

Growth index of matter perturbations in running vacuum modelsSpyros Basilakos^{1,*} and Joan Solà^{2,†}¹*Academy of Athens, Research Center for Astronomy and Applied Mathematics,
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We derive for the first time the growth index of matter perturbations of the Friedmann-Lemaître-Robertson-Walker (FLRW) flat cosmological models in which the vacuum energy depends on redshift. A particularly well-motivated model of this type is the so-called quantum field vacuum, in which, apart from a leading constant term Λ_0 , there is also an H^2 dependence in the functional form of the vacuum, namely, $\Lambda(H) = \Lambda_0 + 3\nu(H^2 - H_0^2)$. Since $|\nu| \ll 1$, this form endows the vacuum energy of a mild dynamics which affects the evolution of the main cosmological observables at the background and perturbation levels. Specifically, at the perturbation level, we find that the growth index of the running vacuum cosmological model is $\gamma_{\Lambda_H} \approx \frac{6+3\nu}{11-12\nu}$, and thus it nicely extends analytically the result of the Λ CDM model, $\gamma_\Lambda \approx 6/11$.

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I. INTRODUCTION

Over the last two decades, studies in cosmology strongly indicate that we are living in a spatially flat Universe that contains $\sim 4\%$ baryonic matter, $\sim 26\%$ dark matter, and $\sim 70\%$ of some sort of dark energy (DE) endowed with large negative pressure. The DE component plays a vital role in the cosmic history because it provides the necessary theoretical platform toward describing the accelerated expansion of the Universe (see Refs. [1,2] and references therein).

Although there is a general agreement concerning the main ingredients of the Universe, there are different proposals regarding the underlying physics which triggers the late cosmic acceleration. In fact, the unknown nature of DE, which challenges the foundations of theoretical physics, has given rise to a plethora of cosmological models. The majority of such scenarios is based either on the existence of new fields in nature or in some modification of the theory of gravity (for review, see Ref. [3]) with the present accelerating epoch appearing as a sort of geometric effect.

An alternative avenue that one can follow in order to explain the cosmic acceleration as well as to overcome, or at least to alleviate, the various cosmological puzzles is to consider a running vacuum $\Lambda(H)$. The idea was proposed some time ago [4–8], and it might be the seed of future important developments of cosmology at a more fundamental level—see, e.g., Refs. [9,10] for a review. This point of view is quite general and can be applied to the entire history of the Universe, including inflation and the “graceful exit” problem in the early Universe [11]. In this

context, we do not need to introduce new fields in the analysis nor modify the theory of standard gravity [General Relativity (GR)]. In this cosmic ideology, the DE equation of state parameter $w \equiv P_{\text{DE}}/\rho_{\text{DE}}$ is by definition identical to -1 , but the vacuum energy density is a function of time. Notice that there is an extensive body of old literature on purely phenomenological models in which the cosmological term is a function of time [12–17] and a more recent series of works in which there is an attempt to connect the time evolution with fundamental aspects of quantum field theory (QFT) in curved spacetime, which include the aforementioned references on running vacuum and others such as Refs. [18–26]. In this extensive body of literature, the time-evolving vacuum has been phenomenologically explored as a function of time in various possible ways, and in the more formal QFT approach, it has been mainly investigated as a possible function of the Hubble parameter. The latter is the basis for what we will refer to as the quantum field vacuum model, in which $\Lambda = \Lambda(H)$ and on which we shall mainly concentrate here.

To test the above cosmological possibilities, it has been proposed that the so-called growth index, γ , of matter perturbations [27] can be used as an observational tool to discriminate between modified gravity models and scalar field DE models which obey general relativity. Nowadays, the accurate estimation of γ is considered one of the most basic tasks in cosmological studies. Not surprisingly, it has become traditional to study, for each proposed cosmological model, its background expansion as well as the growth index of matter perturbations, as in this way one may get an impression of the main cosmological and astrophysical consequences of the model. For example, it has been found that for those DE models based on GR and characterized by

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a constant equation of state parameter, the asymptotic value of the growth index is $\gamma \approx \frac{3(w-1)}{6w-5}$ [28–31]. Obviously, for the concordance Λ CDM model ($w = -1$), we recover the nominal value, namely, $\gamma \approx 6/11$. As far as the modified gravity models are concerned, the situation is as follows. In the case of the braneworld gravity of Ref. [32], we have $\gamma \approx 11/16$ (see Refs. [30,33–35]); for some $f(R)$ gravity models, it has been found that $\gamma \approx 0.415 - 0.21z$ for various parameter values (see Refs. [36,37]); and finally for the Finsler-Randers cosmological model, Basilakos and Stavrinou [38] have shown that $\gamma \approx 9/14$.

In this work, we wish to investigate the growth index of the running vacuum model $\Lambda(H) = \Lambda_0 + 3\nu(H^2 - H_0^2)$. The denomination of running is related to the fact that it can be motivated within the context of QFT in curved spacetime and specifically from the point of view of the renormalization group approach [4–8]; see Refs. [10] and [9] for recent reviews and references therein. This running vacuum model generalizes the traditional Λ CDM model at the background level and can be put to the test. Currently, the value of the dimensionless free parameter ν is observationally allowed to be in the ballpark of $|\nu| \sim \mathcal{O}(10^{-3})$ [18–21,23–26]. From its theoretical interpretation [namely, as being the coefficient of the β function of the running $\Lambda(H)$], it is a natural value, which in addition fits in with the existing theoretical estimates [7,9].

