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Hydrodynamical description of the QCD Dirac spectrum at finite chemical potential

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We present a hydrodynamical description of the QCD Dirac spectrum at finite chemical potential as an incompressible droplet in the complex eigenvalue space. For a large droplet, the fluctuation spectrum around the hydrostatic solution is gapped by a longitudinal Coulomb plasmon and exhibits a frictionless odd viscosity. The stochastic relaxation time for the restoration or breaking of chiral symmetry is set by twice the plasmon frequency. The leading droplet size correction to the relaxation time is fixed by a universal odd viscosity to density ratio $\eta_O/\rho_0 = (\beta - 2)/4$ for the three Dyson ensembles $\beta = 1, 2, 4$.

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I. INTRODUCTION

QCD spontaneously breaks chiral symmetry with the emergence of an octet of light mesons that permeate most of the hadronic processes at low energies [1]. Dedicated lattice simulations are now in full support of this spontaneous breaking [2]. Fundamental light quarks become constitutive and heavy, producing most of the mass of the elements around us.

A remarkable feature of the spontaneous breaking of chiral symmetry is the large accumulation of the eigenvalues of the Dirac operator near zero virtuality with the formation of a finite vacuum chiral condensate [3]. Small eigenvalue virtuality translates to large proper time, as light quarks travel very long in proper time and delocalize. The zero virtuality regime is ergodic, and its neighborhood is diffusive [4]. This behavior is analogous to disordered electrons in mesoscopic systems [5].

The ergodic regime of the QCD Dirac spectrum with its universal spectral oscillation is described by a chiral random matrix model [6]. In short, the model simplifies the Dirac spectrum to its zero-mode zone (ZMZ). The Dirac matrix is composed of hopping between N-zero modes and N-antizero modes because of chirality, which are sampled from Gaussian ensembles thanks to the central limit theorem. The model was initially suggested as a null dynamical limit of the instanton liquid model [7].

QCD at finite chemical potential μ is notoriously difficult to sample on a lattice due to the sign problem [8]. A number of chiral models have been proposed to describe the effects of matter in QCD with light quarks [1]. In vacuum, the chiral random matrix model simplifies the QCD Dirac spectrum to its ZMZ. In matter, the light quark zero modes are involved. Their chiral and cross-hopping in the ZMZ is suppressed exponentially, and the corresponding Dirac matrix is banded and not random. However, large matter effects reduce the banded matrix to its diagonal, localizing the quark zero modes into molecules. In the 1-matrix model the chiral random ensemble is deformed by a constant matrix, leading to a gapped spectrum at large μ [9,10]. In the 2-matrix model the deformation is still random and only generic for moderate μ with no strict banding at large μ [11,12]. The 1-matrix approach to QCD at finite μ has been discussed by many [1,13,14].

The purpose of this paper is to show that the 2-matrix model eigenvalue droplet is amenable to a hydrodynamical description. We will show that the droplet is characterized by a plasmon excitation branch which defines the stochastic relaxation time of the softest modes in the ZMZ. We suggest that this time is dual to the relaxation time for the breaking or restoration of chiral symmetry at finite μ. The difference in details between the matrix models is not important, as we will show that the plasmon branch only depends on the mean density in the droplet and the quark representation at large N.

The chief idea of the paper is to combine the ergodic character of the chiral random matrix model for the low-lying modes, with the universal character of the hydrodynamics approach for the description of the softest modes of a fluid, to describe the relaxation of the QCD Dirac eigenvalues in the ZMZ as a fluid at finite μ. We will obtain the following new results: (1) a hydrodynamical description

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of the Dirac eigenvalues as a droplet in the complex 2-plane, (2) small amplitude deformations in the droplet that are gapped by the emergence of a plasmon with an odd viscosity, and (3) an estimate of the stochastic relaxation time for breaking or restoring chiral symmetry in matter.

In Sec. II we briefly review the 2-matrix model at finite μ. In Sec. III, we show that the joint eigenvalue distribution of the 2-matrix model maps onto a pertinent many-body Hamiltonian in the complex 2-plane. In Sec. IV, we use the collective coordinate method, to rewrite the many-body Hamiltonian in terms of the particle density and velocity as collective and canonically conjugate variables. The ensuing equations of motion are that of a two-dimensional fluid. In Secs. V and VI, we show that the small amplitude fluctuations in the fluid are plasmons with a frictionless viscosity. In Sec. VII, we identify a tunneling minimum or instanton to the fluid equation. We use it to characterize the stochastic relaxation time of an initial droplet with a gapped spectrum to a final hydrostatic droplet with ungapped spectrum. We suggest that this stochastic time is dual to the physical relaxation time from a phase with unbroken chiral symmetry to a phase with broken chiral symmetry. Our conclusions are in Sec. VIII.

II. THE MODEL

The random matrix approach has proven to be a useful tool for understanding aspects of chiral symmetry directly from the QCD Dirac spectra both in vacuum and in matter [1,13,14]. The chief idea is the following: for the purpose of analyzing the spontaneous breaking and/or restoration of chiral symmetry, only the low lying eigenmodes of the QCD Dirac operator are important. For this, the fluctuations of the Dirac operator in the gauge background can be approximated by purely random matrix elements which are chiral (paired spectrum) and fixed by time-reversal symmetry (Dyson ensembles).

