$\mathcal{Q}_m = i\frac{\sqrt{2(m-1)(z-1)}\mu^{\frac{1}{2}\sqrt{2(z-1)}}\Omega_1}{4\pi\sigma z\kappa}\left(\frac{z}{l}\right)^{\frac{1}{z}} \quad (32)$$

and

$$\Phi_m = i\frac{\sqrt{2(m-1)}z\mu^{-\frac{1}{2}\sqrt{2(z-1)}}}{\sqrt{z-1}\,l}\left(\frac{l}{2\pi\sigma z}\right)^{\frac{1}{z}}. \quad (33)$$

It is then easy to verify that the formula (8) with the parameter $\alpha = \frac{z}{z+1}$ correctly fits with the expression of the Wald entropy given by Eq. (22a).

B. Case of Einstein gravity with a nonlinear electrodynamics

In Ref. [22], the authors consider a slight generalization of the previous Lagrangian (19) by introducing a nonlinear term as

$$\mathcal{L} = \frac{1}{2\kappa}\left[R - 2\Lambda - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{\lambda_1\phi}F_{(1)}^2\right.$$
$$\left. + \left(-\frac{1}{4}e^{\lambda_2\phi}F_{(2)}^2\right)^p\right]. \quad (34)$$

We have made some redefinitions of the fields and parameters in the original action [22] such that the Lagrangian (34) reduces to (19) in the linear limiting case $p = 1$. Note that such nonlinear generalization of the Maxwell action is being currently studied; see e.g. Ref. [27].

For an ansatz of the form (14)–(15), the metric function given in Ref. [22] after some redefinitions of the constants reads

$$f(r) = 1 - m\left(\frac{r_h}{r}\right)^{z+1} + (m-1)\left(\frac{r_h}{r}\right)^{2z+\Gamma}, \quad (35)$$

where the constant Γ is defined as

$$\Gamma = -\frac{2(p-1)}{2p-1}. \quad (36)$$

For this solution, the uncharged Abelian field $F_{(1)rt}$, the dilaton, the cosmological constant and the coupling constants are given by the same expressions than in the linear case; see (20b), (20d) and (21). The only changes are concerned with the charged Abelian gauge field $F_{(2)rt}$ and

the coupling constant λ_2 which now take the following forms,

$$F_{(2)rt} = \frac{\sqrt{2}(r_h)^{\frac{(2z-1)p-z+1}{(2p-1)p}}\Sigma^{\frac{1}{2p}}}{l^{\frac{1+(z-1)p}{p}}\mu^{\frac{(z-1)(2p-1)}{\sqrt{2(z-1)p}}}r^{\frac{2(z-1)p-z+2}{2p-1}}},$$

$$\lambda_2 = \frac{2[2(z-1)p-z+1]}{p\sqrt{2(z-1)}},$$

where for simplicity we have defined

$$\Sigma = \frac{(m-1)[2(z-2)p-z+3]}{(2p-1)^2}. \tag{37}$$

The expression of the entropy as well is unchanged and given by

$$\mathcal{S}_W = \frac{2\pi\Omega_1}{\kappa}\left(\frac{r_h}{l}\right), \tag{38}$$

where r_h is now the location of the horizon for the metric function (35). The Hawking temperature for this configuration reads

$$T = \frac{1}{l}\left[\sigma + \frac{(p-1)(m-1)}{2\pi(2p-1)}\right]\left(\frac{r_h}{l}\right)^z = \frac{\bar{\sigma}}{l}\left(\frac{r_h}{l}\right)^z,$$

where we have defined

$$\bar{\sigma} = \sigma + \frac{(p-1)(m-1)}{2\pi(2p-1)}, \tag{39}$$

with σ given by (23). On the other hand, the electric potential together with the electric charge read

$$\Phi_e = \frac{\sqrt{2}(2p-1)(r_h)^{-\frac{[(z-2)p-z+1]}{p}}\Sigma^{\frac{1}{2p}}}{l^{\frac{1+(z-1)p}{p}}[2(z-2)p-z+3]\mu^{\frac{(z-1)(2p-1)}{p\sqrt{2(z-1)}}}}, \tag{40}$$

