

Integrable open spin-chains in AdS₃/CFT₂ correspondences

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We study integrable open boundary conditions for $\mathfrak{d}(2, 1; \alpha)^2$ and $\mathfrak{psu}(1, 1|2)^2$ spin-chains. Magnon excitations of these open spin-chains are mapped to massive excitations of type-IIB open superstrings ending on D-branes in the AdS₃ × S³ × S³ × S¹ and AdS₃ × S³ × T⁴ supergravity geometries with pure R-R flux. We derive reflection matrix solutions of the boundary Yang-Baxter equation which intertwine representations of a variety of boundary coideal subalgebras of the bulk Hopf superalgebra. Many of these integrable boundaries are matched to D1- and D5-brane maximal giant gravitons.

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I. INTRODUCTION

Integrability has been a remarkable discovery in AdS/CFT, leading to the matching of an infinite tower of conserved quantities on the gauge and gravity sides of these dualities [1,2]. In the canonical AdS₅/CFT₄ [3], the infinite-dimensional superalgebra underlying integrability is a Yangian symmetry [4,5] with level-0 Lie superalgebra a central extension of $\mathfrak{su}(2|2)$, denoted $\mathfrak{su}(2|2)_c$. The universal enveloping algebra of $\mathfrak{su}(2|2)_c$ is endowed with the structure of a Hopf algebra. Massive excitations of the world sheet of a closed IIB superstring on AdS₅ × S⁵ map to $\mathfrak{su}(2|2)_c$ -symmetric magnon excitations of a closed spin-chain built from states in a representation of $\mathfrak{psu}(2, 2|4)$, which is the superisometry algebra of AdS₅ × S⁵. The integrable S -matrix describing two-magnon scattering is identified with the R -matrix of the underlying superalgebra [6]. Three-magnon scattering factorizes into a succession of two-magnon scattering processes—a statement of integrability encoded in the Yang-Baxter equation. Massive excitations of an open IIB superstring ending on a D-brane in AdS₅ × S⁵ map to similar magnon excitations of a $\mathfrak{psu}(2, 2|4)$ open spin-chain with a distinguished boundary site [7]. The symmetry of the boundary is determined by the superisometries preserved by the D-brane which are contained in the bulk magnon symmetry algebra $\mathfrak{su}(2|2)_c$. This boundary Lie algebra may be extended to a coideal subalgebra of the bulk Hopf superalgebra. The scattering of a single magnon off an integrable boundary is described by a boundary S -matrix—the matrix part of which is the reflection K -matrix [8–11]. Two-magnon reflections factorize into a succession of single-magnon reflections and bulk two-magnon scattering processes—with this boundary integrability encoded in the boundary Yang-Baxter equation, also called the reflection equation [12].

The R -matrix intertwines representations of the bulk Hopf superalgebra, while the K -matrix intertwines representations of a boundary coideal subalgebra of this bulk superalgebra.

Extensive studies, which were initiated in Ref. [13], have been made of integrability in AdS₃/CFT₂. The dual field theories have been the subject of recent interest [14–17], but much still remains to be understood. On the string theory side, the IIB supergravity backgrounds AdS₃ × S³ × M⁴, with M⁴ = S³ × S¹ or T⁴, are known [18,19]. Both backgrounds are half-BPS, preserving eight left- and eight right-moving supersymmetries, and have a combination of NS-NS or R-R 3-form flux—we focus here exclusively on the case of pure R-R flux. The AdS₃ × S³ × S³ × S¹ supergravity geometry has AdS₃ radius L , and S³ and S³ radii R and R' , which must satisfy [18]

$$\frac{1}{R^2} + \frac{1}{R'^2} = \frac{1}{L^2},$$

implying $R = L \sec \beta$ and $R' = L \csc \beta$. Here $\alpha \equiv \cos^2 \beta$ is a parameter related to the relative size of the 3-spheres. The bosonic isometry algebra is

$$\begin{aligned} & \mathfrak{so}(2, 2) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(4)' \oplus \mathfrak{u}(1) \\ & \sim [\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)']_L \\ & \oplus [\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)']_R \oplus \mathfrak{u}(1), \end{aligned}$$

which splits into two copies (left and right) of the bosonic subalgebra $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)'$ of the Lie superalgebra $\mathfrak{d}(2, 1; \alpha)$. The superisometry algebra of AdS₃ × S³ × S³ × S¹ is $\mathfrak{d}(2, 1; \alpha)^2 \oplus \mathfrak{u}(1)$. The size of the 3-sphere S³ becomes infinite in the $\alpha \rightarrow 1$ limit, and a compactification¹ of the resulting \mathbb{R}^3 gives the IIB supergravity

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¹Here we ignore complications which arise from neglecting winding modes.

background $\text{AdS}_3 \times S^3 \times T^4$. The radii of AdS_3 and S^3 , denoted L , are now the same. The bosonic isometry algebra

$$\mathfrak{so}(2, 2) \oplus \mathfrak{so}(4) \oplus \mathfrak{u}(1)^4 \\ \sim [\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2)]_{\mathbb{L}} \oplus [\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2)]_{\mathbb{R}} \oplus \mathfrak{u}(1)^4$$

contains two copies of the bosonic algebra $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2)$ of the Lie superalgebra $\mathfrak{psu}(1, 1|2)$. The superisometry algebra of $\text{AdS}_3 \times S^3 \times T^4$ is $\mathfrak{psu}(1, 1|2)^2 \oplus \mathfrak{u}(1)^4$.

Massive excitations of the world sheet of a closed IIB superstring on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ map to $\mathfrak{su}(1|1)_{\mathbb{C}}^2$ -symmetric magnon excitations of a closed, alternating spin-chain built from states in two representations of $\mathfrak{d}(2, 1; \alpha)^2$ at odd and even sites. There is symmetry enhancement in the $\alpha \rightarrow 1$ limit. Massive world-sheet excitations of a closed IIB superstring on $\text{AdS}_3 \times S^3 \times T^4$ map to $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$ -symmetric magnon excitations of a closed, homogeneous spin-chain built from states in a representation of $\mathfrak{psu}(1, 1|2)^2$. The universal enveloping algebras of $\mathfrak{su}(1|1)_{\mathbb{C}}^2$ and $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$ can be endowed with Hopf algebra structures. Integrable S -matrices describing two-magnon scattering were derived in Refs. [20,21] (see Ref. [22] for a recent review and Refs. [23–25] for early work). The R -matrix of the $\mathfrak{psu}(1, 1|2)^2$ spin-chain is essentially two copies of the R -matrix of the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain. A variety of results for the string sigma model were obtained in Refs. [26–37], and a proposal for the dressing phases was put forward in Ref. [38] (see also Ref. [39]). The scattering of massless excitations of the superstring world sheet was considered in Refs. [40–44].

Integrability manifests here in the form of infinite-dimensional Yangian symmetries. Hopf algebra structures in $\text{AdS}_5/\text{CFT}_4$ were described in Refs. [45,46], and the full $\mathfrak{su}(2|2)_{\mathbb{C}}$ Yangian symmetry was introduced in Ref. [47] and further explored in Refs. [48,49]. Yangian symmetries in $\text{AdS}_3/\text{CFT}_2$ were explored in Refs. [21,50], but only recently fully described in Ref. [51]. Since the representation theory of $\mathfrak{su}(1|1)_{\mathbb{C}}^2$ is relatively simple, with massive excitations being vectors in a two-dimensional atypical (short) representation and the R -matrix intertwining 2 four-dimensional typical (long) representations, the Yangian is not needed to obtain the R -matrix (see Ref. [52] for the situation in $\text{AdS}_5/\text{CFT}_4$). However, it is necessary to know the full Yangian symmetry to construct the algebraic Bethe ansatz equations. The Bethe equations for $\text{AdS}_3/\text{CFT}_2$ were proposed in Ref. [53] and later derived using the coordinate method in Ref. [54]. Boundary Yangian symmetry for $\text{AdS}_5/\text{CFT}_4$ has undergone an extensive study in Refs. [55–60], and the boundary Bethe equations were constructed in Refs. [11,61–64]. The natural next step in the exploration of integrability and the spectral problem for

open superstrings in $\text{AdS}_3/\text{CFT}_2$ is to find the boundary Yangian symmetries and hence derive the Bethe equations.

A comprehensive study of *open spin-chains with integrable boundaries* in $\text{AdS}_3/\text{CFT}_2$ is presented in this paper. The $\mathfrak{d}(2, 1; \alpha)^2$ and $\mathfrak{psu}(1, 1|2)^2$ open spin-chains map to open IIB superstrings ending on D-branes in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\text{AdS}_3 \times S^3 \times T^4$. In particular, half- and quarter-BPS maximal D1- and D5-brane giant gravitons [65–67] provide a variety of integrable boundaries. We derive reflection matrices which describe single-magnon scattering off singlet and vector boundary states. As for the bulk R -matrix, the K -matrices of the $\mathfrak{psu}(1, 1|2)^2$ spin-chain can be built from two K -matrices of the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain. In the $\mathfrak{d}(2, 1; \alpha)^2$ case, the reflection matrices intertwine representations of totally supersymmetric, half-supersymmetric and non-supersymmetric boundary Lie algebras (symmetries of the D-branes), which can be extended to coideal subalgebras of the bulk Hopf superalgebra. Several of our reflection matrices coincide with certain $\mathfrak{su}(1|1)$ subsectors of the reflection matrices of $\mathfrak{psu}(2, 2|4)$ open spin-chains. These map to open IIB superstrings ending on D3-brane $Y = 0$ and $Z = 0$ giant gravitons [8] and D7 branes [9] in $\text{AdS}_5 \times S^5$. We uncover novel hidden boundary symmetries of a chiral reflection matrix with a non-supersymmetric boundary Lie algebra—these have no known analogue in $\text{AdS}_5/\text{CFT}_4$. We also derive an achiral reflection matrix for a non-supersymmetric boundary Lie algebra generated by the magnon Hamiltonian.

The structure of this paper is as follows: D1 and D5-brane maximal giant gravitons and their symmetries are described in Sec. II. Sections II A and II B therein focus on maximal giant gravitons in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\text{AdS}_3 \times S^3 \times T^4$, respectively. Sections III and IV describe the $\mathfrak{d}(2, 1; \alpha)^2$ closed and open spin-chains. Section III contains a review of the $\mathfrak{d}(2, 1; \alpha)^2$ closed spin-chain and its R -matrices. Here we choose a different frame for the \mathfrak{U} -deformation of the bulk Hopf superalgebra from the one used in Ref. [20]; this frame is more convenient for the boundary scattering theory. Section IV presents our novel results for the $\mathfrak{d}(2, 1; \alpha)^2$ open spin-chain. We derive various K -matrices and describe the associated boundary coideal subalgebras. The $\mathfrak{psu}(1, 1|2)^2$ closed and open spin-chains are the subject of Secs. V and VI. Section V contains a brief review of the $\mathfrak{psu}(1, 1|2)^2$ closed spin-chain and its R -matrices. Our new results concerning $\mathfrak{psu}(1, 1|2)^2$ open spin-chains, K -matrices and boundary coideal subalgebras are presented in Sec. VI. Concluding remarks are contained in Sec. VII. Appendix A states our spinor conventions. Appendix B describes the relevant representation theory of $\mathfrak{d}(2, 1; \alpha)$ and $\mathfrak{psu}(1, 1|2)$. Appendix C shows various useful expressions relating to the $SO(2, 2)$ and $SO(4)$ bosonic isometry groups of the supergravity backgrounds.

II. MAXIMAL GIANT GRAVITONS

A. Maximal giant gravitons on

$$\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$$

We start by giving the details of the type-IIB supergravity background AdS₃ × S³ × S^{3'} × S¹ with pure R-R 3-form flux, and describe D1 and D5-brane maximal giant gravitons based on Ref. [67].

1. AdS₃ × S³ × S^{3'} × S¹ with pure R-R flux

IIB supergravity solution.—The AdS₃ × S³ × S^{3'} × S¹ background has the metric

$$\begin{aligned} ds^2 = & L^2(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2) \\ & + R^2(d\theta^2 + \cos^2 \theta d\chi^2 + \sin^2 \theta d\phi^2) \\ & + R'^2(d\theta'^2 + \cos^2 \theta' d\chi'^2 + \sin^2 \theta' d\phi'^2) \\ & + \ell^2 d\xi^2, \end{aligned} \quad (2.1)$$

with $R = L \sec \beta$ and $R' = L \csc \beta$, where $\alpha \equiv \cos^2 \beta$. This geometry is symmetric under $\alpha \rightarrow 1 - \alpha$ and an interchange of S^3 and $S^{3'}$. The 3-form field strength $F_{(3)} = dC_{(2)}$ is given by

$$\begin{aligned} F_{(3)} = & 2L^2 dt \wedge (\sinh \rho \cosh \rho d\rho) \wedge d\varphi \\ & + 2R^2 (\sin \theta \cos \theta d\theta) \wedge d\chi \wedge d\phi \\ & + 2R'^2 (\sin \theta' \cos \theta' d\theta') \wedge d\chi' \wedge d\phi' \end{aligned} \quad (2.2)$$

in the case of pure R-R flux. The Hodge dual 7-form field strength $F_{(7)} = dC_{(6)} = *F_{(3)}$ is

$$\begin{aligned} F_{(7)} = & -\frac{2R^3 R'^3 \ell}{L} (\sin \theta \cos \theta d\theta) \wedge d\chi \wedge d\phi \\ & \wedge (\cos \theta' \sin \theta' d\theta') \wedge d\chi' \wedge d\phi' \wedge d\xi \\ & -\frac{2L^3 R'^3 \ell}{R} dt \wedge (\sinh \rho \cosh \rho d\rho) \wedge d\varphi \\ & \wedge (\sin \theta' \cos \theta' d\theta') \wedge d\chi' \wedge d\phi' \wedge d\xi \\ & +\frac{2L^3 R^3 \ell}{R'} dt \wedge (\sinh \rho \cosh \rho d\rho) \wedge d\varphi \\ & \wedge (\sin \theta \cos \theta d\theta) \wedge d\chi \wedge d\phi \wedge d\xi. \end{aligned} \quad (2.3)$$

The 3-form and 5-form fluxes couple to D1- and D5-branes. Dynamically stable giant gravitons with angular momentum on both 3-spheres were shown to exist in Ref. [67]. D1- and D5-brane maximal giant gravitons provide integrable boundary conditions for open IIB superstrings on AdS₃ × S³ × S^{3'} × S¹.

Supersymmetry.—The supersymmetry variations of the dilatino and gravitino are²

$$\begin{aligned} \delta\lambda &= \frac{i}{4} F_{(3)}(B\varepsilon)^*, \\ \delta\Psi_M &= \nabla_M \varepsilon - \frac{i}{8} F_{(3)} \Gamma_M (B\varepsilon)^*, \end{aligned} \quad (2.4)$$

parametrized by $\varepsilon = \varepsilon^1 + i\varepsilon^2$, with the 32-component Weyl-Majorana spinors ε^I satisfying $\Gamma \varepsilon^I = \varepsilon^I$ and $(B\varepsilon^I)^* = \varepsilon^I$, for $I \in \{1, 2\}$. The charge conjugation matrix is $C = B\Gamma^0$. Also,³

$$\begin{aligned} F_{(3)} &= \frac{2}{L} \Gamma^{012} (\mathbb{1} + \sqrt{\alpha} \Gamma^{012} \Gamma^{345} + \sqrt{1-\alpha} \Gamma^{012} \Gamma^{678}) \\ &= \frac{4}{L} \Gamma^{012} K^+(\alpha), \end{aligned} \quad (2.5)$$

with our gamma matrix conventions shown in Appendix A. Here

$$K^\pm(\alpha) \equiv \frac{1}{2} [\mathbb{1} \pm (\sqrt{\alpha} \Gamma^{012} \Gamma^{345} + \sqrt{1-\alpha} \Gamma^{012} \Gamma^{678})]. \quad (2.6)$$

The gravitino Killing-spinor equation $\delta\Psi_M = 0$ implies a solution of the form [67]

$$\begin{aligned} \varepsilon(x^M) &= \mathcal{M}^+(x^M) \varepsilon^+ + \mathcal{M}^-(x^M) \varepsilon^- \\ &= \mathcal{M}^+(x^M) (1+i) \varepsilon^L + \mathcal{M}^-(x^M) (1+i) \varepsilon^R, \end{aligned} \quad (2.7)$$

decomposed into left- and right-movers, with

$$\begin{aligned} \mathcal{M}^\pm(x^M) &= e^{\pm \frac{1}{2} \rho \Gamma_{02}} e^{\frac{1}{2}(\varphi \pm t) \Gamma_{12}} e^{\pm \frac{1}{2} \theta \Gamma_{45}} e^{\frac{1}{2}(\phi \mp \chi) \Gamma_{35}} \\ &\times e^{\pm \frac{1}{2} \theta' \Gamma_{78}} e^{\frac{1}{2}(\phi' \mp \chi') \Gamma_{68}}. \end{aligned} \quad (2.8)$$

²The vielbeins $\hat{E}^A = E_M^A dx^M$ are given by

$$\begin{aligned} \hat{E}^0 &= L \cosh \rho dt, & \hat{E}^1 &= L d\rho, & \hat{E}^2 &= L \sinh \rho d\varphi, \\ \hat{E}^3 &= R d\theta, & \hat{E}^4 &= R \cos \theta d\chi, & \hat{E}^5 &= R \sin \theta d\phi, \\ \hat{E}^6 &= R' d\theta', & \hat{E}^7 &= R' \cos \theta' d\chi', & \hat{E}^8 &= R' \sin \theta' d\phi', & \hat{E}^9 &= \ell d\xi. \end{aligned}$$

The supercovariant derivatives $\nabla_M = \partial_M + \Omega_M^{AB} \Gamma_{AB}$, with $\hat{\Omega}^{AB} = \Omega_M^{AB} dx^M$ satisfying $d\hat{E}^A + \hat{\Omega}_B^A \wedge \hat{E}^B$, are

$$\begin{aligned} \nabla_t &= \partial_t + \frac{1}{2} \sinh \rho \Gamma_{01}, & \nabla_\rho &= \partial_\rho, \\ \nabla_\varphi &= \partial_\varphi - \frac{1}{2} \cosh \rho \Gamma_{12}, & \nabla_\theta &= \partial_\theta, \\ \nabla_\chi &= \partial_\chi + \frac{1}{2} \sin \theta \Gamma_{34}, & \nabla_\phi &= \partial_\phi - \frac{1}{2} \cos \theta \Gamma_{35}, & \nabla_{\theta'} &= \partial_{\theta'}, \\ \nabla_{\chi'} &= \partial_{\chi'} + \frac{1}{2} \sin \theta' \Gamma_{67}, & \nabla_{\phi'} &= \partial_{\phi'} - \frac{1}{2} \cos \theta' \Gamma_{68}, & \nabla_\xi &= \partial_\xi. \end{aligned}$$

³Here we use the notation $F_{(3)} \equiv \frac{1}{3!} F_{(3)NRS} E_A^N E_B^R E_C^S \Gamma_{ABC}$.

Here $i(B\epsilon^\pm)^* = \pm\epsilon^\pm$ and $\Gamma\epsilon^\pm = \epsilon^\pm$, with $\epsilon^+ = (1+i)\epsilon^L$ and $\epsilon^- = (1+i)\epsilon^R$. The left- and right-movers satisfy $(B\epsilon^L)^* = \epsilon^L$ and $(B\epsilon^R)^* = -\epsilon^R$, with $\Gamma\epsilon^a = \epsilon^a$, for $a \in \{L, R\}$. The Weyl-Majorana spinors are written in terms of these left- and right-movers as $\epsilon^I = \epsilon^L - i(-1)^I\epsilon^R$.

The dilatino Killing-spinor equation $\delta\lambda = 0$ now further implies $K^+(\alpha)\epsilon^a = 0$, which halves the number of left- and right-moving degrees of freedom from 16 to 8, yielding a half-BPS geometry.

The spinors ϵ^L and ϵ^R can be decomposed into eigenstates $\epsilon^{Lb\beta\dot{\beta}}$ and $\epsilon^{Rb\beta\dot{\beta}}$, which have eigenvalues $(b, \beta, \dot{\beta}) = (\pm, \pm, \pm)$ of the Dirac bilinears given in (A6):

$$\begin{aligned} i\Gamma_{12}\epsilon^{ab\beta\dot{\beta}} &= b\epsilon^{ab\beta\dot{\beta}}, & i\Gamma_{35}\epsilon^{ab\beta\dot{\beta}} &= -\beta\epsilon^{ab\beta\dot{\beta}}, \\ i\Gamma_{68}\epsilon^{ab\beta\dot{\beta}} &= -\dot{\beta}\epsilon^{ab\beta\dot{\beta}}. \end{aligned} \quad (2.9)$$

The IIB supergravity background $\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$ is thus invariant under eight left- and eight right-moving supersymmetry transformations, parametrized by $\epsilon^{ab\beta\dot{\beta}}$, which satisfy

$$\begin{aligned} K^+(\alpha)\epsilon^{ab\beta\dot{\beta}} &= 0 \\ \text{and so } K^-(\alpha)\epsilon^{ab\beta\dot{\beta}} &= \epsilon^{ab\beta\dot{\beta}}. \end{aligned} \quad (2.10)$$

This gives rise to the kappa symmetry condition $K^+(\alpha)\Theta^{ab\beta\dot{\beta}}(\tau, \sigma) = 0$ of Ref. [13] when the target space superfields are pulled back to the superstring world sheet to give $X^M(\tau, \sigma)$ and $\Theta^{ab\beta\dot{\beta}}(\tau, \sigma)$. Here the spinors $\epsilon^{ab\beta\dot{\beta}}$ parametrize translations in these spinor directions in superspace, which are generated by the supercharges $\mathcal{Q}_{ab\beta\dot{\beta}}$. The full superisometry algebra is $\mathfrak{d}(2, 1; \alpha)_L \oplus \mathfrak{d}(2, 1; \alpha)_R \oplus \mathfrak{u}(1)$, with the details of the exceptional Lie superalgebra $\mathfrak{d}(2, 1; \alpha)$ given in Appendix B. It was noted in Ref. [13] that this is the correct kappa symmetry gauge choice to allow for comparison between the Green-Schwarz action and their \mathbb{Z}_4 -graded $(\mathfrak{d}(2, 1; \alpha)_L \oplus \mathfrak{d}(2, 1; \alpha)_R) / (\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2))$ integrable coset model which describes a closed IIB superstring on $\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$.

2. Maximal giant gravitons and boundary algebras

Massive excitations of a closed IIB superstring on $\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$ map to magnon excitations of a $\mathfrak{d}(2, 1; \alpha)^2$ closed spin-chain. Section III discusses how choosing a vacuum \mathcal{Z} breaks the $\mathfrak{d}(2, 1; \alpha)^2$ symmetry to $\mathfrak{su}(1|1)^2$, centrally extended to $\mathfrak{su}(1|1)_c^2$. An open IIB superstring ending on a D-brane maps to a $\mathfrak{d}(2, 1; \alpha)^2$ open spin-chain with boundaries. The boundary Lie algebra is a subalgebra of $\mathfrak{su}(1|1)_c^2$ determined by the D-brane symmetries which survive the choice of spin-chain vacuum. This can be extended to a coideal subalgebra of the Hopf superalgebra, as explained in Sec. IV. We classify boundary

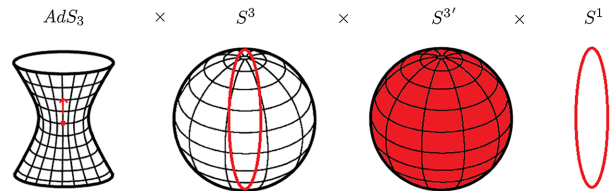


FIG. 1 (color online). Half of the D5-brane maximal giant graviton wrapping $S^1 \times S^{3'} \times S^1$ in $\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$.

