

**Above-threshold poles in model-independent form factor parametrizations**Benjamín Grinstein<sup>\*</sup>*Department of Physics, University of California, San Diego, La Jolla, California 92093, USA*Richard F. Lebed<sup>†</sup>*Department of Physics, Arizona State University, Tempe, Arizona 85287-1504, USA*

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The model-independent parametrization for exclusive hadronic form factors commonly used for semileptonic decays is generalized to allow for the inclusion of above-threshold resonant poles of known mass and width. We discuss the interpretation of such poles, particularly with respect to the analytic structure of the relevant two-point Green's function in which they reside. Their presence has a remarkably small effect on the parametrization, as we show explicitly for the case of  $D \rightarrow \pi e^+ \nu_e$ .

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**I. INTRODUCTION**

Studies of the analytic structure of Green's functions in quantum field theory (QFT) have a long and illustrious history. Here we merely outline, in the briefest possible description, one specific line of inquiry on two-point Green's functions, ultimately stretching back to QFT fundamentals like the optical theorem and the Källén–Lehmann spectral decomposition, and ending with a practical yet rigorous parametrization for the form factors of semileptonic decays of hadrons, in which a heavy quark flavor ( $s, c, b$ ) decays to a lighter one.

The starting point is the two-point Green's function of two currents, in our case a conjugate pair  $JJ^\dagger$  of weak-interaction currents  $J^\mu \equiv \bar{q}\Gamma^\mu Q$ , where  $\Gamma^\mu$  is the  $V - A$  weak interaction Lorentz structure (at least at leading perturbative order) responsible for changing the heavy quark flavor  $Q$ . This two-point function is analytic everywhere in the plane of complex momentum  $q^2$ , except at poles corresponding to resonances and cuts corresponding to collections of particles going onto the mass shell. The most important one is the so-called *unitarity cut* corresponding to the production of the lightest pair of hadrons (of flavor content  $Q\bar{q}$  plus its conjugate) from the currents, since it has the lowest branch point on the real  $q^2$  axis. In 1963, Meiman [1] was the first to consider the conformal mapping of the entire cut  $q^2$  plane to the unit disk in a variable  $z$ , and to note the existence of a bound on the coefficients of the powers of  $z$  for any function derived from the two-point function. Some years later, Okubo [2–5] applied the  $z$ -variable transformation to the two-point function relevant to the semileptonic process  $K_{\ell 3}$ , to obtain bounds on certain moments of the form factors. In 1980, Bourrely *et al.* [6] showed how to obtain bounds for the form factors by using the evaluation of the two-point

function in a region where perturbative QCD is applicable. Finally, in the mid-1990s, Boyd *et al.* [7–12] showed how below-threshold poles—essential to properly treating the analytic structure of the two-point function—can be incorporated into the  $z$ -expansion by means of a well-known trick of complex analysis called *Blaschke factors* (the analytic significance of which for heavy-hadron form factors was first noted by Caprini [13,14]), and applied the  $z$ -parametrization thus derived to a number of heavy-quark semileptonic decays.

It is then natural to ask whether the *above-threshold* poles, for which the corresponding resonances can decay to on-shell pairs of mesons with the quantum numbers of  $JJ^\dagger$ , can loosen or perhaps even fatally weaken these bounds and the parametrization following from them. In fact, as to be shown here, even the most extreme case of a prominent resonance just barely above threshold does not significantly damage the quality of the parametrization. As a *formal* matter, only poles on the first Riemann sheet in  $q^2$  change the analytic structure of the form factor, while the (optional) inclusion of poles on other sheets (in particular, above-threshold resonances) changes only the unitarity bound, and therefore does *not* affect the analytic structure. Nevertheless, we argue in this paper that the maximum effect of such an above-threshold pole on the unitarity bound is such that it can be treated as though it arises from a first-sheet pole at the corresponding  $q^2$  location. In order to support this conclusion rigorously, one must develop a technology in which the above-threshold poles can be treated solely according to their analytic structure within the two-point function.

The contribution of this paper is to show that above-threshold poles, corresponding to resonances of known mass and width, can also be accommodated into the parametrization by using Blaschke factors. The essential mathematical point is that, by virtue of possessing a finite width, the poles lie off the unitarity cut and therefore can be treated as if they reside inside the unit circle, where Blaschke factors are

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applicable. An important subtlety to be discussed below is the sense in which resonant poles, which first appear on the second Riemann sheet, can be handled in this way. The essential phenomenological point is that, by virtue of the widths being sufficiently small compared to the resonant mass (which in turn lies above the heavy-quark hadronic threshold), the poles lie *barely* inside the unit circle, and the corresponding corrections from the Blaschke factors weaken the bounds on coefficients of the semileptonic form factor parametrization very little. As a specific example, one might expect the  $D^*$  resonance, which lies very close to the  $D\pi$  threshold, to have a pronounced effect on  $D^{+,0} \rightarrow \pi^{0,-}\ell^+\nu_\ell$  form factors, but we show below that the effect is only at the level of 1 part in  $10^{-3}$ . The loosening of the bounds actually becomes more prominent for lighter quarks; but even in the case of  $K_{\ell 3}$ , the  $K^*$  pole is seen only to loosen the bounds by a few percent.