Specifically, at finite μ the Dirac spectrum on the lattice is complex [15]. The matrix models at finite μ [9,11] capture this essential aspect of the lattice spectra and the nature of the chiral phase transition [1,13,14]. For a 2-matrix model, the partition function is [11,12]

$$Z_2[m_f] = \int dAdBe^{-aN\mathrm{Tr}(A^\dagger A)}e^{-aN\mathrm{Tr}(B^\dagger B)} \times \det\begin{pmatrix} -im_f & A - i\mu B \\ A^\dagger - i\mu B^\dagger & -im_f \end{pmatrix}^{N_f} \quad (1)$$

for equal quark masses m_f in the complex representation. Here A, B are $C^{(N+\nu)\times N}$ valued. ν accounts for the difference between the number of zero modes and antizero modes. At $\mu = 0$ the parameter $\sqrt{a} = |q^\dagger q|_0/\mathbf{n}$ is fixed by the massless quark condensate in with $\mathbf{n} = N/V_4$ the density of zero modes, by the Banks-Casher formula [3].

The Dirac matrix in (1) has ν unpaired zero modes and N paired eigenvalues $\pm iz_j$ in the massless limit. The paired eigenvalues delocalize and are represented by (1). The unpaired zero modes decouple. In terms of the paired eigenvalues and at large N, (1) simplifies [11]

$$Z_\beta[m_f] = \int \prod_{i=1}^{N} d^2 z_i |z_i|^\alpha \prod_{i<j}^{N} |z_i^2 - z_j^2|^\beta \times (z_i^2 + m_f^2)^{N_f} e^{-W(z_i)}, \quad (2)$$

with $\beta = 2$ and $\alpha = \beta(\nu+1) - 1$. The potential is

$$W(z) = \frac{Na\beta}{2l^2}\left(|z|^2 - \frac{\tau}{2}(z^2 + \bar{z}^2)\right), \quad (3)$$

with $l^2 \equiv 1 - \tau = 2\mu^2/(1+\mu^2)$. For $\mu \to 0$, $\tau \approx 1$ and $l^2 \approx 2\mu^2$, so that $W(z) \approx -(N/\mu^2)(z-\bar{z})^2$, which restricts the eigenvalues to the real axis. Throughout, the dimensionful scale a will be set to 1 and reinstated when needed.

In Fig. 1 we display the distribution of eigenvalues following from the 2-matrix model with A and B sampled from a Gaussian ensemble of 200×200 matrices with $\nu = 0$ and $\mu = 0.3$. The boundary curves follow from the analysis in [13,16]. The domain is an ellipse $x^2/a_+^2 + y^2/a_-^2 = 1$ with semiaxes $a_\pm^2/2l^2 = 1 \pm \tau/1 \mp \tau$ as shown in Fig. 1. The ellipse remains un-split with area $\mathcal{A} = \pi a_+ a_- = 2\pi l^2$ for all values of μ. For the other quark representations with $\beta = 1, 4$ the joint distribution in the 2-matrix model is more subtle [12]. Throughout, (2) will be assumed for $\beta = 2$, but all results extend to $\beta = 1, 2, 4$ for large N.

For comparison, Fig. 2 shows the distribution of the eigenvalues from the 1-matrix model with $B = \mathbf{1}$ and the same Gaussian sampling for A. The boundary curves are from [9,10]. The eigenvalues form a connected droplet in the z plane for $\mu < \mu_c$ and split to two symmetric droplets for $\mu > \mu_c$, restoring chiral symmetry [9,10]. Similar droplets follow from the QCD Dirac spectra at finite μ

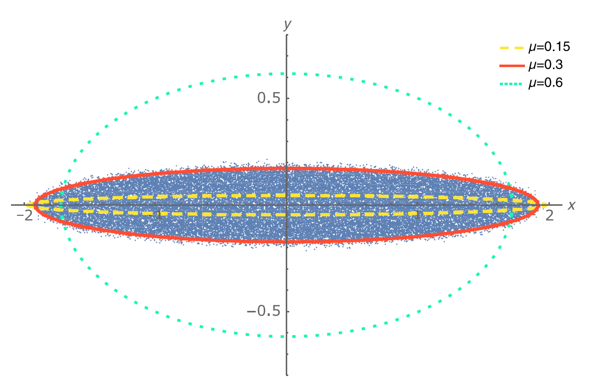

FIG. 1 (color online). Eigenvalue distribution from a 2-matrix model.

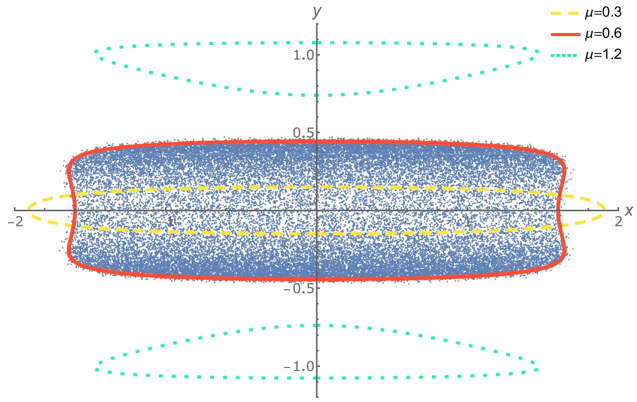

FIG. 2 (color online). Eigenvalue distribution from a 1-matrix model.

on the lattice [15]. In the spontaneously broken phase, all droplets are connected and symmetric about the real axis.