To the best of our knowledge, we are unaware of any previous analysis of this kind applied to dynamical vacuum models, and for this reason we consider that it can be of theoretical interest, and maybe we can also extract some practical consequences.

The structure of the article is as follows. Initially in Sec. II, we briefly present the background cosmological equations. The basic theoretical elements of the linear growth are discussed in Sec. III, while in Sec. IV, we provide the growth index analysis in the case of the running vacuum. In Sec. IV, we compare different $\gamma(z)$ parametrizations, and, finally, in Sec. V, we provide some discussion and finish with our main conclusions.

II. BACKGROUND EVOLUTION

The physics of the dynamical vacuum model under consideration is based on the renormalization group (RG) in QFT according to the aforementioned references. Within this framework, the evolution of the vacuum in the current epoch is given by

$$\Lambda(H) = \Lambda_0 + 3\nu(H^2 - H_0^2), \quad (2.1)$$

where $\Lambda_0 \equiv \Lambda(H_0) = 3\Omega_{\Lambda_0}H_0^2$ and ν is provided in the RG context as a β -function which determines the running of the cosmological ‘constant’ within QFT in curved spacetime [9]. The value of ν is estimated through the upper bound $|\nu| \lesssim 1/(12\pi) \approx 2.6 \times 10^{-2}$, which is approximate and is

valid as an order of magnitude. It ensues from assuming that the effective masses of the heaviest particles involved in the loops for calculating the β function is of the order of the Planck mass [4,18,19], but in general it is not completely fixed. If there is a large multiplicity of heavy particles at a grand unified scale below the Planck mass, it could as well stay of the order 10^{-2} . Notice that the observational bound on ν depends on the particular implementation of the model (2.1), namely, on whether, e.g., the vacuum exchanges energy with matter or not. In the simplest cases, the natural theoretical estimate yields $\nu = 10^{-5} - 10^{-3}$ (see Solà’s Ref. [7] for details), and in these cases, ν is generally significantly smaller than the original upper bound. At the same time, using cosmological data, it has been found that $|\nu| = \mathcal{O}(10^{-3})$ [20,21,23–26], which is in agreement with the aforementioned theoretical expectations. Let us, however, point out that in models in which matter and DE are self-conserved, the observational limits are weaker, and they tolerate the order of magnitude $\nu \sim 10^{-2}$ from the original estimate, as shown in the recent work [39].

Dynamically speaking, since $|\nu| \ll 1$ in all theoretical implementations, it is easy to check that prior to the present epoch the low-energy behavior of the model tends to the usual Λ CDM model, but it is by no means identical (for a recent review, see Refs. [10,25] and references therein).

Considering Eq. (2.1), let us now focus on the derivation of the Friedmann equations. Such a procedure is perfectly allowed by the cosmological principle embedded in the FLRW metric. Namely, the Λ term may perfectly evolve with the cosmic expansion, meaning that ultimately evolves with the cosmic time, t , but in general it depends on an intermediate variable, which in our case is the Hubble function, $H = \dot{a}/a$, where $a(t)$ is the scale factor and the overdot denotes a derivative with respect to t . The corresponding generalization of the Friedmann equations reads

$$8\pi G\rho_{\text{tot}} \equiv 8\pi G\rho_m + \Lambda(H) = 3H^2, \quad (2.2)$$

$$8\pi GP_{\text{tot}} \equiv 8\pi GP_m - \Lambda(H) = -2\dot{H} - 3H^2, \quad (2.3)$$

where the total energy density is $\rho_{\text{tot}} = \rho_m + \rho_\Lambda$ (with $\rho_\Lambda = \Lambda/8\pi G$ the vacuum component of it) and $P_{\text{tot}} = P_m + P_\Lambda$ is the total pressure. For the matter-dominated epoch, and of course also in our days, $(P_m, P_\Lambda) = (0, -\rho_\Lambda)$, where the dynamical character of the vacuum does not alter the usual equation of state that it satisfies. This is an important point to remark. In fact, this observation explains why the dynamics of the vacuum entails a corresponding modification of the local energy conservation for matter at fixed G . The outcome is that matter must exchange energy with the vacuum in order to fulfill the Bianchi identity, and this translates into the following generalized conservation law involving both matter and vacuum energy densities:

$$\dot{\rho}_m + 3H\rho_m = -\dot{\rho}_\Lambda. \quad (2.4)$$

This equation is actually not independent of (2.2) and (2.5), and therefore, using any two of them, it is easy to derive the equation of motion for the Hubble rate:

$$\dot{H} + \frac{3}{2}H^2 = 4\pi G\rho_\Lambda = \frac{\Lambda(H)}{2}. \quad (2.5)$$

From (2.5) and the vacuum model equation (2.1), we are able to determine the explicit form of H as a function of time,

$$H(t) = H_0 \sqrt{\frac{\Omega_{\Lambda 0} - \nu}{1 - \nu}} \coth \left[\frac{3}{2} H_0 \sqrt{(\Omega_{\Lambda 0} - \nu)(1 - \nu)} t \right], \quad (2.6)$$

where $\Omega_{\Lambda 0} = 1 - \Omega_{m 0}$ and H_0 is the Hubble constant. Utilizing $H = \dot{a}/a$, the cosmic time, $t(a)$, follows:

$$t(a) = \frac{2}{3H_0} \frac{\sinh^{-1} \left(\sqrt{\frac{\Omega_{\Lambda 0} - \nu}{\Omega_{m 0}}} a^{3(1-\nu)/2} \right)}{\sqrt{(\Omega_{\Lambda 0} - \nu)(1 - \nu)}}. \quad (2.7)$$

