Dimension-5 *CP*-odd operators: QCD mixing and renormalization

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We study the off-shell mixing and renormalization of flavor-diagonal dimension-five *T*- and *P*-odd operators involving quarks, gluons, and photons, including quark electric dipole and chromoelectric dipole operators. We present the renormalization matrix to one loop in the $\overline{\text{MS}}$ scheme. We also provide a definition of the quark chromoelectric dipole operator in a regularization-independent momentum-subtraction scheme suitable for nonperturbative lattice calculations and present the matching coefficients with the $\overline{\text{MS}}$ scheme to one loop in perturbation theory, using both the naïve dimensional regularization and 't Hooft–Veltman prescriptions for γ_5.

I. INTRODUCTION

Permanent electric dipole moments (EDMs) of non-degenerate systems violate invariance under parity (*P*) and time reversal (*T*) or, equivalently [1], *CP*, the combination of charge conjugation and parity. Given the smallness of Standard Model (SM) *CP*-violating (CPV) contributions induced by quark mixing [2] (for a review see [3]), nucleon, nuclear, and atomic/molecular EDMs [4–8] are very deep probes of the SM θ term (already constrained at the level of $\theta \sim 10^{-10}$) and of possible new sources of *CP* violation beyond the Standard Model (BSM).

In fact, EDMs at the sensitivity level of ongoing and planned experiments probe BSM CPV interactions originating at the TeV scale or above (up to hundreds of TeV depending on assumptions about the BSM scenario). These new CPV interactions may be a key ingredient of relatively low-scale baryogenesis mechanisms such as electroweak baryogenesis (see [9] and references therein), making the study of EDMs all the more interesting. EDMs of the nucleon, nuclei, and atoms are sensitive to a number of new sources of *CP* violation, in a complementary way [10], so that a broad experimental program to search for EDMs in various systems is called for (a summary of current status and prospects can be found in Refs. [11,12]).

Extracting robust information on the new CPV sources from the (non)observation of EDMs is a challenging theoretical problem that involves physics at scales ranging from the TeV (or higher) down to the hadronic, nuclear, and atomic scales, depending on the system under consideration. The relevant physics at the hadronic and nuclear scale involves strong interactions, and requires the calculation of nonperturbative matrix elements. While interesting model-independent statements can be made within a nucleon-level chiral effective theory approach [13–18], ultimately the computation of a number of hadronic matrix elements is necessary. Existing calculations of the impact of BSM operators on hadronic EDMs typically rely on modeling the strong dynamics in ways consistent with the

quantum chromodynamics (QCD) symmetries, using methods such as QCD sum rules [19–22] and the Dyson-Schwinger equations [23,24] (see Refs. [3,12] for reviews). Since models do not rely on systematic approximations to the strong dynamics of quarks and gluons in the nucleons, current results represent in some cases only crude estimates, with different model calculations differing by up to an order of magnitude, depending on the operator under study [12]. Needless to say, this state of affairs greatly dilutes the impact of EDM experimental searches in probing short-distance physics. Moreover, the uncertainties affect the robustness of the phenomenological studies relating new sources of *CP* violation to baryogenesis mechanisms (depending on what is the dominant mechanism and operator generating the EDM).

In this context, lattice QCD calculations offer the opportunity to perform systematically improvable calculations of the CPV hadronic dynamics. Historically, lattice QCD efforts have mostly focused on the determination of the nucleon EDM induced by the SM θ term [25–33]. Only recently there has been interest in studying the impact of the leading *CP*-odd operators on the nucleon EDM [34–36] and the *T*-odd pion nucleon couplings [37].

This program, however, comes with several challenges, ranging from controlling the signal-to-noise ratio on the lattice to studying operator mixing, and matching suitably renormalized lattice operators to the minimally subtracted operators typically used in phenomenological applications. In this paper we focus on defining UV finite *CP*-odd operators of dimension five and lower, using a renormalization scheme suitable for implementation on the lattice, and matching this scheme to the perturbative $\overline{\text{MS}}$ scheme to one loop.

The paper is organized as follows. In Sec. II, we describe the effective theory framework parametrizing BSM effects at low-energy and identify the leading dimension-five CPV operators. In Sec. III, we construct the basis of operators needed to study the renormalization of the quark chromo-electric dipole moment (CEDM) operator in an off-shell

momentum subtraction scheme with nonexceptional momenta. In Sec. IV, we present the one-loop calculations needed to determine the full mixing matrix to $O(\alpha_s)$ for the operator basis discussed in Sec. III. In Sec. V, we give our results for the matrix of renormalization constants in the $\overline{\text{MS}}$ scheme, while in Sec. VI we specify the renormalization conditions that define a regularization-independent (RI) momentum subtraction scheme and provide the $O(\alpha_s)$ matching coefficients to the $\overline{\text{MS}}$ scheme. In Sec. VII, we discuss the consistency of our renormalization conditions with the singlet axial Ward identities. In Sec. VIII, we compare our results to recent related work [38] that studies the renormalization of the strangeness changing chromomagnetic quark operator. We end with our conclusions and outlook in Sec. IX.

A number of technical issues are discussed in the Appendixes. In Appendix A we summarize our choice of phase convention used to define the CP transformations. The regularization-independent calculation is done using off-shell matrix elements with quarks and gluons as external states in a fixed gauge. In Appendix B we derive the constraints on the mixing with gauge dependent and off-shell operators imposed by BRST symmetry. The Peccei-Quinn mechanism and its implications for CPV operators are discussed in Appendix C. In Appendix D we discuss the subtleties that arise in the isospin symmetry limit. Finally, in Appendix E, we summarize the matching coefficients between the $\overline{\text{MS}}$ and the RI scheme.

II. FRAMEWORK

In this section we describe in some detail the hadronic-scale CPV effective Lagrangian induced by BSM physics at the high scale. The identification of the CPV combinations of short-distance parameters involves several steps. We start our discussion in Sec. II A by classifying the leading BSM-induced operators that can lead to CPV effects at the quark and gluon level. We then discuss in Sec. II B the relation between CP and chiral symmetry in presence of operators that explicitly break chiral symmetry: the CP symmetry that remains unbroken by the vacuum takes the standard form given in Appendix A only after performing an appropriate chiral rotation of the fields ("vacuum alignment") that eliminates pion tadpoles [39–41]. In Sec. II C we implement the vacuum alignment in presence of higher-dimensional operators induced by BSM physics and in Sec. II D we summarize the vacuum-aligned effective Lagrangian including operators up to dimension five.

A. *CP* violation in the Standard Model and beyond

Assuming the existence of new physics beyond the Standard Model (BSM) at a scale $\Lambda_{\text{BSM}} \gg v_{\text{ew}}$, we can parametrize the BSM effects in terms of local operators of dimension five and higher, suppressed by powers of the scale Λ_{BSM}. The new operators are built out of SM fields

and respect the $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge symmetries of the SM. The leading CP-violating operators appear only at dimension six [42,43]. Their renormalization group evolution from the new physics scale down to the hadronic scale has been studied in several papers, most recently in Refs. [44,45], and the resulting effective chiral Lagrangian at the hadronic level has been discussed in Refs. [14,16] (for a review see Ref. [12]).

In this work, we are primarily interested in the structure of the effective Lagrangian including new sources of CP violation below the weak scale. After integrating out the top quark, the Higgs boson, and the $W^{\pm}$ and Z gauge bosons, the needed operators are invariant under the $SU(3)_C \times U(1)_{\text{EM}}$ gauge group. At a scale $\mu < M_{W,Z}$, the effective Lagrangian including the leading (i.e., originating at dimension six) flavor-conserving CP-violating effects at the quark- and gluon-level can be written as follows[1]:

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} = {} & \mathcal{L}_{\text{SM}}|_{m_i=0} - m_i \bar{\psi}_{Li} \psi_{Ri} - m_i^* \bar{\psi}_{Ri} \psi_{Li} - \frac{g^2}{32\pi^2} \theta G \tilde{G} \\
& - \frac{v_{\text{ew}}}{2\Lambda_{\text{BSM}}^2} e (d_i^{(\gamma)} \bar{\psi}_{Li} \sigma_{\mu\nu} F^{\mu\nu} \psi_{Ri} + d_i^{(\gamma)*} \bar{\psi}_{Ri} \sigma_{\mu\nu} F^{\mu\nu} \psi_{Li}) \\
& - \frac{v_{\text{ew}}}{2\Lambda_{\text{BSM}}^2} g (d_i^{(g)} \bar{\psi}_{Li} \sigma_{\mu\nu} G^{\mu\nu} \psi_{Ri} + d_i^{(g)*} \bar{\psi}_{Ri} \sigma_{\mu\nu} G^{\mu\nu} \psi_{Li}) \\
& + \frac{d_G}{\Lambda_{\text{BSM}}^2} f^{abc} G_{\mu\nu}^a \tilde{G}^{\nu\beta,b} G_{\beta}^{\mu,c} + \text{4-quark operators},
\end{aligned}
$$
$$(1)$$

where e and g are the electric and color charges, v_{ew} is the Higgs VEV (vacuum expectation value), the index i runs over the active quark flavors (at $\mu \sim 1$ GeV one has $i \in \{u, d, s\}$), and $\tilde{G}^{\mu\nu,b} = \varepsilon^{\mu\nu\alpha\beta} G_{\alpha\beta}^b / 2$.

The first line in Eq. (1) contains the Standard Model dimension-four operators, including the mass matrix put in the standard diagonal form and a common phase, and the QCD θ term. Because of the anomalous Ward identity, a choice of fermion phases can be used to rotate the θ term into a CP-odd pseudoscalar quark mass term, instead.

The second and third lines in Eq. (1) contain the BSM contribution due to the quark magnetic (MDM) and electric dipole moment (EDM), and chromomagnetic (CMDM) and chromoelectric dipole moment (CEDM) operators, respectively. Below the weak scale these operators are of mass-dimension five: their origin as dimension-six operators at the high scale is hidden in the overall dimensionless factor of $v_{\text{ew}}/\Lambda_{\text{BSM}}$. It is important to note that the physical

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[1] Without loss of generality, we have performed a $SU(n_F)_L \times SU(n_F)_R$ transformation to put the quark mass matrix in diagonal form, with complex masses sharing a common phase ρ, namely, $m_i = |m_i| e^{i\rho}$. Moreover, note that the masses m_i and θ include (i) possible threshold corrections, i.e., effects that originate from higher-dimensional operators, such as $H^\dagger H G \tilde{G}$ and $H^\dagger H \bar{q}_L q_R' H$, and (ii) corrections induced by mixing with the chromoelectric dipole moment at finite quark mass.

meaning of these operators as *CP*-violating electric or *CP*-conserving magnetic moments relies on an implicit chiral phase convention. Similar to the quark mass terms, however, these operators explicitly break chiral symmetry, thus contributing to vacuum alignment [39]. As discussed in Sec. II B, the vacuum alignment, in turn, determines the unbroken *CP* symmetry, and a chiral rotation—which mixes the EDM and MDM, as well as CEDM and CMDM operators—may be needed to put the symmetry transformation in the standard form. If no complex phases appear in the Lagrangian after such a rotation, no physical *CP* violation can arise. In Sec. II C we, therefore, discuss the combinations of m_i, θ, $d_i^{(\gamma)}$, and $d_i^{(g)}$ that are independent of such phase choices, and give *CP*-violating contributions to observables.

Finally, the fourth line in Eq. (1) contains the *CP*-odd BSM operators that are genuinely of mass-dimension six at low energy, such as the Weinberg three-gluon operator and four-quark operators.

In order to convert experimental results on nucleon and nuclear EDMs into bounds or ranges for the short-distance *CP*-odd couplings, one needs to compute the effect of the *CP*-odd operators in Eq. (1) on hadronic observables, such as the nucleon EDM and the *T*-odd πNN couplings. One essential step in connecting the short-distance physics to hadronic observables involves defining UV finite operators in a suitable scheme, whose matrix elements can then be computed nonperturbatively using lattice QCD. In this work, we focus on the ultraviolet divergences and mixing structure of the leading gauge-invariant *CP*-odd dimension-five operators, namely, the quark CEDM and EDM. These operators are of great phenomenological interest, being the leading sources of flavor-diagonal *CP* violation in several extensions of the SM [3,12]. Moreover, since dimension-five operators can mix only with operators of dimension up to five (mixing with lower-dimensional operators occurs in mass-dependent renormalization schemes), we can consistently ignore operators of dimension six and higher, which we leave for future work.

B. *CP* symmetry and chiral symmetry breaking

In this subsection we discuss the connection between *CP* and chiral symmetries. The main point is that explicit chiral symmetry breaking selects the vacuum of the theory [39], as well as the unbroken *CP* symmetry. The unbroken *CP* symmetry takes the standard form given in Appendix A only after a chiral rotation that eliminates pion tadpoles, i.e., after implementing vacuum alignment [39] discussed in Sec. II C.

The *CP* transformation interchanges left-chiral particles with right-chiral antiparticles. It is implemented on chiral fermion fields by

$$
\begin{aligned}
\mathcal{CP}^{-1}\psi_L \mathcal{CP} &= i\gamma_2 \overline{\psi_L}^T \\
\mathcal{CP}^{-1}\psi_R \mathcal{CP} &= i\gamma_2 \overline{\psi_R}^T,
\end{aligned}
\tag{2}
$$

where $\mathcal{CP}$ is the *CP* operator (see Appendix A for details). The *CP* operation does not commute with chiral rotations, so we can consider its outer automorphisms. In fact, defining the chiral rotation operator $\hat{\chi}$ via (*i* labels quark flavors)

$$
\begin{aligned}
\hat{\chi}^{-1}\psi_{L,i}\hat{\chi} &= e^{-i\chi_i/2}\psi_{L,i} \\
\hat{\chi}^{-1}\psi_{R,i}\hat{\chi} &= e^{i\chi_i/2}\psi_{R,i},
\end{aligned}
\tag{3}
$$

one finds

$$
\begin{aligned}
\mathcal{CP}_\chi^{-1}\psi_{L,i}\mathcal{CP}_\chi &= ie^{i\chi_i}\gamma_2\overline{\psi_{L,i}}^T \\
\mathcal{CP}_\chi^{-1}\psi_{R,i}\mathcal{CP}_\chi &= ie^{-i\chi_i}\gamma_2\overline{\psi_{R,i}}^T,
\end{aligned}
\tag{4}
$$

where $\mathcal{CP}_\chi \equiv \hat{\chi}^{-1}\mathcal{CP}\hat{\chi}$. If chiral symmetry is a good symmetry of the Lagrangian $\mathcal{L}_0$, then each of these is an equivalent *CP* symmetry.

Because of the spontaneous breaking of chiral symmetry, almost all the $\mathcal{CP}_\chi$ are spontaneously broken by the vacuum of the theory. In this case, it is convenient to make a chiral phase choice such that the vacuum has a zero expectation value for all the flavor bilinears of the form $\langle \bar{\psi}_i\gamma_5\psi_j\rangle$. In fact, it is only with this phase choice that the pions, the Goldstone modes of the broken chiral symmetry, correspond to the operator $\bar{\psi}_i\gamma_5\psi_j$. With this choice of phases, in the "reference vacuum," the *CP* symmetry $\mathcal{CP}_0$ stays unbroken by the vacuum; we implicitly make this choice throughout this paper.

We next consider the effect of explicit chiral symmetry breaking. For a small explicit breaking of chiral symmetry, encoded in a new term $\delta\mathcal{L}$ in the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \varepsilon\delta\mathcal{L}$ (with $\varepsilon \ll 1$), chiral perturbation theory is expected to be a good guide to understanding the structure of the theory. But, because of the explicit breaking of the chiral symmetry, the vacuum is no longer degenerate: the explicit breaking chooses a direction in chiral space with which the vacuum aligns [39]. If this does not match the "reference vacuum", large corrections appear due to degenerate perturbation theory.

To avoid this problem, it is convenient to perform a chiral transformation $\hat{\chi}$ so that the explicit chiral symmetry breaking $\delta\mathcal{L}$ selects the reference vacuum, in which the unbroken *CP* symmetry takes the standard form, namely, $\mathcal{CP}_0$. The way to do this is to impose the condition that the vacuum state does not mix with the Goldstone state [39–41], i.e.,

$$
\langle \pi | \delta\mathcal{L} | \Omega \rangle = 0,
\tag{5}
$$

where $\delta\mathcal{L}$ are the chiral breaking terms after such a rotation and $|\Omega\rangle$ and $|\pi\rangle$ are the reference vacuum and Goldstone pion states respectively. If the only chiral breaking comes from the mass terms, this can be accomplished by rotating away the flavor nonsinglet *CP*-violating mass terms in Eq. (1) by the appropriate chiral transformation $\hat{\chi}$.

C. Vacuum alignment in presence of higher-dimensional operators

We now discuss the vacuum alignment in presence of higher-dimensional operators induced by BSM physics. After a general discussion of the chiral transformation needed to enforce Eq. (5), we specialize to the case in which the dominant source of chiral symmetry breaking is provided by the quark masses, and the dominant BSM operators are the quark (C)EDM and (C)MDM. In this case we present the vacuum-aligned effective Lagrangian in both scenarios with and without the Peccei-Quinn (PQ) mechanism [46].

Except for $G\tilde{G}$, all terms of dimension five and lower in the Lagrangian defined in Eq. (1) that violate CP are fermion bilinears that also violate chiral symmetry. Each of these terms mixes with a CP-conserving one under chiral rotation, and it is conventional to treat the two as real and imaginary parts of a single operator. Generalizing Eq. (1) let us write the chiral and CP-violating part of the Lagrangian involving quark bilinears as

$$\delta\mathcal{L} = -\sum_{i,\alpha}[d_i^\alpha O_i^\alpha + \text{H.c.}]$$

$$= -\sum_{i,\alpha}[\text{Re}\, d_i^\alpha \,\text{Re} O_i^\alpha + \text{Im}\, d_i^\alpha \,\text{Im}\, O_i^\alpha] \quad (6)$$

where

$$O_i^\alpha = \bar{\psi}_{L,i}\Gamma^\alpha\psi_{R,i},$$
$$\text{Re}\, O_i^\alpha = O_i^\alpha + O_i^{\alpha\dagger},$$
$$\text{Im}\, O_i^\alpha = i[O_i^\alpha - O_i^{\alpha\dagger}], \quad (7)$$

i is a flavor index and α parametrizes the different operators, characterized by the structure Γ^α. The first few operators are the mass term ($\alpha = 0$), the quark CEDM ($\alpha = 1$) and the quark EDM ($\alpha = 2$)

$$d_i^0 O_i^0 + \text{H.c.} = \bar{\psi}_i[(\text{Re}\, m_i) + i(\text{Im}\, m_i)\gamma_5]\psi_i \quad (8)$$

$$d_i^1 O_i^1 + \text{H.c.} = \frac{v_{\text{ew}}}{2\Lambda_{\text{BSM}}^2}g\bar{\psi}_i[(\text{Re}\, d_i^{(g)})\sigma_{\mu\nu}G^{\mu\nu}$$
$$+ i(\text{Im}\, d_i^{(g)})\sigma_{\mu\nu}G^{\mu\nu}\gamma_5]\psi_i \quad (9)$$

$$d_i^2 O_i^2 + \text{H.c.} = \frac{v_{\text{ew}}}{2\Lambda_{\text{BSM}}^2}e\bar{\psi}_i[(\text{Re}\, d_i^{(\gamma)})\sigma_{\mu\nu}F^{\mu\nu}$$
$$+ i(\text{Im}\, d_i^{(\gamma)})\sigma_{\mu\nu}F^{\mu\nu}\gamma_5]\psi_i. \quad (10)$$

In this notation, under a chiral rotation $\hat{\chi}$ (parametrized by χ_i)

$$d_i^\alpha \to d_i^\alpha e^{i\chi_i}$$
$$\theta \to \theta + \chi_1 + \cdots + \chi_{n_F}, \quad (11)$$

and we seek a chiral rotation such that Eq. (5) holds and at the same time $\theta \to 0$.