Lie algebras for D5 and D1-brane maximal giant gravitons in $\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$.

Maximal D5-brane giant gravitons.—The D5-brane giant graviton of Ref. [67] factorizes at maximum size into two D5-branes, which wrap $S^1 \times S^{3'} \times S^1$ and $S^3 \times S^{1'} \times S^1$, and move only along the time direction in AdS_3 . Each half-BPS D5-brane preserves four left- and four right-moving supersymmetries on its world volume. We focus on the first half of the maximal giant (see Fig. 1). This breaks the bosonic isometry algebra to

$$\begin{aligned} &(\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus (\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus \mathfrak{so}(4)' \oplus \mathfrak{u}(1) \\ &\subset \mathfrak{so}(2, 2) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(4)' \oplus \mathfrak{u}(1). \end{aligned}$$

$Z = 0$ giant.—Let us take the \mathbf{C}^2 and \mathbf{C}'^2 embedding coordinates of the 3-spheres to be

$$\begin{aligned} (Z, Y) &= (x_1 + ix_2, x_3 + ix_4) = (R \cos \theta e^{i\chi}, R \sin \theta e^{i\phi}), \\ (Z', Y') &= (x'_1 + ix'_2, x'_3 + ix'_4) = (R' \cos \theta' e^{i\chi'}, R' \sin \theta' e^{i\phi'}), \end{aligned} \quad (2.11)$$

as in Ref. [67]. The maximal D5-brane giant graviton consists of $Z = 0$ and $Z' = 0$ halves. Let us focus on the $Z = 0$ giant which wraps the ϕ great circle in S^3 and the $S^{3'} \times S^1$ space. The world-volume metric, obtained by setting $\rho = 0$ and $\theta = \frac{\pi}{2}$, is

$$\begin{aligned} ds^2 &= -L^2 dt^2 + R^2 d\phi^2 + R'^2 (d\theta'^2 + \cos^2 \theta' d\chi'^2 + \sin^2 \theta' d\phi'^2) \\ &\quad + \ell^2 d\xi^2. \end{aligned}$$

The bosonic symmetries of this D5-brane include time translations and rotations in AdS_3 , rotations in S^3 in the x_1x_2 and x_3x_4 planes, and all rotations in $S^{3'}$, generated by

$$\begin{aligned} \mathfrak{F}_{a0} &\in \mathfrak{u}(1)_a \subset \mathfrak{su}(1, 1)_a, & \mathfrak{L}_{a5} &\in \mathfrak{u}(1)_a \subset \mathfrak{su}(2)_a, \\ \{\mathfrak{R}_{a8}, \mathfrak{R}_{a\pm}\} &\in \mathfrak{su}(2)'_a, & \text{for } a &\in \{L, R\}, \end{aligned}$$

with the splitting of $\mathfrak{so}(2, 2)$, $\mathfrak{so}(4)$ and $\mathfrak{so}(4)'$ into left and right algebras shown in Appendix C. The Cartan elements of the $\mathfrak{su}(1|1)^2$ superalgebra, denoted $\mathfrak{H}_a = -\mathfrak{F}_{a0} - \alpha\mathfrak{L}_{a5} - (1 - \alpha)\mathfrak{R}_{a8}$, are thus included in the generators of bosonic symmetries of the $Z = 0$ giant.

Kappa symmetry on the world volume of the D5-brane requires

$$\Gamma_{056789}\varepsilon = -i(B\varepsilon)^*, \quad (2.12)$$

with the pullback of the Killing spinor (2.7) given by

$$\begin{aligned} \varepsilon &= (1+i)\mathcal{M}^+(t, \phi, \theta', \chi', \phi', \xi)e^L \\ &+ (1+i)\mathcal{M}^-(t, \phi, \theta', \chi', \phi', \xi)e^R. \end{aligned} \quad (2.13)$$

Here $K^+(\alpha)\varepsilon^a = 0$, for $\mathbf{a} \in \{\mathbf{L}, \mathbf{R}\}$. This kappa symmetry condition reduces to

$$\Gamma_{12}\varepsilon^a = \Gamma_{35}\varepsilon^a, \quad (2.14)$$

and hence $\varepsilon^{ab\dot{\beta}}$ has labels satisfying $-b = \beta$, with $\dot{\beta}$ free. The 4 + 4 supersymmetries compatible with kappa symmetry on the world volume of the $Z = 0$ giant are generated by

$$\{\mathfrak{Q}_{\mathbf{a}^{---}} \equiv \mathfrak{Q}_{\mathbf{a}}, \mathfrak{Q}_{\mathbf{a}^{--+}}, \mathfrak{Q}_{\mathbf{a}^{+--}} \equiv \mathfrak{C}_{\mathbf{a}}, \mathfrak{Q}_{\mathbf{a}^{+++}}\}.$$

The symmetries of this D5-brane include the $\mathfrak{su}(1|1)^2$ superalgebra generated by $\{\mathfrak{Q}_{\mathbf{a}}, \mathfrak{C}_{\mathbf{a}}, \mathfrak{H}_{\mathbf{a}}\}$, which acts on magnon excitations of the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain. The boundary Lie algebra is the centrally extended $\mathfrak{su}(1|1)_\mathbb{C}^2$. We say the $Z = 0$ giant is completely aligned with the spin-chain vacuum \mathcal{Z} .

$Y = 0$ giant: An $SO(4)$ transformation taking Z to Y corresponds to simultaneous rotations by $\theta_{13} = \theta_{24} = \frac{\pi}{2}$ in the x_1x_3 and x_2x_4 planes, achieved by $U_L = i\sigma_1$ and $U_R = \mathbb{1}$ (see Appendix C). The 4 + 4 supersymmetries on the world volume of the $Y = 0$ giant are hence

$$\begin{aligned} \{\mathfrak{Q}_{\mathbf{L}^{--+}}, \mathfrak{Q}_{\mathbf{L}^{---}}, \mathfrak{Q}_{\mathbf{L}^{+++}}, \mathfrak{Q}_{\mathbf{L}^{+++}}\}, \\ \{\mathfrak{Q}_{\mathbf{R}^{--+}} \equiv \mathfrak{Q}_{\mathbf{R}}, \mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{+--}} \equiv \mathfrak{C}_{\mathbf{R}}, \mathfrak{Q}_{\mathbf{R}^{+++}}\}. \end{aligned}$$

Thus $\{\mathfrak{Q}_{\mathbf{R}}, \mathfrak{C}_{\mathbf{R}}, \mathfrak{H}_{\mathbf{R}}, \mathfrak{H}_{\mathbf{L}}\}$ are both world-volume symmetries and in the $\mathfrak{su}(1|1)_\mathbb{C}^2$ superalgebra of magnon excitations, and they generate the right half-supersymmetric boundary Lie algebra $\mathfrak{u}(1)_L \oplus \mathfrak{su}(1|1)_R$.

$\bar{Y} = 0$ giant: An $SO(4)$ transformation which takes Z to \bar{Y} is obtained by setting $\theta_{13} = -\theta_{24} = \frac{\pi}{2}$, corresponding to $U_L = \mathbb{1}$ and $U_R = -i\sigma_1$. The 4 + 4 supersymmetries of the $\bar{Y} = 0$ giant are

$$\begin{aligned} \{\mathfrak{Q}_{\mathbf{L}^{--+}} \equiv \mathfrak{Q}_{\mathbf{L}}, \mathfrak{Q}_{\mathbf{L}^{--+}}, \mathfrak{Q}_{\mathbf{L}^{+--}} \equiv \mathfrak{C}_{\mathbf{L}}, \mathfrak{Q}_{\mathbf{L}^{+++}}\}, \\ \{\mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{---}}, \mathfrak{Q}_{\mathbf{R}^{+--}}, \mathfrak{Q}_{\mathbf{R}^{+++}}\}. \end{aligned}$$

Here $\{\mathfrak{Q}_{\mathbf{L}}, \mathfrak{C}_{\mathbf{L}}, \mathfrak{H}_{\mathbf{L}}, \mathfrak{H}_{\mathbf{R}}\}$ generate the left half-supersymmetric boundary Lie algebra $\mathfrak{su}(1|1)_L \oplus \mathfrak{u}(1)_R$.

$\bar{Z} = 0$ giant: An $SO(4)$ rotation taking Z to \bar{Z} is obtained by setting $\theta_{24} = \pi$, which corresponds to $U_L = i\sigma_1$ and $U_R = i\sigma_1$. The 4 + 4 supersymmetries of the $\bar{Z} = 0$ giant are thus

$$\begin{aligned} \{\mathfrak{Q}_{\mathbf{L}^{--+}}, \mathfrak{Q}_{\mathbf{L}^{---}}, \mathfrak{Q}_{\mathbf{L}^{+++}}, \mathfrak{Q}_{\mathbf{L}^{+++}}\}, \\ \{\mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{---}}, \mathfrak{Q}_{\mathbf{R}^{+--}}, \mathfrak{Q}_{\mathbf{R}^{+++}}\}, \end{aligned}$$

none of which are in the $\mathfrak{su}(1|1)_\mathbb{C}^2$ superalgebra, although the Cartan elements $\mathfrak{H}_{\mathbf{a}}$ remain world-volume symmetries. The boundary Lie algebra $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$ is therefore non-supersymmetric.

We also consider the $Z' = 0$ giant which is completely aligned with the spin-chain vacuum \mathcal{Z} and has totally supersymmetric boundary algebra $\mathfrak{su}(1|1)_\mathbb{C}^2$. Then $SO(4)'$ transformations yield the $Y' = 0$, $\bar{Y}' = 0$ and $\bar{Z}' = 0$ giants, which give right and left half-supersymmetric, and non-supersymmetric boundary algebras, $\mathfrak{u}(1)_L \oplus \mathfrak{su}(1|1)_R$, $\mathfrak{su}(1|1)_L \oplus \mathfrak{u}(1)_R$ and $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$ (see Table I).

Maximal D1-brane giant gravitons.—The maximal D1-brane giant graviton of Ref. [67] wraps a 1-cycle wound around a torus $S^1 \times S^{1'}$ made up of two great circles of radii R and R' on the 3-spheres. This quarter-BPS D1-brane preserves two left- and two right-moving supersymmetries on its world volume. As shown in Fig. 2, the bosonic isometry algebra is broken by our choice of torus to

$$\begin{aligned} (\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus (\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus (\mathfrak{u}(1)' \oplus \mathfrak{u}(1)') \\ \subset \mathfrak{so}(2, 2) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(4)' \oplus \mathfrak{u}(1), \end{aligned}$$

and further by the D1-brane to

TABLE I. Boundary Lie algebras for D5-brane maximal giant gravitons on AdS₃ × S³ × S^{3'} × S¹.

D5 giant	Bosonic generators	Supersymmetry generators	Boundary algebra
$Z = 0$	$\{\mathfrak{T}_{\mathbf{a}0}, \mathfrak{Q}_{\mathbf{a}5}, \mathfrak{R}_{\mathbf{a}\pm}, \mathfrak{R}_{\mathbf{a}8}\}$	$\{\mathfrak{Q}_{\mathbf{a}^{--+}}, \mathfrak{Q}_{\mathbf{a}^{+--}}\}$	$\mathfrak{su}(1 1)_\mathbb{C}^2$
$Y = 0$	$\{\mathfrak{T}_{\mathbf{a}0}, \mathfrak{Q}_{\mathbf{a}5}, \mathfrak{R}_{\mathbf{a}\pm}, \mathfrak{R}_{\mathbf{a}8}\}$	$\{\mathfrak{Q}_{\mathbf{L}^{--+}}, \mathfrak{Q}_{\mathbf{L}^{+++}}, \mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{+--}}\}$	$\mathfrak{u}(1)_L \oplus \mathfrak{su}(1 1)_R$
$\bar{Y} = 0$	$\{\mathfrak{T}_{\mathbf{a}0}, \mathfrak{Q}_{\mathbf{a}5}, \mathfrak{R}_{\mathbf{a}\pm}, \mathfrak{R}_{\mathbf{a}8}\}$	$\{\mathfrak{Q}_{\mathbf{L}^{+--}}, \mathfrak{Q}_{\mathbf{L}^{+--}}, \mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{+--}}\}$	$\mathfrak{su}(1 1)_L \oplus \mathfrak{u}(1)_R$
$\bar{Z} = 0$	$\{\mathfrak{T}_{\mathbf{a}0}, \mathfrak{Q}_{\mathbf{a}5}, \mathfrak{R}_{\mathbf{a}\pm}, \mathfrak{R}_{\mathbf{a}8}\}$	$\{\mathfrak{Q}_{\mathbf{L}^{--+}}, \mathfrak{Q}_{\mathbf{L}^{+++}}, \mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{+--}}\}$	$\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$
$Z' = 0$	$\{\mathfrak{T}_{\mathbf{a}0}, \mathfrak{Q}_{\mathbf{a}\pm}, \mathfrak{Q}_{\mathbf{a}5}, \mathfrak{R}_{\mathbf{a}8}\}$	$\{\mathfrak{Q}_{\mathbf{a}^{--+}}, \mathfrak{Q}_{\mathbf{a}^{+--}}\}$	$\mathfrak{su}(1 1)_\mathbb{C}^2$
$Y' = 0$	$\{\mathfrak{T}_{\mathbf{a}0}, \mathfrak{Q}_{\mathbf{a}\pm}, \mathfrak{Q}_{\mathbf{a}5}, \mathfrak{R}_{\mathbf{a}8}\}$	$\{\mathfrak{Q}_{\mathbf{L}^{+--}}, \mathfrak{Q}_{\mathbf{L}^{+++}}, \mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{+--}}\}$	$\mathfrak{u}(1)_L \oplus \mathfrak{su}(1 1)_R$
$\bar{Y}' = 0$	$\{\mathfrak{T}_{\mathbf{a}0}, \mathfrak{Q}_{\mathbf{a}\pm}, \mathfrak{Q}_{\mathbf{a}5}, \mathfrak{R}_{\mathbf{a}8}\}$	$\{\mathfrak{Q}_{\mathbf{L}^{--+}}, \mathfrak{Q}_{\mathbf{L}^{+--}}, \mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{+--}}\}$	$\mathfrak{su}(1 1)_L \oplus \mathfrak{u}(1)_R$
$\bar{Z}' = 0$	$\{\mathfrak{T}_{\mathbf{a}0}, \mathfrak{Q}_{\mathbf{a}\pm}, \mathfrak{Q}_{\mathbf{a}5}, \mathfrak{R}_{\mathbf{a}8}\}$	$\{\mathfrak{Q}_{\mathbf{L}^{+--}}, \mathfrak{Q}_{\mathbf{L}^{+++}}, \mathfrak{Q}_{\mathbf{R}^{--+}}, \mathfrak{Q}_{\mathbf{R}^{+--}}\}$	$\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$

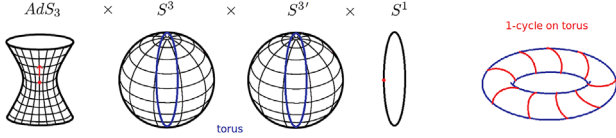


FIG. 2 (color online). The D1-brane maximal giant graviton wraps a 1-cycle on a $S^1 \times S^{1'}$ in $\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$.

$$(\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus (\mathfrak{u}(1)_\sigma \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)').$$

$Z = Z' = 0$ giant: Let us again make use of the coordinates (2.11). The maximal D1-brane giant graviton is specified in Ref. [67] by $\rho = 0$ and $Z = Z' = 0$, which gives $\theta = \theta' = \frac{\pi}{2}$. The world volume is parametrized by (t, σ) , where we define $\phi = \alpha\sigma$ and $\phi' = (1 - \alpha)\sigma$. The D1-brane wraps a 1-cycle on the torus (ϕ, ϕ') . The world-volume metric is

$$ds^2 = -L^2 dt^2 + L^2 d\sigma^2.$$

The bosonic symmetries of the torus are generated by all the Cartan elements $\{\mathfrak{F}_{a0}, \mathfrak{L}_{a5}, \mathfrak{R}_{a8}\}$. The D1-brane itself wraps a 1-cycle on this torus. Its symmetries are time translations and rotations in AdS_3 , rotations in S^3 and $S^{3'}$ in the $x_1 x_2$ and $x'_1 x'_2$ planes transverse to the torus, and translations along the 1-cycle in the world-volume direction σ . These bosonic symmetries of the $Z = Z' = 0$ giant are generated by

$$\begin{aligned} \mathfrak{F}_{a0} &\in \mathfrak{u}(1)_a \subset \mathfrak{su}(1, 1)_a, \\ \mathfrak{L}_{L5} - \mathfrak{L}_{R5} &\in \mathfrak{u}(1), \quad \mathfrak{R}_{L8} - \mathfrak{R}_{R8} \in \mathfrak{u}(1)', \\ -\alpha(\mathfrak{L}_{L5} - \mathfrak{L}_{R5}) - (1 - \alpha)(\mathfrak{R}_{L8} - \mathfrak{R}_{R8}) &\in \mathfrak{u}(1)_\sigma \end{aligned}$$

or, equivalently, by the generators

$$\{\mathfrak{F}_{a0}, \mathfrak{h}_a, \mathfrak{R}_{L8} - \mathfrak{R}_{R8}\} \quad \text{for} \quad a \in \{L, R\}.$$

Kappa symmetry on the world volume of the D1-brane requires

$$\Gamma_0(\sqrt{\alpha}\Gamma_5 + \sqrt{1 - \alpha}\Gamma_8)\varepsilon = -i(B\varepsilon)^*, \quad (2.15)$$

with the pullback of the Killing spinor (2.7) given by

$$\varepsilon = \mathcal{M}^+(t, \sigma)(1 + i)\varepsilon^L + \mathcal{M}^-(t, \sigma)(1 + i)\varepsilon^R. \quad (2.16)$$

Here $K^+(\alpha)\varepsilon^a = 0$, for $a \in \{L, R\}$. This kappa symmetry condition reduces to

$$\Gamma_{12}\varepsilon^a = \Gamma_{35}\varepsilon^a = \Gamma_{68}\varepsilon^a. \quad (2.17)$$

The left- and right-moving supersymmetries preserved on the D1-brane world volume are parametrized by $\varepsilon^{ab\dot{\beta}\dot{\beta}}$, with labels satisfying $-b = \beta = \dot{\beta}$. The 2 + 2 supersymmetries

compatible with kappa symmetry on the world volume of the $Z = Z' = 0$ maximal D1 giant are thus generated by

$$\{\mathfrak{Q}_{a-++} \equiv \mathfrak{Q}_a, \mathfrak{Q}_{a+--} \equiv \mathfrak{Q}_a\}.$$

The generators of the symmetries of this D1-brane include all the generators $\{\mathfrak{H}_a, \mathfrak{Q}_a, \mathfrak{Q}_a\}$ of the $\mathfrak{su}(1|1)^2$ superalgebra of magnon excitations of the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain, which is centrally extended to $\mathfrak{su}(1|1)_\mathbb{C}^2$. The $Z = Z' = 0$ giant is thus totally aligned with the spin-chain vacuum \mathcal{Z} , and the boundary Lie algebra is the full superalgebra $\mathfrak{su}(1|1)_\mathbb{C}^2$.

Various giant gravitons and boundary Lie algebras may again be obtained by $SO(4)$ or (and) $SO(4)$ transformations on one (both) 3-spheres. The results are summarized in Table II. Here $\mathfrak{u}(1)_\pm$ are generated by one Cartan element $\mathfrak{H}_L \pm \mathfrak{H}_R$. In Sec. IV, we will show that totally supersymmetric, right and left half-supersymmetric and non-supersymmetric boundary algebras, $\mathfrak{su}(1|1)_\mathbb{C}^2$, $\mathfrak{u}(1)_L \oplus \mathfrak{su}(1|1)_R$, $\mathfrak{su}(1|1)_L \oplus \mathfrak{u}(1)_R$, $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$ and $\mathfrak{u}(1)_+$, provide integrable open boundary conditions for the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain; that is, we find reflection matrices which intertwine representations of these boundary Lie algebras and satisfy the boundary Yang-Baxter (reflection) equation. However, the boundary Lie algebras $\mathfrak{su}(1|1)_L$, $\mathfrak{su}(1|1)_R$, $\mathfrak{u}(1)_L$, $\mathfrak{u}(1)_R$ and $\mathfrak{u}(1)_-$ will not give rise to reflection matrices without enhanced boundary symmetry.

B. Maximal giant gravitons on $\text{AdS}_3 \times S^3 \times T^4$

We now give the details of the type-IIB supergravity background $\text{AdS}_3 \times S^3 \times T^4$ with pure R-R 3-form flux, and discuss D1 and D5-brane maximal giant gravitons based on Refs. [65,66].

1. $\text{AdS}_3 \times S^3 \times T^4$ with pure R-R flux

IIB supergravity solution.—The metric of the $\text{AdS}_3 \times S^3 \times T^4$ background is

$$\begin{aligned} ds^2 &= L^2(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2) \\ &+ L^2(d\theta^2 + \cos^2 \theta d\chi^2 + \sin^2 \theta d\phi^2) + \ell_i^2 d\xi_i^2. \end{aligned} \quad (2.18)$$

The 3-form field strength $F_{(3)} = dC_{(2)}$ is given by

$$\begin{aligned} F_{(3)} &= 2L^2 dt \wedge (\sinh \rho \cosh \rho d\rho) \wedge d\varphi \\ &+ 2L^2 (\sin \theta \cos \theta d\theta) \wedge d\chi \wedge d\phi \end{aligned} \quad (2.19)$$

in the case of pure R-R flux. The Hodge dual 7-form field strength $F_{(7)} = dC_{(6)} = *F_{(3)}$ is

TABLE II. Boundary Lie algebras for D1-brane maximal giant gravitons on AdS₃ × S³ × S^{3′} × S¹.