III. MANY-BODY SYSTEM

Equation (2) expressed in terms of the complex eigenvalues z_i can be thought of as the partition function of N charged particles in the complex 2-plane trapped in a harmonic potential (Gaussian weight) with Coulomb-type repulsions (Vandermonde term). Alternatively, Eq. (2) can be written as

$$Z_\beta[m_f] = \int \prod_{i=1}^{N} d^2 z_i (z_i^2 + m_f^2)^{N_f} |\mathbf{\Psi}_0[z]|^2, \qquad (4)$$

which is now viewed as the normalization of an N-particle wave function $\mathbf{\Psi}_0[z]$ with a complex measure. $\mathbf{\Psi}_0[z]$ is the zero-mode solution to the Schrodinger equation $H_0\Psi_0 = 0$ with the self-adjoint Hamiltonian,

$$H_0 \equiv \frac{1}{2m} \sum_{i=1}^{N} |\partial_i + \mathbf{a}_i|^2. \qquad (5)$$

Here $\partial_i \equiv \partial/\partial z_i$ and the potential is $\mathbf{a}_i \equiv \partial_i S$ with $S[z] = -\ln \Psi_0[z]$. In (5) the mass parameter is $m = 1/2$. We note that the canonical dimensions of (5) and m follow through a pertinent rescaling by reinstating the dimensions of a.

Following [17,18], we observe that the Vandermonde determinant $\Delta = \prod_{i<j} |z_{ij}^2|^\beta$ with $z_{ij} \equiv z_i - z_j$ induces a diverging 2-body part in H_0. Using a similarity transformation, we can reabsorb it in $\Psi = \Psi_0/\sqrt{\Delta}$, and the new many-body Hamiltonian is

$$H = \frac{1}{\sqrt{\Delta}} H_0 \sqrt{\Delta}. \qquad (6)$$

We will refer to (6) as the "quenched" Hamiltonian following from the omission of the N_f contribution in deriving (6) from (4), which is not to be confused with the standard denomination. The "phase-quenched" Hamiltonian follows a similar reasoning by rewriting (4) as

$$Z_\beta[m_f] = \int \prod_{i=1}^{N} d^2 z_i \left(\frac{z_i^2 + m_f^2}{\bar{z}_i^2 + m_f^2} \right)^{\frac{N_f}{2}} |\mathbf{\Psi}_f[z]|^2. \qquad (7)$$

Below, we will note that the difference between $\Psi_0[z]$ and $\Psi_f[z]$ are subleading terms of order $N_f N^0$ in comparison to the leading contribution of order N.

IV. HYDRODYNAMICS

In the limit of a large number of eigenvalues N, the interacting and quantum many-body system described by Eqs. (5)–(6) is characterized by collective as well as single particle excitations. In the spirit of the liquid drop model in nuclear physics [19], we can describe the low-lying collective excitations of this many-body system by using the collective coordinate method in [17,19]. The idea is to map the Hamiltonian (6) onto the paired eigenvalues as a collective variable $\rho(z) = \sum_{i=1}^{N} \delta^2(z - z_i)$ and its conjugate velocity $\pi(z)$. The result is a semiclassical fluid description of the low-lying collective excitations of (6).

The details of the mapping of (6) onto the collective variables following the construction in [17] are given in Appendix A. The result for the collective Hamiltonian is

$$H = \int d^2 z \rho(z) \frac{1}{2m} ((\vec{\nabla}\pi)^2 + (\vec{\mathbf{A}})^2) \equiv \int d^2 z \mathbf{h}, \qquad (8)$$

with the pair π, ρ canonically conjugate. Defining the even density $\rho^\chi(z) = \rho(z) + \rho(-z)$, we have

$$\vec{\mathbf{A}} = \vec{A} + \frac{1}{2} \vec{\nabla} (\beta \rho_L^\chi(z) + (\beta - 2) \ln \sqrt{\rho}). \qquad (9)$$

Here ρ_L is the logarithmic transform of ρ,

$$[\rho]_L \equiv \rho_L(z) = \int dz' \ln |z - z'| \rho(z'), \qquad (10)$$

and the vector potential ($\tau_\pm = 1 \pm \tau$),

$$\vec{A} \equiv -\frac{N\beta}{2l^2} (\tau_- x, \tau_+ y) + \frac{\alpha}{2|z|^2} (x, y). \qquad (11)$$

We will restrict our discussion to the semiclassical limit with the pair π, ρ obeying the Poisson brackets $\{\pi(z), \rho(z')\} = \delta^2(z - z')$. The semiclassical limit is exact in leading order in $1/N$ and resums a class of subleading-order effects in $1/N$. Quantum corrections follow by expanding around the semiclassical solution, say in one loop.

The equation of motion for ρ yields the current conservation law and the Euler equation for $\vec{v}$. Defining $m\vec{v} = \vec{\nabla}\pi$, they are specifically given by

$$\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$
$$\partial_t\pi + \frac{1}{2}m\vec{v}^2 + \frac{\vec{\mathbf{A}}^2}{2m} - \frac{\beta-2}{4m\rho}\vec{\nabla}\cdot(\rho\vec{\mathbf{A}}) - \frac{\beta}{2m}\vec{\nabla}\cdot[\rho^\chi\vec{\mathbf{A}}]_L = 0. \quad (12)$$

Current conservation follows from $\partial_t\rho = \{\rho, H\}$. The Euler equation follows from $\partial_t\pi = \{\pi, , H\}$, using

$$\{H, \pi(z)\} = \frac{(\vec{\nabla}\pi)^2 + |\vec{\mathbf{A}}|^2}{2m} + \int d^2z' \frac{\rho(z')}{2m} 2\vec{\mathbf{A}}(z')\cdot\{\vec{\mathbf{A}}(z'), \pi(z)\}, \quad (13)$$

and the commutation rule

$$\{\vec{\mathbf{A}}(z'), \pi(z)\} = \frac{\beta}{2}\vec{\nabla}_{z'}\ln|z'-z| + \frac{\beta-2}{4}\vec{\nabla}_{z'}\frac{\delta^2(z-z')}{\rho} \quad (14)$$

The steady state flow from (12) corresponds to Bernoulli law with $\partial_t\pi = C$ a fixed constant. The hydrostatic solution is $\mathbf{A}(z) = 0$ and $\pi = 0$. Using the formal identity $\rho_L = (2\pi/\nabla^2)\rho$, we have

$$\rho(z) = \frac{\kappa N}{\mathcal{A}} - \frac{\alpha}{2\beta}\delta^2(z) - \frac{\beta-2}{8\pi\beta}\nabla^2\ln\rho, \quad (15)$$

where the integration constant $\kappa = 1 + \alpha/(2N\beta)$ is fixed by the density in leading order, and $\mathcal{A}$ is the area of the eigenvalue density. The resummed semiclassical contributions in (15) are of order N^0.