To implement Eq. (5), we need to introduce the non-perturbative matrix elements

$$\Delta_{ij}^\alpha \equiv \langle\pi_j|\text{Im}O_i^\alpha|\Omega\rangle \quad (12)$$

where the state $|\pi_j\rangle$ is interpolated by the field $\bar{\psi}_j i\gamma_5\psi_j$. Then the mixing of the vacuum with the neutral Goldstone modes $(|\pi_j\rangle - |\pi_k\rangle)/\sqrt{2}$ is proportional to $\sum_{i\alpha}\text{Im}d_i^\alpha(\Delta_{ij}^\alpha - \Delta_{ik}^\alpha)$. The condition in Eq. (5) for each neutral Goldstone mode $(|\pi_j\rangle - |\pi_k\rangle)/\sqrt{2}$ becomes

$$\sum_{i,\alpha}\text{Im}(d_i^\alpha e^{i\chi_i})[\Delta_{ij}^\alpha - \Delta_{ik}^\alpha] = 0, \qquad k = 1, j = 2, \ldots, n_F.$$
$$(13)$$

Since the unperturbed Lagrangian $\mathcal{L}_0$ is $SU(n_F)_V$ symmetric, the matrix elements can be written in terms of two constants, the diagonal Δ_S^α and the off-diagonal Δ_V^α, defined by $\Delta_{ij}^\alpha = \Delta_S^\alpha\delta_{ij} + \Delta_V^\alpha(1 - \delta_{ij})$. Eq. (13) implies, for each flavor $i = 1, \ldots, n_F$,

$$\sum_\alpha\text{Im}(d_i^\alpha e^{i\chi_i})r^{(\alpha)} = \kappa \quad (14)$$

where $r^{(\alpha)} \equiv (\Delta_S^\alpha - \Delta_V^\alpha)/(\Delta_S^0 - \Delta_V^0)$ (we divided out the matrix elements of the dimension-three operator $\text{Im}O_i^0 = \bar{\psi}_i i\gamma_5\psi_i$) and κ is a flavor-independent constant. Defining

$$d_i \equiv |d_i|e^{i\phi_i} \equiv \sum_\alpha d_i^\alpha r^{(\alpha)}, \quad (15)$$

the chiral rotation we want needs to satisfy, for each i,

$$|d_i|\sin(\chi_i + \phi_i) = \kappa. \quad (16)$$

Moreover, to implement $\theta \to 0$, one needs $\theta + \sum_i\chi_i = 0$, or, equivalently, the constant κ needs to satisfy

$$\theta - \sum_i\phi_i + \sum_i\sin^{-1}(\kappa|d_i|^{-1}) = 0. \quad (17)$$

Eqs. (16) and (17) provide a system of equations for χ_i and κ, which does not have a closed form solution for $n_F > 2$. On making the chiral transformation dictated by Eqs. (16) and (17) we find that CP violation is proportional to

$$\delta\mathcal{L}_{\text{CPV}} = \sum_{i,\alpha}\left[\kappa\text{Re}\frac{d_i^\alpha}{d_i} + \sqrt{|d_i|^2 - \kappa^2}\text{Im}\frac{d_i^\alpha}{d_i}\right]\text{Im}O_i^\alpha$$

$$\approx \sum_{i,\alpha}\left[-\bar{d}(\theta - \phi_{\text{tot}})\text{Re}\frac{d_i^\alpha}{d_i} + |d_i|\text{Im}\frac{d_i^\alpha}{d_i}\right]\text{Im}O_i^\alpha, \quad (18)$$

where $\bar{d}^{-1} \equiv \sum_i|d_i|^{-1}$ and $\phi_{\text{tot}} \equiv \sum_i\phi_i$, and the second line is obtained by solving Eq. (17) for small $\kappa/|d_i|$, which is appropriate when θ is small and the dominant chiral

violation comes from a real mass term (the latter condition implies $\phi_i \ll 1$).

Notice that if there is a single operator O_i^α that is the only source of *CP* violation, then $d_i \propto d_i^\alpha$, and the second term is zero. This is because in this case this term is also the only term that explicitly breaks the chiral symmetry and the vacuum aligns itself with this direction. As a result, performing a chiral rotation to make the vacuum have the conventional chiral phase removes any imaginary part from the operator, and *CP* violation can only come from the anomalous chiral rotations. In this case, however, the *CP* violation is proportional to the harmonic sum of the chiral violations from each flavor, and therefore vanishes if any flavor remains chirally symmetric.

In what follows, we will instead consider the situation where the dominant chiral breaking is always due to the $\alpha = 0$ mass term, i.e., $d_i \propto d_i^0$ approximately, and consider the case where all flavors are massive. Only in this case, the dominant source of *CP* violation is proportional to Imd_i^α. Consistent with this assumption, when studying mixing and renormalization we will keep in the operator basis terms proportional to the quark mass matrix.

With these assumptions and after vacuum alignment, the explicit form of Eq. (18), specialized to the case of a Lagrangian containing a mass term, quark EDM, and quark CEDM, is

$$\delta\mathcal{L}_{\text{CPV}} = \bar{\psi} i\gamma_5 \psi m_* \left(\bar{\theta} - \frac{r}{2}\text{Tr}[\mathcal{M}^{-1}([d_{CE}] - m_*\bar{\theta}\mathcal{M}^{-1}[d_{CM}])] \right)$$

$$+ \frac{r}{2}\bar{\psi} i\gamma_5 ([d_{CE}] - m_*\bar{\theta}\mathcal{M}^{-1}[d_{CM}])\psi$$

$$- \frac{ig}{2}\bar{\psi}\sigma_{\mu\nu}\gamma_5 G^{\mu\nu}([d_{CE}] - m_*\bar{\theta}\mathcal{M}^{-1}[d_{CM}])\psi$$

$$- \frac{ie}{2}\bar{\psi}\sigma_{\mu\nu}\gamma_5 F^{\mu\nu}([d_E] - m_*\bar{\theta}\mathcal{M}^{-1}[d_M])\psi, \quad (19)$$

where we defined $r \equiv r^{(1)} = (\Delta_S^1 - \Delta_V^1)/(\Delta_S^0 - \Delta_V^0)$ and neglected $r^{(2)} = O(\alpha_{EM}r^{(1)})$. We further defined

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \quad (20)$$

and the matrix-valued CEDM and CMDM couplings as

$$[d_{CE}] = \frac{v_{\text{ew}}}{\Lambda_{\text{BSM}}^2} \begin{pmatrix} \text{Im}d_u^{(g)} & 0 & 0 \\ 0 & \text{Im}d_d^{(g)} & 0 \\ 0 & 0 & \text{Im}d_s^{(g)} \end{pmatrix},$$

$$[d_{CM}] = \frac{v_{\text{ew}}}{\Lambda_{\text{BSM}}^2} \begin{pmatrix} \text{Re}d_u^{(g)} & 0 & 0 \\ 0 & \text{Re}d_d^{(g)} & 0 \\ 0 & 0 & \text{Re}d_s^{(g)} \end{pmatrix}, \quad (21)$$

with analogous definitions for the electric $[d_E]$ and magnetic $[d_M]$ couplings. Finally, $\bar{\theta} = \theta - n_F\rho$ with $n_F\rho$ the phase of the determinant of the mass matrix before the anomalous chiral rotation renders it real, and m_* is the reduced quark mass

$$m_* = \frac{m_s m_d m_u}{m_s(m_u + m_d) + m_u m_d}. \quad (22)$$

The first term in Eq. (19) is the familiar $\bar{\theta}$ term, shifted by a correction proportional to the quark CEDM and a second correction, proportional to the coefficients of the CMDM multiplied by $\bar{\theta}$. The third and fourth lines of Eq. (19) contains the quark (C)EDM operators, which after vacuum alignment receive a correction proportional to the (C)MDM coefficient multiplied by $\bar{\theta}$. Moreover, vacuum alignment causes the appearance of a complex mass term, proportional to the same combination of the CEDM and CMDM coefficients [second line of Eq. (19)].

The above discussion is valid in absence of PQ mechanism [46]. As we review in Appendix C, if *CP* violation arises only from the mass term, the PQ mechanism dynamically relaxes $\bar{\theta}$ to zero. In the presence of other *CP*-violating sources, like the quark CEDM, the Peccei-Quinn (PQ) mechanism causes $\bar{\theta}$ to relax to a nonzero value $\bar{\theta}_{\text{ind}}$, proportional to the new source of *CP* violation. In particular, as we discuss in further detail in Appendix C, in the presence of the quark CEDM

$$\bar{\theta}_{\text{ind}} = \frac{r}{2}\text{Tr}[\mathcal{M}^{-1}[d_{CE}]], \quad (23)$$

thus enforcing a cancellation between the first two terms in Eq. (19). Since $\bar{\theta}_{\text{ind}}$ is suppressed by two powers of Λ_{BSM}, terms proportional to $\bar{\theta}[d_{CM}]$ in Eq. (19) become effectively dimension eight and can be neglected. Thus, if the PQ mechanism is at work, the first line of Eq. (19) vanishes and the terms proportional to $\bar{\theta}$ in the second and third line of Eq. (19) can be neglected, leading to

$$\delta\mathcal{L}_{\text{CPV}}^{\text{PQ}} = \frac{r}{2}\bar{\psi} i\gamma_5 [d_{CE}]\psi - \frac{ig}{2}\bar{\psi}\sigma_{\mu\nu}\gamma_5 G^{\mu\nu}[d_{CE}]\psi$$

$$- \frac{ie}{2}\bar{\psi}\sigma_{\mu\nu}\gamma_5 F^{\mu\nu}[d_E]\psi, \quad (24)$$

with both CEDM and pseudoscalar quark density with flavor structure dictated by $[d_{CE}]$.

Equation (19) and Eq. (24) provide the vacuum-aligned low-energy Lagrangians, in presence of BSM sources of *CP* and chiral symmetry violation. They are particularly useful within the chiral perturbation theory framework, as they guarantee the cancellation of tadpole diagrams in which Goldstone modes are absorbed by the vacuum. This form of the *CP*-violating perturbation allows one to identify what nonperturbative matrix elements are needed in order to address the impact of a BSM-induced CEDM

operator on the nucleon EDM, i.e., the dependence of d_n on $[d_{CE}]$. Both with and without PQ mechanism the effective Lagrangian involves the CEDM operator as well as flavor singlet and nonsinglet pseudoscalar quark operators. Moreover, at the lowest order, the effect of flavor nonsinglet $\bar{\psi} i \gamma_5 t^{3,8} \psi$ operators is proportional to insertions of the flavor-singlet density $\bar{\psi} i \gamma_5 \psi$. This is very simple to see within the functional integral approach, in which $\bar{\psi} i \gamma_5 t^{3,8} \psi$ can be eliminated through a nonanomalous axial rotation. The same result can be obtained within an operator approach. In this framework, using soft-pion techniques, one can show that a cancellation occurs between nontadpole and tadpole diagrams with insertion of $\bar{\psi} i \gamma_5 t^{3,8} \psi$, leaving a term proportional to the insertion of $\bar{\psi} i \gamma_5 \psi$. In absence of PQ mechanism, the resulting flavor-singlet pseudoscalar insertion proportional to $[d_{CE}]$ cancels exactly the existing singlet term in Eq. (19). If the PQ mechanism is operative, the resulting flavor-singlet pseudoscalar insertion is proportional to $m_* \bar{\theta}_{\mathrm{ind}}$. The net effect is equivalent to replacing Eqs. (19) and (24) with

$$
\begin{aligned}
\delta \mathcal{L}_{\mathrm{CPV}} = {} & m_* \bar{\theta} \, \bar{\psi} \, i \gamma_5 \psi - \frac{ig}{2} \bar{\psi} \sigma_{\mu\nu} \gamma_5 G^{\mu\nu} ([d_{CE}] \\
& - m_* \bar{\theta} \mathcal{M}^{-1} [d_{CM}]) \psi \\
& - \frac{ie}{2} \bar{\psi} \sigma_{\mu\nu} \gamma_5 F^{\mu\nu} ([d_E] - m_* \bar{\theta} \mathcal{M}^{-1} [d_M]) \psi
\end{aligned} \tag{25a}
$$

$$
\begin{aligned}
\delta \mathcal{L}_{\mathrm{CPV}}^{\mathrm{PQ}} = {} & m_* \bar{\theta}_{\mathrm{ind}} \bar{\psi} i \gamma_5 \psi - \frac{ig}{2} \bar{\psi} \sigma_{\mu\nu} \gamma_5 G^{\mu\nu} [d_{CE}] \psi \\
& - \frac{ie}{2} \bar{\psi} \sigma_{\mu\nu} \gamma_5 F^{\mu\nu} [d_E] \psi .
\end{aligned} \tag{25b}
$$

These can be regarded as partially aligned effective Lagrangians, in which only the dominant mass term has been aligned to eliminate pion tadpoles, while the BSM perturbation is not aligned. While the physics cannot depend on the choice of equivalent parametrization of Eq. (19), Eq. (24) and Eq. (25), use of different effective Lagrangians is a matter of convenience, depending on the nonperturbative approach employed to study hadronic physics. Starting from the Lagrangian in Eq. (25), in the chiral effective theory approach tadpole diagrams arise, that can be dealt within perturbation theory [14]. On the other hand, in a nonperturbative approach based on the functional integral, such as lattice QCD, the partially aligned Lagrangian can be more convenient: it shows that the only needed nonperturbative matrix elements involve the (C)EDM operator and the singlet pseudoscalar density (or equivalently $G\tilde{G}$).

D. *CP*-violating effective Lagrangian at the hadronic scale

To summarize the above discussion, at the hadronic scale ($\mu \sim 1$ GeV) the vacuum-aligned flavor-conserving effective Lagrangian including the leading BSM sources of *CP* violation (up to dimension five) can be written as follows,

$$
\begin{aligned}
\mathcal{L} = {} & \mathcal{L}_{\mathrm{QCD+QED}} - \bar{\psi} \mathcal{M} \psi - \bar{\psi} [\delta \mathcal{M}] i \gamma_5 \psi \\
& - \frac{ie}{2} \bar{\psi} \sigma_{\mu\nu} \gamma_5 F^{\mu\nu} [D_E] Q \psi - \frac{ig}{2} \bar{\psi} \sigma_{\mu\nu} \gamma_5 G^{\mu\nu} [D_{CE}] \psi,
\end{aligned} \tag{26}
$$

where

$$
Q = \begin{pmatrix} q_u & 0 & 0 \\ 0 & q_d & 0 \\ 0 & 0 & q_s \end{pmatrix}. \tag{27}
$$

Here we are neglecting operators that are total derivatives and/or vanish by using the equations of motion (EOM), needed later on when we impose off-shell renormalization conditions at finite momentum insertion. The matrix-valued *CP*-violating couplings $[\delta \mathcal{M}]$, $[D_{CE}]$, $[D_E]$ are related to the short-distance couplings of Eq. (1) via Eq. (21) and Eq. (19) or Eq. (24), depending on whether or not the PQ mechanism is assumed. The pseudoscalar mass term $[\delta \mathcal{M}]$ in general has a nonsinglet structure in flavor space, though at leading order, its physical effects can be related to a flavor-singlet mass term as discussed in Sec. II C [see Eqs. (25)].

III. *CP*-ODD OPERATOR OF DIMENSION ≤5

The only *T*-odd and *P*-odd operators of dimension five appearing in the low-energy effective Lagrangian Eq. (26) are the quark EDM and CEDM, whose mixing and renormalization we wish to discuss.

The analysis of the quark EDM is relatively simple: this operator is a quark bilinear from the point of view of strong interactions, and it is simply related to the tensor density. Knowledge of the nucleon tensor charges immediately allows one to extract the contribution of the quark EDM to the nucleon EDM [47,48]. To lowest (zeroth) order in electroweak interactions, this operator renormalizes diagonally, precisely as the tensor density. Since we are not interested in the hadronic matrix elements to a precision of order $\alpha_{EM}/\pi < 1\%$, we neglect the quark EDM mixing with any other operator.

On the other hand, the quark CEDM operator does not renormalize diagonally: it mixes with the quark EDM and other operators of dimension five or lower. The mixing structure is particularly rich if one considers renormalization within a so-called regularization-independent (RI), momentum subtraction (MOM) scheme, amenable to nonperturbative calculations in lattice QCD [49]. In this family of schemes, the renormalization conditions are imposed on off-shell quark matrix elements in a fixed gauge, thus requiring the inclusion of operators that do not contribute to physical matrix elements, such as total derivatives and operators that vanish on-shell by using the equations of motion (EOM). We next discuss the relevant operator basis, the mixing structure, and the strategy to determine the renormalization matrix.

A. Operator basis

The implementation of RI momentum subtraction schemes requires working in a fixed gauge. With gauge fixing, full gauge invariance is lost and the action is only invariant under BRST transformations [50,51]. A given gauge invariant operator O (we have in mind the quark CEDM) mixes under renormalization with two classes of operators of the same (or lower) dimension [52,53]: (i) gauge-invariant and ghost-free operators with the same symmetry properties as O (Lorentz, CP, P) that do not vanish by the EOM and (ii) "nuisance" operators allowed by the solution to the Ward identities associated with the BRST symmetry: these vanish by the EOM and need not be gauge invariant. The "nuisance" operators can be constructed as off-shell BRST variation of operators that have ghost number -1, but otherwise with same symmetry properties as O, as discussed in Ref. [52] and detailed in Appendix B.[2]

Following the above general prescription, we have constructed the basis of CP-odd (T- and P-odd) operators that mix with the quark CEDM operator (the CP transformation properties of fields are reviewed in Appendix A). We present our results for $n_F = 3$. To restrict the possible structures in flavor space we use the spurion method. While the effective Lagrangian in Eq. (26) is not invariant under chiral transformations on the quark fields $\psi_{L,R} \rightarrow U_{L,R}\psi_{L,R}$ with $U_{L,R} \in SU(3)_{L,R}$, one can formally recover chiral invariance by assigning spurion transformation properties to the CEDM coupling matrix ($[D_{CE}] \rightarrow U_L[D_{CE}]U_R^\dagger$), the mass matrix ($\mathcal{M} \rightarrow U_L\mathcal{M}U_R^\dagger$), and the charge matrix ($Q \rightarrow U_{L,R}QU_{L,R}^\dagger$). One then includes in the basis operators that are chirally invariant in the spurion sense, and are linear in the CEDM spurion $[D_{CE}]$. Eventually, we set $[D_{CE}] \rightarrow t^a$ ($a = 0, 3, 8$), where $t^0 = 1/\sqrt{6}I_{3\times3}$ is proportional to the identity matrix in flavor space, while for $a = 3, 8$, $t^a = \lambda^a/2$, with λ^a the SU(3) Gell-Mann matrices (normalizations are such that $\mathrm{Tr}_F(t^a t^a) = 1/2$ for $a = 0, 3, 8$).

In our basis we include operators proportional to the quark mass matrix for two reasons: (i) the identification of the CPV terms in Eq. (26) assumed the quark mass to be the dominant source of explicit chiral symmetry breaking, and (ii) we wish to include the effect of the strange quark, for which $m_s/\Lambda_{\mathrm{QCD}}$ is not a big suppression parameter.

Finally, in order to present the operators that vanish by the EOM in a compact form, we introduce the combinations:

$$\psi_E \equiv (iD^\mu\gamma_\mu - \mathcal{M})\psi,$$
$$D_\mu = \partial_\mu - igA_\mu^a T^a - ieQA_\mu^{(\gamma)} \tag{28}$$

$$\bar{\psi}_E \equiv -\bar{\psi}(i\overleftarrow{D}^\mu\gamma_\mu + \mathcal{M}),$$
$$\overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu + igA_\mu^a T^a + ieQA_\mu^{(\gamma)}. \tag{29}$$

Note that ψ_E transforms under CP in the same way as ψ (see Appendix A).

Next, we enumerate the operators of dimension five and lower that can mix with the quark CEDM:

$$C = ig\bar{\psi}t^a\sigma^{\mu\nu}\gamma_5 G_{\mu\nu}\psi, \tag{30}$$

labeled by the flavor-diagonal structure t^a ($a = 0, 3, 8$). We will use the notation $O_i^{(d)}$ to indicate the ith operator of dimension d. If the regularization breaks chiral symmetry, i.e., an additional left-right spurion (proportional to the identity in the case of Wilson fermions) is present in the effective Lagrangian, the CEDM operator can mix with additional operators. While we will restrict our analysis to the case of good chiral symmetry (which can be attained on the lattice by using domain-wall [55] or overlap [56] fermions), we will nonetheless identify the additional operators appearing at a given dimensionality. Finally, note that there are no CP-odd operators containing ghost-antighost fields up to and including dimension five.

1. Dimension three

At dimension three there is only one operator allowed by the symmetries:

$$O^{(3)} \equiv P = \bar{\psi}i\gamma_5 t^a\psi. \tag{31}$$

This operator mixes with the quark CEDM even in the absence of other sources of chiral symmetry breaking, such as mass terms or regularization artifacts. Therefore, the lattice operator C_L requires subtraction of power divergences due to mixing with the lower-dimensional operator P_L.[3] Defining the subtracted operator $C \equiv C_L - \tilde{Z}P_L$, one can determine $\tilde{Z}$ by requiring that the quark two-point function $\Gamma_C^{(2)}$ defined in Eq. (52) vanishes at a given symmetric kinematic point $p^2 = p'^2 = q^2 = -\Lambda_0^2$ for $m_q \rightarrow 0$, namely, $\mathrm{Tr}(\Gamma_C^{(2)}\gamma_5 t^a)_{\Lambda_0} = 0$.

2. Dimension four

Assuming good chiral symmetry, there are no dimension-four operators that mix with the quark CEDM operator. If the regularization breaks chiral symmetry in a flavor blind fashion, the CEDM can mix with the following operators:

---

[2]See Ref. [54] for an application of this formalism to the CP-even sector of QCD. There is a one-to-one correspondence between our operator basis and the one of Ref. [54], provided we drop the total-derivative operators from our basis and set $m = 0$, as done in Ref. [54].