D1 giant	Bosonic generators	Supersymmetry generators	Boundary algebra
$Z = Z' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_a, \mathfrak{Q}_{L5} - \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{a-++}, \mathfrak{Q}_{a+--}\}$	$\mathfrak{su}(1 1)_C^2$
$Y = Z' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L + 2\alpha\mathfrak{Q}_{L5}, \mathfrak{H}_R, \mathfrak{Q}_{L5} + \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L++-}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{su}(1 1)_R$
$Z = Y' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L + 2\alpha\mathfrak{Q}_{L5}, \mathfrak{H}_R, \mathfrak{Q}_{L5} - \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L+++}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{su}(1 1)_R$
$\bar{Y} = Z' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L, \mathfrak{H}_R + 2\alpha\mathfrak{Q}_{R5}, \mathfrak{Q}_{L5} + \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L+--}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{su}(1 1)_L$
$Z = \bar{Y}' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L, \mathfrak{H}_R + 2\alpha\mathfrak{Q}_{R5}, \mathfrak{Q}_{L5} - \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L+--}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{su}(1 1)_L$
$\bar{Z} = Z' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L - \mathfrak{H}_R, \mathfrak{H}_R + 2\alpha\mathfrak{Q}_{R5}, \mathfrak{Q}_{L5} - \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{a-+-}, \mathfrak{Q}_{a+--}\}$	$\mathfrak{u}(1)_-$
$Z = \bar{Z}' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L - \mathfrak{H}_R, \mathfrak{H}_R + 2\alpha\mathfrak{Q}_{R5}, \mathfrak{Q}_{L5} - \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{a-+-}, \mathfrak{Q}_{a+--}\}$	$\mathfrak{u}(1)_-$
$Y = Y' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_a, \mathfrak{Q}_{L5} + \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L----}, \mathfrak{Q}_{L+++}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{u}(1)_L \oplus \mathfrak{su}(1 1)_R$
$\bar{Y} = \bar{Y}' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_a, \mathfrak{Q}_{L5} + \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L+--}, \mathfrak{Q}_{R----}, \mathfrak{Q}_{R++++}\}$	$\mathfrak{su}(1 1)_L \oplus \mathfrak{u}(1)_R$
$\bar{Z} = \bar{Z}' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_a, \mathfrak{Q}_{L5} - \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{a----}, \mathfrak{Q}_{a++++}\}$	$\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$
$\bar{Y} = Y' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L + \mathfrak{H}_R, \mathfrak{H}_R + 2\alpha\mathfrak{Q}_{R5}, \mathfrak{Q}_{L5} + \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L+++}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{u}(1)_+$
$Y = \bar{Y}' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L + \mathfrak{H}_R, \mathfrak{H}_R + 2\alpha\mathfrak{Q}_{R5}, \mathfrak{Q}_{L5} + \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L+++}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{u}(1)_+$
$Y = \bar{Z}' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L, \mathfrak{H}_R + 2\alpha\mathfrak{Q}_{R5}, \mathfrak{Q}_{L5} + \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L----}, \mathfrak{Q}_{L+++}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{u}(1)_L$
$\bar{Z} = Y' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L, \mathfrak{H}_R + 2\alpha\mathfrak{Q}_{R5}, \mathfrak{Q}_{L5} - \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L----}, \mathfrak{Q}_{L+++}, \mathfrak{Q}_{R-+-}, \mathfrak{Q}_{R+--}\}$	$\mathfrak{u}(1)_L$
$\bar{Y} = \bar{Z}' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L + 2\alpha\mathfrak{Q}_{L5}, \mathfrak{H}_R, \mathfrak{Q}_{L5} + \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L+++}, \mathfrak{Q}_{R----}, \mathfrak{Q}_{R++++}\}$	$\mathfrak{u}(1)_R$
$\bar{Z} = \bar{Y}' = 0$	$\{\mathfrak{F}_{a0}, \mathfrak{H}_L + 2\alpha\mathfrak{Q}_{L5}, \mathfrak{H}_R, \mathfrak{Q}_{L5} - \mathfrak{Q}_{R5}\}$	$\{\mathfrak{Q}_{L-+-}, \mathfrak{Q}_{L+++}, \mathfrak{Q}_{R----}, \mathfrak{Q}_{R++++}\}$	$\mathfrak{u}(1)_R$

$$\begin{aligned}
 F_{(7)} = & -2L^2 \ell_1 \ell_2 \ell_3 \ell_4 (\sin \theta \cos \theta d\theta) \wedge d\chi \wedge d\phi \\
 & \wedge d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \wedge d\xi_4 \\
 & - 2L^2 \ell_1 \ell_2 \ell_3 \ell_4 dt \wedge (\sinh \rho \cosh \rho d\rho) \\
 & \wedge d\phi \wedge d\xi_1 \wedge d\xi_2 \wedge d\xi_3 \wedge d\xi_4.
 \end{aligned} \tag{2.20}$$

These 3-form and 5-form fluxes couple to the D5- and D1-brane giant gravitons of Refs. [65,66] which have angular momentum on the 3-sphere. Maximal D1- and D5-brane giant gravitons provide integrable boundaries for open IIB superstrings on AdS₃ × S³ × T⁴.

Supersymmetry.—The supersymmetry variations of the dilatino and gravitino (2.4) are now written⁴ in terms of

$$\begin{aligned}
 \mathcal{F}_{(3)} = & \frac{2}{L} \Gamma^{012} (\mathbb{1} + \Gamma^{012} \Gamma^{345}) = \frac{4}{L} \Gamma^{012} K^+, \\
 \text{where } K^\pm = & \frac{1}{2} (\mathbb{1} \pm \Gamma^{012} \Gamma^{345}).
 \end{aligned} \tag{2.21}$$

⁴The vielbeins $\hat{E}^A = E_M^A dx^M$ are

$$\begin{aligned}
 \hat{E}^0 = L \cosh \rho dt, \quad \hat{E}^1 = L d\rho, \quad \hat{E}^2 = L \sinh \rho d\phi, \\
 \hat{E}^3 = L d\theta, \quad \hat{E}^4 = L \cos \theta d\chi, \quad \hat{E}^5 = L \sin \theta d\phi, \\
 \hat{E}^6 = \ell_1 d\xi_1, \quad \hat{E}^7 = \ell_2 d\xi_2, \quad \hat{E}^8 = \ell_3 d\xi_3, \quad \hat{E}^9 = \ell_4 d\xi_4,
 \end{aligned}$$

and the supercovariant derivatives are given by

$$\begin{aligned}
 \nabla_t = \partial_t + \frac{1}{2} \sinh \rho \Gamma_{01}, \quad \nabla_\rho = \partial_\rho, \\
 \nabla_\phi = \partial_\phi - \frac{1}{2} \cosh \rho \Gamma_{12}, \quad \nabla_\theta = \partial_\theta, \quad \nabla_\chi = \partial_\chi + \frac{1}{2} \sin \theta \Gamma_{34}, \\
 \nabla_\phi = \partial_\phi - \frac{1}{2} \cos \theta \Gamma_{35}, \quad \nabla_{\xi_i} = \partial_{\xi_i}.
 \end{aligned}$$

The *gravitino Killing-spinor equation* $\delta\Psi_M = 0$ implies a solution of the form

$$\varepsilon(x^M) = \mathcal{M}^+(x^M)(1+i)\varepsilon^L + \mathcal{M}^-(x^M)(1+i)\varepsilon^R, \tag{2.22}$$

decomposed into left and right-movers, with

$$\mathcal{M}^\pm(x^M) = e^{\pm\frac{1}{2}\rho\Gamma_{02}} e^{\frac{1}{2}(\phi\pm t)\Gamma_{12}} e^{\pm\frac{1}{2}\theta\Gamma_{45}} e^{\frac{1}{2}(\phi\mp\chi)\Gamma_{35}}. \tag{2.23}$$

The *dilatino Killing-spinor equation* $\delta\lambda = 0$ further implies $K^+\varepsilon^a = 0$, which halves the number of left- and right-moving degrees of freedom. The spinors ε^L and ε^R can be decomposed into eigenstates $\varepsilon^{Lb\beta\dot{\beta}}$ and $\varepsilon^{Rb\beta\dot{\beta}}$. This Killing spinor (2.22) can be seen as the $\alpha \rightarrow 1$ limit of (2.7).

The IIB supergravity background AdS₃ × S³ × T⁴ is thus invariant under eight left- and eight right-moving supersymmetry transformations, parametrized by $\varepsilon^{ab\beta\dot{\beta}}$, which satisfy

$$K^+ \varepsilon^{ab\beta\dot{\beta}} = 0 \quad \text{and hence} \quad K^- \varepsilon^{ab\beta\dot{\beta}} = \varepsilon^{ab\beta\dot{\beta}}. \tag{2.24}$$

These supersymmetry transformations are generated by the supercharges $\mathfrak{Q}_{ab\beta\dot{\beta}}$. The superisometry algebra is $\mathfrak{psu}(1, 1|2)_L \oplus \mathfrak{psu}(1, 1|2)_R \oplus \mathfrak{u}(1)^4$ with the details of the Lie superalgebra $\mathfrak{psu}(1, 1|2)$ given in Appendix C.

2. Maximal giant gravitons and boundary algebras

A closed IIB superstring on AdS₃ × S³ × T⁴ maps to a closed $\mathfrak{psu}(1, 1|2)^2$ spin-chain, as described in Sec. V, with magnon excitations transforming under a centrally extended $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_C^2$ superalgebra specified by our choice of vacuum \mathcal{Z} . An open IIB superstring ending on a D-brane maps to a $\mathfrak{psu}(1, 1|2)^2$ open spin-chain with the boundary Lie algebra determined by the D-brane

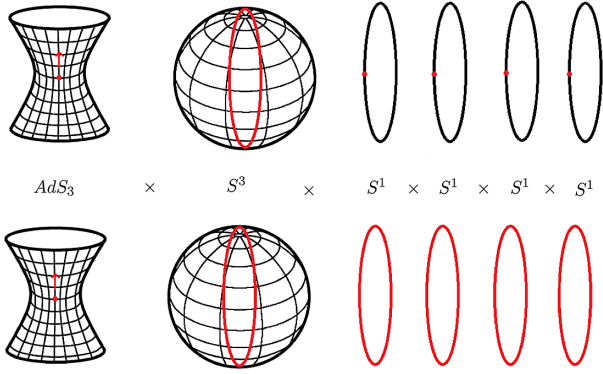


FIG. 3 (color online). The D1- and D5-brane maximal giant gravitons wrapping S^1 and $S^1 \times T^4$ in $AdS_3 \times S^3 \times T^4$.

symmetries which survive the choice of vacuum. We consider D1- and D5-brane maximal giant gravitons which yield boundary Lie algebras, extended to the coideal subalgebras in Sec. VI.

Maximal D1- and D5-brane giant gravitons.—Both the maximal giant gravitons wrap a great circle in S^3 , with the D1-brane pointlike in the T^4 and the D5-brane wrapping the entire T^4 space [65,66]. We consider these D1- and D5-brane giants simultaneously. As shown in Fig. 3, the bosonic isometry algebra breaks to

$$\begin{aligned} & (\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus (\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \\ & \subset \mathfrak{so}(2,2) \oplus \mathfrak{so}(4) \oplus \mathfrak{u}(1)^4 \end{aligned}$$

and

$$\begin{aligned} & (\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus (\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \oplus \mathfrak{u}(1)^4 \\ & \subset \mathfrak{so}(2,2) \oplus \mathfrak{so}(4) \oplus \mathfrak{u}(1)^4. \end{aligned}$$

Z = 0 giants.—Let us take the \mathbf{C}^2 embedding coordinates of the 3-sphere to be

$$\begin{aligned} (Z, Y) &= (x_1 + ix_2, x_3 + ix_4) \\ &= (L \cos \theta e^{i\chi}, L \sin \theta e^{i\phi}). \end{aligned} \quad (2.25)$$

The $Z = 0$ giant wraps the ϕ -circle in S^3 , and is obtained by setting $\rho = 0$ and $\theta = \frac{\pi}{2}$. The D1- and D5-brane world-volume coordinates are (t, ϕ) and (t, ϕ, ξ_i) . The world-volume metrics are

$$\begin{aligned} ds^2 &= -L^2 dt^2 + L^2 d\phi^2 \quad \text{and} \\ ds^2 &= -L^2 dt^2 + L^2 d\phi^2 + \ell_i^2 d\xi_i^2. \end{aligned}$$

The bosonic symmetries of these D-branes include time translations and rotations in AdS_3 , and rotations in S^3 in the x_1x_2 and x_3x_4 planes, generated by

$$\mathfrak{J}_{a0} \in \mathfrak{u}(1)_a \subset \mathfrak{su}(1,1)_a,$$

$$\mathfrak{L}_{a5} \in \mathfrak{u}(1)_a \subset \mathfrak{su}(2)_a, \quad \text{for } a \in \{\mathbf{L}, \mathbf{R}\},$$

with the splitting of $\mathfrak{so}(2,2)$ and $\mathfrak{so}(4)$ in Appendix C. The Cartan elements of $(\mathfrak{psu}(1,1)^2 \oplus \mathfrak{u}(1))^2$, denoted $\mathfrak{H}_a = -\mathfrak{J}_{a0} - \mathfrak{L}_{a5}$, are thus generators of bosonic symmetries of the $Z = 0$ giants.

Kappa symmetry on the world volumes of the D1- and D5-branes requires

$$\Gamma_{05}\epsilon = -i(B\epsilon)^* \quad \text{and} \quad \Gamma_{056789}\epsilon = -i(B\epsilon)^*, \quad (2.26)$$

respectively, with the pullback of the Killing spinor (2.22) to the world volume given by

$$\epsilon = \mathcal{M}^+(t, \phi)(1+i)\epsilon^{\mathbf{L}} + \mathcal{M}^-(t, \phi)(1+i)\epsilon^{\mathbf{R}} \quad (2.27)$$

in both cases. Here $K^+ \epsilon^a = 0$, for $a \in \{\mathbf{L}, \mathbf{R}\}$. This kappa symmetry condition reduces to

$$\Gamma_{12}\epsilon^a = \Gamma_{35}\epsilon^a. \quad (2.28)$$

Hence, $\epsilon^{ab\dot{\beta}}$ satisfies $-b = \dot{\beta}$, with $\dot{\beta}$ a free label. The $4 + 4$ supersymmetries compatible with kappa symmetry on the world volume of the D1- and D5-brane $Z = 0$ giants are generated by

$$\begin{aligned} \{\mathfrak{Q}_{a-++} \equiv \mathfrak{Q}_{a1}, \mathfrak{Q}_{a-+-} \equiv \mathfrak{Q}_{a2}, \\ \mathfrak{Q}_{a+--} \equiv \mathfrak{S}_{a1}, \mathfrak{Q}_{a++-} \equiv \mathfrak{S}_{a2}\}. \end{aligned}$$

The generators of the symmetries of these D1- and D5-branes include all the generators $\{\mathfrak{H}_a, \mathfrak{Q}_{ai}, \mathfrak{S}_{ai}\}$ of the $(\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1))^2$ superalgebra of magnon excitations of the $\mathfrak{psu}(1,1|2)^2$ spin-chain, centrally extended to $(\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1))_{\mathbf{C}}^2$. These $Z = 0$ giants are therefore aligned with the \mathcal{Z} vacuum of the spin-chain. We expect the boundary Lie algebra to be the full superalgebra $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbf{C}}^2$.

Y = 0 giants.—An $SO(4)$ rotation with $\theta_{13} = \theta_{24} = \frac{\pi}{2}$ takes Z to Y , corresponding to $U_{\mathbf{L}} = i\sigma_1$ and $U_{\mathbf{R}} = \mathbb{1}$. The $4 + 4$ supersymmetries on the world volume of the $Y = 0$ giants are generated by

$$\begin{aligned} \{\mathfrak{Q}_{\mathbf{L}---}, \mathfrak{Q}_{\mathbf{L}----}, \mathfrak{Q}_{\mathbf{L}+++}, \mathfrak{Q}_{\mathbf{L}++++}, \mathfrak{Q}_{\mathbf{R}+++} \equiv \mathfrak{Q}_{\mathbf{R}1}, \\ \mathfrak{Q}_{\mathbf{R}+-+} \equiv \mathfrak{Q}_{\mathbf{R}2}, \mathfrak{Q}_{\mathbf{R}+--} \equiv \mathfrak{S}_{\mathbf{R}1}, \mathfrak{Q}_{\mathbf{R}++-} \equiv \mathfrak{S}_{\mathbf{R}2}\}. \end{aligned}$$

The boundary Lie algebra $\mathfrak{u}(1)_{\mathbf{L}} \oplus [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbf{R}}$ is right half-supersymmetric.

$\bar{Y} = 0$ giants.—An $SO(4)$ transformation with $\theta_{13} = -\theta_{24} = \frac{\pi}{2}$ takes Z to \bar{Y} , and corresponds to $U_{\mathbf{L}} = \mathbb{1}$ and $U_{\mathbf{R}} = -i\sigma_1$. The $4 + 4$ supersymmetries of the $\bar{Y} = 0$ giants are generated by

TABLE III. Boundary Lie algebras for D1- and D5-brane maximal giant gravitons on AdS₃ × S³ × T⁴.

D1/D5 giant	Bosonic generators	Supersymmetry generators	Boundary algebra
Z = 0	{ $\mathfrak{F}_{a0}, \mathfrak{Q}_{a5}$ }	{ $\mathfrak{Q}_{a--\pm}, \mathfrak{Q}_{a++\pm}$ }	$[\mathfrak{psu}(1 1)^2 \oplus \mathfrak{u}(1)]_c^2$
Y = 0	{ $\mathfrak{F}_{a0}, \mathfrak{Q}_{a5}$ }	{ $\mathfrak{Q}_{L--\pm}, \mathfrak{Q}_{L++\pm}, \mathfrak{Q}_{R--\pm}, \mathfrak{Q}_{R++\pm}$ }	$\mathfrak{u}(1)_L \oplus [\mathfrak{psu}(1 1)^2 \oplus \mathfrak{u}(1)]_R$
$\bar{Y} = 0$	{ $\mathfrak{F}_{a0}, \mathfrak{Q}_{a5}$ }	{ $\mathfrak{Q}_{L-+\pm}, \mathfrak{Q}_{L+-\pm}, \mathfrak{Q}_{R--\pm}, \mathfrak{Q}_{R++\pm}$ }	$[\mathfrak{psu}(1 1)^2 \oplus \mathfrak{u}(1)]_L \oplus \mathfrak{u}(1)_R$
$\bar{Z} = 0$	{ $\mathfrak{F}_{a0}, \mathfrak{Q}_{a5}$ }	{ $\mathfrak{Q}_{L--\pm}, \mathfrak{Q}_{L++\pm}, \mathfrak{Q}_{R--\pm}, \mathfrak{Q}_{R++\pm}$ }	$\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$

$$\{\mathfrak{Q}_{L--\pm} \equiv \mathfrak{Q}_{L1}, \mathfrak{Q}_{L-+-\pm} \equiv \mathfrak{Q}_{L2}, \mathfrak{Q}_{L+-\pm} \equiv \mathfrak{Q}_{L1}, \\ \mathfrak{Q}_{L++\pm} \equiv \mathfrak{Q}_{L2}, \mathfrak{Q}_{R--\pm}, \mathfrak{Q}_{R-+-\pm}, \mathfrak{Q}_{R+-\pm}, \mathfrak{Q}_{R++\pm}\}.$$

The boundary Lie algebra $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_L \oplus \mathfrak{u}(1)_R$ is left half-supersymmetric.

$\bar{Z} = 0$ giants.—An $SO(4)$ rotation with $\theta_{24} = \pi$ takes Z to \bar{Z} , and corresponds to $U_L = i\sigma_1$ and $U_R = i\sigma_1$. The 4 + 4 supersymmetries of the $\bar{Z} = 0$ giants are generated by

$$\{\mathfrak{Q}_{L--\pm}, \mathfrak{Q}_{L---\pm}, \mathfrak{Q}_{L+-\pm}, \mathfrak{Q}_{L+++}, \\ \mathfrak{Q}_{R--\pm}, \mathfrak{Q}_{R---\pm}, \mathfrak{Q}_{R+-\pm}, \mathfrak{Q}_{R+++}\}.$$

The boundary Lie algebra $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$ is non-supersymmetric.

The above results for both D1- and D5-brane maximal giant gravitons are summarized in Table III. The totally supersymmetric, right and left half-supersymmetric, and non-supersymmetric boundary Lie algebras, $(\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1))_c^2$, $\mathfrak{u}(1)_L \oplus \mathfrak{psu}(1|1)_R^2 \oplus \mathfrak{u}(1)_R$, $\mathfrak{psu}(1|1)_L^2 \oplus \mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$ and $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$, are consistent with reflection matrices that are solutions of the boundary Yang-Baxter (reflection) equation for the $\mathfrak{psu}(1, 1|2)^2$ open spin-chain, as described in Sec. VI.

III. INTEGRABLE CLOSED $\mathfrak{d}(2, 1; \alpha)^2$ SPIN-CHAIN AND SCATTERING MATRICES IN AdS₃ × S³ × S^{3'} × S¹

The bosonic isometry group of the AdS₃ × S³ × S^{3'} × S¹ supergravity background is

$$SO(2, 2) \times SO(4) \times SO(4)' \times U(1),$$

whose Lie algebra splits into left- and right-movers

$$\mathfrak{so}(2, 2) \sim \mathfrak{su}(1, 1)_L \oplus \mathfrak{su}(1, 1)_R, \\ \mathfrak{so}(4) \sim \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R, \\ \mathfrak{so}(4)' \sim \mathfrak{su}(2)'_L \oplus \mathfrak{su}(2)'_R.$$

According to this splitting, the bosonic isometries can be rearranged into

$$[\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)']_L \\ \oplus [\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)']_R \oplus \mathfrak{u}(1),$$

which constitutes the bosonic part of the full superisometry algebra

$$\mathfrak{d}(2, 1; \alpha)_L \oplus \mathfrak{d}(2, 1; \alpha)_R \oplus \mathfrak{u}(1).$$

Massive excitations of the world sheet of a closed IIB superstring propagating on AdS₃ × S³ × S^{3'} × S¹ can be identified with the magnon excitations of an alternating double-row $\mathfrak{d}(2, 1; \alpha)^2$ closed spin-chain which transform under a centrally extended $\mathfrak{su}(1|1)_c^2$ algebra [53]. The left- and right-moving excitations⁵ decouple in the weak coupling limit. This section contains a review based on Refs. [20,22,53] of this integrable $\mathfrak{d}(2, 1; \alpha)^2$ closed spin-chain and the S -matrix describing two-magnon scattering.

A. $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain with $\mathfrak{su}(1|1)^2$ excitations

1. Single-row $\mathfrak{d}(2, 1; \alpha)$ closed spin-chain with $\mathfrak{su}(1|1)$ excitations

Symmetry generators.—The $\mathfrak{d}(2, 1; \alpha)$ superalgebra shown in Appendix B has bosonic generators

$$\mathfrak{F}_0, \mathfrak{F}_b \in \mathfrak{su}(1, 1), \quad \mathfrak{Q}_\beta, \mathfrak{Q}_5 \in \mathfrak{su}(2), \quad \mathfrak{R}_\beta, \mathfrak{R}_8 \in \mathfrak{su}(2)'$$

of $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)'$, and fermionic generators $\mathfrak{Q}_{b\beta\dot{\beta}}$ labeled by $b, \beta, \dot{\beta} = \pm$ indices.