In the "phase-quenched" approximation for $\beta = 2$, the vector potential (11) is shifted,

$$\vec{A} \to \vec{A} + \frac{N_f}{2}\vec{\nabla}\ln|z^2 + m_f^2|, \quad (16)$$

with the hydrodynamical equations (12) unchanged. The corresponding "phase-quenched" hydrostatic density (15) is modified:

$$\rho(z) \to \rho(z) - \frac{N_f}{8\pi\beta}\nabla^2\ln|z^2 + m_f^2|. \quad (17)$$

As indicated earlier, the correction is of order $N_f N^0$.

V. PLASMONS

To characterize the low-lying collective excitations of the hydrostatic droplet of eigenvalues, it is useful to analyze the small deformations in the density and velocity profile by linearizing the current conservation law in (12), i.e. $\partial_t\delta\rho + \rho_0\nabla^2\delta\pi = 0$, which is readily solved using $\delta\rho = -\rho_0\nabla^2\phi$ and $\delta\pi = \partial_t\phi$. Inserting the latter in the canonical action $\mathbf{S} = \int d^2z dt(\pi\partial_t\rho - \mathbf{h})$ yields, in the quadratic approximation,

$$\mathbf{S} \approx \int d^2z dt \frac{\rho_0}{2m}((\partial_t\vec{\nabla}\phi)^2 - W[\phi]^2) \quad (18)$$

with

$$W[\phi] = \left|\vec{\nabla}\left(\frac{\beta}{2}[\delta\rho]_L^\chi + \frac{\beta-2}{4}\frac{\delta\rho}{\rho_0}\right)\right|^2. \quad (19)$$

Using again the formal identity $f_L = (2\pi/\nabla^2)f$ and defining the small longitudinal field $\vec{\varphi} \equiv \vec{\nabla}\phi$, we obtain

$$\mathbf{S} \approx N\int d^2z dt \frac{\rho_0}{2m} \times \left((\partial_t\vec{\varphi})^2 - \left(\frac{\pi\beta\rho_0}{N}\vec{\varphi}^\chi + \frac{\beta-2}{4}\nabla^2\vec{\varphi}\right)^2\right) \quad (20)$$

after the rescaling $Nt \to t$. The small longitudinal excitations in $\vec{\varphi}$ are gapped by the plasmon frequency $\omega_p = 2\pi\beta\rho_0/N$. The emergence of a plasmon branch was expected since the Vandermonde contribution in (2) gives rise to Coulomb law in two dimensions.

For an elliptic droplet of large area $\mathcal{A}$, (20) by Fourier transform, leads to the quadratic dispersion law

$$\omega(k) \approx \pm\left|\omega_p - \frac{\beta-2}{4}\vec{k}^2\right| \quad (21)$$

Here $|k|$ is conjugate to $|z|$. The gapped spectrum means that the droplet is incompressible. For $\beta = 1, 2$ with quarks in the real and complex representation the branch (21) describes a plasma fluid. For $\beta = 4$ with quarks in the quaternion representation, (21) shows the start of a roton-like branch a possible indication of superfluidity [20].

VI. ODD VISCOSITY

There is an interesting analogy between the droplet of Dirac eigenvalues at finite chemical potential, and the quantum Hall effect as a fluid of neutralized charged electrons in the plane [21,22]. To illustrate the analogy, we first note that (11) sources the magnetic field $B(z) \equiv \vec{\nabla}\times\vec{A}^\star \approx N\beta/l^2$, with the dual notation $V_i^\star = \epsilon_{ij}V_j$ subsumed. Amusingly, (5) describes a Coulomb fluid in a magnetic field. In large N the density of eigenvalues is uniform

$$\rho(z) \approx \frac{N}{2\pi l^2} \approx \frac{\nu B}{2\pi} \tag{22}$$

which is the density of a quantum Hall droplet with filling fraction $\nu = 1/\beta$. The plasmon frequency is the cyclotron frequency $\omega_p \equiv B/M$ with $M = N$ the analogue of the effective mass. l identifies with the magnetic length.

The k^2 contribution in (21) is reminiscent of the odd viscosity in the fractional quantum Hall effect. To show this, let $\tilde{\pi} \equiv i\pi$ and define the collective velocity $m\tilde{v} = \vec{\nabla}\tilde{\pi} + \vec{\mathbf{A}}^\star$, then (8) is a free-flow-like Hamiltonian modulo ultralocal terms:

$$H \to \int d^2z \rho(z) \frac{m}{2} \tilde{v}^+ \cdot \tilde{v}. \tag{23}$$

In our case $\tilde{v}^+ \neq \tilde{v}$ but in the fractional quantum Hall effect they are equal, making $\mathbf{A}^\star$ a real gauge-field and $\vec{\tilde{v}}$ a real and gauge-invariant flow velocity for flux-riding quasiparticles [21,22]. In the semiclassical limit $\tilde{\pi}$ and ρ are canonical and after some algebra, the Euler equation following from (23) yields the momentum conservation law $\partial_t(\rho m\tilde{v}_i) + \nabla_j \mathbf{T}_{ij} = 0$, with the stress tensor

$$\mathbf{T}_{ij} = m\rho\tilde{v}_i\tilde{v}_j + \frac{\beta - 2}{4}\rho(\nabla_i \tilde{v}_j^\star + \nabla_i^\star \tilde{v}_j). \tag{24}$$