$$\mathcal{Q}_e = \frac{\sqrt{2}p\mu^{\frac{(z-1)(2p-1)}{p\sqrt{2(z-1)}}}\Sigma^{\frac{2p-1}{2p}}(r_h)^{\frac{(2z-1)p-z+1}{p}}\Omega_1}{2\kappa l^{\frac{3p-1}{p}}}, \tag{41}$$

with Σ given by the expression (37).

Let us now compute the mass of this solution through the quasilocal formalism. For the timelike Killing vector $\xi^t = (1, 0, 0)$, and after some tedious but straightforward computations, the surface term together with the variation of the Noether potential (12)–(13) are given by

$$\int_0^1 ds\Theta^r = -\frac{(m-1)(r_h)^{\frac{2(2z-1)p-2z+2}{2p-1}}}{\kappa l^{z+2}}r^{-\frac{[2(z-2)p-z+3]}{2p-1}} + \frac{zm}{2\kappa l}\left(\frac{r_h}{l}\right)^{1+z},$$

$$\delta K^{rt} = \frac{(m-1)(r_h)^{\frac{2(2z-1)p-2z+2}{2p-1}}}{\kappa l^{z+2}}r^{-\frac{[2(z-2)p-z+3]}{2p-1}} - \frac{(z-1)m}{2\kappa l}\left(\frac{r_h}{l}\right)^{1+z}.$$

This implies that the mass of the Lifshitz black hole is given by

$$\Delta = \frac{m\Omega_1}{2\kappa l}\left(\frac{r_h}{l}\right)^{z+1},$$

and it is simple to verify that the first law (27) still holds. Additionally, the Smarr formula turns to be

$$\Delta = \frac{1}{z+1}T\mathcal{S}_W + \left[\frac{z}{1+z} + \frac{(z-1)(p-1)}{(1+z)p}\right]\Phi_e\mathcal{Q}_e$$

and corresponds to the expression (7) with

$$\alpha = \frac{z(2p-1)-(p-1)}{p(z+1)}. \tag{42}$$

As in the linear case, operating the same diffeomorphism (16), the metric function of the corresponding soliton reads

$$h(\bar{r}) = 1 - \frac{m}{(2\pi\bar{\sigma})^{\frac{z+1}{z}}}\left(\frac{l}{z\bar{r}}\right)^{\frac{z+1}{z}} + \frac{(m-1)}{(2\pi\bar{\sigma})^{2+\frac{\Gamma}{z}}}\left(\frac{l}{z\bar{r}}\right)^{2+\frac{\Gamma}{z}}, \tag{43}$$

where $\bar{\sigma}$ is defined in (39) and Γ is given in (36). As before, the Abelian gauge field $F_{(1)\bar{r}\bar{\varphi}}$ together with the dilaton are given by (30a) and (30c),) respectively, while $F_{(2)\bar{r}\bar{\varphi}}$ yields

$$F_{(2)\bar{r}\bar{\varphi}} = i\sqrt{2}\left(\frac{z}{l}\right)^{\frac{p-1}{(2p-1)zp}}\frac{\bar{\Sigma}^{\frac{1}{2p}}}{\mu^{\frac{(z-1)(2p-1)}{p\sqrt{2(z-1)}}}\bar{r}^{\frac{4(z-1)p+3-2z}{(2p-1)z}}},$$

where

$$\bar{\Sigma} = \frac{[2(z-2)p+3-z](m-1)}{(2\pi\bar{\sigma})^{2+\frac{\Gamma}{z}}z^2(2p-1)^2}.$$