Sites.—Two neighboring sites, called odd and even, in the alternating single-row $\mathfrak{d}(2, 1; \alpha)$ spin-chain are the modules

$$M^{(\alpha)} \equiv M\left(-\frac{\alpha}{2}, \frac{1}{2}, 0\right) = \text{span}_{\mathbb{C}}\{|\phi_\beta^{(n)}\rangle, |\psi_\beta^{(n)}\rangle\}, \\ M'^{(1-\alpha)} \equiv M\left(-\frac{1-\alpha}{2}, 0, \frac{1}{2}\right) = \text{span}_{\mathbb{C}}\{|\phi_\beta^{(n)}\rangle, |\psi_\beta^{(n)}\rangle\}.$$

States at these sites are vectors transforming under half-BPS representations of $\mathfrak{d}(2, 1; \alpha)$ described in Appendix B. The odd and even sites together form the module $M = M^{(\alpha)} \otimes M'^{(1-\alpha)}$, which carries a quarter-BPS

⁵These left- and right-movers are not related to the actual left- and right-moving (clockwise and counterclockwise) modes of a closed string, but are rather string excitations charged under generators of different copies of $\mathfrak{d}(2, 1; \alpha)$.

representation of $\mathfrak{d}(2, 1; \alpha)$. The vacuum state is identified with the highest weight vector

$$|\mathcal{Z}\rangle = |\phi_+^{(0)} \phi_+^{\prime(0)}\rangle, \quad (3.1)$$

and the four fundamental excitations $|\varphi^r\rangle = \{|\phi\rangle, |\psi\rangle, |\phi'\rangle, |\psi'\rangle\}$, with $r = 1 \dots 4$, are defined by

$$\begin{aligned} |\phi\rangle &= |\phi_-^{(0)} \phi_+^{\prime(0)}\rangle, & |\psi\rangle &= |\psi_+^{(0)} \phi_+^{\prime(0)}\rangle, \\ |\phi'\rangle &= |\phi_+^{(0)} \phi_-^{\prime(0)}\rangle, & |\psi'\rangle &= |\phi_+^{(0)} \psi_+^{\prime(0)}\rangle. \end{aligned} \quad (3.2)$$

Here $\{|\phi\rangle, |\psi\rangle\}$ and $\{|\phi'\rangle, |\psi'\rangle\}$ span modules of a closed $\mathfrak{su}(1|1)$ subalgebra⁶ with fermionic generators $\mathfrak{Q} \equiv \mathfrak{Q}_{-++}$ and $\mathfrak{S} \equiv \mathfrak{Q}_{+--}$, and bosonic Cartan element $\mathfrak{H} = \{\mathfrak{Q}, \mathfrak{S}\} = -\mathfrak{J}_0 - \alpha \mathfrak{L}_5 - (1 - \alpha) \mathfrak{R}_8$ the magnon Hamiltonian. These unprimed and primed modules are associated with energies α and $1 - \alpha$. We note this $\mathfrak{su}(1|1)$ subalgebra can be extended to a $\mathfrak{u}(1|1) = \mathfrak{u}(1) \times \mathfrak{su}(1|1)$ algebra

$$\{\mathfrak{Q}, \mathfrak{S}\} = \mathfrak{H}, \quad [\mathfrak{X}, \mathfrak{Q}] = -\frac{1}{2} \mathfrak{Q}, \quad [\mathfrak{X}, \mathfrak{S}] = \frac{1}{2} \mathfrak{S}$$

by the inclusion of $\mathfrak{X} = -\frac{1}{2} \mathfrak{L}_5 - \frac{1}{2} \mathfrak{R}_8$, which does not annihilate the ground state.

Spin-chain.—The alternating single-row $\mathfrak{d}(2, 1; \alpha)$ spin-chain with $2J$ sites can be identified with the module $M^{\otimes J} = (M^{(\alpha)} \otimes M^{(1-\alpha)})^{\otimes J}$. The spin-chain vacuum and fundamental excitations are

$$|0\rangle = |\mathcal{Z}^J\rangle, \quad |\varphi_{(n)}^r\rangle = |\mathcal{Z}^{n-1} \varphi^r \mathcal{Z}^{J-n}\rangle. \quad (3.3)$$

We construct vectors in momentum space using the standard approach to obtain low-lying single-magnon excitations

$$|\varphi_p^r\rangle = \sum_{n=1}^J e^{ipn} |\varphi_{(n)}^r\rangle. \quad (3.4)$$

Here $\{|\phi_p\rangle, |\psi_p\rangle\}$ and $\{|\phi'_p\rangle, |\psi'_p\rangle\}$ are modules of $\mathfrak{su}(1|1)$ with energies α and $1 - \alpha$. The action of the fermionic generators on the unprimed single-magnon excitations is given by

$$\begin{aligned} \mathfrak{Q}|\phi_p\rangle &= \sqrt{\alpha} |\psi_p\rangle, & \mathfrak{S}|\phi_p\rangle &= 0, \\ \mathfrak{Q}|\psi_p\rangle &= 0, & \mathfrak{S}|\psi_p\rangle &= \sqrt{\alpha} |\phi_p\rangle, \end{aligned} \quad (3.5)$$

where also

⁶There are a number of other closed subsectors (see Sec. VI in Ref. [40]).

$$\begin{aligned} \mathfrak{X}|0\rangle &= -\frac{J}{2} |0\rangle, & \mathfrak{X}|\phi_p\rangle &= \left(-\frac{J}{2} + \frac{1}{2}\right) |\phi_p\rangle, \\ \mathfrak{X}|\psi_p\rangle &= -\frac{J}{2} |\psi_p\rangle, \end{aligned}$$

and similarly for the primed single-magnon excitations with $\alpha \rightarrow 1 - \alpha$. Multimagnon excitations are obtained using the generalized standard approach

$$\begin{aligned} &|\varphi_{p_1}^{r_1} \varphi_{p_2}^{r_2} \dots \varphi_{p_k}^{r_k}\rangle \\ &= \sum_{1 \leq n_1 < n_2 < \dots < n_k \leq J} e^{i(p_1 x_{n_1} + p_2 x_{n_2} + \dots + p_k x_{n_k})} \\ &\times |\mathcal{Z}^{n_1-1} \varphi_{(n_1)}^{r_1} \mathcal{Z}^{n_2-n_1-1} \varphi_{(n_2)}^{r_2} \dots \varphi_{(n_k)}^{r_k} \mathcal{Z}^{J-n_k}\rangle \end{aligned}$$

for $2 \leq k \ll J$. The individual excitations $\varphi_{(n_i)}^{r_i}$ are known as impurities or fields in the spin-chain, and are assumed to be well separated.

2. Double-row $\mathfrak{d}(2, 1; \alpha)^2$ closed spin-chain with $\mathfrak{su}(1|1)^2$ excitations

The alternating double-row $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain is made up of left- and right-moving $\mathfrak{d}(2, 1; \alpha)_L$ and $\mathfrak{d}(2, 1; \alpha)_R$ spin-chains which decouple in the weak coupling limit.

Sites.—Odd and even sites of the left- and right-moving spin-chains together form the module $M_L \otimes M_R = M_L^{(\alpha)} \otimes M_L^{(1-\alpha)} \otimes M_R^{(\alpha)} \otimes M_R^{(1-\alpha)}$. The ground state and fundamental excitations are

$$|\mathcal{Z}\rangle = \left| \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix} \right\rangle, \quad |\varphi^r\rangle = \left| \begin{pmatrix} \varphi_L^r \\ \mathcal{Z}_R \end{pmatrix} \right\rangle, \quad |\bar{\varphi}^r\rangle = \left| \begin{pmatrix} \mathcal{Z}_L \\ \varphi_R^r \end{pmatrix} \right\rangle, \quad (3.6)$$

which transform under the $\mathfrak{u}(1|1)^2$ algebra

$$\begin{aligned} \{\mathfrak{Q}_a, \mathfrak{S}_b\} &= \mathfrak{H}_a \delta_{ab}, & [\mathfrak{X}_a, \mathfrak{Q}_b] &= -\frac{1}{2} \mathfrak{Q}_a \delta_{ab}, \\ [\mathfrak{X}_a, \mathfrak{S}_b] &= \frac{1}{2} \mathfrak{S}_a \delta_{ab}, \end{aligned} \quad (3.7)$$

with $\mathbf{a}, \mathbf{b} \in \{L, R\}$. Notice that $\mathfrak{X} = \mathfrak{X}_L - \mathfrak{X}_R$ now does annihilate the ground state, although \mathfrak{X}_L and \mathfrak{X}_R individually do not. We define $\mathfrak{H} = \mathfrak{H}_L + \mathfrak{H}_R$ and $\mathfrak{M} = \mathfrak{H}_L - \mathfrak{H}_R$, with \mathfrak{H} the magnon Hamiltonian. We do not need to consider excitations for which the left- and right-moving excitations φ_L^r and φ_R^r coincide, since we focus on well-separated excitations in the $J \rightarrow \infty$ limit.

Spin-chain.—The alternating double-row $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain can be identified with the module $(M_L \otimes M_R)^{\otimes J}$. The ground state is

$$|0\rangle = |\mathcal{Z}^J\rangle = \left| \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^J \right\rangle, \quad (3.8)$$

and left- and right-moving fundamental excitations are

$$\begin{aligned} |\varphi_{(n)}^r\rangle &= |\mathcal{Z}^{n-1}\varphi^r\mathcal{Z}^{J-n}\rangle = \left| \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^{n-1} \begin{pmatrix} \varphi_L^r \\ \mathcal{Z}_R \end{pmatrix} \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^{J-n} \right\rangle, \\ |\bar{\varphi}_{(n)}^r\rangle &= |\mathcal{Z}^{n-1}\bar{\varphi}^r\mathcal{Z}^{J-n}\rangle = \left| \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^{n-1} \begin{pmatrix} \mathcal{Z}_L \\ \varphi_R^r \end{pmatrix} \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^{J-n} \right\rangle, \end{aligned} \quad (3.9)$$

with low-lying left- and right-moving single-magnon excitations

$$|\varphi_p^r\rangle = \sum_{n=1}^J e^{ipn} |\varphi_{(n)}^r\rangle, \quad |\bar{\varphi}_p^r\rangle = \sum_{n=1}^J e^{ipn} |\bar{\varphi}_{(n)}^r\rangle. \quad (3.10)$$

The unprimed and primed left- and right-moving magnon excitations $\{|\phi_p\rangle, |\psi_p\rangle\}$ and $\{|\bar{\phi}_p\rangle, |\bar{\psi}_p\rangle\}$, and $\{|\phi'_p\rangle, |\psi'_p\rangle\}$ and $\{|\bar{\phi}'_p\rangle, |\bar{\psi}'_p\rangle\}$ have energies α and $1-\alpha$ of the magnon Hamiltonian \mathfrak{H} . The left/right-movers have mass eigenvalues $\pm\alpha$ and $\pm(1-\alpha)$ of \mathfrak{M} . The nontrivial action of the fermionic generators of the $\mathfrak{su}(1|1)^2$ algebra on these left- and right-moving magnon excitations is

$$\begin{aligned} \mathfrak{Q}_L|\phi_p\rangle &= \sqrt{\alpha}|\psi_p\rangle, & \mathfrak{E}_L|\psi_p\rangle &= \sqrt{\alpha}|\phi_p\rangle, \\ \mathfrak{Q}_R|\bar{\phi}_p\rangle &= \sqrt{\alpha}|\bar{\psi}_p\rangle, & \mathfrak{E}_R|\bar{\psi}_p\rangle &= \sqrt{\alpha}|\bar{\phi}_p\rangle, \end{aligned} \quad (3.11)$$

with the nontrivial action of the additional $\mathfrak{u}(1)$ generator of the $\mathfrak{u}(1) \times \mathfrak{su}(1|1)^2$ algebra, which annihilates the ground state, given by

$$\mathfrak{X}|\phi_p\rangle = \frac{1}{2}|\phi_p\rangle, \quad \mathfrak{X}|\bar{\phi}_p\rangle = -\frac{1}{2}|\bar{\phi}_p\rangle,$$

and similarly for the primed left- and right-moving magnon excitations with $\alpha \rightarrow 1-\alpha$. The generalization to multi-magnon states is again straightforward.

B. $\mathfrak{d}(2,1;\alpha)^2$ spin-chain with centrally extended $\mathfrak{su}(1|1)_\mathbb{C}^2$ excitations

We have so far considered only the weak-coupling limit of the $\mathfrak{d}(2,1;\alpha)^2$ spin-chain in which the left- and right-moving excitations decouple. Beyond this regime, interactions must be taken into account by the introduction of a centrally extended $\mathfrak{su}(1|1)^2$ algebra which links the two otherwise independent $\mathfrak{d}(2,1;\alpha)_L$ and $\mathfrak{d}(2,1;\alpha)_R$ spin-chains. Let us denote this extended algebra by $\mathfrak{su}(1|1)_\mathbb{C}^2$.

The algebra $\mathfrak{su}(1|1)_\mathbb{C}^2$ is generated by the fermionic generators \mathfrak{Q}_a , \mathfrak{E}_a , and bosonic generators \mathfrak{H}_a and the central elements \mathfrak{P} , \mathfrak{P}^\dagger , where $a \in \{L, R\}$, satisfying⁷

$$\begin{aligned} \{\mathfrak{Q}_a, \mathfrak{E}_b\} &= \delta_{ab}\mathfrak{H}_a, & \{\mathfrak{Q}_L, \mathfrak{Q}_R\} &= \mathfrak{P}, \\ \{\mathfrak{E}_L, \mathfrak{E}_R\} &= \mathfrak{P}^\dagger, \end{aligned} \quad (3.12)$$

with the remaining relations being trivial. We may further extend this algebra by the inclusion of the element \mathfrak{X} , which annihilates the ground state and has nontrivial commutation relations:

$$\begin{aligned} [\mathfrak{X}, \mathfrak{Q}_L] &= -\frac{1}{2}\mathfrak{Q}_L, & [\mathfrak{X}, \mathfrak{E}_L] &= \frac{1}{2}\mathfrak{E}_L, \\ [\mathfrak{X}, \mathfrak{Q}_R] &= \frac{1}{2}\mathfrak{Q}_R, & [\mathfrak{X}, \mathfrak{E}_R] &= -\frac{1}{2}\mathfrak{E}_R. \end{aligned}$$

A dynamic $\mathfrak{d}(2,1;\alpha)^2$ spin-chain with $\mathfrak{su}(1|1)_\mathbb{C}^2$ -symmetric massive excitations was constructed in Ref. [20] and, subsequently, a nondynamic $\mathfrak{d}(2,1;\alpha)^2$ spin-chain with an additional Hopf algebra structure was introduced in Ref. [21]. We now briefly review these constructions.

1. Finite spin-chain with length-changing effects

Here we allow the additional bosonic central elements \mathfrak{P} and \mathfrak{P}^\dagger of the $\mathfrak{su}(1|1)_\mathbb{C}^2$ algebra to have a length-changing effect on the spin-chain. Let us introduce some additional notation: \mathcal{Z}^+ and \mathcal{Z}^- denote the insertion or removal of a vacuum state (if possible) at the specific spin-chain site in the left-moving magnon excitation $|\varphi_p^r\rangle$:

$$\begin{aligned} |\mathcal{Z}^\pm\varphi_p^r\rangle &= \sum_{n=1}^J e^{ipn} |\mathcal{Z}^{n-1\pm 1}\varphi^r\mathcal{Z}^{J-n}\rangle, \\ |\varphi_p^r\mathcal{Z}^\pm\rangle &= \sum_{n=1}^J e^{ipn} |\mathcal{Z}^{n-1}\varphi^r\mathcal{Z}^{J-n\pm 1}\rangle, \end{aligned}$$

where we define $|\mathcal{Z}^{-1}\varphi^r \dots\rangle \equiv |\varphi^r \dots\rangle$ and $|\dots\varphi^r\mathcal{Z}^{-1}\rangle \equiv |\dots\varphi^r\rangle$ (that is, if there is no vacuum state before or after the field φ^r to remove, then the state remains unchanged). Imposing periodic boundary conditions $e^{ipJ}|\mathcal{Z}^J\varphi\rangle = |\varphi\mathcal{Z}^J\rangle$ for a closed spin-chain now gives

$$|\varphi_p^r\mathcal{Z}^\pm\rangle = e^{\pm ip} |\mathcal{Z}^\pm\varphi_p^r\rangle, \quad (3.13)$$

and similarly for right-moving magnon excitations. For two left-moving magnons, we define

⁷Setting $e_1 = \mathfrak{Q}_L$, $e_2 = \mathfrak{E}_R$, $f_1 = \mathfrak{E}_L$, $f_2 = \mathfrak{Q}_R$, $h_1 = \mathfrak{H}_L$, $h_2 = \mathfrak{H}_R$, $k_1 = \mathfrak{P}$, $k_2 = \mathfrak{P}^\dagger$, the anticommutation relations (3.12) can be written in a more compact form as follows: $\{e_i, f_j\} = \delta_{ij}h_i + (1-\delta_{ij})k_i$ (see Sec. II in Ref. [51]).

$$\begin{aligned}
 |\mathcal{Z}^\pm \varphi_p^r \varphi_q^s\rangle &= \sum_{1 \leq n < m \leq J} e^{i(pn+qm)} |\mathcal{Z}^{n-1 \pm 1} \varphi^r \mathcal{Z}^{m-n-1} \varphi^s \mathcal{Z}^{J-m}\rangle, \\
 |\varphi_p^r \mathcal{Z}^\pm \varphi_q^s\rangle &= \sum_{1 \leq n < m \leq J} e^{i(pn+qm)} |\mathcal{Z}^{n-1} \varphi^r \mathcal{Z}^{m-n-1 \pm 1} \varphi^s \mathcal{Z}^{J-m}\rangle,
 \end{aligned}$$

and hence, using the periodic boundary conditions,

$$|\varphi_p^r \mathcal{Z}^\pm \varphi_q^s\rangle = e^{\pm ip} |\mathcal{Z}^\pm \varphi_p^r \varphi_q^s\rangle, \quad (3.14)$$

and similarly for two right-moving magnons, or for left- and right-moving magnon excitations.

Single-magnon excitations.—The action of the fermionic generators of the $\mathfrak{su}(1|1)_\mathbb{C}^2$ algebra on the left- and right-moving magnon excitations was proposed in Ref. [20]. Here we will consider a slightly different action,⁸ with the insertion or removal of \mathcal{Z} from the left side:

$$\begin{aligned}
 \mathfrak{Q}_L |\phi_p\rangle &= a_p |\psi_p\rangle, & \mathfrak{E}_L |\psi_p\rangle &= b_p |\phi_p\rangle, \\
 \mathfrak{Q}_R |\psi_p\rangle &= c_p |\mathcal{Z}^+ \phi_p\rangle, & \mathfrak{E}_R |\phi_p\rangle &= d_p |\mathcal{Z}^- \psi_p\rangle, \\
 \mathfrak{Q}_R |\bar{\phi}_p\rangle &= \bar{a}_p |\bar{\psi}_p\rangle, & \mathfrak{E}_R |\bar{\psi}_p\rangle &= \bar{b}_p |\bar{\phi}_p\rangle, \\
 \mathfrak{Q}_L |\bar{\psi}_p\rangle &= \bar{c}_p |\mathcal{Z}^+ \bar{\phi}_p\rangle, & \mathfrak{E}_L |\bar{\phi}_p\rangle &= \bar{d}_p |\mathcal{Z}^- \bar{\psi}_p\rangle,
 \end{aligned} \quad (3.15)$$

with the unbarred and barred parameters for left- and right-movers. The action of the central elements is deduced from the algebra (3.12). The energy eigenvalues of $\mathfrak{H} = \mathfrak{H}_L + \mathfrak{H}_R$ of a left- or right-moving magnon excitation are

$$E_p = a_p b_p + c_p d_p, \quad \bar{E}_p = \bar{a}_p \bar{b}_p + \bar{c}_p \bar{d}_p,$$

while the eigenvalues of $\mathfrak{M} = \mathfrak{H}_L - \mathfrak{H}_R$, related to the masses of the magnon excitations, are

$$m = a_p b_p - c_p d_p, \quad -\bar{m} = -(\bar{a}_p \bar{b}_p - \bar{c}_p \bar{d}_p),$$

taken to be independent of the momentum p (but dependent on the unprimed or primed flavor). Here $m = \bar{m} = \alpha$ and $m' = \bar{m}' = 1 - \alpha$. Now, for one physical left-moving magnon excitation, we require the eigenvalues \mathfrak{P} and \mathfrak{P}^\dagger to vanish to return to a representation of the nonextended $\mathfrak{su}(1|1)^2$ symmetry preserved by the vacuum. This leads to the conditions $a_p c_p = b_p d_p = 0$ and $\bar{a}_p \bar{c}_p = \bar{b}_p \bar{d}_p = 0$ with solution $c_p = d_p = \bar{c}_p = \bar{d}_p = 0$ resulting again in decoupled spin-chains.

Two-magnon excitations.—Let us now consider two left-moving magnon excitations $|\varphi_p^r \varphi_q^s\rangle$ of momenta p and q .

⁸This action is equivalent to that of Ref. [20] with a rescaling $c_p \rightarrow e^{-ip} c_p$ and $d_p \rightarrow e^{ip} d_p$ for the left-moving excitations, and a similar rescaling of $\bar{c}_p \rightarrow e^{-ip} \bar{c}_p$ and $\bar{d}_p \rightarrow e^{ip} \bar{d}_p$ for the right-moving excitations, as can be seen from (3.13), and will be convenient when we subsequently consider a semi-infinite open spin-chain with a boundary on the right.

The nontrivial action of the fermionic generators of the $\mathfrak{su}(1|1)_\mathbb{C}^2$ algebra is

$$\begin{aligned}
 \mathfrak{Q}_a |\phi_p \phi_q\rangle &= \delta_{aL} a_p |\psi_p \phi_q\rangle + \delta_{aL} a_q |\phi_p \psi_q\rangle, \\
 \mathfrak{Q}_a |\phi_p \psi_q\rangle &= \delta_{aL} a_p |\psi_p \phi_q\rangle + \delta_{aR} e^{ip} c_q |\mathcal{Z}^+ \phi_p \phi_q\rangle, \\
 \mathfrak{Q}_a |\psi_p \phi_q\rangle &= \delta_{aR} c_p |\mathcal{Z}^+ \phi_p \phi_q\rangle - \delta_{aL} a_q |\psi_p \psi_q\rangle, \\
 \mathfrak{Q}_a |\psi_p \psi_q\rangle &= \delta_{aR} c_p |\mathcal{Z}^+ \phi_p \psi_q\rangle - \delta_{aR} e^{ip} c_q |\mathcal{Z}^+ \psi_p \phi_q\rangle, \\
 \mathfrak{E}_a |\phi_p \phi_q\rangle &= \delta_{aR} d_p |\mathcal{Z}^- \psi_p \phi_q\rangle + \delta_{aR} e^{-ip} d_q |\mathcal{Z}^- \phi_p \psi_q\rangle, \\
 \mathfrak{E}_a |\phi_p \psi_q\rangle &= \delta_{aR} d_p |\mathcal{Z}^- \psi_p \phi_q\rangle + \delta_{aL} b_q |\phi_p \phi_q\rangle, \\
 \mathfrak{E}_a |\psi_p \phi_q\rangle &= \delta_{aL} b_p |\phi_p \phi_q\rangle - \delta_{aR} e^{-ip} d_q |\mathcal{Z}^- \psi_p \psi_q\rangle, \\
 \mathfrak{E}_a |\psi_p \psi_q\rangle &= \delta_{aL} b_p |\phi_p \psi_q\rangle - \delta_{aL} b_q |\psi_p \phi_q\rangle,
 \end{aligned} \quad (3.16)$$

and similarly for primed left-moving magnon excitations. The action on two right-moving magnon excitations $|\bar{\varphi}_p^r \bar{\varphi}_q^s\rangle$, or a left- and a right-mover, $|\varphi_p^r \bar{\varphi}_q^s\rangle$ or $|\bar{\varphi}_p^r \varphi_q^s\rangle$, is obtained by interchanging $L \leftrightarrow R$ indices and using barred notation for the action on the right-movers. The energy eigenvalue of the magnon Hamiltonian \mathfrak{H} is $E_p + E_q$. Now \mathfrak{P} and \mathfrak{P}^\dagger must annihilate physical two-magnon states, which implies

$$\begin{aligned}
 a_p c_p + e^{ip} a_q c_q &= h(1 - e^{i(p+q)}) = 0, \\
 b_p d_p + e^{-ip} b_q d_q &= h(1 - e^{-i(p+q)}) = 0,
 \end{aligned}$$

and hence $e^{i(p+q)} = 1$, with our choice of parameters

$$\begin{aligned}
 a_p &= \sqrt{h} \eta_p, & b_p &= \sqrt{h} \eta_p, & c_p &= -\sqrt{h} \frac{i \eta_p}{x_p^-}, \\
 d_p &= \sqrt{h} \frac{i \eta_p}{x_p^+}, & \text{with } \eta_p^2 &= i(x_p^- - x_p^+).
 \end{aligned} \quad (3.17)$$

Exactly the same choice of barred parameters \bar{a}_p , \bar{b}_p , \bar{c}_p and \bar{d}_p must apply for the right-moving magnon representation. Here x_p^\pm are the Zhukovski variables which satisfy

$$\frac{x_p^+}{x_p^-} = e^{ip}, \quad \left(x_p^+ + \frac{1}{x_p^+}\right) - \left(x_p^- + \frac{1}{x_p^-}\right) = \frac{im}{h}. \quad (3.18)$$

The energy of a single-magnon excitation of momentum p and mass m is given by

$$E_p = \sqrt{m^2 + 16h^2 \sin^2 \frac{p}{2}}. \quad (3.19)$$

2. Infinite spin-chain with Hopf algebra structure

In the $J \rightarrow \infty$ infinite spin-chain limit, we can drop the \mathcal{Z}^\pm symbols on the left, and thus obtain one- and two-magnon representations of the $\mathfrak{su}(1|1)_\mathbb{C}^2$ algebra. The

length-changing effects can be encoded in a \mathcal{U} -braided Hopf algebra structure for the $\mathfrak{su}(1|1)_\mathbb{C}^2$ algebra, similar to that of Appendix B in Ref. [21].