Inverting Eq. (2.7), we easily determine the scale factor $a = a(t)$. Therefore, inserting Eq. (2.7) into Eq. (2.6), we arrive at

$$E^2(a) \equiv \frac{H^2(a)}{H_0^2} = \tilde{\Omega}_{\Lambda 0} + \tilde{\Omega}_{m 0} a^{-3(1-\nu)}, \quad (2.8)$$

where we have rescaled

$$\tilde{\Omega}_{m 0} \equiv \frac{\Omega_{m 0}}{1 - \nu}, \quad \tilde{\Omega}_{\Lambda 0} \equiv \frac{\Omega_{\Lambda 0} - \nu}{1 - \nu}. \quad (2.9)$$

Notice that for $\nu = 0$ all the above formulas correctly reduce to the standard ones for the Λ CDM, and the rescaled parameters become the ordinary ones, $\tilde{\Omega}_{i 0} \rightarrow \Omega_{i 0}$. Moreover, whether in rescaled form or not, the cosmological parameters obey the standard cosmic sum rule, namely, $\tilde{\Omega}_{m 0} + \tilde{\Omega}_{\Lambda 0} = 1 = \Omega_{m 0} + \Omega_{\Lambda 0}$.

Regarding the matter evolution, from Eqs. (2.2) and (2.3), we find $\dot{H} = -4\pi G\rho_m$, and combining the latter with Eqs. (2.1) and (2.4), we obtain a differential equation for the matter density: $\dot{\rho}_m + 3H\rho_m = 3\nu H\rho_m$. Integrating it (using $\dot{\rho}_m = aHd\rho_m/da$), we find

$$\rho_m(a) = \rho_{m 0} a^{-3(1-\nu)}. \quad (2.10)$$

Notice that $\rho_{m 0}$ is the matter density at the present time ($a = 1$), and therefore $\Omega_{m 0} = \rho_{m 0}/\rho_{c 0}$, where $\rho_{c 0} = 3H_0^2/8\pi G$ is the current critical density. As expected, we recover the standard matter conservation law $\rho_m \sim a^{-3}$ only

for $\nu = 0$. However, thanks to $\nu \neq 0$, we can have a mild dynamical vacuum evolution; see Eq. (2.1).

Defining $\Omega_m(a) \equiv \rho_m(a)/\rho_c(a)$, it is easy to obtain, with the aid of Eqs. (2.10) and (2.8),

$$\Omega_m(a) = \frac{\Omega_{m 0} a^{-3(1-\nu)}}{E^2(a)}. \quad (2.11)$$

For convenience, we also define

$$\tilde{\Omega}_m(a) = \frac{\tilde{\Omega}_{m 0} a^{-3(1-\nu)}}{E^2(a)} = \frac{\Omega_m(a)}{1 - \nu}. \quad (2.12)$$

Differentiating Eq. (2.12) and utilizing (2.8), we find that

$$\frac{d\tilde{\Omega}_m}{d \ln a} = -3(1 - \nu)\tilde{\Omega}_m(a)[1 - \tilde{\Omega}_m(a)]. \quad (2.13)$$

Subsequently, upon substituting Eq. (2.10) into Eq. (2.4) and integrating once more in the scale factor variable, we are led to the evolution of the vacuum energy density:

$$\rho_\Lambda(a) = \rho_{\Lambda 0} + \frac{\nu\rho_{m 0}}{1 - \nu} [a^{-3(1-\nu)} - 1]. \quad (2.14)$$

Once more for $\nu = 0$, the cosmological solutions of the running vacuum model under study boil down to the concordance Λ CDM cosmology, and in this case, $\rho_\Lambda = \rho_{\Lambda 0}$ at all times.

Finally, the observational viability of the current vacuum model has been tested previously, and in the most recent analysis provided in Gómez-Valent *et al.* [23], it is found that $(\Omega_m, \nu) = (0.282 \pm 0.012, 0.0048 \pm 0.0032)$ (see also Table 1 in Ref. [25]). For the rest of the paper, we shall take this result as the basis for our estimates. Notice that, due to the rescaling [see the first equality of (2.9)], we have $\tilde{\Omega}_{m 0} = 0.283 \pm 0.012$. Recall that for the concordance Λ model, ($\nu = 0$) $\Omega_{m 0}^{(\Lambda)} = 0.291 \pm 0.011$, which is in agreement with the recent *Planck 2015* results [2].

III. LINEAR GROWTH

In this section, we concentrate on the basic linear equation that governs the evolution of the matter perturbations. Following Refs. [16,20,23], we write the following equation for the matter density contrast $D \equiv \delta\rho_m/\rho_m$:

$$\ddot{D} + (2H + Q)\dot{D} - (4\pi G\rho_m - 2HQ - \dot{Q})D = 0, \quad (3.1)$$

where $Q(t) = -\dot{\rho}_\Lambda/\rho_m$. For a formal proof of this equation in relativistic cosmology, see Refs. [22,23]. It assumes that the DE perturbations are very small and that the divergence of the perturbed matter velocity is also negligible. Obviously, the running vacuum energy still affects the growth factor through the function $Q(t)$, and therefore it affects the background evolution of the matter perturbations.