This paper is organized as follows: In Sec. II we review the current technology of the  $z$ -expansion. Section III generalizes the expansion to the case of a pole lying above the pair-production threshold of the two-point function but slightly off the unitarity cut due to a finite imaginary part. In Sec. IV we address the question of what sense in which an observed resonance can be treated as such a pole contributing to the dispersion relation from which the expansion is derived. Section V shows the effect of this approach in two sample cases, the decays  $D^{+,0} \rightarrow \pi^{0,-}\ell^+\nu_\ell$  and  $K^{+,0} \rightarrow \pi^{0,-}\ell^+\nu_\ell$ , and in Sec. VI we offer concluding remarks.

## II. REVIEW OF THE EXPANSION

We reprise here the key formulas relevant to the form factor parametrization in a description almost identical to that in Ref. [12], and incorporate minor modifications relevant to the inclusion of above-threshold resonant poles. Starting with the heavy-light ( $Q \rightarrow q$ ) vectorlike ( $V$ ,  $A$ , or  $V - A$ ) quark transition current

$$J^\mu \equiv \bar{Q}\Gamma^\mu q, \quad (1)$$

the two-point momentum-space Green's function  $\Pi_J^{\mu\nu}$  is defined, and separated into manifestly spin-1 ( $\Pi_J^T$ ) and spin-0 ( $\Pi_J^L$ ) pieces, by

$$\begin{aligned} \Pi_J^{\mu\nu}(q) &\equiv i \int d^4x e^{iqx} \langle 0 | T J^\mu(x) J^{\dagger\nu}(0) | 0 \rangle \\ &= \frac{1}{q^2} (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_J^T(q^2) + \frac{q^\mu q^\nu}{q^2} \Pi_J^L(q^2). \end{aligned} \quad (2)$$

In QCD, the functions  $\Pi_J^{L,T}$  contain divergences of different degrees and must undergo subtractions (one and two, respectively) to appear in finite dispersion relations:

$$\begin{aligned} \chi_J^L(q^2) &\equiv \frac{\partial \Pi_J^L}{\partial q^2} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi_J^L(t)}{(t - q^2)^2}, \\ \chi_J^T(q^2) &\equiv \frac{1}{2} \frac{\partial^2 \Pi_J^T}{\partial (q^2)^2} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi_J^T(t)}{(t - q^2)^3}. \end{aligned} \quad (3)$$

Perturbative QCD (or more thoroughly, QCD sum rules) may be used to compute the functions  $\chi(q^2)$  at values of  $q^2$  far from the region where  $J$  can produce manifestly nonperturbative effects like resonances. This condition specifically reads  $(m_Q + m_q)\Lambda_{\text{QCD}} \ll (m_Q + m_q)^2 - q^2$ .  $q^2 = 0$  is sufficient for  $Q = c, b$ , while  $Q = s$  might require a slightly negative value, say  $q^2 = -1 \text{ GeV}^2$ .

The functions  $\text{Im} \Pi_J$  are evaluated by inserting into the dispersion relation a complete set of states  $X$  that couple the current  $J$  to the vacuum, leading to

$$\text{Im} \Pi_J^{T,L}(q^2) = \frac{1}{2} \sum_X (2\pi)^4 \delta^4(q - p_X) |\langle 0 | J | X \rangle|^2. \quad (4)$$

This relation shares a common origin with the optical theorem and the Källén–Lehmann spectral decomposition, but it refers particularly to matrix elements of a specific current  $J$  (in our case, the amplitudes for the weak processes  $W^* \rightarrow X$ ) rather than those of a single field or a full transition operator. The dispersion relations Eqs. (3) indicate the equality of the perturbatively evaluated function  $\chi(q^2)$  with an integral over the production rate as a function in momentum of the processes  $W^* \rightarrow X$ , which includes phase space and other smooth functions. Since each term in the sum is semipositive definite, one obtains a strict inequality for each  $X$ , which may further be restricted if one chooses to include other states in the sum. In our case, we choose  $X$  to be the two-particle states consisting of the lightest meson pair in which one of them contains a  $Q$  quark (mass  $M$ ) and the other a  $\bar{q}$  (mass  $m$ ). For  $D_{\ell 3}$  ( $K_{\ell 3}$ ) decays,  $X$  is  $D\pi$  ( $K\pi$ ). Defining

$$t_\pm \equiv (M \pm m)^2, \quad (5)$$

and choosing, for definiteness, the form factor  $F(t)$  to be the one coupling to  $\Pi^T$ , one has

$$\frac{1}{\pi \chi^T(q^2)} \int_{t_+}^\infty dt \frac{W(t) |F(t)|^2}{(t - q^2)^3} \leq 1, \quad (6)$$

where  $W(t)$  is a simple, computable nonnegative function (largely phase space factors). An analogous expression holds for  $\Pi^L$ .