Single-magnon representations.—We want to rewrite the action (3.15) in terms of matrix representations of the $\mathfrak{su}(1|1)_\mathbb{C}^2$ algebra. Let us introduce vector spaces

$$V_p = \text{span}_{\mathbb{C}}\{|\phi_p\rangle, |\psi_p\rangle\}, \quad V'_p = \text{span}_{\mathbb{C}}\{|\phi'_p\rangle, |\psi'_p\rangle\}$$

for the left-moving magnons, and

$$\bar{V}_p = \text{span}_{\mathbb{C}}\{|\bar{\phi}_p\rangle, |\bar{\psi}_p\rangle\}, \quad \bar{V}'_p = \text{span}_{\mathbb{C}}\{|\bar{\phi}'_p\rangle, |\bar{\psi}'_p\rangle\}$$

for the right-moving magnons. We can identify these vector spaces with $\mathbb{C}^{1|1}$ in the natural way. Now the action (3.15) can be defined in terms of the usual supermatrices $\mathbb{E}_{ij} \in \text{End}(\mathbb{C}^{1|1})$ which span the \mathbf{Z}_2 -graded $\mathfrak{gl}(1|1)$ Lie superalgebra,⁹ and the identity matrix is $\mathbb{1} = \mathbb{E}_{11} + \mathbb{E}_{22}$. The left-moving representation $\pi_p: \mathfrak{su}(1|1)_\mathbb{C}^2 \rightarrow \text{End}(\mathbb{C}^{1|1})$ is

$$\begin{aligned} \pi_p(\mathfrak{Q}_L) &= a_p \mathbb{E}_{21}, & \pi_p(\mathfrak{Q}_R) &= c_p \mathbb{E}_{12}, \\ \pi_p(\mathfrak{S}_L) &= b_p \mathbb{E}_{12}, & \pi_p(\mathfrak{S}_R) &= d_p \mathbb{E}_{21}, \\ \pi_p(\mathfrak{H}_L) &= a_p b_p \mathbb{1}, & \pi_p(\mathfrak{H}_R) &= c_p d_p \mathbb{1}, \\ \pi_p(\mathfrak{P}) &= a_p c_p \mathbb{1}, & \pi_p(\mathfrak{P}^\dagger) &= b_p d_p \mathbb{1}, \end{aligned} \quad (3.20)$$

with the parameters given by (3.17). The right-moving representation $\bar{\pi}_p: \mathfrak{su}(1|1)_\mathbb{C}^2 \rightarrow \text{End}(\mathbb{C}^{1|1})$ is

⁹The supermatrices $\mathbb{E}_{ij} \in \text{End}(\mathbb{C}^{1|1})$ have matrix elements $(\mathbb{E}_{ij})_{ab} = \delta_{ia} \delta_{jb}$, with \mathbb{E}_{11} and \mathbb{E}_{22} even elements of degree 0, and \mathbb{E}_{12} and \mathbb{E}_{21} odd elements of degree 1. These supermatrices satisfy

$$[\mathbb{E}_{ij}, \mathbb{E}_{kl}] = \delta_{jk} \mathbb{E}_{il} - (-1)^{\text{deg } \mathbb{E}_{ij} \text{ deg } \mathbb{E}_{kl}} \delta_{il} \mathbb{E}_{kj},$$

with $[\cdot, \cdot]$ the supercommutator. The multiplication of the tensor product of supermatrices is

$$\begin{aligned} (X \otimes Y)(Z \otimes W) &= (-1)^{\text{deg } Z \text{ deg } Y} XZ \otimes YW, \\ \text{deg}(X \otimes Y) &= \text{deg } X + \text{deg } Y \end{aligned}$$

for any $X, Y, Z, W \in \mathfrak{gl}(1|1)$ and, in particular,

$$(\mathbb{E}_{ij} \otimes \mathbb{E}_{kl})(\mathbb{E}_{pr} \otimes \mathbb{E}_{st}) = (-1)^{\text{deg } \mathbb{E}_{kl} \text{ deg } \mathbb{E}_{pr}} \delta_{jp} \delta_{ls} \mathbb{E}_{ir} \otimes \mathbb{E}_{kt}.$$

The graded permutation operator $\mathbb{P} \in \text{End}(\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1})$ is

$$\begin{aligned} \mathbb{P} &= \sum_{i,j} (-1)^{(j-1)} \mathbb{E}_{ij} \otimes \mathbb{E}_{ji} \\ &= \mathbb{E}_{11} \otimes \mathbb{E}_{11} - \mathbb{E}_{12} \otimes \mathbb{E}_{21} + \mathbb{E}_{21} \otimes \mathbb{E}_{12} - \mathbb{E}_{22} \otimes \mathbb{E}_{22}. \end{aligned}$$

The generalization to supermatrices in $\text{End}(\mathbb{C}^{2|2})$ is straightforward.

$$\begin{aligned} \bar{\pi}_p(\mathfrak{Q}_R) &= a_p \mathbb{E}_{\bar{2}\bar{1}}, & \bar{\pi}_p(\mathfrak{Q}_L) &= c_p \mathbb{E}_{\bar{1}\bar{2}}, \\ \bar{\pi}_p(\mathfrak{S}_R) &= b_p \mathbb{E}_{\bar{1}\bar{2}}, & \bar{\pi}_p(\mathfrak{S}_L) &= d_p \mathbb{E}_{\bar{2}\bar{1}}, \\ \bar{\pi}_p(\mathfrak{H}_R) &= a_p b_p \mathbb{1}, & \bar{\pi}_p(\mathfrak{H}_L) &= c_p d_p \mathbb{1}, \\ \bar{\pi}_p(\mathfrak{P}) &= a_p c_p \mathbb{1}, & \bar{\pi}_p(\mathfrak{P}^\dagger) &= b_p d_p \mathbb{1}, \end{aligned} \quad (3.21)$$

obtained by interchanging $L \leftrightarrow R$ indices; we have also replaced the indices $1 \rightarrow \bar{1}$ and $2 \rightarrow \bar{2}$ to distinguish left and right vector spaces. The primed representations are similarly defined. We will not add subscripts to distinguish the identity matrices; we hope it will always be clear from the context in which space the identity matrix lives.

Hopf algebra.—We introduce an additional grouplike generator \mathcal{U} , which is central with respect to the $\mathfrak{su}(1|1)_\mathbb{C}^2$ algebra. The action on any single-magnon excitation is

$$\mathcal{U}|\varphi_p^r\rangle = e^{i\frac{p}{2}}|\varphi_p^r\rangle, \quad \mathcal{U}|\bar{\varphi}_p^r\rangle = e^{i\frac{p}{2}}|\bar{\varphi}_p^r\rangle, \quad (3.22)$$

so that

$$\begin{aligned} \pi_p(\mathcal{U}) &= -c_p d_p^{-1} \mathbb{1} = e^{i\frac{p}{2}}, \\ \bar{\pi}_p(\mathcal{U}) &= -\bar{c}_p \bar{d}_p^{-1} \mathbb{1} = e^{i\frac{p}{2}} \end{aligned} \quad (3.23)$$

for our left- and right-moving single-magnon representations. We are now ready to define a Hopf algebra structure on $\mathfrak{su}(1|1)_\mathbb{C}^2$. We denote this Hopf superalgebra by \mathcal{A} throughout Secs. III and IV so that

$$L(\mathcal{A}) = \mathfrak{su}(1|1)_\mathbb{C}^2$$

is the associated Lie superalgebra.

Let $\mathbf{1}$ denote the unit of \mathcal{A} . The coproduct Δ_L corresponding to the action on the left-moving magnon excitations in this spin-chain frame is given by

$$\begin{aligned} \Delta_L(\mathfrak{Q}_a) &= \mathfrak{Q}_a \otimes \mathbf{1} + \mathcal{U}^{2\delta_{aR}} \otimes \mathfrak{Q}_a, \\ \Delta_L(\mathfrak{P}) &= \mathfrak{P} \otimes \mathbf{1} + \mathcal{U}^2 \otimes \mathfrak{P}, \\ \Delta_L(\mathfrak{S}_a) &= \mathfrak{S}_a \otimes \mathbf{1} + \mathcal{U}^{-2\delta_{aR}} \otimes \mathfrak{S}_a, \\ \Delta_L(\mathfrak{P}^\dagger) &= \mathfrak{P}^\dagger \otimes \mathbf{1} + \mathcal{U}^{-2} \otimes \mathfrak{P}^\dagger, \\ \Delta_L(\mathfrak{H}_a) &= \mathfrak{H}_a \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{H}_a, \\ \Delta_L(\mathcal{U}^{\pm 1}) &= \mathcal{U}^{\pm 1} \otimes \mathcal{U}^{\pm 1}, \end{aligned}$$

and the coproduct Δ_R giving the action on the right-moving magnon excitations is obtained by interchanging indices $L \leftrightarrow R$. In the representation $\pi_p \otimes \pi_q$, this yields (3.16) in the infinite spin-chain limit. It is convenient to switch to a symmetric frame,¹⁰ similar to that of Ref. [21], in which the

¹⁰In the representation $\pi_p \otimes \pi_q$, we twist the left and right coproducts $\Delta_L \rightarrow T_L^{-1} \Delta_L T_L$ and $\Delta_R \rightarrow T_R^{-1} \Delta_R T_R$ using the twist matrices

$$\mathbb{T}_L = e^{i\frac{p}{2}} \mathbb{1} \otimes \mathbb{E}_{11} + \mathbb{1} \otimes \mathbb{E}_{22}, \quad \mathbb{T}_R = e^{i\frac{p}{2}} \mathbb{1} \otimes \mathbb{E}_{\bar{1}\bar{1}} + \mathbb{1} \otimes \mathbb{E}_{\bar{2}\bar{2}}.$$

coproduct Δ is the same for both left- and right-moving sectors¹¹:

$$\begin{aligned}\Delta(\mathfrak{Q}_a) &= \mathfrak{Q}_a \otimes \mathbf{1} + \mathfrak{U} \otimes \mathfrak{Q}_a, \\ \Delta(\mathfrak{P}) &= \mathfrak{P} \otimes \mathbf{1} + \mathfrak{U}^2 \otimes \mathfrak{P}, \\ \Delta(\mathfrak{C}_a) &= \mathfrak{C}_a \otimes \mathbf{1} + \mathfrak{U}^{-1} \otimes \mathfrak{C}_a, \\ \Delta(\mathfrak{P}^\dagger) &= \mathfrak{P}^\dagger \otimes \mathbf{1} + \mathfrak{U}^{-2} \otimes \mathfrak{P}^\dagger, \\ \Delta(\mathfrak{H}_a) &= \mathfrak{H}_a \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{H}_a, \\ \Delta(\mathfrak{U}^{\pm 1}) &= \mathfrak{U}^{\pm 1} \otimes \mathfrak{U}^{\pm 1}.\end{aligned}\quad (3.24)$$

The central elements of the superalgebra $\mathfrak{C} \in \{\mathfrak{H}_a, \mathfrak{P}, \mathfrak{P}^\dagger\}$ must be co-commutative, $\Delta(\mathfrak{C}) = \Delta^{\text{op}}(\mathfrak{C})$, where $\Delta^{\text{op}}(\mathfrak{a}) = \mathcal{P}\Delta(\mathfrak{a})$ for $\mathfrak{a} \in \mathcal{A}$, with \mathcal{P} the graded permutation operator which permutes the elements of the superalgebra in the coproduct. This is true if¹²:

$$\begin{aligned}\mathfrak{P} &= \nu_1(1 - \mathfrak{U}^2) \quad \text{and} \quad \mathfrak{P}^\dagger = \nu_2(1 - \mathfrak{U}^{-2}), \\ \text{with } \nu_1, \nu_2 &\in \mathbb{C} \setminus \{0\},\end{aligned}\quad (3.25)$$

where we choose $\nu_1 = \nu_2 = h$ to obtain our previous unitary representation.

Let $\mu: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ be the usual associative multiplication of elements of the superalgebra \mathcal{A} . Moreover, let $\iota: \mathbb{C} \rightarrow \mathcal{A}$ be the unit, which maps $\iota(1) = \mathbf{1}$, and $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ the counit, defined by

$$\epsilon(\mathfrak{U}^{\pm 1}) = 1 \quad \text{and} \quad \epsilon(\mathfrak{F}) = 0 \quad (3.26)$$

for all $\mathfrak{F} \in \mathfrak{su}(1|1)_\mathbb{C}^2$. The antipode $\mathcal{S}: \mathcal{A} \rightarrow \mathcal{A}$ must then satisfy $\mu(\mathcal{S} \otimes id)\Delta = \iota\epsilon$, which gives

$$\begin{aligned}\mathcal{S}(\mathfrak{Q}_a) &= -\mathfrak{U}^{-1}\mathfrak{Q}_a, & \mathcal{S}(\mathfrak{C}_a) &= -\mathfrak{U}\mathfrak{C}_a, & \mathcal{S}(\mathfrak{H}_a) &= -\mathfrak{H}_a, \\ \mathcal{S}(\mathfrak{P}) &= -\mathfrak{U}^{-2}\mathfrak{P}, & \mathcal{S}(\mathfrak{P}^\dagger) &= -\mathfrak{U}^2\mathfrak{P}^\dagger, & \mathcal{S}(\mathfrak{U}) &= \mathfrak{U}^{-1}.\end{aligned}\quad (3.27)$$

This antipode relates left- and right-movers in the representations π_p and $\pi_{\bar{p}}$:

$$\pi_p(\mathcal{S}(\mathfrak{a})) = (\pi_{\bar{p}}(\bar{\mathfrak{a}}))^{\text{str}}, \quad (3.28)$$

with the charge conjugation matrix trivial. Here $\mathfrak{a} \in \mathcal{A}$, with $\bar{\mathfrak{a}} \in \mathcal{A}$ defined by

¹¹This coproduct exhibits a \mathbf{Z} -grading (see Remark 2.1 in Ref. [51]).

¹²To be precise, we extend $\mathfrak{su}(1|1)_\mathbb{C}^2$ by \mathfrak{U} and its inverse \mathfrak{U}^{-1} , and then consider the Hopf algebra over the quotient of the enveloping algebra of this double-extended algebra by the ideal $\langle \mathfrak{P} - \nu_1(1 - \mathfrak{U}^2), \mathfrak{P}^\dagger - \nu_2(1 - \mathfrak{U}^{-2}) \rangle$.

$$\begin{aligned}\bar{\mathfrak{Q}}_a &= \delta_{aL}\mathfrak{Q}_R + \delta_{aR}\mathfrak{Q}_L, & \bar{\mathfrak{P}} &= \mathfrak{P}, & \bar{\mathfrak{H}}_a &= \delta_{aL}\mathfrak{H}_R + \delta_{aR}\mathfrak{H}_L, \\ \bar{\mathfrak{C}}_a &= \delta_{aL}\mathfrak{C}_R + \delta_{aR}\mathfrak{C}_L, & \bar{\mathfrak{P}}^\dagger &= \mathfrak{P}^\dagger, & \bar{\mathfrak{U}}^{\pm 1} &= \mathfrak{U}^{\pm 1}.\end{aligned}\quad (3.29)$$

The representation $\pi_{\bar{p}}$ has Zhukovski variables

$$x_{\bar{p}}^\pm = \frac{1}{x_p^\pm}. \quad (3.30)$$

C. Two-magnon scattering and R -matrices

We are interested in the scattering of magnon excitations. Let $\mathcal{H}_{(\text{in})}$ denote the space of all asymptotic incoming states, and let $\mathcal{H}_{(\text{out})}$ be the space of all asymptotic outgoing states. We consider the limit in which the spin-chain is infinitely long and the number of excitations n is much smaller than the number of spin-chain sites L . This allows us to treat the asymptotic states as well separated and noninteracting. Integrability implies that any scattering process factorizes into two-magnon scattering events, in which the only dynamical process allowed is the interchange of magnon momenta and flavors. We need thus only consider such scattering of two-magnon asymptotic states.

The two-magnon scattering matrix $S(p, q)$ is a map from $\mathcal{H}_{(\text{in})}$ to $\mathcal{H}_{(\text{out})}$ which takes an incoming two-magnon state to an outgoing two-magnon state:

$$S(p, q): \mathcal{H}_{(\text{in})} \rightarrow \mathcal{H}_{(\text{out})}, \quad |\Phi_p^{1(\text{in})}\Phi_q^{2(\text{in})}\rangle \mapsto |\Phi_q^{2(\text{out})}\Phi_p^{1(\text{out})}\rangle,$$

where these asymptotic states can be represented by

$$\begin{aligned}|\Phi_p^{1(\text{in})}\Phi_q^{2(\text{in})}\rangle &\equiv |\Phi_p^1\rangle \otimes |\Phi_q^2\rangle, \\ |\Phi_q^{2(\text{out})}\Phi_p^{1(\text{out})}\rangle &\equiv |\Phi_q^2\rangle \otimes |\Phi_p^1\rangle.\end{aligned}$$

We shall consider integrable two-magnon scattering on the $\mathfrak{d}(2, 1; \alpha)^2$ double-row spin-chain. Hence the magnons $|\Phi_p\rangle$ are vectors in one of the spaces

$$\begin{aligned}W_p &= V_p \oplus \bar{V}_p = \text{span}_{\mathbb{C}}\{|\phi_p\rangle, |\psi_p\rangle, |\bar{\phi}_p\rangle, |\bar{\psi}_p\rangle\}, \\ W'_p &= V'_p \oplus \bar{V}'_p = \text{span}_{\mathbb{C}}\{|\phi'_p\rangle, |\psi'_p\rangle, |\bar{\phi}'_p\rangle, |\bar{\psi}'_p\rangle\},\end{aligned}$$

both isomorphic to $\mathbb{C}^{2|2}$. The scattering matrix then acts as

$$S(p, q): \mathcal{W}_p \otimes \mathcal{W}_q \rightarrow \mathcal{W}_q \otimes \mathcal{W}_p, \quad \text{with } \mathcal{W}_p \in \{W_p, W'_p\},$$

and also $\mathcal{V}_p \in \{V_p, V'_p\}$ and $\bar{\mathcal{V}}_p \in \{\bar{V}_p, \bar{V}'_p\}$. This simplifying notation takes into account both primed and unprimed magnon states. The Zhukovski variables x_p^\pm and x_q^\pm satisfy

$$\frac{x_p^+}{x_p^-} = e^{ip} \equiv u_p^2, \quad \frac{x_q^+}{x_q^-} = e^{iq} \equiv u_q^2, \quad (3.31)$$

and the mass-shell constraints

$$\begin{aligned} \left(x_p^+ + \frac{1}{x_p^+}\right) - \left(x_p^- + \frac{1}{x_p^-}\right) &= \frac{im_p}{h}, \\ \left(x_q^+ + \frac{1}{x_q^+}\right) - \left(x_q^- + \frac{1}{x_q^-}\right) &= \frac{im_q}{h}, \end{aligned} \quad (3.32)$$

where $m_p, m_q \in \{m = \alpha, m' = 1 - \alpha\}$ are the masses of the two magnons with momenta p and q .

The scattering matrix can be written as

$$S(p, q) = PR(p, q), \quad \text{with} \quad R(p, q) \in \text{End}(\mathcal{W}_p \otimes \mathcal{W}_q), \quad (3.33)$$

in terms of the graded permutation operator P . Therefore $\mathbb{S}(p, q) = \mathbb{P}R(p, q)$ with the R -matrix $\mathbb{R}(p, q) \in \text{End}(\mathbf{C}^{2|2} \otimes \mathbf{C}^{2|2})$. The \mathbb{S} -matrix commutes with all the symmetries of magnon excitations:

$$\begin{aligned} [\mathbb{S}(p, q), ((\pi_p \oplus \bar{\pi}_p) \otimes (\pi_q \oplus \bar{\pi}_q))(\Delta(\mathbf{a}))] &= 0, \\ \text{for all } \mathbf{a} \in \mathcal{A}, \end{aligned}$$

which implies the intertwining equations on the R -matrix:

$$\begin{aligned} ((\pi_p \oplus \bar{\pi}_p) \otimes (\pi_q \oplus \bar{\pi}_q))(\Delta^{\text{op}}(\mathbf{a}))\mathbb{R}(p, q) \\ = \mathbb{R}(p, q)((\pi_p \oplus \bar{\pi}_p) \otimes (\pi_q \oplus \bar{\pi}_q))(\Delta(\mathbf{a})), \quad \text{for all } \mathbf{a} \in \mathcal{A}. \end{aligned} \quad (3.34)$$

We change π_p and π_q to representations π'_p and π'_q for primed magnon excitations. The \mathbb{S} -matrix is unitary, $\mathbb{S}(q, p)\mathbb{S}(p, q) = \mathbb{1}$, implying $(\mathbb{R}(q, p))^{\text{op}}\mathbb{R}(p, q) = \mathbb{1}$ with $(\mathbb{R}(p, q))^{\text{op}} = \mathbb{P}R(p, q)\mathbb{P}$.