In this context, we can rewrite the homogeneous form of Eq. (3.1) in terms of the scale factor (using $d/dt = aH(a)d/da$) as follows:

$$\begin{aligned} a^2 \frac{d^2 D}{da^2} + \left(3a + a \frac{d \ln H}{d \ln a} + \frac{aQ}{H} \right) \frac{dD}{da} \\ = \left(\frac{3}{2} \Omega_m - \frac{2Q}{H} - \frac{a}{H} \frac{dQ}{da} \right) D \\ = \left[\frac{3}{2} (1 - \nu) \tilde{\Omega}_m - \frac{2Q}{H} - \frac{a}{H} \frac{dQ}{da} \right] D, \end{aligned} \quad (3.2)$$

where

$$\frac{d \ln H}{d \ln a} = \frac{d \ln E}{d \ln a} = -\frac{3}{2} (1 - \nu) \tilde{\Omega}_m(a), \quad (3.3)$$

$$\frac{Q(a)}{H(a)} = -\frac{\dot{\rho}_\Lambda(a)}{H(a)\rho_m(a)} = 3\nu \quad (3.4)$$

and

$$\frac{a}{H(a)} \frac{dQ}{da} = -\frac{9}{2} \nu (1 - \nu) \tilde{\Omega}_m(a). \quad (3.5)$$

Notice, that, in order to derive the above expressions, we have utilized Eqs. (2.8)–(2.14). The growing mode solution of Eq. (3.2) is written as (for more details, see Ref. [20])

$$D(a) = C_1 a^{\frac{9\xi-4}{2}} E(a) F\left(\frac{1}{3\xi} + \frac{1}{2}, \frac{3}{2}, \frac{1}{3\xi} + \frac{3}{2}; -\frac{\tilde{\Omega}_{\Lambda 0}}{\tilde{\Omega}_{m 0}} a^{3\xi}\right), \quad (3.6)$$

where $\xi = 1 - \nu$, C_1 is an integration constant to be constrained by an initial condition and F is the hypergeometric function [40].

Of course for the concordance Λ CDM model [$\nu = 0$ or $Q(t) = 0$], Eq. (3.2) reduces to the standard perturbation equation, a solution of which is (see Refs. [27,41])

$$D_\Lambda(a) = \frac{5\Omega_{m0}E(a)}{2} \int_0^a \frac{dx}{x^3 E^3(x)}. \quad (3.7)$$

IV. GROWTH INDEX

For any type of dark energy, a useful parametrization of the matter perturbations is based on the growth rate of clustering [27]. In our framework, the natural parametrization is¹

¹In the running vacuum model (2.1), one may check from (2.11) that, at large redshifts $z \gg 1$, $\Omega_m(z) \sim 1 - \nu$. Therefore, if the growth rate of clustering is modeled as a power law, then it is more appropriate to use $f(a) \simeq \tilde{\Omega}_m(z)^\gamma$, because for $z \gg 1$ we achieve the correct normalization $\tilde{\Omega}_m(z) \sim 1$.

$$f(a) = \frac{d \ln D}{d \ln a} \simeq \tilde{\Omega}_m^\gamma(a), \quad (4.1)$$

with $\tilde{\Omega}_m(a)$ defined in Eq. (2.12). The exponent γ is the so-called growth index (see Refs. [28–31,41,42]), and it plays an important role in cosmological studies as we discussed in the Introduction. Inserting the first equality of (4.1) into Eq. (3.2), we derive after some calculations

$$\begin{aligned} \frac{df}{d \ln a} + \left(2 + \frac{Q}{H} + \frac{d \ln H}{d \ln a} \right) f + f^2 \\ = \frac{3(1 - \nu) \tilde{\Omega}_m}{2} - \frac{2Q}{H} - \frac{a}{H} \frac{dQ}{da} \end{aligned} \quad (4.2)$$

or

$$\begin{aligned} \frac{df}{d \ln a} + \left[\frac{1}{2} + \frac{3}{2} (1 - \nu) \tilde{\Omega}_\Lambda + \frac{9\nu}{2} \right] f + f^2 \\ = \frac{3(1 - \nu) \tilde{\Omega}_m}{2} - 6\nu + \frac{9\nu(1 - \nu)}{2} \tilde{\Omega}_m, \end{aligned} \quad (4.3)$$

where $\tilde{\Omega}_\Lambda(a) \equiv 1 - \tilde{\Omega}_m(a)$. At this point, it is interesting to mention that there have been many theoretical speculations concerning the functional form of the growth index, and indeed various candidates have been proposed in the literature. In this work, we phenomenologically parametrize $\gamma(a)$ by the following general relation [43]:

$$\gamma(a) = \gamma_0 + \gamma_1 y(a). \quad (4.4)$$

In other words, Eq. (4.4) can be seen as a first-order Taylor expansion around some cosmological function $y(a)$. The following options have been considered in the literature: $\omega(a) = \ln \tilde{\Omega}_m(a)$ (hereafter Γ_0 parametrization: Ref. [44]), $a(z)$ (hereafter Γ_1 parametrization: Ref. [45]), and z (hereafter Γ_2 parametrization: Ref. [46]). Below, we briefly present various forms of $\gamma(a)$ for the running vacuum model in these various parametrizations.