For the same timelike Killing vector, the variation of the Noether potential and the surface term yield

$$\delta K^{\bar{r}\bar{t}} = \frac{(m-1)}{\kappa(2\pi\bar{\sigma})^{2+\frac{\Gamma}{z}}z\bar{r}}\left(\frac{z\bar{r}}{l}\right)^{\frac{4p-3}{z(2p-1)}} - \frac{m}{\kappa(2\pi\bar{\sigma})^{\frac{z+1}{z}}l},$$

$$\int_0^1 ds\Theta^{\bar{r}} = -\frac{(m-1)}{\kappa(2\pi\bar{\sigma})^{2+\frac{\Gamma}{z}}z\bar{r}}\left(\frac{z\bar{r}}{l}\right)^{\frac{4p-3}{z(2p-1)}} + \frac{m(z+2)}{2\kappa(2\pi\bar{\sigma})^{\frac{z+1}{z}}l},$$

yielding to

$$\Delta_0 = \frac{zm\Omega_1}{2\kappa l(2\pi\bar{\sigma})^{\frac{z+1}{z}}}.$$

The magnetic charge and potential, as before, are purely imaginary and read

$$\mathcal{Q}_m = \frac{ip\sqrt{2}}{2\kappa}\left(\frac{z}{l}\right)^{\frac{2p-1}{zp}}\bar{\Sigma}^{\frac{2p-1}{2p}}\mu^{\frac{(z-1)(2p-1)}{p\sqrt{2(z-1)}}}\Omega_1, \tag{44}$$

$$\Phi_m = \frac{i\sqrt{2}(2p-1)z}{[2(z-2)p+3-z]}\left(\frac{z}{l}\right)^{\frac{p-1}{(2p-1)zp}}\bar{\Sigma}^{\frac{1}{2p}}\mu^{-\frac{(z-1)(2p-1)}{p\sqrt{2(z-1)}}}$$
$$\times\left(\frac{l}{2\pi\bar{\sigma}z}\right)^{-\frac{[2(z-2)p+3-z]}{(2p-1)z}}. \tag{45}$$

As a matter of check, one can see that all the expressions involved in the nonlinear case reduce to those obtained in the previous subsection in the linear limiting case $p = 1$.

Finally, it is straightforward to check that the formula (8) with α given by (42) fits perfectly with the Wald formula (38).

C. Case of Einstein gravity with a Proca and Maxwell fields

We now consider the case of Einstein gravity with a Proca field $A_{(1)_\mu}$ together with a Maxwell field $A_{(2)_\mu}$ of which the Lagrangian is given by

$$\mathcal{L} = R - 2\Lambda - \frac{1}{4}F_{(1)\alpha\beta}F_{(1)}^{\alpha\beta} - \frac{1}{2}m^2 A_{(1)\alpha}A_{(1)}^{\alpha} - \frac{1}{4}F_{(2)\alpha\beta}F_{(2)}^{\alpha\beta} \tag{46}$$

with $F_{(i)\alpha\beta} = \partial_\alpha A_{(i)\beta} - \partial_\beta A_{(i)\alpha}$ for $i = 1, 2$.

In this case, the electrically charged Lifshitz black hole solution exists only for $z = 2$; the metric function (14) and the Proca and Maxwell fields read [20]

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^2, \qquad A_{(1)t}(r) = \left(\frac{r}{l}\right)^2 f(r),$$

$$F_{(2)tr}(r) = \frac{\sqrt{2}}{l^2}r_h, \tag{47}$$

while the parameters must be fixed as follows:

$$m = \frac{\sqrt{2}}{l}, \qquad \Lambda = -\frac{5}{2l^2}.$$

For this solution, the Wald entropy $\mathcal{S}_\text{W}$ and the Hawking temperature are given by

$$\mathcal{S}_\text{W} = \frac{4\pi r_h \Omega_1}{l}, \qquad T = \frac{r_h^2}{2\pi l^3}. \tag{48}$$