[3]Since dimensionally regularized operators do not mix with lower-dimensional operators at any finite order in perturbation theory, we will, when necessary, use a subscript L for operators regularized in a scheme, like the lattice, that includes a hard cutoff.

$$G\tilde{G}, \qquad \partial_\mu(\bar{\psi}\gamma^\mu\gamma_5 t^a\psi), \qquad \bar{\psi}i\gamma_5\{\mathcal{M},t^a\}\psi,$$

$$\text{Tr}[\mathcal{M}t^a]\bar{\psi}i\gamma_5\psi, \qquad \text{Tr}[\mathcal{M}]\bar{\psi}i\gamma_5 t^a\psi. \qquad (32)$$

3. Dimension five

At dimension five, 14 Hermitian operators are present. The first ten operators are gauge-invariant and do not vanish by the EOM. The latter four are "nuisance" operators. To all operators we assign a number and also a more suggestive name:

$$O_1^{(5)} \equiv C = ig\bar{\psi}\tilde{\sigma}^{\mu\nu}G_{\mu\nu}t^a\psi \quad \tilde{\sigma}^{\mu\nu} \equiv \frac{1}{2}(\sigma^{\mu\nu}\gamma_5 + \gamma_5\sigma^{\mu\nu}) \tag{33}$$

$$O_2^{(5)} \equiv \partial^2 P = \partial^2(\bar{\psi}i\gamma_5 t^a\psi) \tag{34}$$

$$O_3^{(5)} \equiv E = \frac{ie}{2}\bar{\psi}\tilde{\sigma}^{\mu\nu}F_{\mu\nu}\{Q,t^a\}\psi \tag{35}$$

$$O_4^{(5)} \equiv (mF\tilde{F}) = \text{Tr}[\mathcal{M}Q^2 t^a]\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{36}$$

$$O_5^{(5)} \equiv (mG\tilde{G}) = \text{Tr}[\mathcal{M}t^a]\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}G_{\mu\nu}^b G_{\alpha\beta}^b \tag{37}$$

$$O_6^{(5)} \equiv (m\partial\cdot A)_1 = \text{Tr}[\mathcal{M}t^a]\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) \tag{38}$$

$$O_7^{(5)} \equiv (m\partial\cdot A)_2 = \frac{1}{2}\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\{\mathcal{M},t^a\}\psi)$$
$$- \frac{1}{3}\text{Tr}[\mathcal{M}t^a]\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) \tag{39}$$

$$O_8^{(5)} \equiv (m^2 P)_1 = \frac{1}{2}\bar{\psi}i\gamma_5\{\mathcal{M}^2,t^a\}\psi \tag{40}$$

$$O_9^{(5)} \equiv (m^2 P)_2 = \text{Tr}[\mathcal{M}^2]\bar{\psi}i\gamma_5 t^a\psi \tag{41}$$

$$O_{10}^{(5)} \equiv (m^2 P)_3 = \text{Tr}[\mathcal{M}t^a]\bar{\psi}i\gamma_5\mathcal{M}\psi \tag{42}$$

$$O_{11}^{(5)} \equiv P_{EE} = i\bar{\psi}_E\gamma_5 t^a\psi_E \tag{43}$$

$$O_{12}^{(5)} \equiv \partial\cdot A_E = \partial_\mu[\bar{\psi}_E\gamma^\mu\gamma_5 t^a\psi + \bar{\psi}\gamma^\mu\gamma_5 t^a\psi_E] \tag{44}$$

$$O_{13}^{(5)} \equiv A_\partial = \bar{\psi}\gamma_5\overrightarrow{\partial}t^a\psi_E - \bar{\psi}_E\overleftarrow{\partial}\gamma_5 t^a\psi \tag{45}$$

$$O_{14}^{(5)} \equiv A_{A^{(\gamma)}} = \frac{ie}{2}(\bar{\psi}\{Q,t^a\}\slashed{A}^{(\gamma)}\gamma_5\psi_E - \bar{\psi}_E\{Q,t^a\}\slashed{A}^{(\gamma)}\gamma_5\psi). \tag{46}$$

With a flavor blind breaking of chiral symmetry, the CEDM can mix with additional dimension-five operators, namely,

$$\text{Tr}[\mathcal{M}]\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5 t^a\psi), \qquad \text{Tr}[\mathcal{M}]\bar{\psi}i\gamma_5\mathcal{M}t^a\psi,$$

$$(\text{Tr}\mathcal{M})^2\bar{\psi}i\gamma_5 t^a\psi, \qquad \text{Tr}[\mathcal{M}^2 t^a]\bar{\psi}i\gamma_5\psi,$$

$$\text{Tr}[\mathcal{M}]\text{Tr}[\mathcal{M}t^a]\bar{\psi}i\gamma_5\psi. \tag{47}$$

In the perturbative analysis presented below, we will use dimensional regularization. For γ_5 we will present results for both the naïve anticommuting scheme known as naïve dimensional regularization (NDR) and the consistent 't Hooft–Veltman (HV) scheme (see [53] and references therein). It is important that the regulator does not break the Hermiticity of the operator basis: when considering operator insertions in the dimensionally regulated theory, care must be taken to ensure that the operators remain Hermitian for arbitrary spacetime dimension d. This is essential in order to obtain correct results for the finite parts of the diagrams. In what follows we will need to insert $O_1^{(5)}$ in loop diagrams, so we provide in Eq. (33) the explicit Hermitian form of $O_1^{(5)}$, valid both in HV and NDR schemes.

B. Mixing structure and Renormalization Conventions

The relation between renormalized operators (O_i) in any given scheme and bare operators ($O_i^{(0)}$) (expressed in terms of the bare fields) can be written as:

$$O_i^{(0)} = Z_{ij}O_j. \tag{48}$$

The renormalization mixing matrix Z_{ij} is scheme-dependent and has the general structure given in Table I.[4] This structure is dictated by several considerations, including (i) power-counting (some operators are effectively of dimension three and four with either factors of masses or external derivatives and cannot mix with genuinely dimension-five operators), (ii) BRST invariance [52], and (iii) vanishing by EOM or at zero four-momentum injection. Indicating the gauge-invariant operators that do not vanish on using the EOM ($O_i^{(5)}$ for $i = 1, ..., 10$) by O_α and the "nuisance" operators ($O_i^{(5)}$ for $i = 11, ..., 14$) by N_α, the renormalization matrix has the block-structure

$$\begin{pmatrix} O^{(0)} \\ N^{(0)} \end{pmatrix} = \begin{pmatrix} Z_O & Z_{ON} \\ 0 & Z_N \end{pmatrix} \begin{pmatrix} O \\ N \end{pmatrix}. \tag{49}$$

The divergent part of Z_O (proportional to $1/(d-4)$ in dimensional regularization or $\log\Lambda^2$ in a cutoff theory), controlling the physical anomalous dimension, is independent of the gauge-fixing choice [52].

In the following we will provide Z_O in the $\overline{\text{MS}}$ scheme and in a momentum subtraction scheme to one-loop order.

---

[4]Working to first order in insertions of the new physics operator, each sector labeled by the diagonal flavor structure t^a ($a = 0, 3, 8$) renormalizes independently, so that the renormalization matrix has a block-diagonal form in flavor space.

TABLE I.   Mixing structure of the dimension-fivee operators, with "x" representing nonzero entries. Throughout, we neglect effects proportional to the electroweak coupling α_{EW}.

	C	$\partial^2 P$	E	$mF\tilde{F}$	$mG\tilde{G}$	$(m\partial\cdot A)_1$	$(m\partial\cdot A)_2$	$(m^2P)_1$	$(m^2P)_2$	$(m^2\hat{P})_3$	P_{EE}	$\partial\cdot A_E$	A_∂	$A_{A^{(\gamma)}}$
C	x	x	x	x	x	x	x	x	x	x	x	x	x	x
$\partial^2 P$		x												
E			x											
$mF\tilde{F}$				x										
$mG\tilde{G}$					x	x								
$(m\partial\cdot A)_1$						x								
$(m\partial\cdot A)_2$							x							
$(m^2P)_1$								x						
$(m^2\hat{P})_2$									x					
$(m^2\hat{P})_3$										x				
P_{EE}											x	x	x	
$\partial\cdot A_E$												x		
A_∂											x	x	x	x
$A_{A^{(\gamma)}}$														x

We will perform the calculations in dimensional regularization ($d = 4 - 2\epsilon$) and will present results in both the HV and NDR schemes [53] for γ_5 and the γ-matrix algebra (we use the definition of γ_5 and $\epsilon^{\mu\nu\alpha\beta}$ given in Ref. [57]). To extract the operator renormalization matrix we define the field, coupling, and mass renormalization constants as

$$\psi^{(0)} = \sqrt{Z_\psi}\psi \qquad (50a)$$

$$A_\mu^{(0)} = \sqrt{Z_G}A_\mu \qquad (50b)$$

$$g^{(0)} = Z_g g\mu_{\overline{\mathrm{MS}}}^\epsilon \quad \mu_{\overline{\mathrm{MS}}} \equiv \mu \frac{e^{\gamma_E/2}}{(4\pi)^{1/2}} \qquad (50c)$$

$$m^{(0)} = Z_m m. \qquad (50d)$$

Here, as usual, μ denotes an arbitrary parameter with dimensions of mass, introduced to keep the renormalized coupling g dimensionless ($[g] = 0$), while $[m] = 1$, $[\psi] = 3/2 - \epsilon$, and $[A_\mu] = 1 - \epsilon$. Note that g and $\alpha_s \equiv g^2/(4\pi)$ depend on both μ and ϵ, so that $d\alpha_s/d(\log\mu) = -2\epsilon\alpha_s + O(\alpha_s^2)$.

Finally, let us discuss different conventions for the renormalization factors for fields, couplings, masses, and operators, generically denoted by Z. Our definitions in Eqs. (48) and (50) follow the notation typically used in the perturbative QCD literature (see for example [58]). However, we warn the reader that the lattice community typically uses a different convention (fleshed out explicitly in Ref. [59]), which is related to the one followed here by replacing everywhere $Z \to Z^{-1}$.

IV. GREEN'S FUNCTION CALCULATIONS

In order to determine Z_{ij} and the relation between $\overline{\mathrm{MS}}$ and the RI-$\tilde{\mathrm{S}}$MOM scheme to be defined in Sec. VI below,

we will study amputated two- and three-point functions[5] with operator insertion. These are shown in Fig. 1 and defined as follows:

$$\int d^4x e^{-iq\cdot x}\langle g(p',\epsilon^{*'})|O(x)|g(p,\epsilon)\rangle$$
$$= (2\pi)^4\delta^{(4)}(q+p-p')\epsilon_\mu^{*'}(p')\Gamma_O^{\mu\nu}(p,p')\epsilon_\nu(p) \qquad (51)$$

$$\int d^4x e^{-iq\cdot x}\langle q(p')|O(x)|q(p)\rangle$$
$$= (2\pi)^4\delta^{(4)}(q+p-p')\bar{u}(p')\Gamma_O^{(2)}(p,p')u(p) \qquad (52)$$

$$\int d^4x e^{-iq\cdot x}\langle q(p'),g(k,\epsilon^*)|O(x)|q(p)\rangle$$
$$= (2\pi)^4\delta^{(4)}(q+p-p'-k)\bar{u}(p')\Gamma_O^{(3)}(p,p',k)u(p). \qquad (53)$$

To minimize notational clutter in the above equations and throughout the paper we will suppress the color indices, which can be restored as follows. The gluon two-point function $\Gamma_O^{\mu\nu}$ carries the color structure $\delta^{cc'}$, where c, c' are the octet color indices labeling the two amputated gluon external legs. The quark two-point function $\Gamma_O^{(2)}$ carries the color structure δ_{ij}, where i, j are the color indices labeling the two amputated quark external legs. The quark-quark-gluon three-point function $\Gamma_O^{(3)}$ carries the color structure T_{ij}^b, where b is the octet color index labeling the amputated gluon external leg and i, j are the color indices labeling the amputated quark external legs. Moreover, in our notation

---

[5] Since the terminology of lattice simulations also counts the points at which the operator is inserted, these correspond to three- and four- point functions in that terminology.
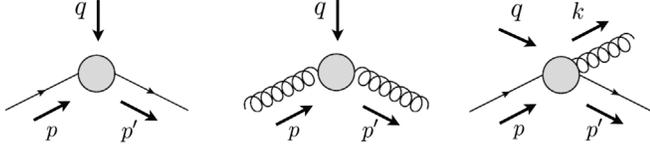
FIG. 1. Momentum flow of generic diagrams contributing to the quark-quark, gluon-gluon, and quark-quark-gluon Green's functions with operator insertion. The shaded blob represents the operator insertion with incoming 4-momentum q and higher order corrections. In the four-point function, the gluon (photon) momentum is labeled by k.

$\Gamma_O^{(3)}$ is linear in the gluon polarization vector, i.e., $\Gamma_O^{(3)} = \epsilon_\mu^{b*}(k)\Gamma_O^{(3)\mu}$. Analogous definitions exist for the photon two-point function and the quark-quark-photon three-point function, which we will denote by $\Gamma_O^{\mu\nu(\gamma)}(p, p')$ and $\Gamma_O^{(3,\gamma)}(p, p', k)$ (the latter carries color structure δ_{ij}). The assignment of momentum flow in these two- and three-point functions with operator insertion is shown in Fig. 1.

In any scheme, the renormalization factors Z_{ij} introduced in Eq. (48) can then be determined by imposing conditions on the two- and three-point functions defined above. Working to first order in α_s, the needed Green's functions with insertion of $O_1^{(5)} \equiv C$ read:

$$\Gamma_C^{(2)} = \Gamma_C^{(2)}\Big|_{1\text{-loop}} + \sum_{j\neq 1}(Z^{-1})_{1j}\Gamma_{O_j^{(5)}}^{(2)}\Big|_{\text{tree}} \quad (54)$$

$$\Gamma_C^{(3)} = \Gamma_C^{(3)}\Big|_{1\text{-loop}} + (Z_\psi Z_g \sqrt{Z_G}(Z^{-1})_{11} - 1)\Gamma_C^{(3)}\Big|_{\text{tree}}$$
$$+ \sum_{j\neq 1}(Z^{-1})_{1j}\Gamma_{O_j^{(5)}}^{(3)}\Big|_{\text{tree}} \quad (55)$$

$$\Gamma_C^{\mu\nu} = \Gamma_C^{\mu\nu}\Big|_{1\text{-loop}} + \sum_{j\neq 1}(Z^{-1})_{1j}\Gamma_{O_j^{(5)}}^{\mu\nu}\Big|_{\text{tree}}. \quad (56)$$

The simplest perturbative scheme is $\overline{\text{MS}}$, in which one determines the $(Z^{-1})_{1j}$ by requiring cancellation of the poles in $\epsilon = (4-d)/2$. Similar relations involving insertions of $O_{i\neq 1}^{(5)}$ allow one to determine the remaining entries Z_{ij} of the renormalization matrix. This program requires computing the tree-level and one-loop results for the two- and three-point functions, to which we turn next.

A. Tree level matrix elements

In this section we give tree-level results for the gluon two-point functions $\Gamma_O^{\mu\nu}(p, p')$, the quark two-point functions $\Gamma_O^{(2)}(p, p')$, the gluon-quark-quark three-point functions $\Gamma_O^{(3)}(k, p, p')$, and the photon-quark-quark three-point functions $\Gamma_O^{(3,\gamma)}(k, p, p')$, for all the relevant operators $O_i^{(d)}$.

The only operator with nonzero two-gluon matrix element at tree level is $O_5^{(5)} \equiv mG\tilde{G}$:

$$\Gamma_{O_5^{(5)}}^{\mu\nu}(p, p') = \text{Tr}[\mathcal{M}t^a] \times 4\epsilon^{\mu\nu\alpha\beta}p_\alpha p'_\beta. \quad (57)$$

An analogous result holds for the photon two-point function $\Gamma_{O_4^{(5)}}^{\mu\nu(\gamma)}(p, p')$.

In Tables II, III, and IV, we give the tree-level 1-particle irreducible (1PI) matrix elements $\Gamma_O^{(2)}(p, p')$, $\Gamma_O^{(3)}(k, p, p')$, and $\Gamma_O^{(3\gamma)}(k, p, p')$ for each operator. Throughout, we use the notation:

TABLE II. Nonvanishing tree-level 2-point functions with operator insertion. For notational conventions and momentum flow, see discussion below Eq. (53).

O	$\Gamma_O^{(2)}$
$O^{(3)} = P$	$i\gamma_5 t^a$
$O_2^{(5)} = \partial^2 P$	$-iq^2\gamma_5 t^a$
$O_6^{(5)} = (m\partial \cdot A)_1$	$\text{Tr}[\mathcal{M}t^a]i q\!\!\!/ \gamma_5$
$O_7^{(5)} = (m\partial \cdot A)_2$	$(\frac{1}{2}\{\mathcal{M}, t^a\} - \frac{1}{3}\text{Tr}[\mathcal{M}t^a])i q\!\!\!/ \gamma_5$
$O_8^{(5)} = (m^2 P)_1$	$\frac{1}{2}\{\mathcal{M}^2, t^a\}i\gamma_5$
$O_9^{(5)} = (m^2 P)_2$	$\text{Tr}[\mathcal{M}^2 t^a]i\gamma_5$
$O_{10}^{(5)} = (m^2 P)_3$	$\text{Tr}[\mathcal{M}t^a]i\mathcal{M}\gamma_5$
$O_{11}^{(5)} = P_{EE}$	$-i[p \cdot p' t^a - \frac{1}{2}\{\mathcal{M}^2, t^a\} + i\sigma(p, p')t^a$ $+\frac{1}{2}\{\mathcal{M}, t^a\}q\!\!\!/]\gamma_5$
$O_{12}^{(5)} = \partial \cdot A_E$	$i[q^2 t^a - \{\mathcal{M}, t^a\}q\!\!\!/ - 2i\sigma(p, p')t^a]\gamma_5$
$O_{13}^{(5)} = A_\partial$	$-i[(p^2 + p'^2)t^a - \frac{1}{2}\{\mathcal{M}, t^a\}q\!\!\!/]\gamma_5$

TABLE III. Nonvanishing tree-level 1PI quark-quark-gluon 3-point functions. For notational conventions and momentum flow, see discussion below Eq. (53).

O	$\Gamma_O^{(3)}$ (1PI)
$O_1^{(5)} = C$	$2g\sigma(\epsilon^*, k)\gamma_5 t^a$
$O_{11}^{(5)} = P_{EE}$	$-ig[\epsilon^* \cdot (p + p') - i\sigma(\epsilon^*, p - p')]\gamma_5 t^a$
$O_{12}^{(5)} = \partial \cdot A_E$	$2g\sigma(\epsilon^*, q)\gamma_5 t^a$
$O_{13}^{(5)} = A_\partial$	$-ig[\epsilon^* \cdot (p + p') + i\sigma(\epsilon^*, p - p' - 2k)]\gamma_5 t^a$

TABLE IV. Nonvanishing tree-level 1PI quark-quark-photon 3-point functions. For notational conventions and momentum flow, see discussion below Eq. (53).

O	$\Gamma_O^{(3,\gamma)}$ (1PI)
$O_3^{(5)} = E$	$e\{Q, t^a\}\sigma(\epsilon^*, k)\gamma_5$
$O_{11}^{(5)} = P_{EE}$	$-\frac{ie}{2}\{Q, t^a\}[\epsilon^* \cdot (p + p') - i\sigma(\epsilon^*, p - p')]\gamma_5$
$O_{12}^{(5)} = \partial \cdot A_E$	$e\{Q, t^a\}\sigma(\epsilon^*, q)\gamma_5$
$O_{13}^{(5)} = A_\partial$	$-\frac{ie}{2}\{Q, t^a\}[\epsilon^* \cdot (p + p') + i\sigma(\epsilon^*, p - p' - 2k)]\gamma_5$
$O_{14}^{(5)} = A_{A^{(\gamma)}}$	$-\frac{ie}{2}\{Q, t^a\}[\epsilon^* \cdot (p + p') - i\sigma(\epsilon^*, p - p')]\gamma_5$
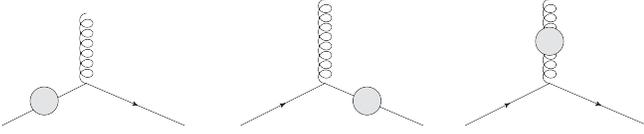
FIG. 2. Non-1PI diagrams contributing to the quark three-point function. The shaded blob represents the 1PI contribution to the relevant two-point function with operator insertion.

$$\sigma(a, b) \equiv a_\mu \sigma^{\mu\nu} b_\nu$$
$$\epsilon_\mu(a, b, c) \equiv \epsilon_{\alpha\beta\rho\mu} a^\alpha b^\beta c^\rho. \tag{58}$$

Finally, for a given operator O, non-1PI tree level contributions to the three-point functions (see Fig. 2) can be expressed in terms of quark and gluon two-point functions as follows,

$$\Gamma_O^{(3)}(p, p', k) = -g\epsilon^* \frac{\not{k} + \not{p}' + m}{s - m^2} \Gamma_O^{(2)}(p, k + p')$$
$$- g\Gamma_O^{(2)}(p - k, p') \frac{\not{p} - \not{k} + m}{u - m^2} \epsilon^*$$
$$- \frac{g}{t} \gamma_\mu \Gamma_O^{\mu\nu}(p - p', k)\epsilon_\nu^*, \tag{59}$$

where $s = (p' + k)^2$, $u = (p - k)^2$, $t = (p' - p)^2$.