This result is checked in details in Appendix B using explicitly the equations of motion. The first contribution is the classical free fluid part. The second contribution is the odd viscosity contribution following from the breaking of parity in two dimensions [23], with

$$\frac{\eta_O}{\rho} = \frac{\beta - 2}{4} \to -\frac{1}{4}, 0, \frac{1}{2}, \tag{25}$$

which is the coefficient of the k^2 term in (21). A recent and direct calculation confirms this interpretation [20]. We do not have a physical interpretation for why $\eta_O = 0$ for $\beta = 2$.

In the fractional quantum Hall fluid, η_O originates from a mixed gauge-gravitational anomaly [24]. We note that the pair $\tilde{v}$, $\tilde{v}^\star$ are orthogonal. This explains that the k^2 contribution in (21) acts as the even (shear) viscosity but without the i for dissipation. No vorticity is therefore expected.

VII. INSTANTON AND RELAXATION TIME

An interesting question regarding the droplet of Dirac eigenvalues is the typical relaxation time for the formation or disappearance of the spontaneous breaking of chiral symmetry. In this section, we answer this question in two steps. First, we identify an instanton or tunneling configuration to the general equations of motion with minimum energy. We then use it to estimate the time it takes for a localized droplet to relax to its hydrostatic limit. Since the relaxation time is a property of the fluid, it is independent of the initial conditions. Indeed, we will show that it is fixed by the plasmon branch.

With this in mind, we identify the zero energy configuration in (8) as an instanton solution with imaginary (tunneling) velocity $\pi \to i\pi$, and minimum energy i.e. $\mathbf{h} \to |\vec{\nabla}\pi|^2 - |\vec{\mathbf{A}}|^2 = 0$, that satisfies the analytically continued in time conservation law $(t \to -it_E)$:

$$-\partial_{t_E}\rho + \vec{\nabla} \cdot (\rho\vec{\nabla}\pi) = 0. \tag{26}$$

Without loss of generality and for simplicity we choose $\tau = 0$ in (3) so that the hydrostatic droplet is circular. To solve (26) we set $\rho(0, z) = K/\pi \gg \rho_0$, which corresponds to all eigenvalues localized in a small disc centered around the origin. (26) simplifies by radial symmetry:

$$\partial_r\rho_L(r, t_E) = f(r, t_E)$$
$$r\partial_{t_E} f + r\left(\beta f - \frac{N\beta r}{2l^2}\right)\partial_r f + f\left(\beta f - \frac{N\beta r}{2l^2}\right) = 0. \tag{27}$$

We note that similar nonlinear equations emerge from the diffusion of non-Hermitian matrices [25].

The solution to (27) with a free boundary or large droplet size $\mathcal{A}$ can be obtained using the method of characteristics. Specifically,

$$\frac{dt_E}{ds} = -r$$
$$\frac{dr}{ds} = -r\left(\beta f - \frac{N\beta r}{2l^2}\right)$$
$$\frac{df}{ds} = f\left(\beta f - \frac{N\beta r}{2l^2}\right), \tag{28}$$

with the conditions $t_E(s = 0) = 0$, $r(s = 0) = r_0$ and $f(s = 0) = f(r_0)$. For $f(r, t_E = 0) = Kr$, we obtain by direct integration of (28):

$$t_E = -\mathbf{a}s + \frac{l^2}{N\beta}\ln\left(\frac{r_0 + \mathbf{a} - (r_0 - \mathbf{a})e^{\frac{N\beta}{l^2}\mathbf{a}s}}{2\mathbf{a}}\right)$$
$$r = \mathbf{a}\frac{r_0 + \mathbf{a} + (r_0 - \mathbf{a})e^{\frac{N\beta}{l^2}\mathbf{a}s}}{r_0 + \mathbf{a} - (r_0 - \mathbf{a})e^{\frac{N\beta}{l^2}\mathbf{a}s}}$$
$$\mathbf{a} = \sqrt{2Kr_0^2 l^2/N}$$
$$f = \frac{Kr_0^2}{r}. \tag{29}$$

For $K \gg \rho_0$ and large time, the first equation is approximated by $t_E \approx -\mathbf{a}s$. Inserting the latter in the second equation and using the third equation, we obtain $r_0 = r_0(r, t_E)$. Substituting the result in the fourth equation in (29), we find explicitly $f(r, t_E)$. Its large time asymptotic for $s \to -\infty$ is

$$f(r, t_E) \approx \frac{Nr}{2l^2}\left(\frac{1+\sqrt{\frac{2Kl^2}{N}}-(1-\sqrt{\frac{2Kl^2}{N}})e^{-\frac{N\beta}{l^2}t_E}}{1+\sqrt{\frac{2Kl^2}{N}}+(1-\sqrt{\frac{Kkl^2}{N}})e^{-\frac{N\beta}{l^2}t_E}}\right)^2. \quad (30)$$

Equation (30) relaxes as $e^{-2\omega_p N t_E}$ to $f(r,\infty) = Nr/2l^2$, leading to the hydrostatic density $\rho_0 = N/(2\pi l^2)$. We identify $T_R \approx 1/2\omega_p$ with the relaxation time after rescaling $Nt_E \to t_E$.