A. Γ_0 parametrization

In this parametrization, we use $y(a) = \omega = \ln \tilde{\Omega}_m(a)$. Obviously, for $z \gg 1$, namely, $\tilde{\Omega}_m(a) \rightarrow 1$ (or $\omega \rightarrow 0$), the asymptotic value of the growth index becomes $\gamma_\infty \approx \gamma_0$. Steigerwald *et al.* [44] proposed a general mathematical treatment [see their Eqs. (5)–(12)]² which provides

²The methodology of Steigerwald *et al.* [44] can be applied to the framework of $\gamma(a) = \sum_{n=0}^N \gamma_n \frac{\omega^n(a)}{n!}$. However, for the purpose of our study, we restrict our analysis to $N = 1$. We would like to point that Eqs. (3) and (5) of Steigerwald *et al.* [44] have a typo. Indeed, one has to replace there the quantity $1 + \nu_H$ with $2\nu_H$. Notice, however, that this typo does not alter our results because in our case the coefficient ν_H is a constant [see Eq. (4.12)] which implies that $N_n = 0$; see Eq. (4.11).

compact analytic formulas for the coefficients γ_0 and γ_1 . These authors start from the fact that for a large family of dark energy models (including those of modified gravity) the linear differential equation of the matter perturbations takes on the form

$$\ddot{D} + 2\nu_H H \dot{D} - 4\pi G \mu \rho_m D = 0. \quad (4.5)$$

Naturally, any modification to the Friedmann equation and to the theory of gravity is included in the quantities ν_H and μ . For the nominal scalar field dark energy which adheres to general relativity, we have $\nu_H = \mu = 1$, while for modified gravity models, we get $\nu_H = 1$ and $\mu \neq 1$. Notice that for the latter cases we have $\tilde{\Omega}_m(a) \equiv \Omega_m(a)$ by definition. In the following, we will determine the precise relation between the generic coefficients (ν_H, μ) and our vacuum parameter ν .

To start with, we expect that if we allow interactions in the dark sector, then, in general, the cosmological quantities (ν_H, μ) are different from unity. Furthermore, if the matter component evolves differently from the usual power law a^{-3} , one may expect that the quantity $\Omega_m(a) = \rho_m(a)/\rho_c(a)$ is slightly different from $\tilde{\Omega}_m(a)$ as defined in Eq. (2.12). For example, in our case, due to Eqs. (2.12) and the first equality of (4.1), one can write Eq. (4.5) as follows:

$$\frac{df}{d \ln a} + \left(2\nu_H + \frac{d \ln H}{d \ln a}\right) f + f^2 = \frac{3\tilde{\mu}\tilde{\Omega}_m}{2}, \quad (4.6)$$

where $\tilde{\mu} \equiv \mu(1 - \nu)$. Similar to Ref. [44], let us transform Eq. (4.6) as

$$\frac{d\omega}{d \ln a} \left(\gamma + \omega \frac{d\gamma}{d\omega} \right) + e^{\omega\gamma} + 2\nu_H + \frac{d \ln H}{d \ln a} - \frac{3}{2} \tilde{\mu} e^{\omega(1-\gamma)} = 0, \quad (4.7)$$

where we have set $\omega = \ln \tilde{\Omega}_m(a)$. Within the mathematical framework of Ref. [44], one finds

$$\gamma_0 = \frac{3(M_0 + M_1) - 2(\mathcal{H}_1 + N_1)}{2 + 2X_1 + 3M_0} \quad (4.8)$$

and

$$\gamma_1 = 3 \frac{M_2 + 2M_1 B_1(1 - y_1) + M_0 B_2(1 - y_1, -y_2)}{2(2 + 4X_1 + 3M_0)} - 2 \frac{B_2(y_1, y_2) + X_2 \gamma_0 + \mathcal{H}_2 + N_2}{2(2 + 4X_1 + 3M_0)}. \quad (4.9)$$

The following quantities have been defined:

$$X_n = \frac{d^n(d\omega/d \ln a)}{d\omega^n} \Big|_{\omega=0}, \quad M_n = \frac{d^n \tilde{\mu}}{d\omega^n} \Big|_{\omega=0} \quad (4.10)$$

and

$$N_n = \frac{d^n \nu_H}{d\omega^n} \Big|_{\omega=0}, \quad \mathcal{H}_n = \frac{d^n(d \ln H/d \ln a)}{d\omega^n} \Big|_{\omega=0}, \quad (4.11)$$

with $\frac{d^0}{d\omega^0} \equiv 1$. Also, $B_{1,2}$ are the Bell polynomials of first and second kinds, namely, $B_1(y_1) = y_1$ and $B_2(y_1, y_2) = y_1^2 + y_2$. According to Steigward *et al.* [44] [see their Eq. (10)], the pair (y_1, y_2) is equal to $(\gamma_0, 0)$, implying $B_1(1 - y_1) = 1 - \gamma_0$, $B_2(y_1, y_2) = \gamma_0^2$, and $B_2(1 - y_1, -y_2) = (\gamma_0 - 1)^2$.

For the current running vacuum model, the quantities ν_H and $\tilde{\mu}$ can be easily identified on comparing Eq. (4.2) with Eq. (4.6), and we find

$$\nu_H = 1 + \frac{Q}{2H} = 1 + \frac{3}{2}\nu \quad (4.12)$$

and

$$\begin{aligned} \tilde{\mu} &= 1 - \nu - \frac{4Q}{3\tilde{\Omega}_m H} - \frac{2a}{3\tilde{\Omega}_m H} \frac{dQ}{da} \\ &= 1 - \nu - \frac{4\nu}{\tilde{\Omega}_m} + 3\nu(1 - \nu), \end{aligned} \quad (4.13)$$

where in the derivation of the second equalities we have used Eqs. (3.4) and (3.5). Then, based on Eqs. (2.13), (3.3), (4.12), and (4.13), we obtain

$$\begin{aligned} \{M_0, M_1, M_2, N_1, N_2\} &= \{1 - 2\nu - 3\nu^2, 4\nu, -4\nu, 0, 0\} \\ \{X_1, \mathcal{H}_1\} = \{X_2, \mathcal{H}_2\} &= \left\{ 3(1 - \nu), -\frac{3(1 - \nu)}{2} \right\}, \end{aligned}$$