B. One-loop Green's functions with CEDM insertion

At one-loop level, we regulate the diagrams with dimensional regularization, following the notation introduced in Sec. III B. Working in general covariant gauge (with gauge fixing parameter ξ),[6] we have computed both the divergent and finite parts of the Green's functions at generic kinematic points, before specializing to nonexceptional momentum configurations needed to define the operators in the RI-S̃MOM scheme (see Sec. VI). Specifically, for the two-point functions (with $p + q = p'$) we work at the symmetric point $p^2 = p'^2 = q^2 = -\Lambda^2$. For the three-point functions ($p + q = p' + k$) we work at the nonsymmetric point $\tilde{S}$ characterized by $p^2 = p'^2 = k^2 = q^2 = s = u = t/2 = -\Lambda^2$. We will provide the motivation behind this choice in Sec. VI.

Throughout this work we will denote the $SU(N_C)$ color factors as follows:

$$C_F = \frac{N_C^2 - 1}{2N_C}, \qquad C_A = N_C, \qquad T_F = \frac{1}{2}. \tag{60}$$

---

[6]Feynman gauge corresponds to $\xi = 1$, while Landau gauge corresponds to $\xi = 0$.

1. Quark two-point function

At one loop, $\Gamma_C^{(2)}(p, p')$ receives contributions from the diagrams in Fig. 3 and reads:

$$\Gamma_C^{(2)}(p, p') = \frac{i\alpha_s}{4\pi} \left\{ (p^2 + p'^2)\gamma_5 t^a \left[3C_F \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2} \right) + f_0 \right] \right.$$
$$+ \{\mathcal{M}, t^a\} \not{q}\gamma_5 \left[-\frac{3C_F}{2} \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2} \right) \right.$$
$$+ f_1 + O\left(\frac{m_q^2}{\Lambda^2} \right) \bigg]$$
$$+ \{\mathcal{M}^2, t^a\}\gamma_5 \left[-6C_F \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2} \right) \right.$$
$$\left. + f_2 + O\left(\frac{m_q^2}{\Lambda^2} \right) \bigg] \right\}, \tag{61}$$

where

$$f_0^{\mathrm{HV}} = \frac{22}{9} \times 3C_F, \qquad f_0^{\mathrm{NDR}} = \frac{4}{3} \times 3C_F \tag{62}$$

$$f_1^{\mathrm{HV}} = -3C_F, \qquad f_1^{\mathrm{NDR}} = -\frac{2}{3} \times 3C_F \tag{63}$$

$$f_2^{\mathrm{HV}} = -\frac{10}{3} \times 3C_F, \qquad f_2^{\mathrm{NDR}} = -\frac{2}{3} \times 3C_F. \tag{64}$$

2. Gluon two-point function

As illustrated in Fig. 4, at one-loop level three diagrams contribute to the gluon two-point function with insertion of the CEDM operator, $\Gamma_C^{\mu\nu}(p, p')$, defined in Eq. (51). The third diagram vanishes due to the antisymmetry of $\sigma_{\mu\nu}$, while the other two contribute

$$\Gamma_C^{\mu\nu}(p, p') = \frac{\alpha_s}{4\pi} \mathrm{Tr}[\mathcal{M}t^a]\Gamma_{G\tilde{G}}^{\mu\nu}(p, p')$$
$$\times \left[2\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2} \right) + 4 + O\left(\frac{m_q^2}{\Lambda^2} \right) \right], \tag{65}$$
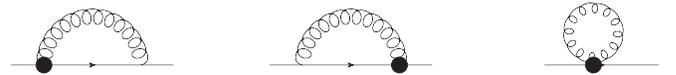
FIG. 3. Diagrams contributing to the quark two-point function. The dot denotes the insertion of the CEDM operator.
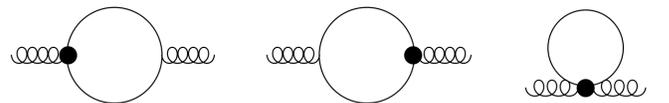
FIG. 4. Diagrams contributing to the gluon two-point function. The dot denotes the insertion of the CEDM operator.
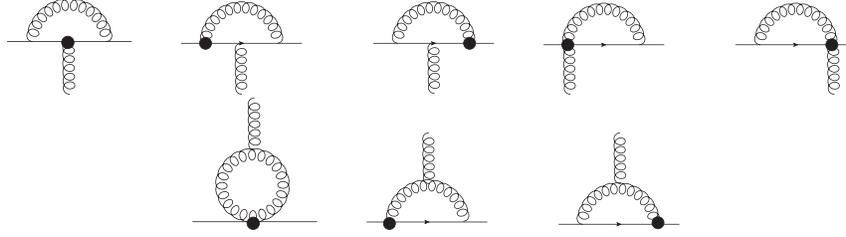
FIG. 5.   1PI diagrams contributing to the quark three-point function. The dot denotes the insertion of the CEDM operator.

where $\Gamma^{\mu\nu}_{G\tilde{G}}(p, p') = 4\epsilon^{\mu\nu\alpha\beta}p_\alpha p'_\beta$ [see Eq. (57)]. This result allows us to identify the mixing between the CEDM operator C and the operator $O_5^{(5)} = mG\tilde{G}$.

3. Quark-quark-gluon three-point function

We now turn to the quark-quark-gluon three-point function with insertion of the chromoelectric operator, $\Gamma_C^{(3)}(p, p', k)$, defined in Eq. (53). In all diagrams we chose to eliminate the four-momentum q^μ in favor of $(k + p' - p)^\mu$. The amputated three-point function receives contributions from 1PI diagrams (see Fig. 5), non-1PI diagrams (see Fig. 2), and quark and gluon wave-function renormalization. In this section we summarize our results for the 1PI diagrams and note that the non-1PI contributions of Fig. 2 are determined by the one-loop results for the quark and gluon two-point functions $\Gamma_C^{(2)}$ and $\Gamma_C^{\mu\nu}$ presented in Eqs. (61) and (65), as detailed in Eq. (59). As we will discuss in Sec. VI, we can choose a kinematic point and appropriate conditions so that the non-1PI diagrams are not needed to determine the RI-S̃MOM renormalization constants.

$\Gamma_C^{(3)}(p, p', k)$ can be decomposed in terms of 16 spinor structures and is characterized by 16 scalar coefficients $c_{1,\dots,16}$[7]

$$\begin{aligned}\Gamma_C^{(3)} = &[c_1\gamma_5 + c_2\epsilon(\epsilon^*, k, p, p') + c_3\epsilon^*\gamma_5 + c_4 k\gamma_5 + c_5 p\gamma_5\\ &+ c_6 p'\gamma_5 + c_7\epsilon_\mu(\epsilon^*, k, p)\gamma^\mu + c_8\epsilon_\mu(\epsilon^*, k, p')\gamma^\mu\\ &+ c_9\epsilon_\mu(\epsilon^*, p, p')\gamma^\mu + c_{10}\epsilon_\mu(k, p, p')\gamma^\mu\\ &+ c_{11}\sigma(\epsilon^*, k)\gamma_5 + c_{12}\sigma(\epsilon^*, p)\gamma_5 + c_{13}\sigma(\epsilon^*, p')\gamma_5\\ &+ c_{14}\sigma(k, p)\gamma_5 + c_{15}\sigma(k, p')\gamma_5 + c_{16}\sigma(p, p')\gamma_5].\end{aligned}$$
(66)

The coefficients c_i are functions of the invariants p^2, p'^2, k^2, q^2, s, t, u, and $\epsilon^* \cdot (p \pm p')$. The c_i's can be expressed in terms of triangle and bubble scalar integrals and their derivatives with respect to the invariants they depend on. For a generic kinematic configuration, the result involves logarithms and dilogarithms of ratios of invariants, and logarithms of ratios of the invariants to the renormalization scale μ. Working at the RI-S̃MOM kinematic point $p^2 = p'^2 = k^2 = q^2 = s = u = t/2 = -\Lambda^2$, and in the massless limit, greatly simplifies the integrals, reducing them to single-scale integrals. At this point, the triangle scalar integrals collapse to constants, and contribute in two forms. First, triangles that are functions of three invariants that become equal at the renormalization point, like p^2, k^2 and s, or p'^2, k^2 and u, are proportional to the constant

$$\psi = \frac{2}{3}\left(\psi^{(1)}\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2\right),$$
(67)

called C_0 in Ref. [59]. Here $\psi^{(1)}$ denotes the first derivative of the digamma function. Second, triangles that depend on the invariants p^2, p'^2 and t are proportional to the Catalan constant, which can also be expressed in terms of the first derivative of the digamma function

$$K = \frac{1}{8}\left(\psi^{(1)}\left(\frac{1}{4}\right) - \pi^2\right).$$
(68)

The only other nonrational number occurring in the result is $\log(2)$, which originates from the choice $t = -2\Lambda^2$.

Next, we give the UV divergent parts of the diagrams in Fig. 5, and the finite pieces of those Dirac structures that give nonvanishing contributions to the projections used to define the quark CEDM operator in the RI-S̃MOM scheme (see Sec. VI). The quark-quark-gluon three-point function is

$$\begin{aligned}\Gamma_C^{(3)}(p, p', k) = g\frac{\alpha_s}{4\pi}\Bigg\{&2\sigma(\epsilon^*, k)\gamma_5\left[\left(C_F(\xi - 2) + C_A\left(\frac{11}{4} + \frac{\xi}{4}\right)\right)\left(\frac{1}{\varepsilon} + \log\frac{\mu^2}{\Lambda^2}\right) + k_1\right]\\ &+ \sigma(\epsilon^*, p - p')\gamma_5\left[-\frac{3C_A}{4}\left(\frac{1}{\varepsilon} + \log\frac{\mu^2}{\Lambda^2}\right) + k_2\right]\\ &+ i(p + p')\cdot\epsilon^*\gamma_5\left[\left(6C_F - \frac{3}{4}C_A\right)\left(\frac{1}{\varepsilon} + \log\frac{\mu^2}{\Lambda^2}\right) + k_3\right]\Bigg\}t^a + \cdots\end{aligned}$$
(69)

---

[7]Hermiticity of the operator implies constraints among the various coefficients, such as $c_{12} = -c_{13}$, which we have used to check our calculation.

where, here and later, … denotes the contribution of the Dirac structures that are not relevant to defining the quark CEDM in RI-$\tilde{\text{S}}$MOM scheme. The constants k_1, k_2 and k_3 depend on the definition of γ_5 in d-dimension. In the 't Hooft–Veltman scheme, they are given by

$$k_1^{\text{HV}} = -2C_F(2-\xi) + \frac{33-\xi^2}{4}C_A + \left(\frac{2}{3}C_F - \frac{5+2\xi}{3}C_A\right)\psi$$
$$+ (2C_F - C_A)(1-\xi)K \qquad (70)$$

$$k_2^{\text{HV}} = C_F(2-\xi) - C_A\left(\frac{13-2\xi}{4} - \frac{\xi}{6}\psi\right)$$
$$+ (2C_F - C_A)(1-\xi)\left(\frac{1}{2}\log(2) - K\right) \qquad (71)$$

$$k_3^{\text{HV}} = \frac{44}{3}C_F - 2C_A + \left(-4C_F + \frac{3+2\xi}{6}C_A\right)\psi$$
$$+ (2C_F - C_A)\frac{(3+\xi)}{2}\log(2), \qquad (72)$$

while in NDR

$$k_1^{\text{NDR}} = k_1^{\text{HV}} - 2(C_F + C_A) \qquad (73)$$

$$k_2^{\text{NDR}} = k_2^{\text{HV}} + \frac{1}{2}C_A \qquad (74)$$

$$k_3^{\text{NDR}} = k_3^{\text{HV}} - \frac{20}{3}C_F + \frac{1}{2}C_A. \qquad (75)$$

4. Quark-quark-photon three-point functions

The quark-quark-photon three-point function with insertion of the quark CEDM gives

$$\Gamma_C^{(3,\gamma)} = \frac{e}{2}\{Q, t^a\}\frac{\alpha_s}{4\pi}\left\{2\sigma(\epsilon^*, k)\gamma_5\left[-C_F\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2}\right) + k_1^{(\gamma)}\right]\right.$$
$$+ C_F\sigma(\epsilon^*, p-p')\gamma_5$$
$$\left. + i(p+p')\cdot\epsilon^*\gamma_5\left[6C_F\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2}\right) + k_3^{(\gamma)}\right]\right\} + \cdots$$
$$(76)$$

with

$$k_1^{(\gamma)\text{HV}} = C_F\left(-2 + \frac{2}{3}\psi\right)$$

$$k_1^{(\gamma)\text{NDR}} = C_F\left(-4 + \frac{2}{3}\psi\right) \qquad (77)$$

$$k_3^{(\gamma)\text{HV}} = C_F\left(\frac{44}{3} - 4\psi\right) \quad k_3^{(\gamma)\text{NDR}} = C_F(8 - 4\psi). \qquad (78)$$

C. One-loop Green's functions with insertions of E, P, $\partial \cdot A$, and $G\tilde{G}$

The determination of the physical block Z_O of the mixing matrix in Eq. (49) requires the calculation of quark and/or gluon two-point functions with insertions of the operators E, $\partial^2 P$, $(m^2 P)_{1,2,3}$, $(m\partial \cdot A)_{1,2}$ and $mG\tilde{G}$. The renormalization of the pseudoscalar and tensor densities, and axial current has been studied in many papers, and the conversion between $\overline{\text{MS}}$-NDR and RI-SMOM to one loop was addressed in Ref. [59]. The renormalization of $G\tilde{G}$ in $\overline{\text{MS}}$ was studied in Ref. [60]. Here we provide one-loop 1PI results for the Green's functions in $\overline{\text{MS}}$-HV and $\overline{\text{MS}}$-NDR.

The relevant projection of the quark-quark-photon 1PI three-point function (this is essentially a quark-quark function) with insertion of the quark EDM operator, evaluated at the symmetric point gives

$$\Gamma_E^{(3,\gamma)} = -e\{Q, t^a\}\frac{\alpha_s}{4\pi}\sigma(\epsilon^*, k)\gamma_5$$
$$\times\left[(1-\xi)C_F\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2}\right) + k_T\right] + \cdots \quad (79)$$

$$k_T = C_F(1-\xi)\left(2 - \frac{5}{6}\psi\right), \qquad (80)$$

both in HV and NDR.

At one loop, the 1PI quark two-point functions with insertions of the operators $\partial^2 P$ and $(m^2 P)_{1,2,3}$, evaluated at the symmetric point, are given by

$$\Gamma_{\partial^2 P,(m^2 P)_{1,2,3}}^{(2)}$$
$$= i\gamma_5\left\{-q^2 t^a, \frac{1}{2}\{\mathcal{M}^2, t^a\}, \text{Tr}[\mathcal{M}^2 t^a]\mathbb{1}, \text{Tr}[\mathcal{M}t^a]\mathcal{M}\right\}$$
$$\times\frac{\alpha_s}{4\pi}\left[k_P + (3+\xi)C_F\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2}\right)\right.$$
$$\left. + C_F\frac{1-\xi}{3\Lambda^2}i\sigma^{\alpha\beta}p_\alpha p'_\beta + O\left(\frac{m_q}{\Lambda}\right)\right], \qquad (81)$$

where k_P depends on the d-dimensional definition of γ_5, namely,

$$k_P^{\text{HV}} = C_F\left[2(6+\xi) - \frac{3+\xi}{2}\psi\right], \qquad k_P^{\text{NDR}} = k_P^{\text{HV}} - 8C_F.$$
$$(82)$$

The gluon and photon two-point functions with insertions of P are finite, and not needed for renormalization. Eq. (81) is in agreement with the result of Ref. [59], where the calculation was carried out in the NDR scheme.

The 1PI quark two-point function with insertion of the operators proportional to the divergence of the axial current is

$$\Gamma^{(2)}_{(m\partial\cdot A)_1,(m\partial\cdot A)_2} = iq\gamma^5\left\{\text{Tr}[\mathcal{M}t^a]\mathbb{1}, \frac{1}{2}\{\mathcal{M}, t^a\} - \frac{1}{3}\text{Tr}[\mathcal{M}t^a]\mathbb{1}\right\}$$

$$\times \frac{\alpha_s}{4\pi}\left[k_A + C_F\xi\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2}\right)\right]$$

$$k_A^{\text{HV}} = C_F(\xi+4) \quad k_A^{\text{NDR}} = C_F\xi. \tag{83}$$

The gluon two-point function with insertion of the operator $(m\partial\cdot A)_1$ is finite, and we find,

$$\Gamma^{(2)\mu\nu}_{(m\partial\cdot A)_1} = \text{Tr}[\mathcal{M}t^a]\frac{\alpha_s n_F}{4\pi}4e^{\mu\nu\alpha\beta}p_\alpha p'_\beta, \tag{84}$$

where $n_F = 3$ is the number of flavors we are considering. The insertion of the operator $(m\partial\cdot A)_2$ vanishes.

Finally, the gluon and quark two-point functions with insertion of $(mG\tilde{G})$ are given by

$$\Gamma^{(2)\mu\nu}_{(mG\tilde{G})} = 4\text{Tr}[\mathcal{M}t^a]\epsilon^{\mu\nu\alpha\beta}p_\alpha p'_\beta\frac{\alpha_s}{4\pi}$$

$$\times\left\{C_A\frac{3+\xi}{2}\left(\frac{1}{\epsilon}+\log\frac{\mu^2}{\Lambda^2}\right)+k_G\right\} \tag{85}$$

$$\Gamma^{(2)}_{(mG\tilde{G})} = i\text{Tr}[\mathcal{M}t^a]q\gamma_5\frac{\alpha_s}{4\pi}\left[6C_F\left(\frac{1}{\epsilon}+\log\frac{\mu^2}{\Lambda^2}\right)+\tilde{k}_G\right]$$

$$+ \mathcal{O}(\mathcal{M}^2) \tag{86}$$

$$k_G = \frac{C_A}{2}\left(17 - \xi^2 - \frac{4}{3}(3+\xi)\psi\right) \tag{87}$$

$$\tilde{k}_G = C_F(16 - 4\psi). \tag{88}$$

These results determine the self-renormalization of $(mG\tilde{G})$ and its mixing with $(m\partial\cdot A)_1$.

V. RENORMALIZATION MATRIX IN $\overline{\text{MS}}$ SCHEME

In this section we provide one-loop results for the Z_O block of the renormalization matrix given in Eq. (49). At various stages of the calculation we need the one-loop results for the mass, couplings, and field renormalization constants in general covariant gauge (recall $d = 4 - 2\epsilon$):

$$Z_m = 1 - \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}3C_F \tag{89}$$

$$Z_q = 1 - \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}\xi C_F \tag{90}$$

$$Z_G = 1 + \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}\left[-\frac{4}{3}n_F T_F + C_A\left(\frac{13}{6}-\frac{\xi}{2}\right)\right] \tag{91}$$

$$Z_g = 1 - \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}\frac{11C_A - 4T_F n_F}{6}, \tag{92}$$

where the color factors are C_F, C_A, and T_F are given in Eq. (60). We will also need the renormalization constants for the pseudoscalar $\bar{\psi}\gamma_5\psi$ and tensor $\bar{\psi}\sigma_{\mu\nu}\psi$ densities, defined by $O_\Gamma^{(0)} = Z_\Gamma O_\Gamma$:

$$Z_P = Z_m^{-1} = 1 + \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}3C_F \tag{93}$$

$$Z_T = 1 - \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}C_F. \tag{94}$$

For the mixing of dimension-five operators, specializing Eq. (48) to the $\overline{\text{MS}}$ scheme at one loop one finds

$$O_i^{\overline{\text{MS}}} = (Z^{-1})_{ij}^{\overline{\text{MS}}}O_j^{(0)}, \qquad Z_{ij}^{\overline{\text{MS}}} \equiv \delta_{ij} - \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}z_{ij}. \tag{95}$$

Note that in the above expressions α_s denotes the d-dimensional renormalized coupling defined in Sec. III B, satisfying $d\alpha_s/d(\log\mu) = -2\epsilon\alpha_s + O(\alpha_s^2)$. So to $O(\alpha_s)$ the anomalous dimension matrix $\gamma \equiv d(\log Z)/d(\log\mu)$ can be immediately read off Eq. (95): $\gamma_{ij} = 2\alpha_s/(4\pi)z_{ij}$.

The various entries of the renormalization matrix are determined as follows:

(i) Finiteness of the quark two-point function $\Gamma_C^{(2)}$, gluon two-point function $\Gamma_C^{\mu\nu}$, quark-quark-gluon $\Gamma_C^{(3)}$ and quark-quark-photon $\Gamma_C^{(3,\gamma)}$ three-point functions implies a set of conditions for z_{1n}, $n = 1, ..., 14$. Note that only the results for $n = 1, ..., 10$ affect physical observables, the rest are given for completeness.

(ii) The operator $O_2^{(5)} = \partial^2 P$ renormalizes diagonally with constant Z_P.

(iii) The quark EDM operator $O_3^{(5)} \equiv E$ renormalizes diagonally (to zeroth order in the fine structure constant) in the same way as the tensor quark bilinear, i.e., $(Z^{-1})_{33} = Z_T^{-1}$.