The complex- $t$  plane contains a branch cut extending from  $t_+ \rightarrow \infty$ . It is mapped to the unit disk in a variable  $z$  (with the two sides of the cut forming the unit circle  $C$ ) using the conformal variable transformation

$$z(t; t_0) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}} = \frac{t_0 - t}{(\sqrt{t_+ - t} + \sqrt{t_+ - t_0})^2}, \quad (7)$$

where  $t_0$  is a parameter chosen later for convenience. In particular,  $z$  is real for  $t \leq t_+$  and a pure phase for  $t \geq t_+$ . In fact, the definition in Eq. (7) can be used in several capacities since, as seen from its second form, multiplying by  $z(t; t_0)$  eliminates a simple pole at  $t = t_0$ . The bound Eq. (6) on  $F(t)$  may then be rewritten as

$$\frac{1}{\pi} \int_{t_+}^{\infty} dt \left| \frac{dz(t; t_0)}{dt} \right| |\phi(t; t_0) P(t) F(t)|^2 \leq 1, \quad (8)$$

where the weight function  $\phi(t; t_0)$  is called an *outer function* in complex analysis. It is given here by

$$\phi(t; t_0) = \tilde{P}(t) \left[ \frac{W(t)}{|dz(t; t_0)/dt| \chi^T(q^2)(t - q^2)^3} \right]^{1/2}, \quad (9)$$

where the function  $\tilde{P}(t)$  is a product of factors  $z(t; t_s)$  or  $\sqrt{z(t; t_s)}$  (and hence unimodular on the unit circle  $|z(t; t_0)| = 1$ ) designed to remove *kinematical* singularities at points  $t = t_s$  from the other factors in Eq. (9). The functions  $\phi(t; t_0)$  for any form factor of spin-0 and spin-1 meson and spin- $\frac{1}{2}$  baryon semileptonic decays are tabulated in Ref. [12]. On the other hand, the function  $P(t)$  in Eq. (8) is a product of Blaschke factors  $z(t; t_p)$  (again unimodular on the unit circle  $|z(t; t_0)| = 1$ ) that remove *dynamical* singularities due to resonant poles in the two-point function.

In total, the analyticity of the two-point function away from the cut and all poles is most efficiently expressed by isolating the factors that encode the nonanalytic behavior of the form factor  $F(t)$  into the functions  $\phi(t; t_0)$  and  $P(t)$  and then transforming to the variable  $z = z(t; t_0)$ , so that the dispersion relation inequality Eq. (6) or (8) becomes

$$\frac{1}{2\pi i} \oint_C \frac{dz}{z} |\phi(z) P(z) F(z)|^2 \leq 1, \quad (10)$$

which in turn allows the expansion

$$F(t) = \frac{1}{|P(t)| \phi(t; t_0)} \sum_{n=0}^{\infty} a_n z(t; t_0)^n, \quad (11)$$

with the bound of Eq. (10) now reading

$$\sum_{n=0}^{\infty} a_n^2 \leq 1. \quad (12)$$

All possible functional dependences of the form factor  $F(t)$  consistent with Eqs. (3) are now incorporated into the

coefficients  $a_n$  of Eq. (11), which are highly constrained by Eq. (12).

The strength of the parametrization Eq. (11) becomes truly apparent when one notes that the kinematical variable  $z$  typically assumes a small range for semileptonic decays, so that the series converges quickly and can be truncated after a small number of terms. To be specific, let us rewrite Eq. (7) in terms of parent and daughter velocity 4-vectors  $v^\mu \equiv p_M^\mu/M$ ,  $v'^\mu \equiv p_m^\mu/m$ . A convenient commonly used invariant is their dot product,

$$w \equiv v \cdot v' = \gamma_m = \frac{E_m}{m} = \frac{M^2 + m^2 - t}{2Mm}, \quad (13)$$

where  $\gamma_m$  is the relativistic dilation factor of the daughter  $m$  in the rest frame of the parent  $M$ . In terms of  $w$ , Eq. (7) becomes

$$z(t; t_0) = z(w; N) = \frac{\sqrt{1+w} - \sqrt{2N}}{\sqrt{1+w} + \sqrt{2N}}, \quad (14)$$

where  $N$  is a free parameter related to  $t_0$  by

$$N = \frac{t_+ - t_0}{t_+ - t_-}. \quad (15)$$

The kinematic limits for the semileptonic decay  $M \rightarrow m \ell \nu_\ell$  are  $t_{\min} = m_\ell^2$ ,  $t_{\max} = t_-$ , which correspond, respectively, to

$$w_{\max} = \frac{1 + r^2 - \delta^2}{2r}, \quad w_{\min} = 1, \quad (16)$$

or

$$z_{\max} = \frac{\sqrt{(1+r)^2 - \delta^2} - 2\sqrt{Nr}}{\sqrt{(1+r)^2 - \delta^2} + 2\sqrt{Nr}}, \quad z_{\min} = -\left( \frac{\sqrt{N} - 1}{\sqrt{N} + 1} \right), \quad (17)$$

using the abbreviations  $r \equiv m/M$ ,  $\delta \equiv m_\ell/M$ . The minimum (optimized) truncation error is achieved when  $z_{\min} = -z_{\max}$ , which occurs when

$$N_{\text{opt}} = \sqrt{\frac{(1+r)^2 - \delta^2}{4r}}, \quad (18)$$

or

$$t_0 = t_+ \left[ 1 - \sqrt{\left( 1 - \frac{t_-}{t_+} \right) \left( 1 - \frac{m_\ell^2}{t_+} \right)} \right]. \quad (19)$$