and thus

$$\gamma_0 = \frac{6 + 3\nu(1 - 3\nu)}{11 - 3\nu(4 + 3\nu)} \quad (4.14)$$

$$\begin{aligned} \gamma_1 &= 3 \frac{-4\nu + 8\nu(1 - \gamma_0) + (1 - 2\nu - 3\nu^2)(\gamma_0 - 1)^2}{2[2 + 12(1 - \nu) + 3(1 - 2\nu - 3\nu^2)]} \\ &\quad - \frac{2\gamma_0^2 + 6(1 - \nu)\gamma_0 - 3(1 - \nu)}{2[2 + 12(1 - \nu) + 3(1 - 2\nu - 3\nu^2)]}. \end{aligned} \quad (4.15)$$

If we take the aforementioned fitting value $\nu = 0.0048$ from Ref. [23], we find $(\gamma_0, \gamma_1) = (0.5496, -0.009)$. Inserting $\nu = 0$ into Eqs. (4.14) and (4.15) we recover the Λ CDM pair $(\gamma_0, \gamma_1) = (\frac{6}{11}, -0.00729)$ as it should [44]. Lastly, since ν is of the order of $\mathcal{O}(10^{-3})$ [20,21,23], it is safe to neglect high-order terms of ν from Eq. (4.14). In that case, the asymptotic value of the growth index becomes

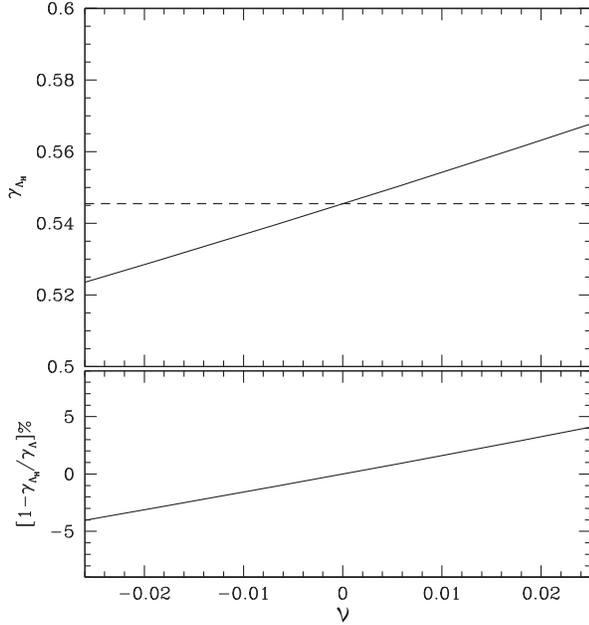


FIG. 1. In the upper panel, we present the asymptotic value of the growth index as a function of ν (solid line). Notice that the dashed curve corresponds to $\gamma_{\Lambda} \approx 6/11$. In the lower panel, we show the relative deviation $[1 - \gamma_{\Lambda_H}/\gamma_{\Lambda}]%$.

$$\gamma_{\infty} \approx \gamma_0 = \frac{6 + 3\nu}{11 - 12\nu} \approx \frac{6}{11} \left(1 + \frac{35}{22}\nu \right), \quad (4.16)$$

where $\gamma_{\infty} \approx [\gamma(a)]_{\tilde{\Omega}_m=1}$.

In the upper panel of Fig. 1, we show the asymptotic value of the growth index as a function of ν (solid curve), where the ν parameter lies in the theoretical interval $[-\frac{1}{12\pi}, \frac{1}{12\pi}]$, which implies that $\gamma_{\Lambda_H} \in [0.5235, 0.5678]$. In the lower panel of Fig. 1, we present the relative deviation $[1 - \gamma_{\Lambda_H}/\gamma_{\Lambda}]%$ of the growth index with respect to $\gamma_{\Lambda} \approx 6/11$. We observe that for negative values of ν the asymptotic value of the growth index becomes less than $6/11$ (the opposite holds for positive values). This deviation can reach $\sim \pm 5%$ when we attain the aforementioned theoretical upper bound of $\nu = \pm 1/12\pi \approx \pm 0.026$. These features can be easily understood from the approximate formula on the rhs of Eq. (4.16). As advanced in Sec. 2, there are DE models based also on Eq. (2.1) and generalizations there (cf. Ref. [39]) in which the observational limits lie at the border of the upper theoretical bound, so in general we can say that $\sim 5%$ corrections to γ are conceivable and could perhaps be within reach in the future. For the cases in which Eq. (2.1) represents a dynamical vacuum model in interaction with matter, however, the corrections are smaller. Using the latest observed value of $\nu = 0.0048 \pm 0.0032$ provided by Gómez-Valent *et al.* [23] (see also Table 1 in Ref. [25]), we find $\gamma_{\Lambda_H} = 0.5496 \pm 0.0028$, to be compared with $\gamma_{\Lambda} \approx 0.5454$ for the concordance model, hence a $\lesssim 1%$ correction.