The expressions of the surface term and Noether potential (12)–(13) read

$$\int_0^1 ds\Theta^r = \frac{2rr_h^2}{l^4}, \qquad \delta K^{rt} = -\frac{2rr_h^2}{l^4}, \tag{49}$$

which in turn implies that the mass $\Delta = 0$. This solution with vanishing mass can be interpreted as an extremal charged Lifshitz black hole as it occurs for examples in Refs. [12,14,28]. Nevertheless, the electric charge $\mathcal{Q}_e$ and the electric potential Φ_e are nonvanishing and given by

$$\mathcal{Q}_e = \frac{\sqrt{2}\Omega_1}{l}\left(\frac{r_h}{l}\right), \qquad \Phi_e = -\sqrt{2}\left(\frac{r_h}{l}\right)^2. \tag{50}$$

It is easy to verify that the first law of thermodynamics holds,

$$d\Delta = 0 = Td\mathcal{S}_\text{W} + \Phi_e d\mathcal{Q}_e, \tag{51}$$

and the Smarr formula (7) reads in this case

$$\Delta = 0 = \frac{1}{3}(T\mathcal{S}_\text{W} + \Phi_e \mathcal{Q}_e); \tag{52}$$

that is, the constant α appearing in the generic formula (7) is $\alpha = \frac{1}{3}$.

The corresponding soliton is given by the line element (18) with $z = 2$ where the metric function and the gauge fields are given by

$$h(\bar{r}) = 1 - \frac{l}{2\bar{r}}, \quad A_{(1)\bar{\varphi}} = 2i\left(\frac{\bar{r}}{l}\right)h(\bar{r}), \quad F_{(2)\bar{r}\bar{\varphi}} = i(l\bar{r})^{-1/2}. \tag{53}$$

Along the same lines as before, the Noether potential together with the surface term take the following forms,

$$\int_0^1 ds\Theta^{\bar{r}} = \frac{2}{l}\left(\frac{2\bar{r}}{l}\right)^{\frac{1}{2}}, \qquad \Delta K^{\bar{r}\bar{t}} = -\frac{2}{l}\left(\frac{2\bar{r}}{l}\right)^{\frac{1}{2}}, \tag{54}$$

and as in the electric case, the mass of the soliton is vanishing $\Delta_0 = 0$. The magnetic charge and potential are purely imaginary and read

$$\mathcal{Q}_m = i\frac{\sqrt{2}}{l}\Omega_1, \qquad \Phi_m = -i\sqrt{2}, \tag{55}$$

and it is a matter of check that the formula (8) with $z = 2$ and $\alpha = 1/3$ fits perfectly with the Wald formula (48).

III. GENERALIZATION FOR CHARGED LIFSHITZ BLACK HOLES WITH HYPERSCALING VIOLATION

In the anisotropic extension of the AdS/CFT correspondence, there exists another dual metric of interest, the so-called hyperscaling violation spacetime of which the line element can be parametrized as follows:

$$ds^2 = \frac{1}{r^{\frac{2\theta}{D-2}}}\left[-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2\right]. \quad (56)$$

In this case, the anisotropic transformations (1) together with $r \to \lambda^{-1}r$ act rather like a conformal transformation, $ds^2 \to \lambda^{2\theta/(D-2)}ds^2$. Note also that this metric reduces to the Lifshitz metric (2) in the limiting case $\theta = 0$.

In Refs. [29,30], it was shown that if the entropy S scales with respect to the temperature T as

$$S \sim T^{\frac{d_{\text{eff}}}{z}}, \quad (57)$$

where d_{eff} is the effective spatial dimensionality, and where z is the dynamical exponent, the formula (5) in the uncharged case becomes

$$S = \frac{2\pi}{d_{\text{eff}}}(z + d_{\text{eff}})\left(\frac{\Delta_0 d_{\text{eff}}}{z}\right)^{\frac{z}{z+d_{\text{eff}}}}\Delta^{\frac{d_{\text{eff}}}{z+d_{\text{eff}}}}. \quad (58)$$

Repeating the same exercise as in the Lifshitz case, the first law $d\Delta = Td\mathcal{S}$ allows one to express the mass as

$$\Delta = (2\pi T)^{\frac{z+d_{\text{eff}}}{z}}\left(\frac{\Delta_0 d_{\text{eff}}}{z}\right), \quad (59)$$

and the Smarr formula becomes

$$\Delta = \frac{d_{\text{eff}}}{z + d_{\text{eff}}}T\mathcal{S}. \quad (60)$$

We may note that the expressions (57)–(60) with $d_{\text{eff}} = 1$ reduce to those obtained in the Lifshitz case.