Evaluating at  $N = N_{\text{opt}}$ , one finds

$$z_{\text{max}} = -z_{\text{min}} = \frac{[(1+r)^2 - \delta^2]^{1/4} - (4r)^{1/4}}{[(1+r)^2 - \delta^2]^{1/4} + (4r)^{1/4}}, \quad (20)$$

While the Blaschke factors due to resonant poles at  $t = t_p$  can be expressed as  $z(t; t_p)$ , it is more convenient to use the form used in previous works:

$$P(z; z_p) = \frac{z - z_p}{1 - z z_p}, \quad (21)$$

and  $z(t; t_p) = P(z; z_p)$  whenever  $t_p < t_+$  (a subthreshold pole) so that  $z_p$  is real. However, the same technique works just as well for any complex value for  $z_p$  inside the unit disk. In that case, the definition of Eq. (21) can be generalized to

$$P(z; z_p) = \frac{|z_p|}{z_p} \frac{z_p - z}{1 - z_p^* z}, \quad (22)$$

which, assuming  $t_0 < t_+$ , equals  $z(t; t_p)$  times the phase of  $t_p - t_0$ , the latter factor being irrelevant in the bound Eq. (10). Note that  $P(0; z_p) = |z_p|$  (i.e., with this definition  $P(0; z_p)$  is manifestly nonnegative), and that all  $z_p$  with  $|z_p| = 1$  give  $P(z) = 1$ . The usefulness of the Blaschke factors for phenomenology is determined by how much they degrade the bound Eq. (11) in the semileptonic region (near  $z = 0$ ): Fewer poles with  $|z_p| < 1$  means a more constrained allowed region for  $F(z)$ .

### III. POLES ABOVE THRESHOLD

Consider a pole at the complex mass value  $M_R - i\Gamma/2$ , such that  $M_R \equiv M + m + \Delta m > M + m = \sqrt{t_+}$  and  $\Gamma > 0$ . Specifically, let us define dimensionless mass excess and width parameters:

$$\mu \equiv \frac{\Delta m}{\sqrt{t_+}} = \frac{M_R}{\sqrt{t_+}} - 1, \quad (23)$$

$$\gamma \equiv \frac{\Gamma}{2\sqrt{t_+}}. \quad (24)$$

It is furthermore advantageous to define the following dimensionless variables:

$$\begin{aligned} a &\equiv \mu(2 + \mu) - \gamma^2, \\ b &\equiv 2\gamma(1 + \mu), \\ c &\equiv \sqrt{a^2 + b^2} = \sqrt{(\mu^2 + \gamma^2)[(2 + \mu)^2 + \gamma^2]}, \\ \beta &\equiv \sqrt{1 - \frac{t_+}{t_0}} = \frac{2\sqrt{Nr}}{1 + r}. \end{aligned} \quad (25)$$

One expects both  $\mu \ll 1$ , indicating that the mass does not lie far above threshold, and  $\gamma \ll 1$ , indicating a narrow

TABLE I. Parameter values for the decay  $D^0 \rightarrow \pi^- e^+ \nu_e$ .

$r$	$7.484 \times 10^{-2}$	$\mu$	$2.919 \times 10^{-3}$
$\delta$	$2.740 \times 10^{-4}$	$\gamma$	$2.080 \times 10^{-5}$
$N_{\text{opt}}$	1.964	$a \approx c$	$5.846 \times 10^{-3}$
$z_{\text{max}} = -z_{\text{min}}$	0.1672	$b$	$4.173 \times 10^{-5}$
$\beta$	0.7135	$z_p$	$-0.9762 + 0.2117i$
		$ z_p $	0.99924
		$\arg z_p$	167.8°

width. The usual narrow-width approximation,  $\Gamma \ll M_R$ , can be enhanced in this case to assume that the width is sufficiently small so as to clearly separate the peak from threshold,  $\gamma \ll \mu$  ( $\Gamma \ll \Delta m$ ). Likewise, one expects  $b \ll a \approx c \ll 1$ , but generically  $\beta = O(1)$ . The specific values for the case of  $D^0 \rightarrow \pi^- e^+ \nu_e$ , for which the  $D^{*+}$  pole lies slightly above the  $D^0 \pi^+$  threshold, are presented in Table I.<sup>1</sup> Similar values hold for  $D^+ \rightarrow \pi^0 e^+ \nu_e$  and for muon channels.