1. Complete and partial R -matrices

We now describe the structure of the R -matrix $\mathbb{R}(p, q)$. Recall that the 4-dimensional vector space $\mathcal{W}_p = \mathcal{V}_p \oplus \bar{\mathcal{V}}_p$ is a direct sum of two 2-dimensional vector spaces $\mathcal{V}_p, \bar{\mathcal{V}}_p \cong \mathbf{C}^{1|1}$, so that $\mathcal{W}_p \cong \mathbf{C}^{2|2}$. Thus $\mathcal{W}_p \otimes \mathcal{W}_q$ can be split into four 4-dimensional subspaces, and consequently $\mathbb{R}(p, q)$ is a 16×16 matrix that can be decomposed into 16 sectors. However, conservation of chirality (the total number of left- and right-moving magnon states) and mass impose additional constraints [20]. Focusing on the case of pure transmission (rather than pure reflection) for the scattering of left and right states, the *complete* R -matrix decomposes into a direct sum of four *partial* R -matrices:

$$\begin{aligned} \mathbb{R}(p, q) &= \mathbb{R}^{\text{LL}}(p, q) \oplus \mathbb{R}^{\text{LR}}(p, q) \oplus \mathbb{R}^{\text{RL}}(p, q) \\ &\oplus \mathbb{R}^{\text{RR}}(p, q), \end{aligned} \quad (3.35)$$

where $\mathbb{R}^{\text{ab}}(p, q) \in \text{End}(\mathbf{C}^{1|1} \otimes \mathbf{C}^{1|1})$. These four partial R -matrices depend on Zhukovski variables satisfying mass-shell constraints which may depend on either primed or unprimed masses.

The partial R -matrices satisfy intertwining equations which are the result of the corresponding partial \mathbb{S} -matrices commuting with the generators of the Hopf superalgebra \mathcal{A} . These R -matrices also satisfy unitarity and crossing symmetry conditions, and a discrete LR symmetry condition made manifest by our choice of a symmetric Hopf algebra. These conditions determine the complete R -matrix up to overall factors (dependent on dressing phases) and imply the Yang-Baxter equation.

Left-left and right-right sectors.—We write the partial R -matrices in the LL and RR sectors as

$$\begin{aligned} \mathbb{R}^{\text{LL}}(p, q) &= \sum_{i,j,k,l=1,2} (R^{\text{LL}}(p, q))_{jl}^{ik} \mathbb{E}_{ij} \otimes \mathbb{E}_{kl}, \\ \mathbb{R}^{\text{RR}}(p, q) &= \sum_{i,j,k,l=\bar{1},\bar{2}} (R^{\text{RR}}(p, q))_{jl}^{ik} \mathbb{E}_{ij} \otimes \mathbb{E}_{kl}, \end{aligned}$$

which depend on the momenta p and q , and on the masses m_p and m_q of the two magnons. The intertwining equations in the LL and RR sectors are

$$\begin{aligned} (\pi_p \otimes \pi_q)(\Delta^{\text{op}}(\mathbf{a}))\mathbb{R}^{\text{LL}}(p, q) \\ = \mathbb{R}^{\text{LL}}(p, q)(\pi_p \otimes \pi_q)(\Delta(\mathbf{a})), \\ (\bar{\pi}_p \otimes \bar{\pi}_q)(\Delta^{\text{op}}(\mathbf{a}))\mathbb{R}^{\text{RR}}(p, q) \\ = \mathbb{R}^{\text{RR}}(p, q)(\bar{\pi}_p \otimes \bar{\pi}_q)(\Delta(\mathbf{a})), \end{aligned}$$

for all $\mathbf{a} \in \mathcal{A}$. We note that $(\pi_p \otimes \pi_q)(\Delta^{\text{op}}(\mathbf{a})) = \mathbb{P}(\pi_p \otimes \pi_q)(\Delta(\mathbf{a}))\mathbb{P}$. These intertwining equations are linear equations which are relatively easy to solve. They determine $\mathbb{R}^{\text{LL}}(p, q)$ and $\mathbb{R}^{\text{RR}}(p, q)$, each up to one complex factor,¹³ called $s^{\text{LL}}(p, q)$ and $s^{\text{RR}}(p, q)$:

$$\begin{aligned} \mathbb{R}^{\text{LL}}(p, q) \\ = s^{\text{LL}}(p, q) \left[\mathbb{E}_{11} \otimes \mathbb{E}_{11} + \frac{(x_p^+ - x_q^+)}{u_p(x_p^- - x_q^+)} \mathbb{E}_{11} \otimes \mathbb{E}_{22} \right. \\ + \frac{u_q(x_p^- - x_q^-)}{(x_p^- - x_q^+)} \mathbb{E}_{22} \otimes \mathbb{E}_{11} + \frac{u_q(x_p^+ - x_q^-)}{u_p(x_p^- - x_q^+)} \mathbb{E}_{22} \otimes \mathbb{E}_{22} \\ \left. + \frac{i\eta_p\eta_q}{(x_p^- - x_q^+)} \mathbb{E}_{12} \otimes \mathbb{E}_{21} - \frac{i u_q \eta_p \eta_q}{u_p(x_p^- - x_q^+)} \mathbb{E}_{21} \otimes \mathbb{E}_{12} \right], \end{aligned} \quad (3.36)$$

¹³This happens because the tensor product of two atypical, two-dimensional, irreducible representations is isomorphic to the typical, four-dimensional, irreducible representation of \mathcal{A} (see, for example, Sec. II. C in Ref. [51]).

$\mathbb{R}^{\text{RR}}(p, q)$

$$\begin{aligned}
 &= s^{\text{RR}}(p, q) \left[\mathbb{E}_{\bar{1}\bar{1}} \otimes \mathbb{E}_{\bar{1}\bar{1}} + \frac{(x_p^+ - x_q^+)}{u_p(x_p^- - x_q^+)} \mathbb{E}_{\bar{1}\bar{1}} \otimes \mathbb{E}_{2\bar{2}} \right. \\
 &\quad + \frac{u_q(x_p^- - x_q^-)}{(x_p^- - x_q^+)} \mathbb{E}_{2\bar{2}} \otimes \mathbb{E}_{\bar{1}\bar{1}} + \frac{u_q(x_p^+ - x_q^-)}{u_p(x_p^- - x_q^+)} \mathbb{E}_{2\bar{2}} \otimes \mathbb{E}_{2\bar{2}} \\
 &\quad \left. + \frac{i\eta_p\eta_q}{(x_p^- - x_q^+)} \mathbb{E}_{\bar{1}\bar{2}} \otimes \mathbb{E}_{2\bar{1}} - \frac{i u_q \eta_p \eta_q}{u_p(x_p^- - x_q^+)} \mathbb{E}_{2\bar{1}} \otimes \mathbb{E}_{\bar{1}\bar{2}} \right].
 \end{aligned} \tag{3.37}$$

The unitarity conditions on the R -matrices take the form

$$(\mathbb{R}^{\text{LL}}(q, p))^{\text{op}} \mathbb{R}^{\text{LL}}(p, q) = \mathbb{I}, \quad (\mathbb{R}^{\text{RR}}(q, p))^{\text{op}} \mathbb{R}^{\text{RR}}(p, q) = \mathbb{I},$$

with $(\mathbb{R}^{\text{ab}}(p, q))^{\text{op}} = \mathbb{P} \mathbb{R}^{\text{ab}}(p, q) \mathbb{P}$,

implying $s^{\text{LL}}(p, q) s^{\text{LL}}(q, p) = s^{\text{RR}}(p, q) s^{\text{RR}}(q, p) = 1$.

$$\begin{aligned}
 \mathbb{R}^{\text{LR}}(p, q) &= s^{\text{LR}}(p, q) [(x_p^+ x_q^+ - 1)(x_p^- x_q^- - 1)]^{-\frac{1}{2}} [(x_p^+ x_q^- - 1) \mathbb{E}_{11} \otimes \mathbb{E}_{\bar{1}\bar{1}} + u_p(x_p^- x_q^- - 1) \mathbb{E}_{11} \otimes \mathbb{E}_{2\bar{2}} \\
 &\quad + u_q^{-1}(x_p^+ x_q^+ - 1) \mathbb{E}_{22} \otimes \mathbb{E}_{\bar{1}\bar{1}} + u_p u_q^{-1}(x_p^- x_q^+ - 1) \mathbb{E}_{22} \otimes \mathbb{E}_{2\bar{2}} + u_p \eta_p \eta_q \mathbb{E}_{12} \otimes \mathbb{E}_{\bar{1}\bar{2}} + \eta_p \eta_q u_q^{-1} \mathbb{E}_{21} \otimes \mathbb{E}_{2\bar{1}}],
 \end{aligned} \tag{3.38}$$

$$\begin{aligned}
 \mathbb{R}^{\text{RL}}(p, q) &= s^{\text{RL}}(p, q) [(x_p^+ x_q^+ - 1)(x_p^- x_q^- - 1)]^{-\frac{1}{2}} [(x_p^+ x_q^- - 1) \mathbb{E}_{\bar{1}\bar{1}} \otimes \mathbb{E}_{11} + u_p(x_p^- x_q^- - 1) \mathbb{E}_{\bar{1}\bar{1}} \otimes \mathbb{E}_{22} \\
 &\quad + u_q^{-1}(x_p^+ x_q^+ - 1) \mathbb{E}_{2\bar{2}} \otimes \mathbb{E}_{11} + u_p u_q^{-1}(x_p^- x_q^+ - 1) \mathbb{E}_{2\bar{2}} \otimes \mathbb{E}_{22} + u_p \eta_p \eta_q \mathbb{E}_{\bar{1}\bar{2}} \otimes \mathbb{E}_{12} + \eta_p \eta_q u_q^{-1} \mathbb{E}_{2\bar{1}} \otimes \mathbb{E}_{21}],
 \end{aligned} \tag{3.39}$$

up to the overall factors $s^{\text{LR}}(p, q)$ and $s^{\text{RL}}(p, q)$. The unitarity conditions are

$$\begin{aligned}
 (\mathbb{R}^{\text{LR}}(q, p))^{\text{op}} \mathbb{R}^{\text{RL}}(p, q) &= \mathbb{I}, \\
 (\mathbb{R}^{\text{RL}}(q, p))^{\text{op}} \mathbb{R}^{\text{LR}}(p, q) &= \mathbb{I},
 \end{aligned}$$

implying $s^{\text{LR}}(p, q) s^{\text{RL}}(q, p) = s^{\text{RL}}(p, q) s^{\text{LR}}(q, p) = 1$.

We require the parity relation $s^{\text{ba}}(-q, -p) = s^{\text{ab}}(p, q)$ for the scale factors in the R -matrix, and also impose a discrete LR symmetry $s^{\text{LL}}(p, q) = s^{\text{RR}}(p, q)$ and $s^{\text{LR}}(p, q) = s^{\text{RL}}(p, q)$ as in Ref. [20]. These conditions will be necessary to derive reflection matrix solutions of the boundary Yang-Baxter equation in Sec. IV. We note, finally, that additional *bulk crossing symmetry conditions* must be imposed which further constrain the scale factors $s^{\text{ab}}(p, q)$. A proposal for the solution to these crossing symmetry conditions has been advanced in Ref. [38].

2. Yang-Baxter equation

The complete R -matrix $\mathbb{R}(p, q)$ must satisfy

$$\begin{aligned}
 &\mathbb{R}_{12}(p, q) \mathbb{R}_{13}(p, r) \mathbb{R}_{23}(q, r) \\
 &= \mathbb{R}_{23}(q, r) \mathbb{R}_{13}(p, r) \mathbb{R}_{12}(p, q),
 \end{aligned} \tag{3.40}$$

Left-right and right-left sectors.—We write the partial R -matrices in the LR and RL sectors as

$$\begin{aligned}
 \mathbb{R}^{\text{LR}}(p, q) &= \sum_{\substack{i,j=1,2; \\ k,l=1,2}} (R^{\text{LR}}(p, q))_{jl}^{ik} \mathbb{E}_{ij} \otimes \mathbb{E}_{kl}, \\
 \mathbb{R}^{\text{RL}}(p, q) &= \sum_{\substack{i,j=1,2; \\ k,l=1,2}} (R^{\text{RL}}(p, q))_{jl}^{ik} \mathbb{E}_{ij} \otimes \mathbb{E}_{kl}.
 \end{aligned}$$

The intertwining equations are given by

$$\begin{aligned}
 &(\pi_p \otimes \bar{\pi}_q)(\Delta^{\text{op}}(\mathbf{a})) \mathbb{R}^{\text{LR}}(p, q) \\
 &= \mathbb{R}^{\text{LR}}(p, q)(\pi_p \otimes \bar{\pi}_q)(\Delta(\mathbf{a})), \\
 &(\bar{\pi}_p \otimes \pi_q)(\Delta^{\text{op}}(\mathbf{a})) \mathbb{R}^{\text{RL}}(p, q) \\
 &= \mathbb{R}^{\text{RL}}(p, q)(\bar{\pi}_p \otimes \pi_q)(\Delta(\mathbf{a})),
 \end{aligned}$$

for all $\mathbf{a} \in \mathcal{A}$, which determine the transmission R -matrices in the LR and RL sectors as

which is the Yang-Baxter equation of an integrable system. The partial R -matrices then satisfy

$$\begin{aligned}
 &\mathbb{R}_{12}^{\text{ab}}(p, q) \mathbb{R}_{13}^{\text{ac}}(p, r) \mathbb{R}_{23}^{\text{bc}}(q, r) \\
 &= \mathbb{R}_{23}^{\text{bc}}(q, r) \mathbb{R}_{13}^{\text{ac}}(p, r) \mathbb{R}_{12}^{\text{ac}}(p, q)
 \end{aligned} \tag{3.41}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{\text{L}, \text{R}\}$ (see Fig. 4). Here we define

$$\begin{aligned}
 \mathbb{R}_{12}^{\text{ab}}(p, q) &= \mathbb{R}^{\text{ab}}(p, q) \otimes \mathbb{I}, \quad \mathbb{R}_{23}^{\text{ab}}(p, q) = \mathbb{I} \otimes \mathbb{R}^{\text{ab}}(p, q), \\
 \mathbb{R}_{13}^{\text{ab}}(p, q) &= (\mathbb{I} \otimes \mathbb{P})(\mathbb{R}^{\text{ab}}(p, q) \otimes \mathbb{I})(\mathbb{I} \otimes \mathbb{P}),
 \end{aligned}$$

and similarly for $\mathbb{R}_{12}(p, q)$, $\mathbb{R}_{13}(p, q)$ and $\mathbb{R}_{23}(p, q)$ in terms of the complete R -matrices, with a change to the identity matrix in $\text{End}(\mathbb{C}^{2|2})$ and the graded permutation matrix in $\text{End}(\mathbb{C}^{2|2} \otimes \mathbb{C}^{2|2})$.

We can check that the R -matrices above satisfy (3.41) for (abc) in the homogeneous (LLL, RRR) and mixed (LLR, LRL, RLL, LRR, RLR, RRL) sectors. Note that the discrete LR symmetry means we need only check half

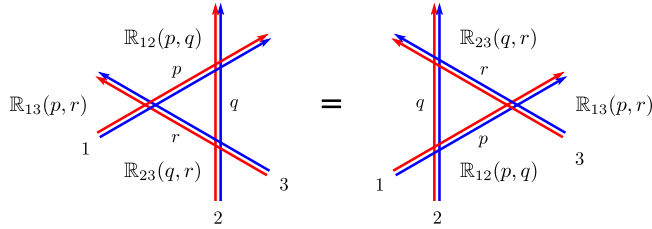


FIG. 4 (color online). Three-magnon scattering factorizes into a succession of two-magnon scattering events. The double red and blue line indicates the *direct sum of left- and right-magnon states* with scattering described by the complete R -matrix. The scattering of individual red or blue lines (left or right magnons) is described by partial R -matrices. This diagram gives both the Yang-Baxter equation for the complete R -matrix (treating double lines as a single composite line) and the Yang-Baxter equation for the partial R -matrices (choosing a red or blue line from each pair to give eight possibilities).

of these equations. This ensures that the complete R -matrix satisfies (3.40).

IV. INTEGRABLE OPEN $\mathfrak{d}(2,1;\alpha)^2$ SPIN-CHAIN AND REFLECTION MATRICES IN $\text{AdS}_3 \times S^3 \times S^3 \times S^1$

In this section, we consider the boundary scattering of magnon excitations of a $\mathfrak{d}(2,1;\alpha)^2$ open spin-chain off an integrable boundary. These correspond to massive excitations of an open superstring ending on a D-brane in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$, such as one of the maximal D1- or D5-brane giant gravitons described in Sec. II A.

A. Open spin-chains and boundary scattering

1. Double-row $\mathfrak{d}(2,1;\alpha)^2$ open spin-chain with $\mathfrak{su}(1|1)_{\mathbb{C}}^2$ excitations

Semi-infinite open spin-chain.—In the infinite $J \rightarrow \infty$ spin-chain limit we can consider one end of the spin-chain at a time—we choose the right end of the open spin-chain. Thus we obtain a semi-infinite spin-chain with a distinguished site at the right end which we call the boundary site. The vector occupying this site transforms in a representation of a boundary subalgebra, dictated by boundary conditions. The ground state of the semi-infinite open spin-chain is

$$|0\rangle = |\mathcal{Z}^J \mathcal{F}_{\text{B}}\rangle, \quad (4.1)$$

where \mathcal{F}_{B} is the right boundary field. This infinitely heavy state represents the whole D-brane to which the open

superstring is attached (to account for conservation of momentum).¹⁴

We will enumerate the sites of the semi-infinite open spin-chain by $-J, -J+1, \dots, -1, 0$, with 0 denoting the boundary field site. In this notation the fundamental excitations are described by the spin-chain state vectors

$$\begin{aligned} |\varphi_{(n)}^r\rangle_{\text{B}} &= |\mathcal{Z}^{J-n} \varphi^r \mathcal{Z}^{n-1} \mathcal{F}_{\text{B}}\rangle, \\ |\bar{\varphi}_{(n)}^r\rangle_{\text{B}} &= |\mathcal{Z}^{n-1} \bar{\varphi}^r \mathcal{Z}^{J-n} \mathcal{F}_{\text{B}}\rangle, \end{aligned} \quad (4.2)$$

where $\varphi^r, \bar{\varphi}^r$ represent the excitations discussed in Sec. III A 2. The low-lying left- and right-moving single-magnon excitations are now given by

$$\begin{aligned} |\varphi_p^r\rangle_{\text{B}} &= \sum_{n=1}^J e^{-ipn} |\varphi_{(n)}^r\rangle_{\text{B}}, \\ |\bar{\varphi}_p^r\rangle_{\text{B}} &= \sum_{n=1}^J e^{-ipn} |\bar{\varphi}_{(n)}^r\rangle_{\text{B}}. \end{aligned} \quad (4.3)$$

The supercharges acting on these states insert or remove a \mathcal{Z} field from the left side. In the $J \rightarrow \infty$ limit, in which the state has infinite length in the left direction, this does not change the length of the spin-chain or the location of the excitation. We can identify magnon states of the semi-infinite open spin-chain with magnon states of the infinite closed spin-chain, with an extra boundary state,

$$|\varphi_p^r\rangle_{\text{B}} = |\varphi_p^r\rangle \otimes |0\rangle_{\text{B}}, \quad |\bar{\varphi}_p^r\rangle_{\text{B}} = |\bar{\varphi}_p^r\rangle \otimes |0\rangle_{\text{B}}, \quad (4.4)$$

where $|\varphi_p^r\rangle$ and $|\bar{\varphi}_p^r\rangle$ are bulk magnon excitations in the vector spaces \mathcal{V}_p and $\bar{\mathcal{V}}_p$, with the action of the Hopf superalgebra \mathcal{A} defined in the usual way. The boundary field \mathcal{F}_{B} is represented by the boundary vacuum state $|0\rangle_{\text{B}}$. The generalization for the multimagnon states is straightforward.

Boundary algebra.—The symmetries of the boundary site are related to those symmetries of the D-brane which survive the choice of vacuum \mathcal{Z} . These boundary symmetries are generated by a subalgebra of the bulk symmetry algebra, which in our case is Hopf superalgebra \mathcal{A} . We will

¹⁴A similar construction is presented in Sec. II B of Ref. [8], in which open superstrings on the $\text{AdS}_5 \times S^5$ background are attached to $Y=0$ and $Z=0$ maximal giant gravitons. In the dual CFT_4 description, an open superstring corresponds to a gauge invariant local operator $\text{Tr}(Z \dots Z)$, with impurities, attached to a $\det(Y)$ or $\det(Z)$ operator dual to a maximal D3-brane giant graviton. The full determinant plays the role of the boundary field \mathcal{F}_{B} . Here the CFT_2 dual of IIB superstring theory on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ is not known, although it is conjectured to arise as the IR limit of the D1-D5-D5' world-volume gauge theory of Ref. [17]. We expect, however, that an open superstring attached to our maximal D1- and D5-brane giant gravitons will be dual to similar determinant-like CFT_2 operators.

denote the boundary symmetry algebra by \mathcal{B} with elements $\mathfrak{b} \in \mathcal{B}$.

Since $\mathcal{B} \subset \mathcal{A}$, the elements \mathfrak{b} inherit all the additional algebraic structures (coproduct, counit, etc.) defined on \mathcal{A} . However, \mathcal{B} is not a Hopf subalgebra of \mathcal{A} . The integrability assumption requires boundary algebra to be a coideal subalgebra of the bulk symmetry algebra:

$$\Delta(\mathfrak{b}) \in \mathcal{A} \otimes \mathcal{B} \quad \text{for all } \mathfrak{b} \in \mathcal{B}. \quad (4.5)$$

The boundary vacuum state $|0\rangle_{\mathcal{B}}$ is taken to be a vector in the trivial representation of the coideal subalgebra \mathcal{B} , defined by the counit map ϵ . We call this the *singlet boundary*. We will later extend this construction to accommodate a vector representation of \mathcal{B} —called the *vector boundary*.

Boundary integrability.—It is known that integrability is preserved if the bulk and boundary Lie algebras form a symmetric pair (see Ref. [5] and references therein for a review of boundary integrability). Recall that a symmetric pair of Lie algebras is a pair $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where \mathfrak{h} is a subalgebra of \mathfrak{g} and the following relations hold:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \quad (4.6)$$

The statement above is true if \mathfrak{g} is a simple complex Lie algebra. Integrability is then ensured by the existence of a twisted Yangian associated with the symmetric pair $(\mathfrak{g}, \mathfrak{h})$, the Cartan subalgebra of which is an infinite-dimensional Abelian algebra. By a quantum extension of the classical Liouville integrability theorem, this is enough to ensure that the system is integrable.¹⁵ The question of boundary integrability becomes much more complicated if \mathfrak{g} is nonsimple, since both the Yangian and twisted Yangian (for a given symmetric pair) are not well defined. Integrability then needs to be examined on a case-by-case basis. Likewise, the integrable structures associated with symmetric pairs of Lie superalgebras often require further investigation. An alternative way of verifying the integrability of boundary scattering is by finding the reflection K -matrix which is a solution of the reflection equation and intertwines the boundary symmetries.