(iv) To zeroth order in the electromagnetic couplings, $O_4^{(5)} = mF\tilde{F}$ renormalizes diagonally with the renormalization constant $(Z^{-1})_{44} = Z_m^{-1}$.

(v) The subset of operators $O_{5,6,10}^{(5)}$ related to the axial anomaly renormalize, to one loop, as follows [60] (recall $Z_m Z_P = 1$):

$$\begin{pmatrix} mG\tilde{G} \\ (m\partial\cdot A)_1 \\ (m^2P)_3 \end{pmatrix}^{\overline{\text{MS}}} = \begin{pmatrix} Z_m^{-1}Z_g^2 & -\frac{1}{\epsilon}\frac{\alpha_s}{4\pi}6C_F & 0 \\ 0 & Z_m^{-1} & 0 \\ 0 & 0 & Z_m^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} mG\tilde{G} \\ (m\partial\cdot A)_1 \\ (m^2P)_3 \end{pmatrix}^{(0)}. \tag{96}$$

To explicitly check Eq. (96) at one loop use Eqs. (81), (83), (85) and (86).

(vi) Finally, $O_7^{(5)} = (m\partial \cdot A)_2$ renormalizes as $(m\partial \cdot A)_1$ and $O_{8,9}^{(5)} = (m^2 P)_{1,2}$ renormalize as $(m^2 P)_3$, thus leading to $(Z^{-1})_{77,88,99} = Z_m^{-1}$.

In summary, the entries in the first row in Eq. (49) are:

$$z_{11} = 5C_F - 2C_A \tag{97a}$$

$$z_{12} = 0 \tag{97b}$$

$$z_{13} = 4C_F \tag{97c}$$

$$z_{14} = 0 \tag{97d}$$

$$z_{15} = -2 \tag{97e}$$

$$z_{16} = C_F - \frac{1}{4}C_A \tag{97f}$$

$$z_{17} = 3C_F - \frac{3}{4}C_A \tag{97g}$$

$$z_{18} = 6C_F + \frac{3}{2}C_A \tag{97h}$$

$$z_{19} = 0 \tag{97i}$$

$$z_{1,10} = 0 \tag{97j}$$

$$z_{1,11} = 6C_F - \frac{3}{2}C_A \tag{97k}$$

$$z_{1,12} = -3C_F + \frac{3}{4}C_A \tag{97l}$$

$$z_{1,13} = \frac{3}{4}C_A \tag{97m}$$

$$z_{1,14} = \frac{3}{4}C_A. \tag{97n}$$

For the remaining nonzero entries we have:

$$z_{22} = -3C_F \tag{98a}$$

$$z_{33} = C_F \tag{98b}$$

$$z_{44} = 3C_F \tag{98c}$$

$$z_{55} = -\frac{11C_A - 4T_F n_F}{3} + 3C_F \tag{98d}$$

$$z_{56} = -6C_F \tag{98e}$$

$$z_{66} = z_{77} = z_{88} = z_{99} = z_{10,10} = 3C_F. \tag{98f}$$

The submatrix z_{11}, z_{13} and z_{33} agrees with the original calculation of Refs. [61–65].

VI. DEFINITION OF R I-ŠMOM OPERATORS AND MATCHING TO $\overline{\text{MS}}$

A consistent phenomenological analysis of BSM-induced *CP* violation in hadronic systems requires computation of the effect of the *CP*-odd operators in Eq. (26) on couplings at the hadronic scale, such as the nucleon EDM and the *T*-odd πNN couplings. This is an intrinsically nonperturbative problem. The first step in this program involves *defining* UV finite operators in a suitable renormalization scheme, whose matrix elements can be then computed nonperturbatively within lattice QCD. Here we will define finite operators within a class of regularization-independent (RI) momentum subtraction (MOM) schemes [49,59]. Next, one converts the matrix elements in the RI-MOM scheme to the $\overline{\text{MS}}$ scheme, commonly adopted to compute the Wilson coefficients and their renormalization-group evolution down to the hadronic scale, using continuum perturbation theory.

In this section we address the following issues:

(1) We provide a set of regularization independent normalization conditions for the amputated Green's functions $\Gamma_{O_i^{(5)}}$ that subtract all the UV divergences and fix the finite parts of the renormalization constants for the gauge-invariant *CP*-odd operators $O_{1,\ldots,10}^{(5)}$. Since we will use subtraction conditions for the three-point functions at a nonsymmetric momentum point, we call this scheme RI-ŠMOM, as opposed to RI-SMOM [59].

(2) We provide the finite matching matrix that relates the RI-ŠMOM and $\overline{\text{MS}}$ operators to one loop in perturbation theory:

$$O_i^{\text{RI-ŠMOM}} = C_{ij} O_j^{\overline{\text{MS}}}. \tag{99}$$

In practice this amounts to finding a linear combination of $\overline{\text{MS}}$ operators $O_i^{\overline{\text{MS}}}$ such that the Green's functions with insertions of $O_i^{\text{RI-ŠMOM}}$ satisfy the normalization conditions that define the scheme (see item 1. above).

A. Defining the R I-ŠMOM scheme

We follow the strategy outlined in Refs. [49,59], with appropriate modifications related to the operators we are dealing with. The content of this scheme can be summarized as follows:

(i) We require that the quark and gluon two-point functions with insertion of the quark CEDM operator $\Gamma_C^{(2)}(p, p')$ and $\Gamma_C^{(2)\mu\nu}(p, p')$ vanish at the symmetric kinematic point S defined by $p^2 = p'^2 = q^2 = -\Lambda^2$.

(ii) We require that certain projections of the three-point functions with quark CEDM insertion $\Gamma_C^{(3)}$ and $\Gamma_C^{(3,\gamma)}$

take the tree-level value at the nonsymmetric kinematic point $\tilde{S}$ (involving only nonexceptional momenta) characterized by $p^2 = p'^2 = k^2 = q^2 = s = u = t/2 = -\Lambda^2$.

With this choice, and by virtue of the normalization condition imposed on $\Gamma_C^{(2)}(p, p')$, the non-1PI diagrams [see Eq. (59)] contributing to the three-point function with insertion of $O_1^{(5)} = C$ on the quark external legs vanish. In other words, the amputated Green's function coincide with the 1PI Green's functions up to a non-1PI term arising from operator insertion on the gluon external leg. This non-1PI term does not project on the spin/Lorentz structures that we use to impose the normalization conditions, so for all practical purposes the renormalization conditions can be imposed on the 1PI vertices.

(iii) We require that the gluon and quark two-point functions with insertion $O_5^{(5)} = (mG\tilde{G})$ take their tree-level value at the symmetric point S given by $p^2 = p'^2 = q^2 = -\Lambda^2$. The condition on the gluon two-point function involves overall factors of the quark masses. While one can use quark masses in any scheme, we choose to use the quark masses in the $\overline{\text{MS}}$ scheme. This leads to the simplest matching factors, and corresponds to imposing the subtraction conditions on the operator $G\tilde{G}$, ignoring the mass factors.

(iv) The remaining operators are related to quark bilinears: $O_{2,8-10}^{(5)}$ are related to the pseudoscalar density, $O_3^{(5)}$ is related to the tensor density, and $O_{6-7}^{(5)}$ are related to the divergence of the axial current. We exploit this factorized structure and impose the "standard" RI-SMOM conditions [59] on the quark bilinear part. The subtraction condition for $O_{6-10}^{(5)}$ involves again overall factors of the quark masses, for which we choose the $\overline{\text{MS}}$ values. This is equivalent to imposing the conditions on the quark bilinears, ignoring the overall quark mass factors.

Throughout, we impose the normalization conditions in the chiral limit $m_q \to 0$. This is achieved as follows: (i) We expand two- and three-point Green's functions in spin-flavor structures, keeping explicit powers of the quark mass. (ii) Through appropriate projections we then isolate the coefficients of the various spin-flavor structures, which are defined for any value of the quark mass. (iii) Finally, we impose normalization conditions on these coefficient functions in the chiral limit. This procedure defines a mass-independent renormalization scheme.

This RI scheme, defined in terms of gauge fixed correlation functions of quark and gluon states in the deep Euclidean region, serves as a useful intermediary for converting nonperturbative results to those required for phenomenology. In this work, we only discuss the matching

of this RI scheme to the perturbative $\overline{\text{MS}}$ scheme in covariant gauges. To complete the program of connecting the $\overline{\text{MS}}$ to a lattice scheme, we also need to calculate the matching between lattice and this RI scheme. Among the covariant gauges, the Landau gauge is the most convenient for lattice calculations. The calculation of the corresponding matrix elements on the lattice can be done either using lattice perturbation theory, or nonperturbatively. In fact, matrix elements with quark external states are used extensively nowadays for renormalizing lattice operators [49]. However, renormalization of the CEDM operator needs extension of such calculations to include gluon external states. Even though gluonic correlators have long been studied on the lattice [66], they are typically noisy. In addition, the matrix elements with two quarks and a gluon external state gives rise to "four-point" functions,[8] and there is little experience with calculating these in the lattice community.

Apart from these difficulties, however, the nonperturbative evaluation of the matrix elements is theoretically straightforward. The large number of off-shell operators does not pose a significant challenge either. In particular, since these operators explicitly involve the equation of motion, an $n + 1$-point function involving them is straightforwardly related to a n-point function obtained by exactly canceling an external propagator using the equation of motion. With such reductions, the number of correlation functions that need to be evaluated nonperturbatively are much fewer than the number of operators in the basis.

1. Subtraction conditions on the quark CEDM

We now give explicitly the 14 conditions needed to determine $Z_{1n}^{\text{RI-\tilde{S}MOM}}$.[9] We begin with the conditions on the two-point functions with external gluons and photons:

$$\epsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta \Gamma_C^{\mu\nu}(p, p')|_S = 0 \qquad (100\text{a})$$

$$\epsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta \Gamma_C^{\mu\nu(\gamma)}(p, p')|_S = 0. \qquad (100\text{b})$$

The quark-quark Green's function has the following spin-flavor structures,

$$\begin{aligned}
\Gamma_C^{(2)} = {}& \alpha_1 \gamma_5 t^a + \alpha_2 \sigma(p, p')\gamma_5 t^a + \alpha_3 \mathcal{M} t^a \slashed{q}\gamma_5 \\
& + \alpha_4 \text{Tr}[\mathcal{M} t^a]\slashed{q}\gamma_5 \\
& + \alpha_5 \mathcal{M}^2 t^a \gamma_5 + \alpha_6 \text{Tr}[\mathcal{M}^2]\gamma_5 t^a \\
& + \alpha_7 \text{Tr}[\mathcal{M} t^a]\mathcal{M}\gamma_5,
\end{aligned} \qquad (101)$$

where the α_i are functions of the kinematic invariants. We impose the RI-$\tilde{\text{S}}$MOM condition that all the α_i vanish at

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[8]It is conventional in the lattice literature to count the point of operator insertion.

[9]This is in addition to the condition for eliminating possible power divergences (see Sec. III A 1).

the symmetric kinematic point S in the chiral limit $m_q \to 0$. This can be achieved with the following projections (traces are over color, spin, and flavor indices):

$$\text{Tr}[\Gamma_C^{(2)}\gamma_5 t^a]_S = 0 \tag{102a}$$

$$\text{Tr}[\Gamma_C^{(2)}\gamma_5\sigma_{\mu\nu}t^a]_S = 0 \tag{102b}$$

$$M_2^{-1}\begin{pmatrix} \text{Tr}[\Gamma_C^{(2)}\gamma_5 \slashed{q}\mathcal{M}t^a] \\ \text{Tr}[\Gamma_C^{(2)}\gamma_5 \slashed{q}]\text{Tr}[\mathcal{M}t^a] \end{pmatrix}_S = 0 \tag{102c}$$

$$M_3^{-1}\begin{pmatrix} \text{Tr}[\Gamma_C^{(2)}\gamma_5\mathcal{M}^2 t^a] \\ \text{Tr}[\Gamma_C^{(2)}\gamma_5 t^a]\text{Tr}[\mathcal{M}^2] \\ \text{Tr}[\Gamma_C^{(2)}\gamma_5\mathcal{M}]\text{Tr}[\mathcal{M}t^a] \end{pmatrix}_S = 0, \tag{102d}$$

where the matrices M_2 and M_3 are given by (here Tr_F denotes the trace over flavor indices only)

$$M_2 = \begin{pmatrix} \text{Tr}_F[(\mathcal{M}t^a)^2] & (\text{Tr}_F[\mathcal{M}t^a])^2 \\ (\text{Tr}_F[\mathcal{M}t^a])^2 & n_F(\text{Tr}_F[\mathcal{M}t^a])^2 \end{pmatrix} \tag{103}$$

and

$$M_3 = \begin{pmatrix} \text{Tr}_F[(\mathcal{M}^2 t^a)^2] & \text{Tr}_F\mathcal{M}^2\text{Tr}_F[\mathcal{M}^2(t^a)^2] & \text{Tr}_F[\mathcal{M}t^a]\text{Tr}_F[\mathcal{M}^3 t^a] \\ \text{Tr}_F\mathcal{M}^2\text{Tr}_F[\mathcal{M}^2(t^a)^2] & (\text{Tr}_F\mathcal{M}^2)^2\text{Tr}_F[t^a t^a] & \text{Tr}_F\mathcal{M}^2(\text{Tr}_F[\mathcal{M}t^a])^2 \\ \text{Tr}_F[\mathcal{M}t^a]\text{Tr}_F[\mathcal{M}^3 t^a] & \text{Tr}_F\mathcal{M}^2(\text{Tr}_F[\mathcal{M}t^a])^2 & \text{Tr}_F\mathcal{M}^2(\text{Tr}_F[\mathcal{M}t^a])^2 \end{pmatrix}. \tag{104}$$

The above projections work for nondegenerate quark masses ($m_u \neq m_d \neq m_s$). In the isospin limit $m_u = m_d$, the matrices M_2 and M_3 become singular. In Appendix D we describe the projections needed in this case.

To express the subtraction conditions on the quark-quark-gluon three-point function, we restore the color and flavor indices of these objects. Recalling that $\Gamma_C^{(3)}$ is proportional to $t^a T^b$ where t^a is a matrix in flavor space[10] while T^b is a color generator, we will use the notation $\Gamma_C^{(3)} \to \Gamma_C^{(3),ab}$. The conditions then read (there is no summation over a and b)

$$\frac{1}{i\epsilon(\epsilon^*, k, p, p')}\text{Tr}[\Gamma_C^{(3),ab}\sigma(p,p')t^a T^b]_{\tilde{S}} = 2g^{\overline{\text{MS}}} \tag{105a}$$

$$\text{Tr}[\Gamma_C^{(3),ab}\sigma(k, p+p')t^a T^b]_{\tilde{S}} = 0 \tag{105b}$$

$$\text{Tr}[\Gamma_C^{(3),ab}\gamma_5 t^a T^b]_{\tilde{S}} = 0. \tag{105c}$$

Note that in the first condition above, we could have used the renormalized value of the strong coupling constant in any renormalization scheme. The use of $g^{\overline{\text{MS}}}$ makes the connection between RI-$\tilde{\text{S}}$MOM and $\overline{\text{MS}}$ schemes simpler.

Finally, we impose the following conditions on the quark-quark-photon three-point function:

$$\text{Tr}[\Gamma_C^{(3,\gamma)}\sigma(p,p')Qt^a]_{\tilde{S}} = 0 \tag{106a}$$

$$\text{Tr}[\Gamma_C^{(3,\gamma)}\gamma_5 Qt^a]_{\tilde{S}} = 0. \tag{106b}$$

2. Subtraction conditions on the remaining operators

We give here the subtraction conditions needed to determine the remaining entries of $Z_{ij}^{\text{RI-}\tilde{\text{S}}\text{MOM}}$. For the operator $O_5^{(5)} = (mG\tilde{G})$, we prescribe

$$-\frac{1}{6\Lambda^4}\epsilon_{\mu\nu\alpha\beta}p^\alpha p'^\beta \Gamma_{O_5^{(5)}}^{\mu\nu}(p,p')|_S = \text{Tr}_F[\mathcal{M}^{\overline{\text{MS}}}t^a] \tag{107a}$$

$$\text{Tr}[\Gamma_{O_5^{(5)}}^{(2)}\gamma_5 \slashed{q}]_S = 0. \tag{107b}$$

The remaining operators $O_{2,3,6-10}^{(5)}$ are related to quark bilinears, and we wish to impose the "standard" RI-SMOM conditions [59]. $O_{2,6-10}^{(5)}$ have a simple factorized form, and the normalization conditions of Ref. [59] are equivalent to

$$\frac{i}{6q^2}\text{Tr}[\Gamma_{O_2^{(5)}}^{(2)}\gamma_5 t^a]_S = 1 \tag{107c}$$

$$\frac{1}{12q^2}\text{Tr}[\Gamma_{O_6^{(5)}}^{(2)}\gamma_5 \slashed{q}]_S = n_F \text{Tr}_F[\mathcal{M}^{\overline{\text{MS}}}t^a] \tag{107d}$$

$$\frac{1}{12q^2}\text{Tr}[\Gamma_{O_7^{(5)}}^{(2)}\gamma_5 \slashed{q}\mathcal{M}]_S$$
$$= \text{Tr}_F[(\mathcal{M}^2)^{\overline{\text{MS}}}t^a - \frac{1}{3}\mathcal{M}^{\overline{\text{MS}}}\text{Tr}_F[\mathcal{M}^{\overline{\text{MS}}}t^a]] \tag{107e}$$

$$\frac{1}{12i}\text{Tr}[\Gamma_{O_8^{(5)}}^{(2)}\gamma_5]_S = \text{Tr}_F[(\mathcal{M}^2)^{\overline{\text{MS}}}t^a] \tag{107f}$$

$$\frac{1}{12i}\text{Tr}[\Gamma_{O_9^{(5)}}^{(2)}\gamma_5 t^a]_S = \frac{1}{2}\text{Tr}_F[(\mathcal{M}^2)^{\overline{\text{MS}}}] \tag{107g}$$

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[10]We use $t^0 = 1/\sqrt{6}I_{3\times3}$ so that $\text{Tr}_F(t^a t^a) = 1/2$ for $a = 0, 3, 8$.

$$\frac{1}{12i}\mathrm{Tr}[\Gamma^{(2)}_{O^{(5)}_{10}}\gamma_5]_S = \mathrm{Tr}_F[\mathcal{M}^{\overline{\mathrm{MS}}}t^a]\mathrm{Tr}_F[\mathcal{M}^{\overline{\mathrm{MS}}}]. \quad (107\mathrm{h})$$

The operator $O_3^{(5)}$ is related to the tensor density but contains an explicit photon field strength. One would be tempted to impose the following condition on the quark-quark-photon matrix element,

$$\frac{1}{12i\epsilon(\epsilon^*, k, p, p')\mathrm{Tr}[(Qt^a)^2]}\mathrm{Tr}[\Gamma^{(3,\gamma)}_{O_3^{(5)}}\sigma(p, p')Qt^a]_S = 2e^{\overline{\mathrm{MS}}}, \quad (108)$$

which effectively fixes the projection on the structure $\sigma(\epsilon^*, k)\gamma_5$ to its tree-level value. However, in terms of matrix elements of the tensor density, this prescription corresponds to

$$\mathrm{Tr}[\Gamma^{\mu\nu}_T(p, p')\gamma_5\sigma(p, p')]_S = 12i\epsilon^{\mu\nu\alpha\beta}p_\alpha p'_\beta, \quad (109)$$

which differs from the standard one [59],

$$\mathrm{Tr}[\Gamma^{\mu\nu}_T(p, p')\sigma_{\mu\nu}]_S = 144, \quad (110)$$

and would lead to a finite difference in the renormalization factors.[11] In our analysis we stick to the standard normalization condition Eq. (110). This can be obtained by imposing Eq. (108) while performing a finite shift δk_T in the loop factor k_T given in Eq. (80), namely,

$$\delta k_T = C_F(1-\xi)\left(\frac{2}{3} - \frac{1}{3}\psi\right). \quad (111)$$

B. Matching R I-S̃MOM and $\overline{\mathrm{MS}}$ operators

We now determine the conversion matrix appearing in Eq. (99),

$$C_{ij} = ((Z^{\mathrm{RI\text{-}\tilde{S}MOM}})^{-1} \cdot Z^{\overline{\mathrm{MS}}})_{ij}, \quad (112)$$

to first order in α_s. Denoting field renormalization and renormalized amputated Green's functions of any operator O in the RI-S̃MOM scheme with $\tilde{Z}_{q,G}$ and $\tilde{\Gamma}_O$, respectively, and the corresponding quantities in the $\overline{\mathrm{MS}}$ scheme with $Z_{q,G}$ and Γ_O, the matching conditions take the form

$$\tilde{\Gamma}^{(2)}_{O_i} = \frac{\tilde{Z}_q}{Z_q}\sum_j C_{ij}\Gamma^{(2)}_{O_j} \quad (113\mathrm{a})$$

$$\tilde{\Gamma}^{\mu\nu}_{O_i} = \frac{\tilde{Z}_G}{Z_G}\sum_j C_{ij}\Gamma^{\mu\nu}_{O_j} \quad (113\mathrm{b})$$

$$\tilde{\Gamma}^{(3)}_{O_i} = \frac{\tilde{Z}_q\tilde{Z}_G^{1/2}}{Z_qZ_G^{1/2}}\sum_j C_{ij}\Gamma^{(3)}_{O_j} \quad (113\mathrm{c})$$

$$\tilde{\Gamma}^{(3,\gamma)}_{O_i} = \frac{\tilde{Z}_q}{Z_q}\sum_j C_{ij}\Gamma^{(3,\gamma)}_{O_j}. \quad (113\mathrm{d})$$

When one imposes that the $\tilde{\Gamma}_{O_i}$ satisfy the RI-S̃MOM subtraction conditions given in subsections VIA1 and VIA2, one obtains a system of linear equations for the C_{ij} matching factors.