Regardless of the smallness of any parameters, one can compute compact closed-form solutions for the position of  $z_p$ . One finds

$$z_p = \frac{-\beta^2 + c + i\beta\sqrt{2(c+a)}}{\beta^2 + c + \beta\sqrt{2c(c-a)}}, \quad (26)$$

from which one obtains

$$|z_p|^2 = 1 - \frac{2\beta\sqrt{2(c-a)}}{\beta^2 + c + \beta\sqrt{2c(c-a)}}, \quad (27)$$

and

$$\arg z_p = \arg [(-\beta^2 + c) + i\beta\sqrt{2(c+a)}]. \quad (28)$$

Using Eq. (28) with  $\beta^2 > c$  (a resonance near threshold), one has

$$\arg z_p = \pi - \tan^{-1} \left( \frac{\beta\sqrt{2(c+a)}}{\beta^2 - c} \right), \quad (29)$$

while for  $\beta^2 < c$  (a resonance far above threshold),

$$\arg z_p = \tan^{-1} \left( \frac{\beta\sqrt{2(c+a)}}{c - \beta^2} \right). \quad (30)$$

Neglecting  $m_\ell$  ( $\delta$ ), using  $N = N_{\text{opt}}$  from Eq. (18), and retaining only the lowest power in  $\Gamma$  ( $\gamma$ ), one obtains

<sup>1</sup>We use central values from [15]. Strictly speaking, only the hadronic part of  $\Gamma_{D^{*+}}$  should be included in these strong-interaction dispersion relations; however, the hadronic branching fraction of  $D^{*+}$  is  $98.4 \pm 0.7\%$ , and therefore is taken to be 1.



$$1 - |z_p| \rightarrow \frac{\Gamma}{2\sqrt{\Delta m}(Mm)^{1/4}} \cdot \frac{\beta^2}{\beta^2 + \mu(2 + \mu)} \cdot \frac{1 + \mu}{\sqrt{1 + \mu/2}}, \quad (31)$$

while the argument of the arctangent in Eqs. (29)–(30) becomes

$$\frac{2\sqrt{\Delta m}}{(Mm)^{1/4}} \cdot \frac{\beta^2 \sqrt{1 + \mu/2}}{|\beta^2 - \mu(2 + \mu)|}, \quad (32)$$

independent of the width to linear order. Additionally taking the near-threshold resonance limit  $\mu \ll 1$ , the latter two factors of Eq. (31) and the second factor of Eq. (32) become unity:

$$1 - |z_p| \rightarrow \frac{\Gamma}{2\sqrt{\Delta m}(Mm)^{1/4}}, \quad (33)$$

and

$$\arg z_p \rightarrow \pi - \arctan \left[ \frac{2\sqrt{\Delta m}}{(Mm)^{1/4}} \right]. \quad (34)$$

The corresponding exact values of  $z_p$ ,  $|z_p|$ , and  $\arg z_p$  for  $D^0 \rightarrow \pi^- e^+ \nu_e$  also appear in Table I. The values obtained from the approximate forms in Eqs. (33) and (34) agree with the exact results of Eqs. (27) and (28) within  $10^{-5}$  and  $0.15^\circ$ , respectively.

The naive effect of such an additional pole is to allow  $|F(z)|$ —and hence each of the coefficients  $a_n$  in Eqs. (11)–(12)—to be larger by a factor of  $1/|P(z; z_p)|$ , where  $z \in [-z_{\max}, z_{\max}]$  for the semileptonic decay. Noting that  $|z_p|$  lies very close to unity—much closer to unity than it does to the allowed semileptonic values of  $z$ —one finds  $1/|P(z; z_p)|$  to lie uniformly close to unity, meaning that *the presence of a pole with  $|z_p| \approx 1$  weakens the model-independent form factor bounds very little*. To give a simple figure of merit, consider the value of  $1/|P(z; z_p)|$  at the center of the semileptonic range,  $z = 0$ ; as we have seen,  $1/|P(0; z_p)| = 1/|z_p|$ . The exact value is given by Eq. (27):

$$\frac{1}{|P(0; z_p)|} = \frac{1}{|z_p|} = \left[ 1 - \frac{2\beta\sqrt{2(c-a)}}{\beta^2 + \beta\sqrt{2(c-a)} + c} \right]^{-1/2}, \quad (35)$$

while its approximate value ( $\mu \ll 1$ ) is given by Eq. (33):

$$\frac{1}{|P(0; z_p)|} \rightarrow 1 + \frac{\Gamma}{2\sqrt{\Delta m}(Mm)^{1/4}}. \quad (36)$$

Again, inasmuch as the pole represents a resonance with a narrow width well separated from threshold,  $\Gamma \ll \Delta m \ll M + m$ , the correction term is quite small; in the case of

$D^0 \rightarrow \pi^- e^+ \nu_e$ , the allowed ranges for the  $a_n$  are expanded by less than 8 parts in  $10^4$ . In summary, the parametrization is exactly as before in Eq. (11), but the allowed range for each  $a_n$  is slightly expanded beyond  $|a_n| \leq 1$ .