Finally, we note that $T_R \approx 1/2\omega_p$ translates to a diffusive time with $l^2 \equiv \mathcal{A}/2\pi \approx 2\beta T_R$. The diffusion constant is $\mathbf{D} = 2\beta$. An estimate of the finite droplet size corrections follows from (21) using the substitution $\omega_p \to \omega(k \approx 1/\sqrt{\mathcal{A}})$. The leading correction is controlled by the odd viscosity to density ratio and is small.

So far, our description of the Dirac spectrum at finite μ is mathematical, with $T_R \approx 1/2\omega_p$ a characteristic of the relaxation of the eigenvalues from an initial and localized distribution of eigenvalues to a final distribution with spontaneous chiral symmetry breaking at finite μ. The choice of initial conditions is not important as the relaxation time is fixed by the low-lying and collective plasmon frequency.

We now suggest that this relaxation in eigenvalue space is dual to a relaxation in physical space under the same conditions. The physical relaxation time for the breaking or restoration of chiral symmetry at finite μ in canonical dimensions is then

$$T_R \approx \frac{1}{2\omega_p} \to \left(\frac{1+\frac{\mathcal{A}a}{2\pi}}{2\beta}\right)\sqrt{a} \quad (31)$$

after reinstating $1 \equiv \sqrt{a} = |q^\dagger q|_0/\mathbf{n}$, and adding the 1 to reproduce the $\mu = 0$ result in [18]. A simple extension to finite temperature amounts to a redefinition of units or $\sqrt{a} \to \sqrt{a_T} = |q^\dagger q|_T/\mathbf{n}_T$ as in [18,26].

VIII. CONCLUSIONS

The hydrodynamical reduction organizes the fluctuations of the eigenvalues around the low-lying collective modes. It supports an instanton that describes the stochastic relaxation of the Dirac eigenvalues as a fluid. The fluid is incompressible and exhibits nondissipative plasmon waves that can be used to estimate the time it takes for a chirally symmetric phase to relax to a chirally broken phase in matter. The time estimate is nonperturbative and gauge independent.

Our starting point was a 2-matrix model of $2N$ paired Dirac eigenvalues in QCD at finite μ, followed by a hydrodynamical reduction using the collective coordinate method. Both the hydrostatic and hydrodynamical solutions capture the large N effects exactly, and resum a class of corrections in $1/N$. Quantum corrections in $1/N$ can be sought by expanding around these solutions say to one loop. In general, these corrections form a trans-series with edge oscillating contributions.

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APPENDIX A: COLLECTIVE HAMILTONIAN

The details for the derivation of the collective Hamiltonian (8) follow the arguments presented in [17]. Here we provide the details for the derivation of (8). Throughout, we will ignore ultralocal contributions. We note that we can recast $\Psi_0[z] \to e^{-S[\rho]/2}$ using the density $\rho(z)$ by noting for instance that

$$\sum_i f[z_i] = \int dz \rho(z) f(z), \quad (A1)$$

for which (5) is formally

$$H_0 = \sum_i (-\nabla_i + \nabla_i S/2)(\nabla_i + \nabla_i S/2), \quad (A2)$$

with

$$-\nabla \equiv -\sum_i \frac{d}{dx_i} = -\sum_{k,i} \frac{d\rho_k}{dx_i}\frac{\delta}{\delta\rho_k} = \sum_k ik\rho_k \frac{\delta}{\delta\rho_k} \quad (A3)$$

$$-\nabla^2 \equiv -\sum_i \nabla_i^2 = \sum_k k^2\rho_k \frac{\delta}{\delta\rho_k} + \sum_{k,k'} k\cdot k' \rho_{k+k'} \frac{\delta^2}{\delta\rho_k \delta\rho_{k'}}, \quad (A4)$$

and the Fourier transform of the collective density,

$$\rho_k = \frac{1}{V}\sum_i e^{-ikx_i}. \quad (A5)$$

The formal result is

$$\begin{aligned} H_0 &= \sum_i (-\nabla_i^2) + V \\ &= \sum_i \left(-(\nabla_i)^2 - \frac{\nabla_i^2 S}{2} + \frac{(\nabla_i S)^2}{4} \right) \\ &= \int d^2 z \rho ((\nabla\pi)^2 + i\nabla\pi\nabla\rho - \nabla\cdot\mathbb{A} + \mathbb{A}^2) \\ &= \int d^2 z \rho \left(\left(\nabla\pi + \frac{i\nabla\rho}{2\rho} \right)^2 + \frac{(\nabla\rho)^2}{4\rho^2} - \nabla\cdot\mathbb{A} + \mathbb{A}^2 \right) \\ &= \int d^2 z \rho \left((\nabla\pi')^2 + \left(\mathbb{A} + \frac{\nabla\rho}{2\rho} \right)^2 \right), \end{aligned} \tag{A6}$$

with $\mathbb{A}_i = \partial_i S/2$. (A6) is identical to the rhs of (8) with

$$\mathbf{A} = -\mathbb{A} - \nabla \ln\sqrt{\rho}. \tag{A7}$$

For (6), we have instead

$$\begin{aligned} H &= \frac{1}{\sqrt{\Delta}} \sum_i (-\nabla_i^2)\sqrt{\Delta} + V \\ &= \sum_i -\nabla_i^2 - 2B_i\nabla_i - B_i^2 - \nabla_i B_i + V, \end{aligned} \tag{A8}$$

with $B_i = \frac{1}{\sqrt{\Delta}}\partial_i\sqrt{\Delta}$. Using (A3) we have

$$-\sum_i 2B_i\nabla_i = -2i\int d^2 z \rho(z) B(z)\cdot\nabla\pi(z), \tag{A9}$$

and

$$\begin{aligned} H &= \int d^2 z (\rho(\nabla\pi)^2 + i\nabla\rho\cdot\nabla\pi - 2i\rho B\cdot\nabla\pi) \\ &\quad - \int d^2 z \rho (B^2 + \nabla\cdot B) + \int d^2 z (-\rho\nabla\cdot\mathbb{A} + \mathbb{A}^2). \end{aligned} \tag{A10}$$