B. $\Gamma_{1,2}$ parametrizations

Here, we generalize the original Polarski and Gannouji [46] work. In particular, changing the variables in Eq. (4.3) from $a(z)$ to redshift $[\frac{df}{da} = -(1+z)^{-2} \frac{df}{dz}]$ and utilizing $f(z) = \tilde{\Omega}_m(z)^{\gamma(z)}$, we find

$$\begin{aligned} & -\tilde{\Omega}_m^{\gamma} [(1+z)\gamma' \ln(\tilde{\Omega}_m) + 3\gamma(1-\nu)\tilde{\Omega}_{\Lambda}] \\ & + \left[\frac{1}{2} + \frac{3}{2}(1-\nu)\tilde{\Omega}_{\Lambda} + \frac{9\nu}{2} \right] \tilde{\Omega}_m^{\gamma} + \tilde{\Omega}_m^{2\gamma} \\ & = \frac{3(1-\nu)\tilde{\Omega}_m}{2} - 6\nu + \frac{9\nu(1-\nu)}{2} \tilde{\Omega}_m, \end{aligned} \quad (4.17)$$

where prime denotes a derivative with respect to the redshift. For those $y(z)$ functions which satisfy the restriction $y(z=0) = 0$ [or $\gamma(z=0) = \gamma_0$],³ we obtain the parameter γ_1 in terms of γ_0 , Ω_{m0} , and ν . Specifically, if we substitute $z=0$ and $\gamma'(0) = \gamma_1 y'(0)$ in Eq. (4.17), then we have

$$\gamma_1 = \frac{\tilde{\Omega}_{m0}^{\gamma_0} - 3(1-\nu)(\gamma_0 - \frac{1}{2})\tilde{\Omega}_{\Lambda 0} + \frac{1}{2} - \frac{3}{2}(1-\nu)\tilde{\Omega}_{m0}^{1-\gamma_0} + \Psi_0}{y'(0) \ln \tilde{\Omega}_{m0}}, \quad (4.18)$$

where

$$\Psi_0 = \frac{9\nu}{2} - \frac{9\nu(1-\nu)\tilde{\Omega}_{m0}^{1-\gamma_0}}{2} + 6\nu\tilde{\Omega}_{m0}^{1-\gamma_0}.$$

Clearly, in the case of the usual Λ cosmology ($\nu = 0$, $\Psi_0 = 0$), the above formula boils down to that of Polarski and Gannouji [46] for $y(z) = z$ (Γ_2 parametrization). Furthermore, based on the Λ CDM cosmological model and for $y(z) = 1 - a(z) = \frac{z}{1+z}$ (Γ_1 parametrization), we also confirm the literature results (see Refs. [47–49]). Now, due to the fact that the function $y(z) = z$ goes to infinity at large redshifts, for the rest of the paper, we focus on the Γ_1 parametrization, which implies $y'(0) = 1$. In this case one can easily see that $\gamma_{\infty} \approx \gamma_0 + \gamma_1$ as long as $z \gg 1$. Therefore, plugging $\gamma_0 = \gamma_{\infty} - \gamma_1$ into Eq. (4.18) and using at the same time $\gamma_{\infty} \approx \frac{6+3\nu}{11-12\nu}$, we can derive the constants $\gamma_{0,1}$ in terms of Ω_{m0} and ν . For example, in the case of $(\Omega_{m0}, \nu) = (0.283, 0.0048)$, we find $(\gamma_0, \gamma_1) = (0.5636, -0.140)$, while for the concordance Λ cosmological model, we get $(\gamma_0, \gamma_1) = (0.5565, -0.110)$ for $(\Omega_{m0}, \nu) = (0.291, 0)$.

³For $\Gamma_{1,2}$ parametrizations $\gamma_0 = \gamma(z=0)$ is different with that of Steigerwald *et al.* [44], namely $\gamma_0 = [\gamma(a)]_{\tilde{\Omega}_m=1} \approx \gamma_{\infty}$ (see section 3A).

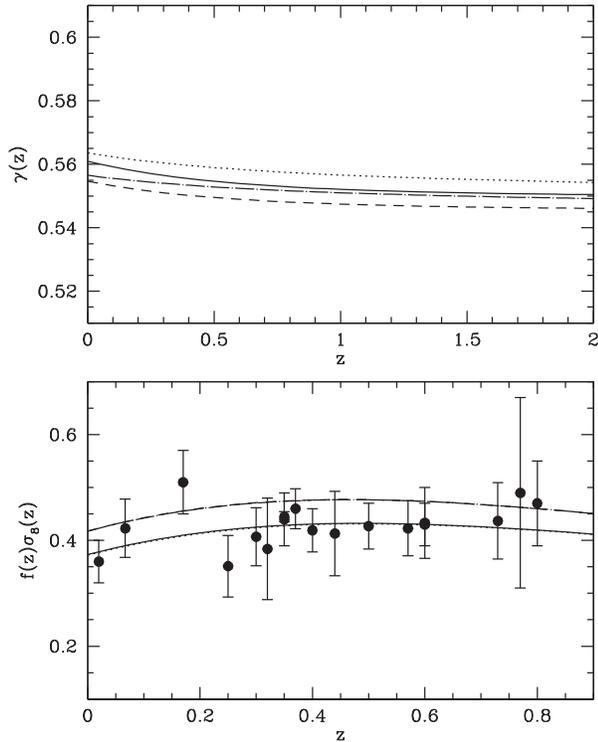


FIG. 2. *Upper Panel:* The evolution of the growth index of matter perturbations. The curves are as follows. The results for $\Lambda(H)$ in the $\Gamma_{0,1}$ parametrizations are given by the solid and dotted lines. Moreover, the dashed and the dot-dashed curves correspond to the Λ CDM in the $\Gamma_{0,1}$ parametrizations respectively. *Lower Panel:* Comparison of the observed and theoretical evolution of the growth rate $f(z)\sigma_8(z)$. The growth data can be found in Ref. [50] (see their Table 1). For the Λ CDM, we use $\sigma_8 = 0.829$, while for the running vacuum model, we have $\sigma_8 = 0.758$ (see Refs. [23,25]).