Of course,  $z = 0$  is just one point in the allowed range for semileptonic decay. Since the poles of interest lie not far above threshold,  $z_p$  lies rather close to  $-1$ ; therefore, from Eq. (22), the largest correction to the  $a_n$  factors occurs at  $z = -z_{\max}$  ( $t = m_\ell^2$ ). In the case of  $D^0 \rightarrow \pi^- e^+ \nu_e$ , the correction is still only about 1 part in  $10^{-3}$ . The effect of the near-threshold pole is truly minimal.

#### IV. EXISTENCE AND NATURE OF ABOVE-THRESHOLD POLES

In the previous section, we have shown that incorporating an above-threshold pole into the two-point function that corresponds to a resonance is mathematically not difficult. Here we discuss in detail issues related to the question of whether such a treatment is appropriate to physical resonances.

The most common approach treats an above-threshold resonance, which is identified by a Lorentzian distribution in energy identified with a Breit-Wigner distribution:

$$|\mathcal{M}|^2 \propto \frac{1}{(s - M_R^2)^2 + s\Gamma^2}, \quad (37)$$

as being associated with a Breit-Wigner pole at the value  $\sqrt{s} = M_R - i\Gamma/2$ , assuming the narrow-width approximation  $\Gamma \ll M_R$ .<sup>2</sup> More generally, the width  $\Gamma$  need not be a constant but can have an energy dependence,  $\Gamma(s)$ . In either case, one anticipates the existence of a pole in the amplitude  $\mathcal{M}$  off the real- $s$  axis.

Nevertheless, as was pointed out long ago [16], an observable line shape arbitrarily close to an idealized Breit-Wigner distribution can be simulated even in the absence of a literal pole off the real- $s$  axis. Inasmuch as most complex energy values are experimentally inaccessible, the only ways to unambiguously detect a literal pole (either measuring at the pole location itself or measuring at points surrounding it and using Cauchy's theorem) are unavailable. So while the presence of a pole in the complex plane is a natural way to interpret the appearance of a narrowly peaked distribution along the real axis, its certainty is not guaranteed [17]. One may model the amplitude along the cut by incorporating an explicit Lorentzian function, including a specific value of residue [18,19]. See also [20], in which the resonance is incorporated into phase and modulus information along the cut.

Even so, the assumption of a pole at a complex value of  $z$  near the unit circle has been seen in the previous section to

<sup>2</sup>Here we use  $q^2 = s$  rather than  $t$ , to emphasize the pair-production origin of the cut.

loosen the bounds on semileptonic form factors very little. Note particularly that the Blaschke factor Eq. (22) makes reference only to the position of the pole and not its residue; therefore, it must work equally well for a pole with residue as large as is allowed by unitarity (which is explicitly built into the dispersion relation) and a pole with vanishing residue—which is, of course, no pole at all. Since, once again, the effect of a complex-valued pole projected along the real axis is to allow for a narrow peak of an arbitrary physically allowed value of residue, one sees that including the Blaschke factor in the two-point function is appropriate for accommodating the effects of a Breit-Wigner line shape along the real-axis cut, but does not actually commit one to demanding the existence of a pole off the real axis.

Another interesting point regarding the above-threshold pole is its appearance in the full Riemann surface for the two-point function. The existence of a cut indicates the existence of at least one additional Riemann sheet. For example, a particle pair created in the  $L$ th partial wave has phase space proportional to  $k^{2L+1}$ , where

$$k = \sqrt{\frac{(s - t_+)(s - t_-)}{4s}} \quad (38)$$

is the center-of-momentum-frame value of the spatial momentum of either particle. Since the discontinuity along the cut is proportional to phase space, one thus obtains a two-sheet Riemann surface, corresponding to the double valuedness of the square root function. The number of sheets doubles each time a distinct two-body threshold is encountered.

The question then becomes, on what sheet do the physical resonances live, and on what sheet or sheets were the dispersion relation integrals obtained? The first question was originally answered by Peierls [21], who argued that a resonant pole must live on the unphysical (second) Riemann sheet below the real axis, just on the other side of the cut from the first sheet. Otherwise, the Schwarz reflection principle would require a pole just below the real axis on the first sheet to have a mirror pole just above the real axis on the first sheet; and since the negative imaginary value ( $-i\Gamma/2$ ) of the former pole is necessary to obtain an exponentially decaying state, the latter mirror pole would correspond to an unphysical runaway state.

On the other hand, the contour bounding the dispersion integral is easily seen to live entirely on the first sheet, since its derivation uses the Schwarz reflection principle to obtain a nonnegative contribution along the cut. So then, one may ask, why worry about poles that are not even encircled by the contour? The answer is simple pragmatism: A pole that lies just below the cut on the second sheet creates a Breit-Wigner projection along the cut identical to the contribution that would be obtained from an unphysical pole just above the cut on the first sheet. While causality knows that the pole lies just below

the cut on the second sheet, the dispersion relation is sensitive to the pole only through its projection along the cut, and this contribution can be obtained from a pole at  $M_R \pm i\Gamma/2$  on any sheet such that its projection along the real axis agrees with data. One sees that treating the pole as if it occurred in the fourth quadrant of the first sheet, as done in Sec. III, leads to the appropriate projection along the cut. In particular, the value of  $\arg z_p$  given in Eq. (29) places it in the second quadrant of the complex- $z$  plane, but the value of  $\arg z_p$  symmetric about  $\pi$  lying in the third quadrant of the complex- $z$  plane is equally valid for the analysis of Sec. III.