Now by a certain analogy with the charged Lifshitz case, the Cardy formula for electrically charged black holes with hyperscaling violation should be

$$S = \frac{2\pi}{d_{\text{eff}}}(z + d_{\text{eff}})\left(\left|\frac{\Delta_0}{z}d_{\text{eff}} + \alpha\Phi_m\mathcal{Q}_m\right|^z |\Delta - \alpha\Phi_e\mathcal{Q}_e|^{d_{\text{eff}}}\right)^{\frac{1}{z+d_{\text{eff}}}}. \quad (61)$$

As before, the constant α is the one appearing in the charged version of the Smarr formula in the hyperscaling case, namely

$$\Delta = \frac{d_{\text{eff}}}{z + d_{\text{eff}}}T\mathcal{S} + \alpha\Phi_e\mathcal{Q}_e. \quad (62)$$

Let us now verify this formula for the charged hyperscaling violation black hole derived in Ref. [31] for which the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2\kappa}\left(R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \sum_{i=1}^{2}\frac{1}{4}e^{\lambda_i\phi}F_{(i)\mu\nu}F_{(i)}^{\mu\nu}\right), \quad (63)$$

where the potential is

$$V(\phi) = -2\Lambda e^{\gamma\phi}.$$

The solution as reported in Ref. [31], again after some redefinitions of the constant, reads

$$ds^2 = \frac{1}{r^{2\theta}}\left[-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{\varphi}^2\right], \quad (64)$$

where

$$f(r) = 1 - m\left(\frac{r_h}{r}\right)^{z+1-\theta} + (m-1)\left(\frac{r_h}{r}\right)^{2z-2\theta}, \quad (65a)$$

$$F_{(1)rt} = \sqrt{2(z-1)(1-\theta+z)}\mu^{2\sqrt{(1-\theta)(z-\theta-1)}}r^{z-\theta}, \quad (65b)$$

$$F_{(2)rt} = \sqrt{2(1-\theta)(z-\theta-1)(m-1)}\mu^{-\frac{\sqrt{z-\theta-1}\sqrt{2}}{2\sqrt{1-\theta}}}\left(\frac{r_h}{r}\right)^{z-\theta}, \quad (65c)$$

$$e^{\phi} = \mu r^{\sqrt{2(1-\theta)(z-\theta-1)}}, \quad (65d)$$

while the parameters are fixed as

$$\lambda_1 = -\sqrt{\frac{2}{(1-\theta)(z-\theta-1)}}, \qquad \lambda_2 = \sqrt{\frac{2(z-\theta-1)}{1-\theta}},$$

$$\Lambda = \frac{1}{2}(1-\theta+z)(z-\theta)\mu^{-\frac{\theta\sqrt{2}}{\sqrt{(1-\theta)(z-\theta-1)}}},$$

$$\gamma = \frac{2\theta}{\sqrt{2(1-\theta)(z-\theta-1)}}. \quad (66)$$

For this solution, the Wald entropy is given by

$$\mathcal{S}_W = \frac{2\pi r_h^{1-\theta}\Omega_1}{\kappa}, \quad (67)$$

while the Hawking temperature is

$$T = \frac{[(m-2)\theta - m(z-1) + 2z]r_h^{z}}{4\pi} = \rho r_h^{z}, \quad (68)$$

where for simplicity we define

$$\rho = \frac{(m-2)\theta}{4\pi} + \sigma, \qquad (69)$$

with σ given in (23). In this case, the electric charge together with the potential read, respectively,

$$\mathcal{Q}_e = \frac{\sqrt{2(1-\theta)(z-\theta-1)(m-1)}\mu^{\frac{\sqrt{2(z-\theta-1)}}{2\sqrt{1-\theta}}}r_h{}^{z-\theta}\Omega_1}{2\kappa}, \qquad (70)$$

$$\Phi_e = \sqrt{\frac{2(1-\theta)(m-1)}{z-\theta-1}}\mu^{-\frac{\sqrt{2(z-\theta-1)}}{2\sqrt{1-\theta}}}r_h. \qquad (71)$$