V. DISCUSSION AND CONCLUSIONS

In the upper panel of Fig. 2, we present the evolution of the growth index for the running vacuum model $\Lambda(H)$ in the Γ_0 parametrization (solid line) and in the Γ_1 parametrization (dotted line). In the same figure, the dashed and the dot-dashed curves correspond to the Λ CDM in the Γ_0 and Γ_1 parametrizations, respectively. The comparison indicates that the growth index of the $\Lambda(H)$ and Λ CDM cosmological models is well approximated by the Γ_0 and Γ_1 parametrizations. Specifically, we find that the corresponding relative deviations are $[1 - \gamma_{\Lambda_H}^{(\Gamma_1)} / \gamma_{\Lambda_H}^{(\Gamma_0)}] \sim 0.7\%$ and $[1 - \gamma_{\Lambda}^{(\Gamma_1)} / \gamma_{\Lambda}^{(\Gamma_0)}] \sim 0.6\%$, which means that essentially both parametrizations are equivalent; namely, they provide the same growth index results. Based on the previous analysis, it is interesting to mention that the differences that we find with respect to the Λ CDM growth index are near the edge of the present experimental limits. Indeed, in a recent analysis of the clustering properties of luminous red galaxies and the growth rate data provided by the various galaxy surveys, it has been found that $\gamma = 0.56 \pm 0.05$ and

$\Omega_{m0} = 0.29 \pm 0.01$ [51]. Obviously, the prediction of γ for all our possible cases lies within 1σ of that range.

In the lower panel of Fig. 2, we show the growth data (solid points; see Ref. [50]) together with the predicted $f(z)\sigma_8(z) \approx \sigma_8 D(z)\Omega_m(z)^{\gamma(z)}$ for the running vacuum $\Lambda(H)$ in the Γ_0 (solid line) and Γ_1 (dotted line) parametrizations and for the Λ CDM (dashed line, Γ_0 , and dot-dashed line, Γ_1). We observe that the running vacuum model reproduces the growth data well, in a way that is compatible with the Λ CDM model. Notice that, in order to obtain the above results for the concordance Λ cosmology, we utilize $\sigma_8 = 0.829$, while for the running vacuum model, we use $\sigma_8 = 0.750$ (see Refs. [23,25]). Also, $D(z)$ is the growth factor normalized to unity at the present time.

As we have already said in the Introduction, the determination of the growth index is important in cosmological studies because it can be used as a tool toward testing the validity of general relativity on extragalactic scales. For those dark energy models that adhere to GR and are characterized by a constant equation of state parameter, it has been found that $\gamma \approx \frac{3(w-1)}{6w-5}$ [28–31], while for the $\Lambda(H)$ +GR case, we obtained $\gamma_{\Lambda_H} \approx \frac{6+3\nu}{11-12\nu}$. Obviously, the growth index reduces to $6/11$ for the usual Λ CDM model ($w = -1$ or $\nu = 0$).

In the case of extended theories of gravity the situation is as follows. Recently, for the holographic dark energy models, it has been found that the asymptotic value of the growth index is $\gamma \approx 4/7$ [52]. For the braneworld gravity of Ref. [32], we have $\gamma \approx 11/16$ (see also Refs. [30,33–35]); for some $f(R)$ gravity models, it has been found that $\gamma \approx 0.415 - 0.21z$ (see Refs. [36,37]); and lastly for the Finsler-Randers cosmological model, Basilakos and Stavrinou [38] have shown that $\gamma \approx 9/14$. Based on the aforementioned results, if the derived value of γ (based on the next generation of surveys, cf. *Euclid*) shows scale or time dependence or it is inconsistent with $6/11$, then this will be a hint that the nature of DE reflects on the physics of gravity.

To conclude, in this work, we have analytically studied the growth index of matter perturbations for the FLRW flat cosmological models in which the vacuum energy density is a function of the Hubble parameter, namely, $\Lambda(H) = \Lambda_0 + 3\nu(H^2 - H_0^2)$. In previous comprehensive studies [20,21,23], we have utilized such a dynamical vacuum model in order to investigate the background expansion and have carefully compared the differences with respect to the concordance Λ CDM cosmological model. We believe that the combination of the works of Refs. [20,21,23] together with the current article provide a rather complete investigation of the observational status of the RG running vacuum model both at the background and perturbation levels.

Within this framework, we have calculated for the first time (to the best of our knowledge) the asymptotic value of the growth index, which is given by $\gamma_{\Lambda_H} \approx \frac{6+3\nu}{11-12\nu}$.

Obviously, the obtained formula analytically extends in a very clear way that of the Λ CDM model, $\gamma_\Lambda \approx 6/11$. In our study, we have applied the two most popular parametrizations for the evolution of the growth index, $\gamma(z) = \gamma_0 + \gamma_1 y(z)$, with $y(z) = \ln \tilde{\Omega}_m(z)$ and $y(z) = z/(1+z)$; we have solved the problem analytically; and we have thus provided for the first time the coefficients γ_0 and γ_1 in terms of Ω_{m0} and ν . The comparison shows that the above $\gamma(z)$ parametrizations are practically equivalent and the corresponding evolution of the growth rate of clustering matches quite well the recent growth data. Finally, we have estimated the numerical corrections that the running $\Lambda(H)$ model could produce on the growth index as

compared to the concordance model and pointed out that they could reach, in some cases, the level of a few percent, hopefully accessible in the future.

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