This point is worth emphasizing. The physical second-sheet poles do *not* literally appear inside the unit circle  $|z| = 1$ . The first-sheet poles examined in Sec. III are strictly unphysical. However, such an unphysical pole near the unitarity cut, were it nevertheless to occur, would create a Breit-Wigner line shape indistinguishable from that created by a physical second-sheet pole equally near the unitarity cut. The unphysical poles of Sec. III must not be thought of as altering the *analytic structure* of the form factor—in other words, of changing the shape of the form factor  $F(z)$  through the  $z$ -dependence of the Blaschke factor  $P(z)$ . Rather, they alter the *unitarity bound* Eq. (12), by allowing the coefficients  $a_n$  to have larger ranges in exchange for the benefit of completely ignoring the effect of an above-threshold resonance, no matter how prominent.

It is interesting to note that the leading-order perturbative expansion of the two-point function  $\chi(q^2)$  in the deep Euclidean region contains logarithmic dependence (and polylogarithmic dependence at higher perturbative order). As is well known, these functions have Riemann surfaces with an infinite number of sheets, in contrast to the two sheets for a function with a half-integer power, such as those previously discussed. Since the perturbative two-point function can be considered as an inclusive sum over all allowed exclusive channels, the mismatch between the sheet counting can be construed as indicating the necessity of including an arbitrarily large number of open channels in order to achieve quark-hadron duality.

## V. EXAMPLES

### A. $D$ semileptonic decays

The unflavored semileptonic decays of  $D$  mesons are particularly interesting for this formalism. First, several such modes ( $D \rightarrow \{\pi, \rho, \omega, \eta, \eta'\}$ ) have been observed, each with an  $O(10^{-3})$  branching fraction. Furthermore, modes with both  $e^+$  and  $\mu^+$  have been seen. Since all of these modes proceed through the  $J^\mu = \bar{q}\Gamma^\mu c$  currents, where  $q$  is a light quark and  $\Gamma^\mu$  represents Lorentz structure, they all serve to saturate the same small set of dispersion relations, leading to stronger bounds on any one of them.

Second, the processes  $D^{*0} \rightarrow \pi^{0,-} \ell^+ \nu_\ell$  are remarkable due to the closeness of the  $D^*$  resonance to the crossed-channel  $D\pi$  threshold in each case. Specifically [15],

$$\begin{aligned} m_{D^{*0}} - m_{D^0} - m_{\pi^+} &= 5.86 \pm 0.07 \text{ MeV}, \\ m_{D^{*+}} - m_{D^+} - m_{\pi^0} &= 5.68 \pm 0.08 \text{ MeV}. \end{aligned} \quad (39)$$

As we have seen, the smallness of these numbers (combined with the small width  $\Gamma_{D^{*+}} = 83.4 \pm 1.8 \text{ keV}$ ) guarantees a minimal modification to the allowed range for the semileptonic form factor coefficients  $a_n$ . Furthermore, isospin symmetry relates the two processes<sup>3</sup> (separately for the  $I = \frac{1}{2}$  and  $\frac{3}{2}$  channels, but with no resonance in the latter channel). As seen in Ref. [12], the presence of separate isospin-related channels increases the function  $\phi(z)$  in Eq. (9) by a Clebsch-Gordan factor  $\sqrt{n_I}$ , where  $n_I = \frac{3}{2}$  for  $D \rightarrow \pi$ . Noting that  $\phi(z)$  appears in the denominator of the parametrization Eq. (11), one finds that the coefficient bound of Eq. (12) effectively has its unity factor replaced by  $\frac{2}{3}$ —a much more dramatic effect than that due to the near-threshold  $D^*$  pole.

### B. $K$ semileptonic decays

The  $K_{\ell 3}$  decays are interesting in this context, partly because they were the ones originally studied by Okubo [2–5], but also because they possess a prominent, fairly narrow resonance  $K^*$  ( $M_R = 891.66 \text{ MeV}$ ,  $\Gamma = 50.8 \text{ MeV}$ ) that lies significantly far above the threshold  $\sqrt{t_+} = m_K + m_\pi$ . It is worth pointing out that the  $K_{\ell 3}$  and  $D_{\ell 3}$  decays have the same form factor and isospin structure. For definiteness, let us consider the specific mode  $K^+ \rightarrow \pi^0 e^+ \nu_e$ , for which the numerical values of the key parameters are presented in Table II, but the corresponding values for the modes  $K_L \rightarrow \pi^- e^+ \nu_e$ ,  $K^+ \rightarrow \pi^0 \mu^+ \nu_\mu$ , and  $K_L \rightarrow \pi^- \mu^+ \nu_\mu$  are very similar. It should also be noted that the decay  $\tau \rightarrow K \pi \nu_\tau$  is bounded by the same dispersion relation, and indeed, can provide a tighter constraint [22].