The variation of the Noether potential together with the surface term are obtained as

$$\delta K^{rt} = -\frac{(z-\theta-1)mr_h{}^{1-\theta+z}}{2\kappa}$$
$$-\frac{(m-1)(\theta-1)r_h{}^{2z-2\theta}}{\kappa}r^{\theta-z+1},$$
$$\int_0^1 ds\Theta^r = \frac{(z-2\theta)mr_h{}^{1-\theta+z}}{2\kappa}$$
$$+\frac{(m-1)(\theta-1)r_h{}^{2z-2\theta}}{\kappa}r^{\theta-z+1}, \qquad (72)$$

yielding to the same expression of the mass as the one found in Ref. [31], namely

$$\Delta = \frac{m(1-\theta)\Omega_1}{2\kappa}r_h{}^{1-\theta+z}. \qquad (73)$$

From all these expressions, one can easily check the validity of the first law while the effective spatial dimensionality is given by

$$d_{\mathrm{eff}} = 1 - \theta, \qquad (74)$$

and the Smarr formula (62) is realized with a constant α chosen as

$$\alpha = \frac{z-\theta}{z+1-\theta}. \qquad (75)$$

On the other hand, the soliton counterpart for the hyperscaling violation metric (64) with the metric function (65a), obtained through a double Wick rotation, has the following form,

$$ds^2 = \frac{1}{r^{2\theta}}\left[-r^{2z}dt^2 + \frac{dr^2}{r^2h(r)} + r^2h(r)d\vec{\varphi}^2\right], \qquad (76)$$

where the metric function h is defined as

$$h(r) = 1 - \frac{m}{(2\pi\rho)^{\frac{1+z-\theta}{z}}r^{1+z-\theta}} + \frac{(m-1)}{(2\pi\rho)^{\frac{2z-2\theta}{z}}r^{2z-2\theta}}, \qquad (77)$$

with ρ being given by Eq. (69). The gauge fields read in this case

$$F_{(1)r\varphi} = i\sqrt{2(z-1)(1-\theta+z)}\mu^{\frac{\sqrt{2}}{2\sqrt{(1-\theta)(z-\theta-1)}}}r^{z-\theta}, \qquad (78)$$

$$F_{(2)r\varphi} = -i\sqrt{\frac{2(1-\theta)(m-1)}{(z-\theta-1)}}\mu^{-\frac{\sqrt{2(z-\theta-1)}}{2\sqrt{1-\theta}}}\left(\frac{1}{2\pi\rho}\right)^{\frac{z-\theta}{z}}r^{-z+\theta+1}, \qquad (79)$$

while the dilaton is given by Eq. (65d).

As before, choosing the Killing vector $\xi^t = (1,0,0)$, the variation of the Noether potential and the surface term are calculated as

$$\delta K^{rt} = -\frac{(\theta-1)(m-1)}{\kappa(2\pi\rho)^{\frac{2z-2\theta}{z}}r^{z-\theta-1}} + \frac{(\theta-1)m}{\kappa(2\pi\rho)^{\frac{1-\theta+z}{z}}},$$
$$\int_0^1 ds\Theta^r = \frac{(\theta-1)(m-1)}{\kappa(2\pi\rho)^{\frac{2z-2\theta}{z}}r^{z-\theta-1}} + \frac{(z+2-2\theta)m}{2\kappa(2\pi\rho)^{\frac{1-\theta+z}{z}}},$$

which, in turn, implies that

$$\Delta_0 = \frac{zm\Omega_1}{2\kappa(2\pi\rho)^{\frac{1+z-\theta}{z}}}. \qquad (80)$$