Using the explicit one-loop results of Secs. IV B and IV C and the ratios of wave-function renormalization factors,

$$\frac{\tilde{Z}_q}{Z_q} \equiv 1 + \frac{\alpha_s}{4\pi}r_q = 1 - \frac{\alpha_s}{4\pi}C_F\xi\left[1 + \log\frac{\mu^2}{\Lambda^2}\right] \quad (114)$$

$$\frac{\tilde{Z}_G}{Z_G} \equiv 1 + \frac{\alpha_s}{4\pi}r_G$$

$$= 1 + \frac{\alpha_s}{4\pi}\left[C_A\left(\frac{97}{36} + \frac{\xi}{2} + \frac{\xi^2}{4}\right) - \frac{20}{9}n_FT_F\right.$$

$$\left. + \left[C_A\left(\frac{13}{6} - \frac{\xi}{2}\right) - \frac{4}{3}n_FT_F\right]\log\frac{\mu^2}{\Lambda^2}\right], \quad (115)$$

we solve for the C_{ij}.

To $O(\alpha_s)$, the matching coefficients have the structure

$$C_{ij} \equiv \delta_{ij} + \frac{\alpha_s}{4\pi}\left[c_{ij} + z_{ij}\log\frac{\mu^2}{\Lambda^2}\right], \quad (116)$$

corresponding to the RI-S̃MOM renormalization matrix

$$Z^{\mathrm{RI\text{-}\tilde{S}MOM}}_{ij} = \delta_{ij} - \frac{\alpha_s}{4\pi}\left[z_{ij}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{\Lambda^2}\right) + c_{ij}\right]. \quad (117)$$

We have given the pole terms z_{ij} in Sec. V, while the constants c_{ij} can be expressed in terms of the loop factors $r_{q,G}$ defined above and $f_{0,1,2}$, $k_{1,2,3}$, $k_{1,3}^{(\gamma)}$, k_G, $\tilde{k}_G$, and $k_{A,P,T}$ defined in Secs. IV B and IV C. We find for the first row c_{1n}:

$$c_{11} = -k_1 - \frac{1}{2}k_2 + \frac{1}{2}k_3 - r_q - \frac{1}{2}r_G \quad (118\mathrm{a})$$

$$c_{12} = 2f_0 + k_2 - k_3 \quad (118\mathrm{b})$$

$$c_{13} = -k_1^{(\gamma)} + \frac{k_3 - k_2}{2} \quad (118\mathrm{c})$$

$$c_{14} = 0 \quad (118\mathrm{d})$$

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[11]Note that in the free theory ($\Gamma^{\mu\nu}_T \to \sigma^{\mu\nu}$), both Eq. (109) and Eq. (110) hold. However, when including interactions a difference arises: the projection Eq. (109) selects the $\sigma^{\mu\nu}$ component of $\Gamma^{\mu\nu}_T$, while the projection Eq. (110) picks up not only $\sigma^{\mu\nu}$ but also additional terms in $\Gamma^{\mu\nu}_T$, such as $\sigma^{\alpha\beta}p_\alpha p'_\beta(p^\mu p'^\nu - p^\nu p'^\mu)$.

$$c_{15} = -4 \tag{118e}$$

$$c_{16} = \frac{k_2 - 2f_1}{3} \tag{118f}$$

$$c_{17} = k_2 - 2f_1 \tag{118g}$$

$$c_{18} = -2f_2 - k_2 - k_3 \tag{118h}$$

$$c_{19} = 0 \tag{118i}$$

$$c_{1,10} = 0 \tag{118j}$$

$$c_{1,11} = k_2 + k_3 \tag{118k}$$

$$c_{1,12} = -\frac{k_2 + k_3}{2} \tag{118l}$$

$$c_{1,13} = -k_2 \tag{118m}$$

$$c_{1,14} = k_3^{(\gamma)} - k_3. \tag{118n}$$

For the remaining nonzero entries of c_{ij}, we find

$$c_{22} = -k_P - r_q \tag{118o}$$

$$c_{33} = -k_T - \delta k_T - r_q \tag{118p}$$

$$c_{55} = -\kappa_G - r_G \tag{118q}$$

$$c_{56} = -\tilde{\kappa}_G \tag{118r}$$

$$c_{66} = c_{77} = -k_A - r_q \tag{118s}$$

$$c_{88} = c_{99} = c_{10,10} = -k_P - r_q. \tag{118t}$$

In Appendix E, we report explicit results for the matching coefficients c_{ij} using both the HV and NDR prescriptions for γ_5.

VII. RENORMALIZATION AND THE AXIAL WARD IDENTITIES

In the previous section we have imposed a set of subtraction conditions on *CP*-odd operators of dimension five, some of which are related to the axial current ($O_{6-7}^{(5)}$), the pseudoscalar density ($O_{8-10}^{(5)}$), and $G\tilde{G}$ ($O_5^{(5)}$). So far we have not discussed whether the resulting finite operators satisfy the nonsinglet and singlet axial Ward identities (WIs). In particular, the normalization conditions on the singlet A, P, and $G\tilde{G}$ may be inconsistent with the singlet WIs. For the nonsinglet case, RI-SMOM subtraction conditions have been shown to be consistent with the WIs [59].

In general, one can obtain properly normalized symmetry currents through the Ward identity method, discussed in Refs. [67,68]. Moreover, in Ref. [59] the RI-SMOM conditions were suitably chosen so that they are consistent with the nonsinglet axial WIs. Here we take a different point of view: we discuss how to define renormalized (singlet and nonsinglet) axial current and pseudoscalar density operators that satisfy the axial WIs, starting from an arbitrary subtraction scheme, such as $\overline{\text{MS}}$ or the RI-$\tilde{\text{S}}$MOM scheme defined in Sec. VI A. We put forward a two-step approach:

(1) Using any regulator and any subtraction scheme, define renormalized (finite) axial (A_μ), pseudoscalar (P) and $G\tilde{G}$ operators.

(2) Starting from any of the above schemes, perform a finite renormalization that leads to operators A_μ, P and $G\tilde{G}$ that obey properly normalized WIs. The resulting A_μ is the "symmetry current" associated with axial transformations. We may call this new scheme the "WI scheme."

In the case of $\overline{\text{MS}}$ and the RI-$\tilde{\text{S}}$MOM scheme defined in the previous section, we provide the explicit matching factors to the WI scheme to $O(\alpha_s)$. We will also describe the procedure to obtain nonperturbative matching factors connecting the RI-$\tilde{\text{S}}$MOM and WI schemes.

Our discussion is inspired by the analysis of Refs. [53,69] for a dimensionally regulated theory and of Refs. [70,71] for a lattice regulated theory. While we give details pertaining to the dimensionally regulated theory, our aim is to point out that the general features of the analysis are "RI," i.e., regularization independent. Therefore, we will draw parallels with discussions of the axial current in various lattice QCD formulations [70,72,73] in appropriate places.

A. PCAC relation in terms of bare operators

We focus on the singlet axial current for concreteness. A discussion of the nonsinglet current in the context of dimensional regularization and minimal subtraction is presented in Ref. [53], and the relevant results are a special case of the analysis presented below. In terms of suitably regularized operators, the PCAC relation takes the form

$$\partial \cdot A = 2(mP) + P_E + X. \tag{119}$$

In dimensional regularization the bare operators take the form $A_\mu = \bar{\psi}(1/2)[\gamma_\mu, \gamma_5]\psi$, $(mP) \equiv \bar{\psi}\mathcal{M}i\gamma_5\psi$, $P_E = \bar{\psi}_E i\gamma_5\psi + \bar{\psi}i\gamma_5\psi_E$. X is the anomaly operator, whose tree-level insertions in Green's functions vanish as one removes the regulator ($d \to 4$ in dimensional regularization or $a \to 0$ in the lattice theory). In the dimensionally regulated theory, with HV prescription for the γ_5, one has

$$X = \frac{1}{2}\bar{\psi}\{\gamma_5, \vec{\slashed{D}} - \overleftarrow{\slashed{D}}\}\psi, \tag{120}$$

which clearly vanishes at the classical level in $d = 4$ due to the anticommutation properties of γ_5. For $d \neq 4$ this operator is nonvanishing and through divergent quantum corrections it can leave a finite remnant in Green's functions, including anomalous terms in the axial current conservation equation. In NDR, X always vanishes. For this reason, NDR does not "see" the axial anomaly and cannot consistently be used for the discussion of the singlet axial current. In the lattice theory with Wilson fermion discretization, X is the variation of the Wilson term under axial transformation [70,72], and its properties are similar to those of X in the HV scheme.

The anomalous term X can be expressed as a linear combination of other regulated operators with same quantum numbers and an evanescent operator $\bar{X}$, whose insertions in Green's functions with arbitrary number of fields vanish at the quantum level as one removes the regulator. To perform the projection on nonevanescent operators one defines [70]

$$\bar{X} = X + \alpha \partial \cdot A + \beta 2(mP) + \gamma G\tilde{G}, \qquad (121)$$

and determines the coefficients α, β, γ perturbatively or nonperturbatively by requiring that appropriate projections of matrix elements of $\bar{X}$ in quark and gluon states (and their derivative with respect to the mass) vanish[12]

$$\langle q|\bar{X}|q\rangle|_{\slashed{q}\gamma_5} = 0 \qquad (122)$$

$$\frac{\partial}{\partial m}\langle q|\bar{X}|q\rangle|_{\gamma_5} = 0 \qquad (123)$$

$$\langle g|\bar{X}|g\rangle = 0. \qquad (124)$$

Analyzing Green's functions of X with two quarks and with two gluons to one loop in perturbation theory in the $\overline{\text{MS}}$-HV scheme, we find

$$\Gamma_X^{(2)} = \frac{\alpha_s}{4\pi} 4C_F[i\slashed{q}\gamma_5 - 4i\mathcal{M}\gamma_5] \qquad (125)$$

$$\Gamma_X^{(2)\mu\nu} = n_F \frac{\alpha_s}{4\pi} 4\epsilon^{\mu\nu\alpha\beta} p_\alpha p'_\beta, \qquad (126)$$

leading to

$$\alpha = -4C_F \frac{\alpha_s}{4\pi} \qquad (127)$$

$$\beta = 8C_F \frac{\alpha_s}{4\pi} \qquad (128)$$

$$\gamma = -n_F \frac{\alpha_s}{4\pi}. \qquad (129)$$

Using Eq. (121) into Eq. (119), one gets the final result

$$(1 + \alpha)\partial \cdot A = 2(1 - \beta)(mP) + P_E - \gamma G\tilde{G} + \bar{X}, \qquad (130)$$

which is still expressed in terms of bare operators and couplings. The nonsinglet case is now straightforward: one finds the same values of α and β, and the nonsinglet anomalous operator $\bar{X}^a$ does not have a $G\tilde{G}$ component.

B. PCAC relation in terms of renormalized operators

We next express the PCAC relation in terms of renormalized operators $[O]_i$, related to bare operators O_j via [note that in this section we use a different notation compared to Eq. (48)],

$$O_i = Z_{ij}[O]_j, \qquad (131)$$

with Z_{ij} given in an arbitrary scheme. For the operators of interest, we have the mixing structure[13]

$$\begin{pmatrix} G\tilde{G} \\ \partial \cdot A \\ (mP) \end{pmatrix} = \begin{pmatrix} Z_{G\tilde{G}} & Z_{G\tilde{G},\partial A} & 0 \\ 0 & Z_A & 0 \\ 0 & 0 & Z_m Z_P \end{pmatrix} \begin{pmatrix} [G\tilde{G}] \\ [\partial \cdot A] \\ [mP] \end{pmatrix}. \qquad (132)$$

Using Eqs. (127), (128), (129), and (130) leads to the renormalized PCAC relation,

$$\bar{C}_1(g^2)[\partial \cdot A] = \bar{C}_2(g^2)2[mP] + \bar{C}_3(g^2)\frac{n_F}{16\pi^2}[g^2 G\tilde{G}] + P_E + \bar{X}, \qquad (133)$$

with coefficients, in terms of the bare coupling g,

$$\bar{C}_1(g^2) = Z_A(1 + \alpha) + \gamma Z_{G\tilde{G},\partial A} \qquad (134a)$$

$$\bar{C}_2(g^2) = Z_P Z_m(1 - \beta) \qquad (134b)$$

$$\bar{C}_3(g^2) = -\frac{16\pi^2\gamma}{n_F g^2} Z_g^2 Z_{G\tilde{G}}, \qquad (134c)$$

satisfying $\bar{C}_{1,2,3}(0) = 1$. As a consequence of the finiteness of the EOM operator, $[P_E] = P_E$, and of the independence of the operators $[\partial \cdot A]$, $[mP]$ and $[G\tilde{G}]$, $\bar{C}_{1,2,3}(g^2)$ must be finite.

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[12]Insertions of X can give nonvanishing results only in Green's functions with positive superficial degree of divergence. Of these, one needs to analyze only the one with two quarks and the one with two gluons, as the others do not provide independent information.

[13]This is valid in schemes in which $m_{u,d,s}$ are multiplicatively renormalized with the same constant Z_m.

C. Finite renormalization and WIs

Equation (133) shows that in a given renormalization scheme (i.e., a choice of $Z_{A,P}$, Z_m, $Z_{G\tilde{G}}$, $Z_{G\tilde{G},\partial\cdot A}$ that makes the operator insertions finite) the renormalized quantities do not necessarily satisfy properly normalized (anomalous) WIs: the $\overline{\text{MS}}$-HV scheme is one example. However, given the scheme-dependent $\bar{C}_{1,2,3}(g^2)$, through a finite renormalization one can restore the WIs, as seen from Eq. (133). Operators in the "WI scheme" are defined by

$$[A^\mu]_{\text{WI}} = \bar{C}_1(g^2)[A^\mu] \tag{135}$$

$$[mP]_{\text{WI}} = \bar{C}_2(g^2)[mP] \tag{136}$$

$$[g^2 G\tilde{G}]_{\text{WI}} = \bar{C}_3(g^2)[g^2 G\tilde{G}]. \tag{137}$$

Applying the operator $d/d(\log\mu)$ to both sides of Eq. (133), using the finiteness of P_E and the independence of the remaining operators, one obtains a set of differential equations for $\bar{C}_{1,2,3}(g^2)$. The solution reveals that the coefficients $\bar{C}_{2,3}(g^2)$ are such that $[mP]_{\text{WI}}$ and $[g^2 G\tilde{G}]_{\text{WI}}$ have vanishing diagonal anomalous dimension to all orders. On the other hand, $\bar{C}_1(g^2)$ is such that $[A_\mu]_{\text{WI}}$ has an anomalous dimension starting at $O(g^4)$, related to the off-diagonal anomalous dimension $\gamma_{G\tilde{G},\partial\cdot A} = -(Z^{-1}dZ/d(\log\mu))_{G\tilde{G},\partial\cdot A}$, namely, $\gamma_{A_{\text{WI}}} = \gamma_{G\tilde{G},\partial\cdot A} \cdot \alpha_s/(4\pi)$. The rescaled operators satisfy the properly normalized PCAC relation:

$$\partial \cdot [A]_{\text{WI}} = 2[mP]_{\text{WI}} + \frac{n_F}{16\pi^2}[g^2 G\tilde{G}]_{\text{WI}} + iP_E + \bar{X}. \tag{138}$$

The coefficients needed to reach the "WI" scheme from the $\overline{\text{MS}}$-HV scheme to $O(g^2)$ are[14]

$$\bar{C}_1 = 1 - 4C_F \frac{\alpha_s}{4\pi} \quad \bar{C}_2 = 1 - 8C_F \frac{\alpha_s}{4\pi} \quad \bar{C}_3 = 1 + O(\alpha_s^2). \tag{139}$$

In the case of the RI-$\tilde{\text{S}}$MOM scheme defined in Sec. VI A, the perturbative values of α, β and γ are still given by Eqs. (127), (128) and (129). In HV, the conditions given in Eqs. (107c), (107d), and (107e), which are the equivalent to the RI-SMOM condition of Ref. [59], give

$$Z_A = 1 + \frac{\alpha_s}{4\pi}(4C_F) \tag{140}$$

$$Z_P Z_m = 1 + \frac{\alpha_s}{4\pi}(8C_F), \tag{141}$$

where we used the value of Z_m obtained in Ref. [59]:

---

[14]To determine $\bar{C}_3$ we rely on the two-loop calculations of Ref. [69].

$$Z_m = 1 - \frac{\alpha_s C_F}{4\pi}\left(4 + \xi - (3+\xi)\frac{\psi}{2}\right). \tag{142}$$

This leads to $\bar{C}_1(g^2) = 1 + \mathcal{O}(g^4)$ and $\bar{C}_2(g^2) = 1 + \mathcal{O}(g^4)$, thus showing that singlet and nonsinglet axial currents and pseudoscalar densities are already correctly normalized, up to corrections of $\mathcal{O}(g^4)$. The RI-$\tilde{\text{S}}$MOM condition Eq. (107a) leads to a $G\tilde{G}$ which is not correctly normalized. However, once a definition of Z_g is given, for example, by fixing the three-gluon or quark-gluon vertex at the symmetric point to its tree-level value [74,75], Eq. (134) allows one to define $[G\tilde{G}]_{\text{WI}}$.

Equations (140) and (141) differ by a finite piece from the results in Ref. [59], which are obtained using NDR and found $Z_A = Z_P Z_m = 1$. The finite pieces in Z_A and $Z_P Z_m$ are crucial in compensating the anomalous dimension of the axial current and pseudoscalar density arising from divergences in the $\overline{\text{MS}}$-HV two-loop calculation, as can be explicitly verified from the results of Ref. [76]. For the nonsinglet axial current, the cancellation is exact, and the RI-$\tilde{\text{S}}$MOM axial current does not have anomalous dimension at $\mathcal{O}(\alpha_s^2)$. In the singlet case, the $\mathcal{O}(\alpha_s)$ finite piece ensures that the relation $\gamma_A = \gamma_{G\tilde{G},\partial\cdot A}\alpha_s/(4\pi)$ is respected.

While we have given explicit results in perturbation theory within the $\overline{\text{MS}}$-HV and RI-$\tilde{\text{S}}$MOM scheme, the above discussion provides the steps needed to determine the coefficients α, β, γ starting from any regulator and any scheme. These, in turn, in combination with the renormalization factors of Eq. (132) determine the finite rescaling factors $\bar{C}_{1,2,3}$ in Eqs. (134) needed to obtain renormalized operators that satisfy the axial Ward identities.

VIII. RELATION TO THE $\Delta S = 1$ CHROMOMAGNETIC OPERATOR

In a recent article [38], the renormalization of the strangeness changing quark chromomagnetic dipole moment (CMDM) operator has been studied. In our notation the *P*- and *CP*-even operator studied in [38] reads

$$O_{CM} = g\bar{\psi}t^{\Delta S}\sigma^{\mu\nu}G_{\mu\nu}\psi,$$
$$t^{\Delta S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{143}$$

Ref. [38] studies the mixing of O_{CM} with lower-dimensional operators nonperturbatively on the lattice, and the mixing of O_{CM} with other dimension-five operators in perturbation theory both in the lattice and in $\overline{\text{MS}}$ schemes. Clearly, a number of common issues arise in our study and in Ref. [38], so a closer comparison of operator basis and mixing results is desirable.

A. Operator basis

First, let us focus on the operator basis. Note that our operator basis was constructed assuming diagonal flavor structures, so that $[\mathcal{M}, t^a] = [Q, t^a] = 0$ and $\mathrm{Tr}(\mathcal{M}t^a) \neq 0$. In the case of flavor off-diagonal generators, such as $t^{\Delta S}$, a number of new operators appears at dimension five, while all the operators involving $\mathrm{Tr}(Mt^a)$ vanish. In what follows we provide (i) the basis of dimension-five operators mixing with the P- and CP-odd CEDM operator $O_{CE} \equiv C$ defined in Eq. (30), with off-diagonal flavor structure $t^a \rightarrow t^{\Delta S}$ and (ii) the corresponding basis for the P- and CP-even sector (mixing with O_{CM}), to be compared with Ref. [38].