After completing the square, we obtain

$$\begin{aligned} H &= \int d^2 z \left(\rho(\nabla\pi'')^2 + \frac{(\nabla\rho - 2B\rho)^2}{4\rho} \right) \\ &\quad - \int d^2 z \rho (B^2 + \nabla\cdot B) + \int d^2 z (-\rho\nabla\cdot\mathbb{A} + \mathbb{A}^2) \\ &= \int d^2 z \rho \left((\nabla\pi'')^2 + \frac{(\nabla\rho)^2}{4\rho^2} - \nabla\cdot\mathbb{A} + \mathbb{A}^2 \right) \\ &= \int d^2 z \rho \left((\nabla\pi'')^2 + \left(-\frac{\nabla\rho}{2\rho} - \mathbb{A} \right)^2 \right), \end{aligned} \tag{A11}$$

again in agreement with (8) after the relabeling $\pi'' \to \pi$. We observe that when reduced to the collective variables, both H and H_0 have the same form.

APPENDIX B: STRESS TENSOR

Here we check that the stress tensor (24) satisfies the conservation law using only the Euler equation following from (23) and the classical canonical rules. Throughout we set in this section $m = 1$, $\alpha = (\beta - 2)/4$ (not to be confused with the one used in the text), and the tildes are omitted for convenience. We recall from Sec. VI that

$$v = \nabla\pi + \nabla^\star(\alpha\ln\rho + \beta\rho_L). \tag{B1}$$

Current conservation and Euler equation follow from the same arguments presented earlier with

$$\begin{aligned} \partial_t\rho + \nabla(\rho v) &= 0 \\ \partial_t\pi + \frac{v^2}{2} - \left(\frac{\alpha}{\rho} + 2\pi\beta\triangle^{-1} \right)\nabla^\star(\rho v) &= 0, \end{aligned} \tag{B2}$$

with $\triangle = \nabla^2$. We now need to verify the conservation law for the stress tensor,

$$\partial_t(\rho v_i) + \partial_k(\rho v_i v_k + \alpha\rho\partial_i v_k^\star + \alpha\rho\partial_i^\star v_k) = 0. \tag{B3}$$

By $O(2)$ symmetry, we only need to check it for the $i = 1$ component. With this in mind, the first contribution in (B3) can be reduced to

$$\begin{aligned} \partial_t\rho v_1 + \rho\partial_t v_1 &= -v_1(\partial_1(\rho v_1) + \partial_2(\rho v_2)) \\ &\quad + \rho\partial_t(\partial_1\pi + \alpha\partial_2\ln\rho + 2\pi\beta\partial_2\triangle^{-1}\partial_t\rho). \end{aligned} \tag{B4}$$

The second line in (B4) can be further transformed to

$$\begin{aligned} &\rho\partial_1\left(-\frac{v^2}{2} + \left(\frac{\alpha}{\rho} + 2\pi\beta\triangle^{-1} \right)\nabla^\star(\rho v) \right) \\ &- \alpha\rho\partial_2\frac{\partial_1(\rho v_1) + \partial_2(\rho v_2)}{\rho} \\ &- 2\pi\beta\rho\triangle^{-1}\partial_2(\partial_1(\rho v_1) + \partial_2(\rho v_2)). \end{aligned} \tag{B5}$$

The term proportional to β can be reduced to

$$-2\pi\rho\beta\triangle^{-1}(\partial_1^2(\rho v_2) + \partial_2^2(\rho v_2)) = -2\pi\rho^2\beta v_2, \tag{B6}$$

so that (B4) now reads

$$\begin{aligned} \partial_t\rho v_1 + \rho\partial_t v_1 &= -v_1(\partial_1(\rho v_1) + \partial_2(\rho v_2)) \\ &\quad - \rho\partial_1\frac{v^2}{2}\alpha\rho\partial_1\frac{\partial_2(\rho v_1) - \partial_1(\rho v_2)}{\rho} \\ &\quad - \alpha\rho\partial_2\frac{\partial_1(\rho v_1) + \partial_2(\rho v_2)}{\rho} - 2\pi\rho^2\beta v_2. \end{aligned} \tag{B7}$$

This should cancel the second contribution in (B3), which is

$$\partial_1(\rho v_1^2) + \partial_2(\rho v_1 v_2) + \alpha\partial_1(\rho\partial_1 v_2) - \alpha\partial_2(\rho\partial_1 v_1)$$
$$\alpha\partial_1(\rho\partial_2 v_1) + \alpha\partial_2(\rho\partial_2 v_2). \tag{B8}$$

This can be proved as follows. First, the terms without any α or β combine to

$$-v_1(\partial_1(\rho v_1) + \partial_2(\rho v_2)) - \rho\partial_1 \frac{v^2}{2}$$
$$+ \partial_1(\rho v_1^2) + \partial_2(\rho v_1 v_2) = \rho v_2(\partial_2 v_1 - \partial_1 v_2). \tag{B9}$$