Finally, the magnetic charge and the magnetic potential read

$$\mathcal{Q}_m = \frac{i\sqrt{2(1-\theta)(z-\theta-1)(m-1)}\mu^{\frac{\sqrt{2(z-\theta-1)}}{2\sqrt{1-\theta}}}\Omega_1}{2\kappa(2\pi\rho)^{\frac{z-\theta}{z}}},$$
$$\Phi_m = i\sqrt{\frac{2(1-\theta)(m-1)}{z-\theta-1}}\mu^{-\frac{\sqrt{2(z-\theta-1)}}{2\sqrt{1-\theta}}}\left(\frac{1}{2\pi\rho}\right)^{\frac{1}{z}}. \qquad (81)$$

Once again, it is easy to verify that the formula (61) with the parameter α given by (75) correctly fits with the Wald entropy defined in (67).

IV. CASE OF ADS CHARGED BLACK HOLES

We now consider the isotropic AdS case which corresponds to a dynamical exponent $z = 1$. There exist examples of electrically charged AdS black holes in three dimensions, and the most popular one is the charged Banados-Teitelboim-Zanelli (BTZ) solution [32]. Unfortunately in this case, because of the logarithmic behavior of the Maxwell electric gauge field, there is not such a Smarr formula (7) encoding the charged BTZ solution. As a direct consequence, the Cardy formula given in (8) is no longer valid in such a situation. Nevertheless, as

shown in Refs. [33,34], considering instead a nonlinear version of the Maxwell action (as the one used in Sec. II. B) and eventually a scalar field nonminimally and conformally coupled, there exists an electrically AdS charged black hole such that the electric gauge field $A_t(r)$ exhibits a Coulombian behavior, that is $A_t(r) \sim r^{-1}$. In what follows, we shall consider such a particular solution that satisfies a Smarr relation (7) and show again that the Cardy formula (8) will reproduce the correct value of the entropy.

We deal with the Lagrangian reported in Ref. [34],

$$\mathcal{L} = \frac{R + 2l^{-2}}{2\kappa} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{16}R\phi^2 - \lambda\phi^6 + \sigma(-F_{\mu\nu}F^{\mu\nu})^{3/4}, \tag{82}$$

where the matter part of the action (the scalar field and the nonlinear electromagnetic action) is chosen such that it enjoys the conformal invariance. The solution we consider for testing the Cardy formula (8) is given by the simplest one found in Refs. [34],

$$f(r) = 1 + \frac{24\lambda b^2 l^2}{r^2}, \quad \phi(r) = \sqrt{\frac{b}{r}}, \quad F_{rt} = \frac{q}{r^2}, \tag{83}$$

where $f(r)$ is the metric function of the line element (14) with $z = 1$ and $|q|^{3/2} = -\lambda b^3$. The constant b is strictly positive, while λ is negative. In this particular case, the quantities of interest read [34]

$$\Delta = \frac{\pi r_h^2}{\kappa l^2}, \quad T = \frac{r_h}{2\pi l^2}, \quad \mathcal{S}_W = \left(1 - \frac{\kappa}{8l\sqrt{-24\lambda}}\right)\frac{4\pi^2 r_h}{\kappa},$$

$$\mathcal{Q}_e = \frac{6\pi(-\lambda)^{1/3}}{\sqrt{-24\lambda}l}r_h, \quad \Phi_e = \frac{r_h}{24l^2(-\lambda)^{1/3}}, \tag{84}$$

where the location of the horizon r_h is defined by $r_h^2 = -24\lambda b^2 l^2$ and for simplicity we have assumed that $q > 0$. Having in hands all these quantities, one easily verifies that a Smarr relation (7) is satisfied with $\alpha = 1/2$.