2. Boundary scattering and K -matrices

Outgoing single-magnon representations π_{-p} and $\bar{\pi}_{-p}$.—Let us take the states (4.3) to be *incoming* magnons with momentum p . To define the boundary scattering theory, we need a notion of *outgoing* states with opposite momentum $-p$. We will denote the corresponding vector

spaces by \mathcal{V}_{-p} and $\bar{\mathcal{V}}_{-p}$. The left- and right-moving outgoing magnon representations $\pi_{-p}: \mathcal{A} \rightarrow \text{End}(\mathbf{C}^{1|1})$ and $\bar{\pi}_{-p}: \mathcal{A} \rightarrow \text{End}(\mathbf{C}^{1|1})$ are parametrized by the labels a_{-p} , b_{-p} , c_{-p} and d_{-p} , with Zhukovski variables x_{-p}^{\pm} . Recall that x_p^{\pm} satisfy (3.18): the first equality implies $x_{-p}^{\pm} = -f_p x_p^{\mp}$, for some phase factor f_p , while the second equality sets $f_p = 1$. Thus we obtain

$$x_{-p}^{\pm} = -x_p^{\mp}, \quad \eta_{-p} = \eta_p. \quad (4.7)$$

The outgoing left-moving representation π_{-p} is (3.20) with the parameters (3.17) replaced by

$$\begin{aligned} a_{-p} &= \sqrt{h}\eta_p, & b_{-p} &= \sqrt{h}\eta_p, & c_{-p} &= \sqrt{h}\frac{i\eta_p}{x_p^+}, \\ d_{-p} &= -\sqrt{h}\frac{i\eta_p}{x_p^-}, & u_{-p}^2 &= \frac{x_p^-}{x_p^+}. \end{aligned} \quad (4.8)$$

The outgoing right-moving representation $\bar{\pi}_{-p}$ is given by (3.21) with an identical replacement.

Singlet boundary scattering.—Boundary scattering on the semi-infinite open spin-chain is described by a boundary scattering matrix $S_{\text{boundary}}(p)$ which maps incoming states to outgoing states, while keeping the boundary fixed. As for bulk scattering, we denote the space of all asymptotic incoming magnon states of the open spin-chain by $\mathcal{H}_{(\text{in})}$, and the space of all outgoing states by $\mathcal{H}_{(\text{out})}$. By the integrability hypothesis, we need only consider the reflection of single magnons off the boundary. The boundary scattering matrix maps $\mathcal{H}_{(\text{in})}$ to $\mathcal{H}_{(\text{out})}$:

$$S_{\text{boundary}}(p): \mathcal{H}_{(\text{in})} \rightarrow \mathcal{H}_{(\text{out})}, \quad |\Phi_p^{(\text{in})}\rangle \mapsto |\Phi_{-p}^{(\text{out})}\rangle.$$

We can write the incoming and outgoing states as

$$|\Phi_p^{(\text{in})}\rangle = |\Phi_p\rangle \otimes |0\rangle_{\mathcal{B}}, \quad |\Phi_{-p}^{(\text{out})}\rangle = |\Phi_{-p}\rangle \otimes |0\rangle_{\mathcal{B}},$$

where $|\Phi_p\rangle \in \mathcal{W}_p$ and $|\Phi_{-p}\rangle \in \mathcal{W}_{-p}$ are bulk magnons, and $|0\rangle_{\mathcal{B}}$ is the singlet boundary state.

For our purposes it will be convenient to introduce the boundary intertwining K -matrix. Let κ denote the natural *reflection* map which acts as the identity map on $|0\rangle_{\mathcal{B}}$ and

$$\kappa: \mathcal{W}_p \rightarrow \mathcal{W}_{-p}, \quad |\Phi_p\rangle \mapsto |\Phi_{-p}\rangle,$$

which is the canonical isomorphism $\mathcal{W}_p \cong \mathcal{W}_{-p}$. The boundary scattering matrix $S_{\text{boundary}}(p)$ is the composition¹⁶

¹⁶This composition is well defined, since $|0\rangle_{\mathcal{B}}$ is a state in a one-dimensional vector space ($\cong \mathbf{C}$) and $\mathcal{W}_p \otimes \mathbf{C} \cong \mathcal{W}_p$.

¹⁵There is no analogue of the Liouville-Arnold theorem for infinite-dimensional quantum systems; thus we consider integrability as the set of constraints that are necessary to ensure factorized scattering.

$$S_{\text{boundary}}(p) = \kappa K(p), \quad \text{with } K(p) \in \text{End}(\mathcal{W}_p). \quad (4.9)$$

Now $\mathcal{W}_p \cong \mathbf{C}^{2|2}$, so the K -matrix corresponds to $\mathbb{K}(p) \in \text{End}(\mathbf{C}^{2|2})$. The boundary S -matrix must commute with all the boundary symmetries, which yields the boundary intertwining equations for the K -matrix. There is also a unitarity condition on the boundary S -matrix.

Vector boundary scattering.—Let us consider a vector representation of the boundary algebra. A boundary vector state is anticipated to have an interpretation as a magnon state¹⁷ $|\Phi_{\mathbf{B}}\rangle \in \mathcal{W}_{\mathbf{B}}$, with maximum total momentum π , absorbed by a singlet boundary state $|0\rangle_{\mathbf{B}}$. The boundary Lie algebra associated with the coideal subalgebra, in this case denoted $\mathcal{B}_{\top} \subset \mathcal{A}$, is thus the totally supersymmetric $\mathfrak{su}(1|1)_{\mathbf{C}}^2$ symmetry of bulk magnon excitations.

In the case of this vector boundary state, the incoming and outgoing states in $\mathcal{H}_{(\text{in})}$ and $\mathcal{H}_{(\text{out})}$ have the following tensor product decomposition:

$$\begin{aligned} |\Phi_p^{(\text{in})}\rangle &= |\Phi_p\rangle \otimes |\Phi_{\mathbf{B}}\rangle \otimes |0\rangle_{\mathbf{B}}, \\ |\Phi_{-p}^{(\text{out})}\rangle &= |\Phi_{-p}\rangle \otimes |\Phi_{\mathbf{B}}\rangle \otimes |0\rangle_{\mathbf{B}}, \end{aligned}$$

where $|\Phi_{\mathbf{B}}\rangle \in \mathcal{W}_{\mathbf{B}}$ is a vector in the boundary space. We will now denote the boundary scattering matrix by $S_{\text{boundary}}(p, \mathbf{B})$. As before, we write this as a composition of the reflection map κ , which now acts as the identity map on the boundary vector state, and a reflection K -matrix:

$$\begin{aligned} S_{\text{boundary}}(p, \mathbf{B}) &= \kappa K(p, \mathbf{B}), \\ \text{with } K(p, \mathbf{B}) &\in \text{End}(\mathcal{W}_p \otimes \mathcal{W}_{\mathbf{B}}). \quad (4.10) \end{aligned}$$

Now $\mathcal{W}_p, \mathcal{W}_{\mathbf{B}} \cong \mathbf{C}^{2|2}$, so the K -matrix $K(p, \mathbf{B})$ corresponds to $\mathbb{K}(p, \mathbf{B}) \in \text{End}(\mathbf{C}^{2|2} \otimes \mathbf{C}^{2|2})$.

We will discuss singlet boundaries in Sec. IV B and the vector boundary in Sec. IV C. Note that only a vector representation (not a singlet) of the boundary algebra \mathcal{B}_{\top} is possible due to the inclusion of the central elements \mathfrak{P} and \mathfrak{P}^{\dagger} . We will explain this argument in detail in subsequent sections.

B. Singlet boundaries

Boundary algebras.—There are four boundary subalgebras \mathcal{B} of the bulk Hopf superalgebra \mathcal{A} which describe the scattering of magnons off singlet boundaries. The

¹⁷A boundary state is created when the boundary absorbs magnon excitations via the so-called boundary *bootstrap* procedure (see, for example, Sec. III in Ref. [68]). Boundary states for open superstrings in AdS₅ × S⁵ were considered in Refs. [52,58,69]. We will give a detailed description of similar boundary states for open superstrings in AdS₃ × S³ × S^{3′} × S¹ in a forthcoming publication.

associated Lie algebras $L(\mathcal{B})$ can be compared with the boundary algebras in Tables I and II. We will denote them as follows:

- (1) *Left and right half-supersymmetric boundary algebras* $\mathcal{B}_{\text{L}}, \mathcal{B}_{\text{R}}$, corresponding to D-branes preserving half of the bulk supersymmetries, and the magnon Hamiltonian \mathfrak{H} and \mathfrak{M} , implying chiral boundary scattering. The boundary Lie algebras are $\mathfrak{su}(1|1)_{\text{L}} \oplus \mathfrak{u}(1)_{\text{R}}$ and $\mathfrak{u}(1)_{\text{L}} \oplus \mathfrak{su}(1|1)_{\text{R}}$.
- (2) *Non-supersymmetric chiral boundary algebra* \mathcal{B}_{NC} , corresponding to D-branes which do not respect any bulk supersymmetries, but preserve both \mathfrak{H} and \mathfrak{M} . The boundary Lie algebra is $\mathfrak{u}(1)_{\text{L}} \oplus \mathfrak{u}(1)_{\text{R}}$. We will show that scattering off the non-supersymmetric chiral boundary has a *hidden symmetry*, denoted \mathcal{B}_{D} , at the level of the Hopf superalgebra.
- (3) *Non-supersymmetric achiral boundary algebra* \mathcal{B}_{NA} , corresponding to D-branes which preserve \mathfrak{H} and no bulk supersymmetries. The associated boundary Lie algebra is $\mathfrak{u}(1)_{+}$.

Boundary intertwining equations.—The K -matrix $\mathbb{K}(p) \in \text{End}(\mathbf{C}^{2|2})$ is the boundary analogue of the bulk R -matrix and is required to satisfy the boundary intertwining equations

$$\begin{aligned} ((\pi_{-p} \oplus \bar{\pi}_{-p}) \otimes \epsilon)(\Delta(\mathfrak{b}))\mathbb{K}(p) \\ = \mathbb{K}(p)((\pi_p \oplus \bar{\pi}_p) \otimes \epsilon)(\Delta(\mathfrak{b})) \end{aligned} \quad (4.11)$$

for all $\mathfrak{b} \in \mathcal{B}$ for a given boundary subalgebra \mathcal{B} . For those $\mathfrak{b} \in \mathfrak{su}(1|1)_{\mathbf{C}}^2$, this simplifies to the form

$$(\pi_{-p} \oplus \bar{\pi}_{-p})(\mathfrak{b})\mathbb{K}(p) = \mathbb{K}(p)(\pi_p \oplus \bar{\pi}_p)(\mathfrak{b}), \quad (4.12)$$

where we have dropped the trivial boundary representation ϵ , since $\epsilon(\mathfrak{b}) = 0$.

The *complete* K -matrix $\mathbb{K}(p)$ can have four sectors which correspond to *chiral* reflections [left-to-left $\mathbb{K}^{\text{L}}(p)$ and right-to-right $\mathbb{K}^{\text{R}}(p)$] and *achiral* reflections [left-to-right $\mathbb{A}^{\text{L}}(p)$ and right-to-left $\mathbb{A}^{\text{R}}(p)$]. We denote these *partial* K -matrices by $\mathbb{K}^{\mathfrak{a}}(p), \mathbb{A}^{\mathfrak{a}}(p) \in \text{End}(\mathbf{C}^{1|1})$. Here the superscript \mathfrak{a} denotes the chirality of the incoming magnon before the reflection. The complete K -matrix is then

$$\begin{aligned} \mathbb{K}(p) &= \begin{pmatrix} \mathbb{K}^{\text{L}}(p) & \mathbb{A}^{\text{R}}(p) \\ \mathbb{A}^{\text{L}}(p) & \mathbb{K}^{\text{R}}(p) \end{pmatrix} \\ &= (\mathbb{K}^{\text{L}}(p) \oplus \mathbb{K}^{\text{R}}(p)) \\ &\quad + \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{pmatrix} (\mathbb{A}^{\text{L}}(p) \oplus \mathbb{A}^{\text{R}}(p)), \end{aligned} \quad (4.13)$$

which is also required to satisfy the unitarity condition $\mathbb{K}(-p)\mathbb{K}(p) = \mathbb{1}$.

Constraints from the central elements.—The central elements of the boundary subalgebra \mathcal{B} play a crucial role

in boundary scattering. Let us explain why. A central element $\mathfrak{C} \in \mathcal{B}$ is required to commute with the boundary scattering matrix and thus must intertwine the K -matrix $\mathbb{K}(p)$ trivially. Since $\mathcal{B} \subset \mathcal{A}$, there are five candidates for central elements in \mathcal{B} . Consider \mathfrak{H}_a with $a \in \{\mathbb{L}, \mathbb{R}\}$. Note that

$$\pi_p(\mathfrak{H}_a) \neq \bar{\pi}_{-p}(\mathfrak{H}_a), \quad \bar{\pi}_p(\mathfrak{H}_a) \neq \pi_{-p}(\mathfrak{H}_a),$$

and thus chirality is conserved if either (or both) \mathfrak{H}_L or \mathfrak{H}_R are in \mathcal{B} . However, chirality is not conserved if only the linear combination $\mathfrak{H} = \mathfrak{H}_L + \mathfrak{H}_R$, but not $\mathfrak{M} = \mathfrak{H}_L - \mathfrak{H}_R$, is in \mathcal{B} , since

$$\pi_p(\mathfrak{H}) = \bar{\pi}_{-p}(\mathfrak{H}), \quad \pi_p(\mathfrak{M}) \neq \bar{\pi}_{-p}(\mathfrak{M}).$$

Now, let us consider $\mathfrak{C} \in \{\mathfrak{P}, \mathfrak{P}^\dagger\}$. Then

$$\pi_p(\mathfrak{C}) \neq \pi_{-p}(\mathfrak{C}), \quad \bar{\pi}_p(\mathfrak{C}) \neq \bar{\pi}_{-p}(\mathfrak{C}),$$

so, if the boundary representation is trivial, then the central elements \mathfrak{P} and \mathfrak{P}^\dagger cannot be in the boundary subalgebra. However, suitable linear and quadratic combinations of \mathfrak{P} and \mathfrak{P}^\dagger are allowed. In particular, $\mathfrak{P}^\pm \equiv \mathfrak{P}^\dagger \pm \mathfrak{P}$ and $\mathfrak{K} \equiv \mathfrak{P}^\dagger \mathfrak{P}$ satisfy

$$\begin{aligned} \pi_p(\mathfrak{P}^+) &= \pi_{-p}(\mathfrak{P}^+), & \pi_p(\mathfrak{P}^-) &\neq \pi_{-p}(\mathfrak{P}^-), \\ \pi_p(\mathfrak{K}) &= \pi_{-p}(\mathfrak{K}), & \bar{\pi}_p(\mathfrak{P}^+) &= \bar{\pi}_{-p}(\mathfrak{P}^+), \\ \bar{\pi}_p(\mathfrak{P}^-) &\neq \bar{\pi}_{-p}(\mathfrak{P}^-), & \bar{\pi}_p(\mathfrak{K}) &= \bar{\pi}_{-p}(\mathfrak{K}), \end{aligned}$$

and thus we may have $\mathfrak{P}^+, \mathfrak{K} \in \mathcal{B}$, but $\mathfrak{P}^- \notin \mathcal{B}$, if the boundary representation is trivial. We also note that $\mathfrak{P}^+, \mathfrak{K} \in \mathcal{B}$ does not imply any constraints on the chirality of the reflection.

The last central element we need to consider is \mathfrak{U} . Since $\epsilon(\mathfrak{U}) = 1$, we should note that

$$((\pi_p \oplus \bar{\pi}_p) \otimes \epsilon)(\Delta(\mathfrak{U})) \neq ((\pi_{-p} \oplus \bar{\pi}_{-p}) \otimes \epsilon)(\Delta(\mathfrak{U})),$$

which implies that \mathfrak{U} cannot be in the boundary algebra \mathcal{B} , if boundary representation is given by the counit ϵ . We will show in the next section that this is also true for the vector boundary.

1. Boundary subalgebras and K -matrices

Left and right half-supersymmetric boundary algebras.— We define the left and right half-supersymmetric boundary superalgebras, \mathcal{B}_L and \mathcal{B}_R , to be coideal subalgebras of \mathcal{A} generated as

$$\begin{aligned} \mathcal{B}_L &= \langle \mathfrak{Q}_L, \mathfrak{S}_L, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\ \mathcal{B}_R &= \langle \mathfrak{Q}_R, \mathfrak{S}_R, \mathfrak{H}_L, \mathfrak{H}_R \rangle. \end{aligned} \quad (4.14)$$

It remains to check the symmetric pair property. Let

$$\begin{aligned} \mathfrak{g} &\equiv L(\mathcal{A}) = \mathfrak{su}(1|1)_{\mathfrak{C}}, \\ \mathfrak{h}_L &\equiv L(\mathcal{B}_L) = \mathfrak{su}(1|1)_L \oplus \mathfrak{u}(1)_R, \\ \mathfrak{h}_R &\equiv L(\mathcal{B}_R) = \mathfrak{u}(1)_L \oplus \mathfrak{su}(1|1)_R \end{aligned} \quad (4.15)$$

denote the associated Lie superalgebras, and \mathfrak{m}_L and \mathfrak{m}_R denote spaces generated by $\{\mathfrak{Q}_R, \mathfrak{S}_R, \mathfrak{P}, \mathfrak{P}^\dagger\}$ and $\{\mathfrak{Q}_L, \mathfrak{S}_L, \mathfrak{P}, \mathfrak{P}^\dagger\}$, respectively. Thus $\mathfrak{g} = \mathfrak{h}_L \oplus \mathfrak{m}_L$ and $\mathfrak{g} = \mathfrak{h}_R \oplus \mathfrak{m}_R$ as vector spaces, and an easy computation shows that the property (4.6) indeed holds. The boundary Lie superalgebra \mathfrak{h}_L is that of an open superstring on $\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$ ending on the $\bar{Y} = 0$ or $\bar{Y}' = 0$ half of the D5-brane maximal giant graviton, or the $\bar{Y} = \bar{Y}' = 0$ D1-brane giant (which is the intersection of $\bar{Y} = 0$ and $\bar{Y}' = 0$ giants). The boundary Lie superalgebra \mathfrak{h}_R is that of an open superstring ending on the $Y = 0$ or $Y' = 0$ half of the D5-brane maximal giant, or the $Y = Y' = 0$ D1-brane giant. These are analogues of open superstrings attached to $\bar{Y} = 0$ and $Y = 0$ giant gravitons in $\text{AdS}_5 \times S^5$ [8].

Let us construct the K -matrices $\mathbb{K}_{\mathcal{B}_a}(p)$ for both $a \in \{\mathbb{L}, \mathbb{R}\}$. Since $\mathfrak{H}_L, \mathfrak{H}_R \in \mathcal{B}_a$, these K -matrices are chiral:

$$\mathbb{K}_{\mathcal{B}_a}(p) = \mathbb{K}_{\mathcal{B}_a}^L(p) \oplus \mathbb{K}_{\mathcal{B}_a}^R(p), \quad (4.16)$$

where

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_a}^L(p) &= \sum_{i,j=1,2} (K_{\mathcal{B}_a}^L(p))_j^i E_{ij}, \\ \mathbb{K}_{\mathcal{B}_a}^R(p) &= \sum_{i,j=\bar{1},\bar{2}} (K_{\mathcal{B}_a}^R(p))_j^i E_{ij}. \end{aligned}$$

The boundary intertwining equations are given by

$$\begin{aligned} \pi_{-p}(\mathfrak{b}) \mathbb{K}_{\mathcal{B}_a}^L(p) &= \mathbb{K}_{\mathcal{B}_a}^L(p) \pi_p(\mathfrak{b}), \\ \bar{\pi}_{-p}(\mathfrak{b}) \mathbb{K}_{\mathcal{B}_a}^R(p) &= \mathbb{K}_{\mathcal{B}_a}^R(p) \bar{\pi}_p(\mathfrak{b}) \end{aligned} \quad (4.17)$$

for all $\mathfrak{b} \in \mathcal{B}_a$. The unitarity conditions are $\mathbb{K}_{\mathcal{B}_a}^L(-p) \mathbb{K}_{\mathcal{B}_a}^L(p) = \mathbb{1}$ and $\mathbb{K}_{\mathcal{B}_a}^R(-p) \mathbb{K}_{\mathcal{B}_a}^R(p) = \mathbb{1}$. The solutions to these intertwining equations take the form

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_L}^L(p) &= k_{\mathcal{B}_L}^L(p) \mathbb{1}, \\ \mathbb{K}_{\mathcal{B}_L}^R(p) &= k_{\mathcal{B}_L}^R(p) (E_{\bar{1}\bar{1}} - u_p^2 E_{\bar{2}\bar{2}}), \end{aligned} \quad (4.18)$$

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_R}^L(p) &= k_{\mathcal{B}_R}^L(p) (E_{11} - u_p^2 E_{22}), \\ \mathbb{K}_{\mathcal{B}_R}^R(p) &= k_{\mathcal{B}_R}^R(p) \mathbb{1}, \end{aligned} \quad (4.19)$$

with both $k_{\mathcal{B}_a}^L(-p) k_{\mathcal{B}_a}^L(p) = 1$ and $k_{\mathcal{B}_a}^R(-p) k_{\mathcal{B}_a}^R(p) = 1$ for unitarity.

Non-supersymmetric chiral boundary algebra.—Let us now consider a boundary algebra containing no supercharges. We set

$$\mathcal{B}_{\text{NC}} = \langle \mathfrak{H}_L, \mathfrak{H}_R \rangle. \quad (4.20)$$

The boundary Lie algebra $L(\mathcal{B}_{\text{NC}}) = \mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$ is from an open superstring on $\text{AdS}_3 \times S^3 \times S^{3'} \times S^1$ attached to the $\bar{Z} = 0$ or $\bar{Z}' = 0$ half of the D5-brane maximal giant or to the $Z = \bar{Z}' = 0$ D1-brane giant. This is analogous to an open superstring ending on the $\bar{Z} = 0$ giant graviton on $\text{AdS}_5 \times S^5$ [8].