The large distance of the resonant mass from threshold is manifested in the angle of  $z_p$  lying much further from  $\pi$  radians, indeed, in the first quadrant of the complex- $z$  plane. One must use Eq. (30), since here  $\beta^2 < c$ .

While  $\Gamma$  is not particularly large, it is much larger than the  $D^*$  width, and the threshold  $\sqrt{t_+}$  is much smaller than for  $D_{\ell 3}$  decays since  $m_s \ll m_c$ . These effects combine to give a much larger value of  $\gamma$  or  $b$ . Table II uses the exact formulas Eqs. (26), (27), and (30); the approximations Eqs. (31), (32), which drop subleading terms in  $\gamma$  or  $b$ , give  $|z_p| = 0.94366$  ( $< 0.2\%$  smaller) and  $\arg z_p = 84.0^\circ$  ( $< 0.2\%$  larger).

Even so,  $|z_p|$  does not lie far from the unit circle, and therefore the typical weakening of the form factor bound

TABLE II. Parameter values for the decay  $K^+ \rightarrow \pi^0 e^+ \nu_e$ .

$r$	0.2734	$\mu$	0.4184
$\delta$	$1.035 \times 10^{-3}$	$\gamma$	$4.040 \times 10^{-2}$
$N_{\text{opt}}$	1.218	$a, c$	1.010, 1.017
$z_{\text{max}} = -z_{\text{min}}$	$4.919 \times 10^{-2}$	$b$	0.1146
$\beta$	0.9062	$z_p$	$0.1006 + 0.9400i$
		$ z_p $	0.94535
		$\arg z_p$	$83.9^\circ$

$1/|P(0; z_p)| = 1/|z_p|$  as given by Eq. (35) is 1.0578. Since  $z_p$  lies in the first quadrant, from Eq. (22) one finds that the largest correction to the  $a_n$  factors occurs at  $z = z_{\text{max}}$  ( $t = m_\ell^2$ ), and it equals 1.0581, i.e., uniformly less than 6%. Even for the extreme case of  $K_{\ell 3}$  decays, where the above-threshold pole lies far from threshold, the effect on the parametrization coefficients is quite minimal.

## VI. DISCUSSION AND CONCLUSIONS

In this paper we have extended the utility of the model-independent form factor parametrization for semileptonic decays to explicitly incorporate the effects of above-threshold resonant poles, such as  $D^{*+}$  in  $D^0 \rightarrow \pi^- e^+ \nu_e$  and  $K^{*+}$  in  $K^+ \rightarrow \pi^0 e^+ \nu_e$ . Since such poles have a finite width, they lie off the unitarity cut along the real axis in momentum-transfer space, and therefore map into the interior of the unit disk in the kinematic variable  $z$ . Inasmuch as the width of such resonances is small compared to the other mass scales in the system, the pole lies just inside the unit  $z$  circle, and as we showed, consequently has a rather small effect on the constraints on the form factor coefficients.

The recipe for calculating the amount of the relaxation of the bounds due to the presence of an above-threshold pole is easily obtained through the following steps: First, compute the dimensionless resonance mass  $\mu$  and width  $\gamma$  factors directly from  $M_R$  and  $\Gamma$  using Eqs. (23)–(24), and from them the dimensionless parameters  $a$ ,  $b$ , and  $c$  using Eq. (25) as well as the dimensionless parameter  $\beta$  derived from the threshold  $\sqrt{t_+} = M + m$  and adjustable optimization parameter  $t_0$  (or  $N$ ) from Eq. (18). The exact position  $z_p$  of the pole is then given by Eqs. (26)–(28). A simple estimate for the amount of the relaxation of the bounds is given by Eq. (35), but the full result is obtained by varying the Blaschke function  $1/|P(z; z_0)|$  of Eq. (22) over the whole allowed semileptonic range for the variable  $z$ , as given by Eq. (20).

The Blaschke pole factors present the tremendous benefit of depending only upon the resonant mass and width, and not upon its residue, a quantity that is usually much harder to obtain experimentally. Such a result is all the more remarkable because models for semileptonic form factors often assume shapes given by pole dominance, introducing a source of potentially unquantifiable uncertainties. If one uses the techniques in this paper to accommodate above-threshold resonances but still wishes to obtain tighter bounds on the

<sup>3</sup>The  $D^{*0}$  width has only a measured upper bound of 2.1 MeV [15], but isospin symmetry predicts it to be close to that of  $D^{*+}$ .

semileptonic form factors by incorporating physics along the cut, then only the much milder multihadron continuum dependence of the cut function needs to be modeled. Alternately, one may take a minimal (and completely model-independent) approach by using only the deep-Euclidean perturbative expression for the relevant two-point function to bound the form factor integral and hence the allowed parameters defining each form factor.

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