By expanding v_1, v_2 in (B9), we note that the contributions $\nabla\pi$ are zero. The contributions $\beta\rho_L$ give

$$\rho v_2(\partial_1^2 + \partial_2^2)\rho_L = 2\pi\rho v_2 \rho = 2\pi\rho^2 v_2, \tag{B10}$$

which cancel the last term in (B7). The contributions α give $\alpha\rho v_2 \triangle \ln\rho$, which cancel the remainder, since

$$\alpha\rho\partial_1 \frac{\partial_2(\rho v_1) - \partial_1(\rho v_2)}{\rho} - \alpha\rho\partial_2 \frac{\partial_1(\rho v_1) + \partial_2(\rho v_2)}{\rho}$$
$$+ \alpha\partial_1(\rho\partial_1 v_2) - \alpha\partial_2(\rho\partial_1 v_1)$$
$$+ \alpha\partial_1(\rho\partial_2 v_1) + \alpha\partial_2(\rho\partial_2 v_2)$$
$$= -\alpha(\partial_1^2\rho + \partial_2^2\rho)v_2 + \alpha\frac{(\partial_1\rho)^2 + (\partial_2\rho)^2}{\rho} v_2 \tag{B11}$$

Thus, we have Eq. (B3).

[1] M. A. Nowak, M. Rho, and I. Zahed, *Chiral Nuclear Dynamics*, (World Scientific, Singapore, 1996).
[2] For a recent review, see L. Giusti, *Proc. Sci.* LATTICE2015 (2015) 001.
[3] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[4] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Phys. Rev. Lett. **81**, 264 (1998).
[5] E. Akkermans and G. Montambaux, *Mesoscopic Physics of Electrons and Photons* (Cambridge University Press, Cambridge, England, 2007).
[6] E. V. Shuryak and J. J. M. Verbaarschot, Nucl. Phys. **A560**, 306 (1993); J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993).
[7] M. A. Nowak, J. J. M. Verbaarschot, and I. Zahed, Phys. Lett. B **217**, 157 (1989).
[8] A. Gocksch, Phys. Rev. Lett. **61**, 2054 (1988); I. Barbour, N. E. Behilil, E. Dagotto, F. Karsch, A. Moreo, M. Stone, and H. W. Wyld, Nucl. Phys. **B275**, 296 (1986); I. M. Barbour, S. E. Morrison, E. G. Klepfish, J. B. Kogut, and M. P. Lombardo, Nucl. Phys. B, Proc. Suppl. **60A**, 220 (1998).
[9] M. A. Stephanov, Phys. Rev. Lett. **76**, 4472 (1996).
[10] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Phys. Rev. Lett. **77**, 4876 (1996).
[11] J. C. Osborn, Phys. Rev. Lett. **93**, 222001 (2004).
[12] G. Akemann, J. Phys. A **36**, 3363 (2003); Acta Phys. Pol. B **38**, 3981 (2007).
[13] R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, in *Proceedings of Zakopane 1997, New developments in quantum field theory*; Acta Phys. Pol. B **28**, 2949 (1997).
[14] A. M. Halasz, J. C. Osborn, and J. J. M. Verbaarschot, Phys. Rev. D **56**, 7059 (1997); J. J. M. Verbaarschot, Nucl. Phys. **A642**, c305 (1998); H. Markum, R. Pullirsch, and T. Wettig, Phys. Rev. Lett. **83**, 484 (1999); F. Basile and G. Akemann, J. High Energy Phys. 12 (2007) 043; K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. D **75**, 116003 (2007); J. Han and M. A. Stephanov, Phys. Rev. D **78**, 054507 (2008); H. Fujii and T. Sano, Phys. Rev. D **83**, 014005 (2011); G. Akemann, T. Kanazawa, M. J. Phillips, and T. Wettig, J. High Energy Phys. 03 (2011) 066; A. Mollgaard and K. Splittorff, Phys. Rev. D **91**, 036007 (2015).
[15] I. Barbour, N. E. Behilil, E. Dagotto, F. Karsch, A. Moreo, M. Stone, and H. W. Wyld, Nucl. Phys. **B275**, 296 (1986).
[16] V. A. Kazakov and A. Marshakov, J. Phys. A **36**, 3107 (2003); R. Teodorescu, E. Bettelheim, O. Agam, A. Zabrodin, and P. Wiegmann, Nucl. Phys. **B704**, 407 (2005).
[17] A. Jevicki and B. Sakita, Nucl. Phys. **B165**, 511 (1980); A. Jevicki, Nucl. Phys. **B376**, 75 (1992).
[18] Y. Liu, P. Warchoł, and I. Zahed, arXiv:1505.02107; arXiv:1506.08787.
[19] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, Berlin, 1980).
[20] Y. Liu and I. Zahed, arXiv:1509.00812.
[21] M. Stone, Phys. Rev. B **42**, 212 (1990).
[22] P. Wiegmann, Phys. Rev. Lett. **108**, 206810 (2012).
[23] J. E. Avron, J. Stat. Phys. **92**, 543 (1998).
[24] T. Can, M. Laskin, and P. Wiegmann, Phys. Rev. Lett. **113**, 046803 (2014); A. Gromov and A. G. Abanov, Phys. Rev. Lett. **113**, 266802 (2014).
[25] Z. Burda, J. Grela, M. A. Nowak, W. Tarnowski, and P. Warchoł, Phys. Rev. Lett. **113**, 104102 (2014); Nucl. Phys. **B897**, 421 (2015); J. Grela, arXiv:1506.05927.
[26] For comparison and in pure Yang-Mills at large number of colors N_c, the relaxation time for the confinement or deconfinement transition at finite temperature T is set by the "speed of sound" v_s in the space of holonomies, i.e. $T_R \approx 1/v_s T = 1/T$ [18].