For ease of comparison with Ref. [38] we omit the operators involving the electromagnetic field, i.e., $O_3^{(5)}$ and $O_{14}^{(5)}$ of Sec. III A 3. With this in mind, for the $\Delta S = 1$ sector we find ten independent CP-odd operators. In the notation of Sec. III A 3 (with $t^a \rightarrow t^{\Delta S}$), we find $C, \partial^2 P, (m\partial \cdot A)_2, (m^2 P)_{1,2}, P_{EE}, \partial \cdot A_E, A_\partial$ and two new structures that vanish for diagonal flavor generators, namely, $(m^2 P)_4 = \bar{\psi} i\gamma_5 [\mathcal{M}, [\mathcal{M}, t^{\Delta S}]]\psi$ and $(mP_E)_1 = \bar{\psi}_E i\gamma_5 [\mathcal{M}, t^{\Delta S}]\psi - \bar{\psi} i\gamma_5 [\mathcal{M}, t^{\Delta S}]\psi_E$. In order to match more closely the operator basis of Ref. [38] we can trade the operator $(m\partial \cdot A)_2$ involving derivatives of the axial current in favor of $(mP_E)_2 = \bar{\psi}_E i\gamma_5 \{\mathcal{M}, t^{\Delta S}\}\psi + \bar{\psi} i\gamma_5 \{\mathcal{M}, t^{\Delta S}\}\psi_E$, via the relation

$$2(m\partial \cdot A)_2 = 4(m^2 P)_1 - (m^2 P)_4 + (mP_E)_2. \qquad (144)$$

So we end up with the dimension-five basis in the P- and CP-odd sector reported in the left column of Table V. The corresponding P- and CP-even sector operators can be obtained from the above ones with the substitution $i\gamma_5 \rightarrow 1$ and are given explicitly in the right column of Table V.

We can now compare our basis to the one in Ref. [38], which consists of ten dimension-five operators $\mathcal{O}_{1,\dots,10}$:

(i) For the gauge-invariant operators that do not vanish by the EOM, after converting the operators in [38] from Euclidean to Minkowski metric, we find the

correspondence: $\mathcal{O}_1 = O_{CM}$, $\mathcal{O}_2 = 2(m^2 S)_1$, $\mathcal{O}_3 = (m^2 S)_1 - 1/2(m^2 S)_4$, $\mathcal{O}_4 = -\partial^2 S$.

(ii) For the operators vanishing by the EOM we find: $\mathcal{O}_5 = S_{EE}$, $\mathcal{O}_7 = 1/2[(mS_E)_1 - (mS_E)_2]$, $\mathcal{O}_8 = -1/2[(mS_E)_1 + (mS_E)_2]$, $\mathcal{O}_9 = -(\partial \cdot V_E + V_\partial)$, $\mathcal{O}_{10} = V_\partial$.

(iii) There is no operator in our basis corresponding to $\mathcal{O}_6$ in [38]. The CP-odd counterpart of $\mathcal{O}_6$ is $\tilde{P}_{EE} = \bar{\psi} i\gamma_5 t^{\Delta S}\psi_{EE} + \bar{\psi}_{EE} i\gamma_5 t^{\Delta S}\psi$, and it can be expressed in terms of operators already present in the basis, via:

$$\partial \cdot A_E = \tilde{P}_{EE} + 2P_{EE} + (mP_E)_2, \qquad (145)$$

A similar linear dependence relation holds in the P- and CP-even sector. Ref. [38] finds at one loop that $\mathcal{O}_6$ is not needed to renormalize O_{CM}. This is consistent with our finding that $\mathcal{O}_6$ is not linearly independent.

(iv) In Ref. [38] the operator $(m^2 S)_2 = (m_u^2 + m_d^2 + m_s^2)\bar{s}d$ is absent. This operator is allowed by the symmetries of the problem. In perturbation theory it can mix with O_{CM} starting at two-loop order, so its omission does not affect the results of Ref. [38]. However, the operator should be included in nonperturbative renormalization treatments.

B. One-loop renormalization factors

Using the CP-odd operator basis of Table V, we have extended our analysis of the two- and three-point functions to include off-diagonal flavor structures and have found the mixing to the additional operators (in the $\overline{\mathrm{MS}}$ scheme)

$$Z_{O_{CE},(m^2 P)_4} = -Z_{O_{CE},(mP_E)_1} = \frac{1}{\epsilon}\frac{\alpha_s}{4\pi}\frac{3}{8}(C_A - 4C_F). \qquad (146)$$

Using (i) the results given in Sec. V for the operator mixing in our original basis [extended to the new structures through Eq. (146)] and (ii) the change of basis implied

TABLE V. Operator basis in the CP-odd and CP-even sectors.

CP-odd	CP-even
$O_{CE} = ig\bar{\psi}\sigma^{\mu\nu}\gamma_5 G_{\mu\nu}t^{\Delta S}\psi$	$O_{CM} = g\bar{\psi}\sigma^{\mu\nu}G_{\mu\nu}t^{\Delta S}\psi$
$\partial^2 P = \partial^2(\bar{\psi} i\gamma_5 t^{\Delta S}\psi)$	$\partial^2 S = \partial^2(\bar{\psi}t^{\Delta S}\psi)$
$(m^2 P)_1 = \frac{1}{2}\bar{\psi} i\gamma_5 \{\mathcal{M}^2, t^{\Delta S}\}\psi$	$(m^2 S)_1 = \frac{1}{2}\bar{\psi}\{\mathcal{M}^2, t^{\Delta S}\}\psi$
$(m^2 P)_2 = \mathrm{Tr}[\mathcal{M}^2]\bar{\psi} i\gamma_5 t^{\Delta S}\psi$	$(m^2 S)_2 = \mathrm{Tr}[\mathcal{M}^2]\bar{\psi}t^{\Delta S}\psi$
$(m^2 P)_4 = \bar{\psi} i\gamma_5 [\mathcal{M}, [\mathcal{M}, t^{\Delta S}]]\psi$	$(m^2 S)_4 = \bar{\psi}[\mathcal{M}, [\mathcal{M}, t^{\Delta S}]]\psi$
$P_{EE} = i\bar{\psi}_E\gamma_5 t^{\Delta S}\psi_E$	$S_{EE} = \bar{\psi}_E t^{\Delta S}\psi_E$
$(mP_E)_1 = \bar{\psi}_E i\gamma_5 [\mathcal{M}, t^{\Delta S}]\psi - \bar{\psi} i\gamma_5 [\mathcal{M}, t^{\Delta S}]\psi_E$	$(mS_E)_1 = \bar{\psi}_E[\mathcal{M}, t^{\Delta S}]\psi - \bar{\psi}[\mathcal{M}, t^{\Delta S}]\psi_E$
$(mP_E)_2 = \bar{\psi}_E i\gamma_5 \{\mathcal{M}, t^{\Delta S}\}\psi + \bar{\psi} i\gamma_5 \{\mathcal{M}, t^{\Delta S}\}\psi_E$	$(mS_E)_2 = \bar{\psi}_E\{\mathcal{M}, t^{\Delta S}\}\psi + \bar{\psi}\{\mathcal{M}, t^{\Delta S}\}\psi_E$
$\partial \cdot A_E = \partial_\mu[\bar{\psi}_E\gamma^\mu\gamma_5 t^{\Delta S}\psi - \bar{\psi}\gamma_5\gamma^\mu t^{\Delta S}\psi_E]$	$\partial \cdot V_E = i\partial_\mu[\bar{\psi}_E\gamma^\mu t^{\Delta S}\psi - \bar{\psi}\gamma^\mu t^{\Delta S}\psi_E]$
$A_\partial = \bar{\psi}\gamma_5\partial t^{\Delta S}\psi_E - \bar{\psi}_E\!\leftarrow\!\partial\gamma_5 t^{\Delta S}\psi$	$V_\partial = \bar{\psi} i\partial t^{\Delta S}\psi_E - \bar{\psi}_E i\!\leftarrow\!\partial t^{\Delta S}\psi$

by Eq. (144), we have computed the renormalization matrix relevant to the *CP*-odd operators in Table V.

In order to compare to Ref. [38], we need the relation between the divergence structure of the *CP*-even and *CP*-odd sectors. At one loop we have verified that the divergences of two- and three-point functions with insertion of O_{CM} and O_{CE} are related by a simple operation $\hat{\tau}$:

$$\Gamma_{O_{CM}} = \hat{\tau}[\Gamma_{O_{CE}}],$$
$$\hat{\tau}: \{i\gamma_5 \rightarrow 1, t^{\Delta S}\mathcal{M}^n \rightarrow (-1)^n t^{\Delta S}\mathcal{M}^n (n = 0, 1, 2)\}. \tag{147}$$

Similarly, the tree-level insertions of the *CP*-even (O_+) and *CP*-odd (O_-) operators appearing in each line of Table V are related by $\Gamma_{O_+} = \hat{\tau}[\Gamma_{O_-}]$, except for the following cases:

$$\Gamma_{(mS_E)_{1,2}} = \hat{\tau}[\Gamma_{(mP_E)_{2,1}}] \tag{148a}$$

$$\Gamma_{S_{EE}} = -\hat{\tau}[\Gamma_{P_{EE}}] \tag{148b}$$

$$\Gamma_{\partial \cdot V_E} = -\hat{\tau}[\Gamma_{\partial \cdot A_E}] \tag{148c}$$

$$\Gamma_{V_\partial} = -\hat{\tau}[\Gamma_{A_\partial}] \tag{148d}$$

From the renormalization matrix in the *CP*-odd sector and the relations (148), we have computed the renormalization factors in the *CP*-even sector, in the basis of Table V. Finally, converting to the basis $\mathcal{O}_{1,\ldots,10}$ of Ref [38] (using the relations given in Sec. VIII A), we find our results for the renormalization coefficients to agree with Eqs. (66)–(75) of Ref. [38].

IX. CONCLUSIONS

In this work we have studied the off-shell renormalization and mixing of *CP*-odd dimension-five operators in QCD in both the $\overline{\text{MS}}$ and RI-$\tilde{\text{S}}$MOM schemes (the latter amenable to implementation in lattice QCD), providing the matching matrix between operators in RI-$\tilde{\text{S}}$MOM and $\overline{\text{MS}}$ to $O(\alpha_s)$.

We have paid special attention to the definition of a finite quark CEDM operator in the RI-$\tilde{\text{S}}$MOM scheme, identifying all the needed subtractions. This is the first step towards a lattice QCD calculation of the impact of the quark CEDM on the nucleon EDM, which is currently afflicted by one order of magnitude uncertainty. This paper sets the stage to perform nonperturbative renormalization of the CEDM. The next steps in the program involve (i) performing exploratory computations of the needed CEDM quark and gluon Green's functions on the lattice, and comparing this method to lattice perturbation theory; (ii) performing exploratory calculation of the CEDM insertion in the neutron state, correlated with the electromagnetic current or in external electric field [34].

Besides inducing nucleon EDM, the quark CEDM induces *T*-odd *P*-odd pion-nucleon couplings that are a key input in the computation of EDMs of both light and heavy nuclei. Chiral symmetry implies that the *T*-odd pion-nucleon coupling induced by the quark CEDM can be extracted (up to chiral corrections) by calculating the baryon mass splittings induced by the quark chromomagnetic dipole moment (CMDM) operator [14]. In a future publication we will explore the nonperturbative renormalization and mixing in the flavor-diagonal CMDM sector and its relation to the CEDM.

Finally, a desirable extension of this work involves studying the nonperturbative renormalization and mixing structure of *CP*-odd dimension-six operators, such as Weinberg's operator [77] and four-quark operators.

APPENDIX A: *CP* TRANSFORMATION

In this Appendix we review the definition and properties of the *CP* transformation. On the fermion fields ψ, *CP* is defined as the linear operator $\mathcal{CP}$:

$$\mathcal{CP}^{-1}\psi\mathcal{CP} \equiv \psi^{CP} = ie^{i\phi}\gamma_2\gamma_0\psi^*$$
$$= -ie^{i\phi}(\bar{\psi}\gamma_2^*)^T = ie^{i\phi}\gamma_2\bar{\psi}^T, \tag{A1}$$

where $\psi^* \equiv \psi^{\dagger T}$, ϕ is an arbitrary phase, and we are using the convention that γ_2 is an anti-Hermitian matrix.[15] Note that the *CP* transformation for the $U(1)$-transformed fermion field $\tilde{\psi} = e^{i\theta}\psi$ looks like Eq. (A1) with $\psi \rightarrow \tilde{\psi}$ and $\phi \rightarrow \phi + 2\theta$.

In addition to this, the *CP* transformation changes all vector operators v^μ to v_μ in the metric with signature $(+ - - -)$, and changes every charge generator T^a to $(T^a)^T$.

Let Γ_M denote a gamma structure with M Lorentz indices and O^N denote an operator involving derivatives with N Lorentz indices. Then

$$(\bar{\chi}\Gamma_M O^N \psi)^{CP} = -(\psi^T O^N \Gamma_M^T \gamma_0^T \chi^*)^{CP}$$
$$= -[(-ie^{i\phi_\psi}\bar{\psi}\gamma_2^*)O_N\Gamma_M^T\gamma_0^T(-ie^{-i\phi_\chi}\gamma_2^*\gamma_0^*\chi)]$$
$$= e^{i\Delta\phi}[\bar{\psi}(\gamma_0^\dagger\gamma_2^\dagger\gamma_0\Gamma_M\gamma_2^\dagger)^T O_N\chi]$$
$$= e^{i\Delta\phi}[\bar{\psi}(-\gamma_2\Gamma_M\gamma_2)^T O_N\chi]$$
$$= e^{i\Delta\phi}\bar{\psi}\Gamma_M^{CP}O_N\chi, \tag{A2}$$

---

[15]Whenever we need an explicit representation for the γ matrices, we use that one provided in Ref. [57].

where $\Gamma_M^{CP} \equiv (-\gamma_2 \Gamma_M \gamma_2)^T$ and we have used the Hermiticity of γ_0, anti-Hermiticity of γ_2, and $\gamma_0^2 = 1$. Using $\gamma_2^T = \gamma_2^{\dagger *} = \gamma_2$, we can now write the simpler expression $\Gamma_M^{CP} \equiv -\gamma_2 \Gamma_M^{\dagger *} \gamma_2$. For the 16 Clifford matrices, we then have

$$1^{CP} = 1 \tag{A3a}$$

$$\gamma_5^{CP} = -\gamma_5 \tag{A3b}$$

$$\gamma_\mu^{CP} = -\gamma^\mu \tag{A3c}$$

$$(\gamma_\mu \gamma_5)^{CP} = \gamma_5 \gamma^\mu = -\gamma^\mu \gamma_5 \tag{A3d}$$

$$\sigma_{\mu\nu}^{CP} = \sigma^{\nu\mu} = -\sigma^{\mu\nu}. \tag{A3e}$$

For the equation of motion field $\psi_E = (iD^\mu \gamma_\mu - m)\psi$, the transformation is

$$
\begin{aligned}
\psi_E^{CP} &\equiv (iD^{\mu CP} \gamma_\mu - m)\psi^{CP} \\
&= ie^{i\phi}(iD_\mu^* \gamma_\mu - m)\gamma_2 \gamma_0 \psi^* \\
&= ie^{i\phi}\gamma_2(-iD_\mu^* \gamma_\mu^* - m)\gamma_0 \psi^* \\
&= ie^{i\phi}\gamma_2\gamma_0(-iD_\mu^* \gamma_\mu^T - m)\psi^* \\
&= ie^{i\phi}\gamma_2\gamma_0(-iD_\mu^* \gamma^{\mu *} - m)\psi^* \\
&= ie^{i\phi}\gamma_2\gamma_0(-iD_\mu^* \gamma^{\mu *} - m)\psi^* \\
&= ie^{i\phi}\gamma_2\gamma_0[(iD_\mu \gamma^\mu - m)\psi]^* \\
&= ie^{i\phi}\gamma_2\gamma_0 \psi_E^*,
\end{aligned}
\tag{A4}
$$

where the conjugate of $D_\mu = \partial_\mu - igA_\mu^a T^a$ is defined as $D_\mu^* = \partial_\mu + igA_\mu^a T^{a*}$ to take into account the opposite gauge charge of the antiparticle. One way to state this result is that the CP phase is the same for the fields ψ and ψ_E, i.e., $\phi_{\psi_E} = \phi_\psi$.

Finally, note that the CP transformation on chiral fields $\psi_{L,R} = (1 \mp \gamma_5)/2\psi$

$$
\begin{aligned}
\mathcal{CP}^{-1}\psi_L \mathcal{CP} &= ie^{i\phi}\gamma_2 \overline{\psi_L}^T \\
\mathcal{CP}^{-1}\psi_R \mathcal{CP} &= ie^{i\phi}\gamma_2 \overline{\psi_R}^T.
\end{aligned}
\tag{A5}
$$

APPENDIX B: BRST SYMMETRY AND OPERATOR BASIS

A given gauge invariant operator O mixes under renormalization with two classes of operators of same (or lower) dimension [52,53]: (i) ghost-free gauge-invariant operators with the same symmetry properties of O that do not vanish by the equations of motion (EOM) and (ii) "nuisance" operators allowed by the solution to the Ward identities associated with the BRST symmetry. These include non-gauge-invariant operators. For completeness,

TABLE VI. Properties for dynamical fields and BRST sources. The first row indicates whether the variable is commuting (+) or anticommuting (−). The second and third row list the transformation under Lorentz and color groups. The fourth row gives the ghost number assignments and the fifth row lists the mass dimension.

	M	$\bar{M}$	$J_\mu - \partial_\mu \bar{c}$	K	ψ	$\bar{\psi}$	A_μ	c	$\bar{c}$	∂_μ
Comm.	+	+	−	+	−	−	+	−	−	+
Lorentz	$\frac{1}{2}$	$\frac{\bar{1}}{2}$	1	0	$\frac{1}{2}$	$\frac{\bar{1}}{2}$	1	0	0	1
Color	3	3*	8	8	3	3*	8	8	8	0
Ghost	−1	−1	−1	−2	0	0	0	1	−1	0
Dim.	$\frac{5}{2}$	$\frac{5}{2}$	3	4	$\frac{3}{2}$	$\frac{3}{2}$	1	0	2	1

we sketch below the procedure to obtain the "nuisance" operators, paraphrasing Ref. [52].

The gauge and fermion Lagrangian density for the $SU(3)_C \times U(1)_{EM}$ group is expressed in terms of physical fields A_μ^a, A_μ^γ, ψ, $\bar{\psi}$, the dynamical ghosts c^a, $\bar{c}^a$, c^γ, $\bar{c}^\gamma$, and the nonpropagating sources for BRST transformations M, $\bar{M}$, J_μ^a, K^a, J_μ^γ, whose properties are summarized in Table VI. This Lagrangian is

$$
\mathcal{L}_0 = -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2\xi}(\partial \cdot A^a)^2 - (J_\mu^a - \partial_\mu \bar{c}^a)D^{\mu,ab}c_b
$$
$$
+ \frac{1}{2}gf^{abc}K^a c^b c^c \tag{B1}
$$

$$
-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi_\gamma}(\partial \cdot A^\gamma)^2 - (J_\mu^\gamma - \partial_\mu \bar{c}^\gamma)\partial^\mu c^\gamma \tag{B2}
$$

$$
+ \bar{\psi}(i\not{D} - m)\psi + \bar{M}(-igc^a T^a - iec^\gamma)\psi
$$
$$
+ \bar{\psi}(-igc^a T^a - iec^\gamma)M, \tag{B3}
$$

where

$$
G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \tag{B4}
$$

$$
F_{\mu\nu}^a = \partial_\mu A_\nu^\gamma - \partial_\nu A_\mu^\gamma \tag{B5}
$$

$$
D_\mu^{ab}c^b = \partial_\mu c^a + gf^{abc}A_\mu^b c^c \tag{B6}
$$

$$
D_\mu \psi = (\partial_\mu - igA_\mu^a T^a - ieQA_\mu^\gamma)\psi. \tag{B7}
$$

The action S obtained by adding to the Lagrangian density a set of infinitesimal sources Φ for gauge-invariant ghost-free operators O

$$
S = \int d^4x \mathcal{L}_0(x) + \int d^4x \Phi(x)O(x) \equiv S_0 + \Phi \cdot O, \tag{B8}
$$

is invariant under the BRST transformations given by:

$$
\Delta A_\mu^a = -\frac{\delta S}{\delta J_\mu^a}\delta\lambda \qquad \Delta A_\mu^\gamma = -\frac{\delta S}{\delta J_\mu^\gamma}\delta\lambda \tag{B9}
$$

$$\Delta c^a = \frac{\delta S}{\delta K^a}\delta\lambda \qquad \Delta c^\gamma = 0 \tag{B10}$$

$$\Delta \bar{c}^a = \frac{1}{\xi}\partial \cdot A^a \delta\lambda \qquad \Delta \bar{c}^\gamma = \frac{1}{\xi_\gamma}\partial \cdot A^\gamma \delta\lambda \tag{B11}$$

$$\Delta \psi_i = \frac{\delta S}{\delta \bar{M}_i}\delta\lambda \tag{B12}$$

$$\Delta \bar{\psi}_i = \frac{\delta S}{\delta M_i}\delta\lambda, \tag{B13}$$

with $\delta\lambda$ an anticommuting infinitesimal parameter. This invariance leads to the Ward identities for the generating functional of 1PI Green's functions, that in particular imply the following identity for $\hat{S} \equiv S + \int d^4x[1/(2\xi)(\partial \cdot A^a)^2 + 1/(2\xi_\gamma)(\partial \cdot A^\gamma)^2]$:

$$\int d^4x \left(\frac{\delta\hat{S}}{\delta A_\mu^\gamma}\frac{\delta\hat{S}}{\delta J_\mu^\gamma} + \frac{\delta\hat{S}}{\delta A_\mu^a}\frac{\delta\hat{S}}{\delta J_\mu^a} + \frac{\delta\hat{S}}{\delta c^a}\frac{\delta\hat{S}}{\delta K^a} + \frac{\delta\hat{S}}{\delta \psi_i}\frac{\delta\hat{S}}{\delta \bar{M}_i} + \frac{\delta\hat{S}}{\delta \bar{\psi}_i}\frac{\delta\hat{S}}{\delta M_i} \right)$$
$$= 0. \tag{B14}$$

While $S = S_0 + \Phi \cdot O$ satisfies the Ward identity Eq. (B14), the general solution involves additional terms. Writing the general solution symbolically as

$$S = S_0 + \Phi \cdot O + \Phi \cdot N, \tag{B15}$$

and working to first order in the external sources (one operator insertion), one finds that the nuisance operators N must satisfy

$$\hat{W}(\Phi \cdot N) = 0, \tag{B16}$$

with the operator

$$\hat{W} = \frac{\delta\hat{S}_0}{\delta A_\mu^\gamma}\frac{\delta}{\delta J_\mu^\gamma} + \frac{\delta\hat{S}_0}{\delta J_\mu^\gamma}\frac{\delta}{\delta A_\mu^\gamma} + \frac{\delta\hat{S}_0}{\delta A_\mu^a}\frac{\delta}{\delta J_\mu^a} + \frac{\delta\hat{S}_0}{\delta J_\mu^a}\frac{\delta}{\delta A_\mu^a} + \frac{\delta\hat{S}_0}{\delta c^a}\frac{\delta}{\delta K^a}$$
$$+ \frac{\delta\hat{S}_0}{\delta K^a}\frac{\delta}{\delta c^a} + \frac{\delta\hat{S}_0}{\delta \psi_i}\frac{\delta}{\delta \bar{M}_i} + \frac{\delta\hat{S}_0}{\delta \bar{M}_i}\frac{\delta}{\delta \psi_i} + \frac{\delta\hat{S}_0}{\delta \bar{\psi}_i}\frac{\delta}{\delta M_i}$$
$$+ \frac{\delta\hat{S}_0}{\delta M_i}\frac{\delta}{\delta \bar{\psi}_i}. \tag{B17}$$

Since $\hat{W}\hat{W} = 0$, it turns out that

$$\Phi \cdot N = \hat{W}(\Phi \cdot F), \tag{B18}$$

where F is a set of operators with the same Lorentz property of O, same dimension, and ghost number -1. After acting with $\hat{W}$ one sets the sources M, $\bar{M}$, K to zero, and J_μ to $-\partial_\mu\bar{c}$.