The corresponding soliton solution is described by

$$g(\bar{r}) = 1 - \frac{l^2}{\bar{r}^2}, \quad \phi(\bar{r}) = \left(\frac{1}{24(-\lambda)}\right)^{1/4}\sqrt{\frac{1}{\bar{r}}},$$

$$F_{\bar{r}\bar{\varphi}}(\bar{r}) = \frac{i}{24(-\lambda)^{1/3}\bar{r}^2}, \tag{85}$$

where $g(\bar{r})$ is the metric function of (18) with $z = 1$. We may note that, as said before, the double Wick rotation does not yield to a complex scalar field for the soliton solution.

Along the same lines as before, the surface term together with the variation of the Noether potential are given by

$$\int_0^1 ds\Theta^{\bar{r}} = \frac{\sqrt{6}\bar{r}}{48l^3\sqrt{-\lambda}} - \frac{\sqrt{6}}{48l\sqrt{-\lambda}\bar{r}} + \frac{3}{2\kappa l},$$

$$\delta K^{\bar{r}\bar{t}} = -\frac{\sqrt{6}\bar{r}}{48l^3\sqrt{-\lambda}} + \frac{\sqrt{6}}{48l\sqrt{-\lambda}\bar{r}} - \frac{1}{\kappa l}, \tag{86}$$

yielding to

$$\Delta_0 = \frac{1}{2\kappa l}\Omega_1 = \frac{\pi}{\kappa}.$$

Finally, the magnetic charge and magnetic potential read

$$\mathcal{Q}_m = \frac{i\sqrt{6}\pi}{2(-\lambda)^{1/6}}, \quad \Phi_m = \frac{i}{24l(-\lambda)^{1/3}}. \tag{87}$$

Hence, as in the anisotropic case, the charged version of the Cardy formula (8) with $\alpha = 1/2$ and $z = 1$ gives the correct value of the entropy (84).

V. CONCLUDING REMARKS

Our starting point was the observation that in the case of Lifshitz black holes of which the only charge is the mass, the general asymptotic formula for the asymptotic growth of number of states derived in Ref. [17] naturally implies the emergence of a Smarr formula given by (4) in $D = 3$. In our search of generalizing the Cardy formula to the case of electrically charged Lifshitz black holes, we have proposed a formula compatible with a charged version of the Smarr formula of the form (7). We have tested the viability of this formula in three different examples where charged Lifshitz black holes obeying a Smarr relation were known. We have extended our analysis to the other class of charged black hole solutions with anisotropic symmetry, namely those exhibiting a hyperscaling violation.

In the case of the isotropic charged AdS black holes, we have shown that the absence of a Smarr relation for the charged BTZ solution renders our formula (8) inappropriate. The absence of a Smarr relation of the form (7) is mainly due to the logarithmic behavior of the Maxwell gauge field. It seems that in this case, the appropriate formula should be the Cardy-Verlinde formula [35] where the Smarr relation is augmented by a pressure term; see Ref. [36]. Nevertheless, charged AdS black holes have a U(1)-corrected Cardy formula which accounts for their entropy; see e.g. Ref. [37].

Nevertheless, replacing the standard Maxwell theory by its nonlinear and conformal generalization, asymptotically charged AdS black holes are known with a gauge field behaving as a Coulomb one. In a simple example of such a solution given in Ref. [34], we have again tested the viability of the Cardy formula after ensuring that this Coulombian solution was as well satisfying a Smarr relation.

As a natural extension of this work, it will be desirable to test this formula in many more examples, particulary those involving higher-order gravity theories in three dimensions. This task can be interesting by itself in the hyperscaling violation case, since as shown in Ref. [30], the spatial effective dimensionality d_{eff} may vary in function of the order of the gravity theories involved in the action.

Also there exists a generalization of the Smarr relation in the case of AdS black holes for which the cosmological constant is viewed as a dynamical variable. In a very recent paper, the authors of Ref. [38] showed that such a generalization of the Smarr relation can be understood from a dual holographic point of view. Extension to the Lifshitz case can also be an interesting work to deal with.

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