We are now ready to classify the F operators and resulting nuisance operators N:

(i) At dimension five, the only Lorentz scalars of ghost number -1 that we can write down are: $\bar{\psi}\chi_\pm A\!\!\!/M$, $\bar{\psi}\chi_\pm A\!\!\!/^\gamma M$, $\bar{M}A\!\!\!/\chi_\pm\psi$, $\bar{M}A\!\!\!/^\gamma\chi_\pm\psi$, $\bar{\psi}\chi_\pm\partial\!\!\!/M$, $\bar{M}\partial\!\!\!/\chi_\pm\psi$, $\bar{M}cM$, $\bar{M}c^\gamma M$, where $\chi_\pm = (1 \pm \gamma_5)/2$ is a chiral projector. Acting on these structures with $\hat{W}$ produces the terms $\bar{\psi}_E A\!\!\!/\chi_\pm\psi$, $\bar{\psi}_E A\!\!\!/^\gamma\chi_\pm\psi$, $\bar{\psi}_E\partial\!\!\!/\chi_\pm\psi$, $\bar{\psi}A\!\!\!/\chi_\pm\psi_E$, $\bar{\psi}A\!\!\!/^\gamma\chi_\pm\psi_E$, $\bar{\psi}\partial\!\!\!/\chi_\pm\psi_E$.

In addition, we have the gauge-invariant ghost-free terms that are not zero by equations of motion in the massless limit: $\bar{\psi}\sigma^{\mu\nu}G_{\mu\nu}\chi_\pm\psi$, $\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\chi_\pm\psi$, $\partial^2(\bar{\psi}\chi_\pm\psi)$, $\partial_\mu(\bar{\psi}\sigma^{\mu\nu}D^*_\nu\chi_\pm\psi)$, $\partial_\mu(\bar{\psi}\sigma^{\mu\nu}D_\nu\chi_\pm\psi)$.

(ii) At dimension four, the only Lorentz scalars of ghost number -1 are: $\bar{M}\chi_\pm\psi$, $\bar{\psi}\chi_\pm M$, $J^{a\mu}A_\mu^a$, $J^{\gamma\mu}A_\mu^\gamma$, and Kc. The variation of these produce $\bar{\psi}_E\chi_\pm\psi$, $\bar{\psi}\chi_\pm\psi_E$, $(D_\nu G^{\nu\mu}A_\mu + g\bar{\psi}A\!\!\!/\psi) - g[\partial_\mu\bar{c}, c]A^\mu$, $(\partial_\nu D^{\nu\mu}A_\mu^\gamma + e\bar{\psi}A\!\!\!/^\gamma\psi)$, $(\partial^\mu\bar{c})D_\mu c$, $(D_\mu\partial^\mu\bar{c})c$.

The only gauge-invariant ghost-free operators not zero by equation of motion in the mass-less limit are: $G_{\mu\nu}G^{\mu\nu}$ and $G_{\mu\nu}\tilde{G}^{\mu\nu}$.

(iii) At dimension three and below, there are no ghost number -1 scalars, so the only operators we need to consider are the gauge-invariant ghost-free operators that are not zero by the massless equation of motion. The only possible such terms are $\bar{\psi}\chi_\pm\psi$.

Selecting the T-odd and P-odd structures, including gauge-invariant ghost-free operators that do not vanish by the EOM, and eliminating linearly dependent operators[16] we arrive at the basis presented in Sec. III A.

APPENDIX C: AXION MECHANISM

A very elegant way to dynamically set $\bar{\theta}$ to zero is the Peccei-Quinn (PQ) mechanism [46], which predicts the existence of a new light particle, the axion [78,79]. We follow here the discussion of the PQ mechanism in the EFT framework of Ref. [80]. A common feature of axion models is the existence of a $U_{PQ}(1)$ symmetry, which is spontaneously broken at high energy. The axion is the Goldstone boson of the symmetry, and, under $U_{PQ}(1)$, it changes by an additive constant, $a \to a + c$, while the SM fields are chosen to be invariant. At low energy, around the QCD scale, the Lagrangian includes derivative couplings of the axion to the quarks, which respect $U_{PQ}(1)$. Furthermore, the symmetry is explicitly broken by the anomalous coupling to $G\tilde{G}$ [80], so that the quark-axion Lagrangian has the form

$$\mathcal{L} = \bar{\psi}iD\!\!\!\!/\psi + \frac{1}{2}\partial_\mu a\partial^\mu a + \bar{\psi}(C_0 + C_1\tau_3)\gamma^5\gamma^\mu\psi\partial_\mu\frac{a}{f_a}$$
$$- c_{a\gamma\gamma}\frac{e^2}{32\pi^2}\frac{a}{f_a}F\tilde{F} - \frac{g^2}{32\pi^2}\left(\theta + \frac{a}{f_a}\right)G\tilde{G}$$
$$- e^{i\rho}\bar{\psi}_L\mathcal{M}\psi_R - e^{-i\rho}\bar{\psi}_R\mathcal{M}\psi_L, \tag{C1}$$

---

[16]We used the relation $\partial_\mu(\bar{\psi}\tilde{\sigma}^{\mu\nu}\overleftrightarrow{D}_\nu\psi) = -(\partial^2 + 4m^2)P - \partial \cdot A_E - 2mP_E$, to eliminate one T-odd, P-odd structure. Moreover, there are no T-odd and P-odd operators containing the ghost fields up to dimension five.

where f_a is the axion decay constant. The couplings $C_{0,1}$ and $c_{a\gamma\gamma}$ are model dependent, while the coupling to gluons is fixed by the $U_A(1)$ anomaly.

As in Sec. II, the $G\tilde{G}$ term can be eliminated in favor of a complex mass term, with the difference that the $U_A(1)$ rotation depends on the axion field. The rotation has the effect of modifying the couplings $C_{0,1}$ and $c_{a\gamma\gamma}$, and, more importantly, affects the mass sector. The discussion of vacuum alignment of Sec. II can be immediately generalized, by replacing θ with $\bar{\theta} + a/f_a$. In this context, vacuum alignment achieves the diagonalization of the pion-axion mass term.

After imposing the vacuum alignment condition, the quark-axion Lagrangian becomes

$$\delta\mathcal{L} = -\bar{\psi}\left[\mathcal{M} - \mathcal{M}^{-1}\frac{m_*^2}{2}\left(\bar{\theta} + \frac{a}{f_a}\right)^2\right]\psi$$
$$+ \bar{\psi}i\gamma_5\psi m_*\left(\bar{\theta} + \frac{a}{f_a}\right), \tag{C2}$$

where we have kept terms quadratic in $\bar{\theta} + a/f_a$. When chiral symmetry is spontaneously broken, $\bar{\psi}\psi$ acquires a nonvanishing vacuum expectation value, $-(m_u + m_d)\langle\bar{\psi}\psi\rangle = 3m_\pi^2 f_\pi^2$, and the *CP*-even quark mass term in Eq. (C2) generates an axion potential

$$V_0\left(\bar{\theta} + \frac{a}{f_a}\right) = \frac{1}{3}\langle\bar{\psi}\psi\rangle\text{Tr}\left[\mathcal{M} - \mathcal{M}^{-1}\frac{m_*^2}{2}\left(\bar{\theta} + \frac{a}{f_a}\right)^2\right]$$
$$= -\frac{m_\pi^2 f_\pi^2}{(m_u + m_d)}$$
$$\times\left(m_s + m_d + m_s - m_*\frac{1}{2}\left(\bar{\theta} + \frac{a}{f_a}\right)^2\right). \tag{C3}$$

V_0 is an even function of $\bar{\theta} + a/f_a$, and it is minimized by

$$\bar{\theta} + \frac{\langle a\rangle}{f_a} = 0, \tag{C4}$$

thus canceling the *CP*-violating effects of the $\bar{\theta}$ term. Oscillations around the minimum determine the axion mass in terms of the pion mass and decay constant, and of the axion decay constant

$$m_a^2 = \frac{m_\pi^2 f_\pi^2}{f_a^2}\frac{m_u m_d}{(m_u + m_d)^2}, \tag{C5}$$

where we neglected small corrections $\sim m_{u,d}/m_s$.

The presence of additional, chiral symmetry breaking sources of *CP* violation has the effect of shifting the minimum of the axion potential, inducing a residual $\bar{\theta}$ term, proportional to the amount of *CP* violation. As an example, we discuss the case of *CP* violation from a quark CEDM. Performing vacuum alignment, as discussed in Sec. II, induces a *CP*-even axion-quark Lagrangian of the form

$$\delta\mathcal{L} = -\frac{g}{2}m_*\left(\bar{\theta} + \frac{a}{f_a}\right)\bar{\psi}\sigma^{\mu\nu}G_{\mu\nu}\mathcal{M}^{-1}[d_{CE}]\psi$$
$$+ \frac{r}{2}m_*\left(\bar{\theta} + \frac{a}{f_a}\right)\bar{\psi}\{\mathcal{M}^{-1}[d_{CE}]$$
$$- m_*\mathcal{M}^{-1}\text{Tr}[\mathcal{M}^{-1}d_{CE}]\}\psi + O(\bar{\theta}^2). \tag{C6}$$

When chiral symmetry is broken, the isoscalar components in Eq. (C6) give a correction to the axion potential Eq. (C3). Up to terms of $[d_{CM}] \times (\bar{\theta} + a/f_a)^2$ that affect the value of the axion mass but do not change the minimum of the potential, the shifted potential reads:

$$V\left(\bar{\theta} + \frac{a}{f_a}\right) = \frac{1}{6}m_*\left(\bar{\theta} + \frac{a}{f_a}\right)^2\langle\bar{\psi}\psi\rangle$$
$$- \frac{1}{6}m_*\left(\bar{\theta} + \frac{a}{f_a}\right)\text{Tr}[\mathcal{M}^{-1}[d_{CE}]]$$
$$\times\langle\bar{\psi}\sigma^{\mu\nu}gG_{\mu\nu}\psi\rangle. \tag{C7}$$

The term proportional to $[d_{CE}]$, odd in $\bar{\theta} + a/f_a$, causes the potential to be minimized at a nonzero value of the $\bar{\theta}$ angle,

$$\bar{\theta} + \frac{\langle a\rangle}{f_a} = \bar{\theta}_{\text{ind}} = \frac{r}{2}\text{Tr}[\mathcal{M}^{-1}[d_C]],$$
$$r = \frac{\langle\bar{\psi}\sigma^{\mu\nu}gG_{\mu\nu}\psi\rangle}{\langle\bar{\psi}\psi\rangle}, \tag{C8}$$

where, by chiral symmetry, r is the same ratio defined in Sec. II.

APPENDIX D: PROJECTIONS IN THE ISOSPIN LIMIT

We now discuss the projections needed to extract $\alpha_3, \ldots, \alpha_7$ in Eq. (101) in the isospin limit $m_u = m_d \equiv m$, in which the matrices M_2 and M_3 defined in Eq. (103) and Eq. (104) become singular.

In the case $a = 3$, the operators $O_6^{(5)}$ and $O_{10}^{(5)}$ (and the structures multiplying α_4 and α_7) vanish. To isolate α_3, it is sufficient to impose

$$\text{Tr}[\Gamma_C^{(2)}\gamma_5\slashed{q}\mathcal{M}t^3]_S = 0. \tag{D1}$$

In the isospin limit, for $a = 3$, α_5 and α_6 are both proportional to t_3, and cannot be disentangled with a flavor projection. However, the different dependence on m^2 and m_s^2 can be exploited, by imposing

$$\text{Tr}\left[\left(\frac{\partial^2}{\partial m^2} - 2\frac{\partial^2}{\partial m_s^2}\right)\Gamma_C^{(2)}\gamma_5 t^3\right]_S = 0 \tag{D2a}$$

$$\text{Tr}\left[\frac{\partial^2}{\partial m_s^2}\Gamma_C^{(2)}\gamma_5 t^3\right]_S = 0. \tag{D2b}$$

The first (second) trace above isolates α_5 (α_6).

For $a = 0$ and 8, M_2 is not singular even in the isospin limit, and the mixing of the CEDM with the divergence of the axial current is found by imposing Eq. (102c).

The structures multiplying α_5, α_6 and α_7 can be projected on the two flavor matrices t^0 and t^8, so that the flavor projections in M_3 are not independent, and the matrix is singular. Also in this case, one can take advantage of the different dependence on m_s^2, m^2 and $m_s m$. Defining $t_l = (t_8 + \sqrt{2}t_0)$ and $t_s = (t_8 - \frac{1}{\sqrt{2}}t_0)$, α_5, α_6 and α_7 can be disentangled by the following projections:

$$\text{Tr}\left[\left(\frac{\partial^2}{\partial m^2} - 2\frac{\partial^2}{\partial m_s^2} - 4\frac{\partial^2}{\partial m_s \partial m}\right)\Gamma_C^{(2)}\gamma_5 t_{l,s}\right] = 0 \quad \text{(D3a)}$$

$$\text{Tr}\left[\frac{\partial^2}{\partial m_s^2}\Gamma_C^{(2)}\gamma_5 t_l\right] = 0 \quad \text{(D3b)}$$

$$\text{Tr}\left[\frac{\partial^2}{\partial m_s \partial m}\Gamma_C^{(2)}\gamma_5 t_s\right] = 0, \quad \text{(D3c)}$$

where in Eq. (D3a) t_l (t_s) is to be used for $a = 0$ ($a = 8$).

APPENDIX E: MATCHING COEFFICIENTS

In this appendix we give explicit results for the matching coefficients from RI-$\widetilde{\text{SMOM}}$ to the $\overline{\text{MS}}$-HV scheme (where ψ is defined in Eq. (67), K in Eq. (68), C_A, C_F and T_F in Eq. (60), ξ is the gauge parameter and n_F are the number of flavors).

$$c_{11} = \frac{C_A(23 + 9\xi) - 32C_F}{12}\psi + \frac{C_A - 2C_F}{2}(1 - \xi)K$$
$$- \frac{C_A - 2C_F}{2}(1 + \xi)\log 2$$
$$+ \frac{10}{9}n_F T_F + C_F\left(\frac{31}{3} - \frac{1}{2}\xi\right)$$
$$+ \frac{C_A}{72}(-646 - 36\xi + 9\xi^2) \quad \text{(E1a)}$$

$$c_{12} = \left(4C_F - \frac{C_A}{6}(3 + \xi)\right)\psi + (C_A - 2C_F)(1 - \xi)K$$
$$+ (C_A - 2C_F)(1 + \xi)\log 2$$
$$+ C_F(2 - \xi) + \frac{C_A}{4}(-5 + 2\xi) \quad \text{(E1b)}$$

$$c_{13} = \left(-\frac{8}{3}C_F + \frac{C_A}{12}(3 + \xi)\right)\psi - \frac{1}{2}(C_A - 2C_F)(1 - \xi)K$$
$$- \frac{1}{2}(C_A - 2C_F)(1 + \xi)\log 2$$
$$+ C_F\left(\frac{25}{3} + \frac{1}{2}\xi\right) + \frac{C_A}{8}(5 - 2\xi) \quad \text{(E1c)}$$

$$c_{14} = 0 \quad \text{(E1d)}$$

$$c_{15} = -4 \quad \text{(E1e)}$$

$$c_{16} = 3c_{17} \quad \text{(E1f)}$$

$$c_{17} = \frac{C_A \xi}{6}\psi + (C_A - 2C_F)(1 - \xi)K$$
$$- \frac{1}{2}(C_A - 2C_F)(1 - \xi)\log 2$$
$$+ C_F(8 - \xi) + C_A\left(-\frac{13}{4} + \frac{1}{2}\xi\right) \quad \text{(E1g)}$$

$$c_{18} = \frac{8C_F - C_A(1 + \xi)}{2}\psi - (C_A - 2C_F)(1 - \xi)K$$
$$+ 2(C_A - 2C_F)\log 2$$
$$+ C_F\left(\frac{10}{3} + \xi\right) + C_A\left(\frac{21}{4} - \frac{1}{2}\xi\right) \quad \text{(E1h)}$$

$$c_{19} = 0 \quad \text{(E1i)}$$

$$c_{1,10} = 0 \quad \text{(E1j)}$$

$$c_{22} = c_{88} = c_{99} = c_{10,10} = \frac{C_F}{2}(3 + \xi)\psi - C_F(12 + \xi) \quad \text{(E1k)}$$

$$c_{33} = C_F\left[(1 - \xi)\left(\frac{4}{3} - \frac{1}{2}\psi\right) + \xi\right] \quad \text{(E1l)}$$

$$c_{55} = 2C_A\left(1 + \frac{1}{3}\xi\right)\psi + \frac{C_A}{36}(-403 - 18\xi + 9\xi^2)$$
$$+ \frac{20}{9}n_F T_F \quad \text{(E1m)}$$

$$c_{56} = -4C_F(4 - \psi) \quad \text{(E1n)}$$

$$c_{66} = c_{77} = -4C_F. \quad \text{(E1o)}$$

The coefficients in the NDR scheme are given by (we report only the cases where $c_{ij}^{\text{NDR}} \neq c_{ij}^{\text{HV}}$)

$$c_{11}^{\text{NDR}} = c_{11}^{\text{HV}} + 2C_A - \frac{4}{3}C_F \quad \text{(E2a)}$$

$$c_{13}^{\text{NDR}} = c_{13}^{\text{HV}} - \frac{4}{3}C_F \quad \text{(E2b)}$$

$$c_{16}^{\text{NDR}} = c_{16}^{\text{HV}} + \frac{C_A - 4C_F}{6} \quad \text{(E2c)}$$

$$c_{17}^{\text{NDR}} = c_{17}^{\text{HV}} + \frac{C_A - 4C_F}{2} \quad \text{(E2d)}$$

$$c_{18}^{\text{NDR}} = c_{18}^{\text{HV}} - C_A - \frac{28}{3}C_F \quad \text{(E2e)}$$

$$c_{66}^{\text{NDR}} = c_{77}^{\text{NDR}} = 0 \quad \text{(E2f)}$$

$$c_{22}^{\text{NDR}} = c_{88}^{\text{NDR}} = c_{99}^{\text{NDR}} = c_{10,10}^{\text{NDR}} = \frac{C_F}{2}(3 + \xi)\psi - C_F(4 + \xi). \quad \text{(E2g)}$$

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