Now the coideal boundary algebra \mathcal{B}_{NC} is a subalgebra of \mathcal{A} , and the constraints from the central elements imply that boundary reflections must be chiral. The K -matrix is thus

$$\mathbb{K}_{\mathcal{B}_{\text{NC}}}(p) = \mathbb{K}_{\mathcal{B}_{\text{NC}}}^L(p) \oplus \mathbb{K}_{\mathcal{B}_{\text{NC}}}^R(p). \quad (4.21)$$

Since there are no additional constraints coming from the intertwining equations of \mathcal{B}_{NC} , we must solve the boundary Yang-Baxter equation or reflection equation (4.31), discussed in the next subsection, directly. The solution to this reflection equation has

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_{\text{NC}}}^L(p) &= k_{\mathcal{B}_{\text{NC}}}^L(p) \left[\mathbb{E}_{11} + \frac{(c - x_p^+)}{(c + x_p^-)} \mathbb{E}_{22} \right], \\ \mathbb{K}_{\mathcal{B}_{\text{NC}}}^R(p) &= k_{\mathcal{B}_{\text{NC}}}^R(p) \left[\mathbb{E}_{\bar{1}\bar{1}} + \frac{(1 + cx_p^+)}{(1 - cx_p^-)} \mathbb{E}_{\bar{2}\bar{2}} \right], \end{aligned} \quad (4.22)$$

where $k_{\mathcal{B}_{\text{NC}}}^L(-p)k_{\mathcal{B}_{\text{NC}}}^L(p) = 1$ and $k_{\mathcal{B}_{\text{NC}}}^R(-p)k_{\mathcal{B}_{\text{NC}}}^R(p) = 1$ for unitarity. The parameter $c \in \mathbb{C}$, which is interpreted as a free boundary parameter, has several interesting values. Setting $c = \tan \theta$ gives

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \mathbb{K}_{\mathcal{B}_{\text{NC}}}(p) = \mathbb{K}_{\mathcal{B}_L}(p), \quad \lim_{\theta \rightarrow 0} \mathbb{K}_{\mathcal{B}_{\text{NC}}}(p) = \mathbb{K}_{\mathcal{B}_R}(p),$$

whereas, for $c^2 = -1$, the partial K -matrices (4.22) become identical, since now

$$\frac{(c - x_p^+)}{(c + x_p^-)} = \frac{(1 + cx_p^+)}{(1 - cx_p^-)}.$$

Now we want to ask: Does there exist a larger supersymmetric subalgebra of \mathcal{A} which yields this K -matrix as a solution of the intertwining equations? The answer is yes. Let us introduce the diagonally supersymmetric boundary algebra

$$\mathcal{B}_{\text{D}} = \langle \mathfrak{q}_+, \mathfrak{q}_-, \mathfrak{d}, \tilde{\mathfrak{d}} \rangle, \quad (4.23)$$

which is a coideal subalgebra of \mathcal{A} . Here

$$\begin{aligned} \mathfrak{q}_+ &= \mathfrak{P}^\dagger \mathfrak{Q}_L + ic \mathfrak{P} \mathfrak{E}_R, \\ \mathfrak{q}_- &= \mathfrak{P} \mathfrak{E}_L + ic \mathfrak{P}^\dagger \mathfrak{Q}_R, \\ \mathfrak{d} &= (\mathfrak{H}_L - c^2 \mathfrak{H}_R + ic(\mathfrak{P} + \mathfrak{P}^\dagger)) \mathfrak{K}, \\ \tilde{\mathfrak{d}} &= (\mathfrak{H}_L - c^2 \mathfrak{H}_R - ic(\mathfrak{P} + \mathfrak{P}^\dagger)) \mathfrak{K}, \end{aligned} \quad (4.24)$$

with $\mathfrak{K} = \mathfrak{P} \mathfrak{P}^\dagger$. We also introduce the space \mathcal{M}_{D} generated by $\{\mathfrak{s}_+, \mathfrak{s}_-, \mathfrak{n}, \tilde{\mathfrak{n}}\}$, where

$$\begin{aligned} \mathfrak{s}_+ &= \mathfrak{P}^\dagger \mathfrak{Q}_L - ic \mathfrak{P} \mathfrak{E}_R, \\ \mathfrak{s}_- &= \mathfrak{P} \mathfrak{E}_L - ic \mathfrak{P}^\dagger \mathfrak{Q}_R, \\ \mathfrak{n} &= (\mathfrak{H}_L + c^2 \mathfrak{H}_R + ic(\mathfrak{P} - \mathfrak{P}^\dagger)) \mathfrak{K}, \\ \tilde{\mathfrak{n}} &= (\mathfrak{H}_L + c^2 \mathfrak{H}_R - ic(\mathfrak{P} - \mathfrak{P}^\dagger)) \mathfrak{K}. \end{aligned} \quad (4.25)$$

The generators of \mathcal{B}_{D} and \mathcal{M}_{D} satisfy the following nontrivial identities:

$$\begin{aligned} \{\mathfrak{q}_+, \mathfrak{q}_-\} &= \mathfrak{d}, & \{\mathfrak{q}_+, \mathfrak{s}_-\} &= \mathfrak{n}, & \{\mathfrak{q}_+, \mathfrak{s}_+\} &= 0, \\ \{\mathfrak{s}_+, \mathfrak{s}_-\} &= \tilde{\mathfrak{d}}, & \{\mathfrak{q}_-, \mathfrak{s}_+\} &= \tilde{\mathfrak{n}}, & \{\mathfrak{q}_-, \mathfrak{s}_-\} &= 0, \end{aligned} \quad (4.26)$$

where \mathfrak{d} , $\tilde{\mathfrak{d}}$ and \mathfrak{n} , $\tilde{\mathfrak{n}}$ are central elements. Notice that these relations are identical to those in (3.12). Thus, the associated Lie superalgebra $L(\mathcal{B}_{\text{D}} \oplus \mathcal{M}_{\text{D}})$ is isomorphic to $L(\mathcal{A})$, while $L(\mathcal{B}_{\text{D}}) = \mathfrak{su}(1|1)_{\text{D}} \oplus \mathfrak{u}(1)_{\text{D}}$ consists of the Lie superalgebra $\mathfrak{su}(1|1)_{\text{D}}$ generated by the triple $\{\mathfrak{q}_+, \mathfrak{q}_-, \mathfrak{d}\}$, and $\mathfrak{u}(1)_{\text{D}}$ generated by $\tilde{\mathfrak{d}}$. Moreover,

$$\begin{aligned} \mathfrak{g} &\equiv L(\mathcal{B}_{\text{D}} \oplus \mathcal{M}_{\text{D}}) \cong L(\mathcal{A}) = \mathfrak{su}(1|1)_{\mathbb{C}}^2, \\ \mathfrak{h}_{\text{D}} &\equiv L(\mathcal{B}_{\text{D}}) = \mathfrak{su}(1|1)_{\text{D}} \oplus \mathfrak{u}(1)_{\text{D}} \end{aligned} \quad (4.27)$$

gives a symmetric pair $(\mathfrak{g}, \mathfrak{h}_{\text{D}})$ of Lie superalgebras.

Solving the intertwining equations of the superalgebra \mathcal{B}_{D} , with $c \in \mathbb{C}$ an arbitrary parameter such that $c^2 \neq -1$, gives precisely the K -matrix $\mathbb{K}_{\mathcal{B}_{\text{NC}}}(p)$ obtained above. Hence, the algebra \mathcal{B}_{D} can be understood as a *hidden symmetry* of the non-supersymmetric chiral boundary for $c^2 \neq -1$.

We note, however, that setting $c^2 = -1$ yields a solution of the intertwining equations for \mathcal{B}_{D} which consists of both chiral parts, $\mathbb{K}_{\mathcal{B}_{\text{NC}}}^L(p)$ and $\mathbb{K}_{\mathcal{B}_{\text{NC}}}^R(p)$, and achiral parts $\mathbb{A}_{\mathcal{B}_{\text{NC}}}^R(p)$ and $\mathbb{A}_{\mathcal{B}_{\text{NC}}}^L(p)$. This solution does not satisfy the reflection equation. There is no obvious hidden symmetry for $c^2 = -1$.

Non-supersymmetric achiral boundary algebra.—Let us now consider the possibility of a boundary algebra containing only the magnon Hamiltonian. Let us assume that

$$\mathcal{B}_{\text{NA}} = \langle \mathfrak{H} \rangle, \quad \text{with } \mathfrak{H} = \mathfrak{H}_L + \mathfrak{H}_R, \quad (4.28)$$

which allows for both chiral and achiral reflections. The associated boundary Lie algebra, given by

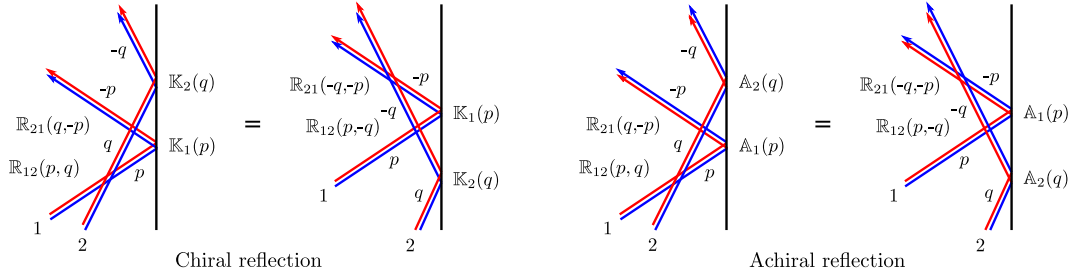


FIG. 5 (color online). A two-magnon reflection off a singlet boundary factorizes into a succession of single-magnon reflections and two-magnon scattering events. The double red and blue line again indicates the *direct sum of left and right magnon states* which scatter by complete R and K -matrices.

$L(\mathcal{B}_{\text{NA}}) = \mathbf{u}(1)_+$, is that of an open superstring on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ ending on the $\bar{Y} = Y' = 0$ or on the $Y = \bar{Y}' = 0$ D1-brane maximal giant graviton. We choose an achiral *ansatz* for the K -matrix

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_{\text{NA}}}(p) &= \begin{pmatrix} 0 & \mathbb{A}_{\mathcal{B}_{\text{NA}}}^{\text{R}}(p) \\ \mathbb{A}_{\mathcal{B}_{\text{NA}}}^{\text{L}}(p) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} (\mathbb{A}_{\mathcal{B}_{\text{NA}}}^{\text{L}}(p) \oplus \mathbb{A}_{\mathcal{B}_{\text{NA}}}^{\text{R}}(p)), \end{aligned} \quad (4.29)$$

where

$$\begin{aligned} \mathbb{A}_{\mathcal{B}_{\text{NA}}}^{\text{R}}(p) &= \sum_{i=1,2} (A_{\mathcal{B}_{\text{NA}}}^{\text{R}}(p))^i E_{ij}, \\ \mathbb{A}_{\mathcal{B}_{\text{NA}}}^{\text{L}}(p) &= \sum_{i=1,2} (A_{\mathcal{B}_{\text{NA}}}^{\text{L}}(p))^i E_{ij}. \end{aligned}$$

There are no constraints from the intertwining equations of \mathcal{B}_{NA} . We must therefore proceed by solving the reflection equation (4.31) directly—which eventually yields a solution of the form

$$\begin{aligned} \mathbb{A}_{\mathcal{B}_{\text{NA}}}^{\text{R}}(p) &= a_{\mathcal{B}_{\text{NA}}}^{\text{R}}(p) (1 + x_p^+ x_p^-)^{-\frac{1}{2}} [-c u_p^{-1} (i + x_p^+) E_{11} \\ &\quad + i \eta_p (E_{12} + E_{21}) - c^{-1} u_p (i - x_p^-) E_{22}], \\ \mathbb{A}_{\mathcal{B}_{\text{NA}}}^{\text{L}}(p) &= a_{\mathcal{B}_{\text{NA}}}^{\text{L}}(p) (1 + x_p^+ x_p^-)^{-\frac{1}{2}} [c^{-1} u_p^{-1} (i + x_p^+) E_{11} \\ &\quad + i \eta_p (E_{12} + E_{21}) + c u_p (i - x_p^-) E_{22}] \end{aligned} \quad (4.30)$$

for any $c \in \mathbb{C}$. Here both $a_{\mathcal{B}_{\text{NA}}}^{\text{R}}(-p) a_{\mathcal{B}_{\text{NA}}}^{\text{L}}(p) = 1$ and $a_{\mathcal{B}_{\text{NA}}}^{\text{L}}(-p) a_{\mathcal{B}_{\text{NA}}}^{\text{R}}(p) = 1$ for unitarity.

We can ask the same question as before: does there exist a coideal subalgebra of \mathcal{A} which yields this K -matrix as a solution of the intertwining equations? We were not able to find such an algebra, but this does not exclude its existence.

2. Reflection equation

A reflection K -matrix must satisfy the boundary Yang-Baxter equation, also called the reflection equation [12],

$$\begin{aligned} \mathbb{K}_2(q) \mathbb{R}_{21}(q, -p) \mathbb{K}_1(p) \mathbb{R}_{12}(p, q) \\ = \mathbb{R}_{21}(-q, -p) \mathbb{K}_1(p) \mathbb{R}_{12}(p, -q) \mathbb{K}_2(q). \end{aligned} \quad (4.31)$$

Let us now consider chiral and achiral reflections separately. This reflection equation is equivalent, for the partial R -matrices and partial chiral K -matrices, to

$$\begin{aligned} \mathbb{K}_2^{\text{b}}(q) \mathbb{R}_{21}^{\text{ba}}(q, -p) \mathbb{K}_1^{\text{a}}(p) \mathbb{R}_{12}^{\text{ab}}(p, q) \\ = \mathbb{R}_{21}^{\text{ba}}(-q, -p) \mathbb{K}_1^{\text{a}}(p) \mathbb{R}_{12}^{\text{ab}}(p, -q) \mathbb{K}_2^{\text{b}}(q), \end{aligned} \quad (4.32)$$

and, for the partial R -matrices and the partial achiral K -matrices, to

$$\begin{aligned} \mathbb{A}_2^{\text{b}}(q) \mathbb{R}_{21}^{\text{ba}}(q, -p) \mathbb{A}_1^{\text{a}}(p) \mathbb{R}_{12}^{\text{ab}}(p, q) \\ = \mathbb{R}_{21}^{\text{ba}}(-q, -p) \mathbb{A}_1^{\text{a}}(p) \mathbb{R}_{12}^{\text{ab}}(p, -q) \mathbb{A}_2^{\text{b}}(q), \end{aligned} \quad (4.33)$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{\text{L}, \text{R}\}$, with $\bar{\text{L}} = \text{R}$ and $\bar{\text{R}} = \text{L}$ (see Fig. 5). Here we define

$$\begin{aligned} \mathbb{R}_{12}^{\text{ab}}(p, q) &= \mathbb{R}^{\text{ab}}(p, q), & \mathbb{R}_{21}^{\text{ab}}(p, q) &= \mathbb{P} \mathbb{R}^{\text{ab}}(p, q) \mathbb{P}, \\ \mathbb{K}_1^{\text{a}}(p) &= \mathbb{K}^{\text{a}}(p) \otimes \mathbb{I}, & \mathbb{K}_2^{\text{a}}(p) &= \mathbb{I} \otimes \mathbb{K}^{\text{a}}(p), \\ \mathbb{A}_1^{\text{a}}(p) &= \mathbb{A}^{\text{a}}(p) \otimes \mathbb{I}, & \mathbb{A}_2^{\text{a}}(p) &= \mathbb{I} \otimes \mathbb{A}^{\text{a}}(p), \end{aligned}$$

and similarly for $\mathbb{R}_{12}(p, q)$ and $\mathbb{R}_{21}(p, q)$, and $\mathbb{K}_1(p)$ and $\mathbb{K}_2(q)$ in terms of the complete R -matrices and K -matrices. All the K -matrices constructed above are found to satisfy the reflection equation, if we impose the parity and discrete LR symmetry constraints on the scale factors $s^{\text{ab}}(p, q)$ in the partial R -matrices, which were described in Sec. III C 1.

C. Vector boundary

The totally supersymmetric boundary superalgebra \mathcal{B}_{T} is the coideal subalgebra of \mathcal{A} generated as

$$\mathcal{B}_{\text{T}} = \langle \mathfrak{Q}_{\text{L}}, \mathfrak{E}_{\text{L}}, \mathfrak{H}_{\text{L}}, \mathfrak{Q}_{\text{R}}, \mathfrak{E}_{\text{R}}, \mathfrak{H}_{\text{R}}, \mathfrak{P}, \mathfrak{P}^{\dagger} \rangle. \quad (4.34)$$

Note that elements \mathfrak{U} , $\mathfrak{U}^{-1} \notin \mathcal{B}_\tau$. Since \mathfrak{U} and \mathfrak{U}^{-1} appear in the left tensor factor of the coproduct only, the coideal property $\Delta(\mathfrak{b}) \in \mathcal{A} \otimes \mathcal{B}_\tau$ for all $\mathfrak{b} \in \mathcal{B}_\tau$ is satisfied.

The vector boundary state transforms in the left- or right-moving representation, π_B or $\bar{\pi}_B$, of the boundary Lie superalgebra $L(\mathcal{B}_\tau) = \mathfrak{su}(1|1)_\mathbb{C}^2$. This boundary superalgebra arises from an open superstring ending on the $Z = 0$ or $Z' = 0$ half of the D5-brane maximal giant graviton, or the $Z = Z' = 0$ D1-brane giant (the intersection of $Z = 0$ and $Z' = 0$ giants), shown in Tables I and II. This is analogous to an open superstring attached to the $Z = 0$ giant graviton in AdS₅ × S⁵ [8].

Boundary representations π_B and $\bar{\pi}_B$.—We define the vector spaces associated with left- and right-moving boundary vector states in the same way as for the magnons in the bulk:

$$\begin{aligned} V_B &= \text{span}_{\mathbb{C}}\{|\phi_B\rangle, |\psi_B\rangle\} \cong \mathbb{C}^{1|1}, \\ \bar{V}_B &= \text{span}_{\mathbb{C}}\{|\bar{\phi}_B\rangle, |\bar{\psi}_B\rangle\} \cong \mathbb{C}^{1|1}, \end{aligned}$$

and set $W_B = V_B \oplus \bar{V}_B \cong \mathbb{C}^{2|2}$. The left boundary representation $\pi_B: \mathcal{B}_\tau \rightarrow \text{End}(\mathbb{C}^{1|1})$ is given by (3.20), with the subindex p replaced by B . The right boundary representation $\bar{\pi}_B: \mathcal{B}_\tau \rightarrow \text{End}(\mathbb{C}^{1|1})$ is analogous to (3.21) subject to the $p \rightarrow B$ replacement. The primed space $W'_B = V'_B \oplus \bar{V}'_B$ and representations π'_B and $\bar{\pi}'_B$ are defined in a similar way. We will use the notation $\mathcal{W}_B \in \{W_B, W'_B\}$.

We may choose the following parametrization of the boundary parameters:

$$\begin{aligned} a_B &= \sqrt{h}\eta_B, & b_B &= \sqrt{h}\eta_B, & c_B &= \sqrt{h}\frac{i\eta_B}{x_B}, \\ d_B &= \sqrt{h}\frac{i\eta_B}{x_B}, & \eta_B^2 &= -ix_B, \end{aligned} \quad (4.35)$$

where x_B is the boundary Zhukovski variable satisfying the boundary mass-shell identity

$$x_B + \frac{1}{x_B} = \frac{im_B}{h}, \quad (4.36)$$

with m_B the boundary mass parameter. This boundary representation can be obtained from π_p by setting the momentum to $p = \pi$ (so that $x_B \equiv x_{p=\pi}^+ = -x_{p=\pi}^-$), defining the boundary mass parameter to be $m_B = m_{p=\pi}$ and rescaling $h \rightarrow h/2$. We expect this representation to describe a magnon state absorbed by the boundary. Notice that the total bulk and boundary momentum before and after the reflection sum to $p + \pi - p + \pi = 2\pi \sim 0$ due to periodicity—which is conservation of momentum for an elastic reflection off an infinitely massive boundary. We will further justify these boundary parameters when we discuss the constraints from the central elements in \mathcal{B}_τ .

Boundary intertwining equations.—For this case of a vector boundary, the intertwining equations for the K-matrix $K_\tau(p, \mathbf{B}) \in \text{End}(\mathcal{W}_p \otimes \mathcal{W}_B)$, corresponding to $\mathbb{K}_\tau(p, \mathbf{B}) \in \text{End}(\mathbb{C}^{2|2} \otimes \mathbb{C}^{2|2})$, are very similar to those imposed on the bulk R -matrix in Sec. III. The boundary scattering involves three states: bulk magnon, boundary vector state and the boundary singlet vacuum state. Since the boundary vacuum state is described by the trivial representation ϵ satisfying $\epsilon(\mathfrak{b}) = 0$ for all $\mathfrak{b} \in \mathcal{B}_\tau$, we can drop it to obtain the boundary intertwining equations for the vector boundary:

$$\begin{aligned} ((\pi_{-p} \oplus \bar{\pi}_{-p}) \otimes (\pi_B \oplus \bar{\pi}_B))(\Delta(\mathfrak{b}))\mathbb{K}_{\mathcal{B}_\tau}(p, \mathbf{B}) \\ = \mathbb{K}_{\mathcal{B}_\tau}(p, \mathbf{B})((\pi_p \oplus \bar{\pi}_p) \otimes (\pi_B \oplus \bar{\pi}_B))(\Delta(\mathfrak{b})) \end{aligned} \quad (4.37)$$

for all $\mathfrak{b} \in \mathcal{B}_\tau$.

Constraints from the central elements for vector boundary.—Central elements of \mathcal{B}_τ imply important constraints for reflections off a vector boundary and for the parameters of the boundary representation π_B . Recall that central elements must intertwine the reflection matrix trivially, which means that, for a chiral reflection,

$$\begin{aligned} (\pi_p \otimes \pi_B)(\Delta(\mathfrak{C})) &= (\pi_{-p} \otimes \pi_B)(\Delta(\mathfrak{C})), \\ (\bar{\pi}_p \otimes \bar{\pi}_B)(\Delta(\mathfrak{C})) &= (\bar{\pi}_{-p} \otimes \bar{\pi}_B)(\Delta(\mathfrak{C})) \end{aligned}$$

for all $\mathfrak{C} \in \{\mathfrak{S}_a, \mathfrak{P}, \mathfrak{P}^\dagger\}$. Suppose that we do not know the boundary representation π_B . Since the elements \mathfrak{P} and \mathfrak{P}^\dagger are central, we must have $\pi_B(\mathfrak{P}) = f_B \mathbb{1}$ and $\pi_B(\mathfrak{P}^\dagger) = f_B^\dagger \mathbb{1}$ for some $f_B, f_B^\dagger \in \mathbb{C}$. Using $\pi_p(\mathfrak{P}) = h(1 - u_p^2)\mathbb{1}$ and $\pi_p(\mathfrak{P}^\dagger) = h(1 - u_p^{-2})\mathbb{1}$, and the boundary intertwining equations, we obtain

$$\begin{aligned} h(1 - u_p^2) + u_p^2 f_B &= h(1 - u_p^{-2}) + u_p^{-2} f_B, \\ h(1 - u_p^{-2}) + u_p^{-2} f_B^\dagger &= h(1 - u_p^2) + u_p^2 f_B^\dagger, \end{aligned}$$

giving $f_B = f_B^\dagger = h$. Now notice that $\pi_p(\mathfrak{P})|_{p=\pi, h \rightarrow h/2} = \pi(\mathfrak{P}^\dagger)|_{p=\pi, h \rightarrow h/2} = h\mathbb{1}$, which justifies our interpretation of the vector boundary as a bulk magnon state with momentum $p = \pi$. A vector state at the boundary is always necessary for the boundary algebra \mathcal{B}_τ , since the central elements \mathfrak{P} and \mathfrak{P}^\dagger cannot be preserved by a singlet boundary.

The next step is to check the intertwining equations for achiral reflections. For example,

$$\begin{aligned} (\pi_p \otimes \pi_B)(\Delta(\mathfrak{S}_a)) &\neq (\bar{\pi}_{-p} \otimes \pi_B)(\Delta(\mathfrak{S}_a)), \\ (\bar{\pi}_p \otimes \bar{\pi}_B)(\Delta(\mathfrak{S}_a)) &\neq (\pi_{-p} \otimes \bar{\pi}_B)(\Delta(\mathfrak{S}_a)). \end{aligned}$$

However,

$$\begin{aligned}(\pi_p \otimes \bar{\pi}_B)(\Delta(\mathfrak{C})) &= (\bar{\pi}_{-p} \otimes \pi_B)(\Delta(\mathfrak{C})), \\(\bar{\pi}_p \otimes \pi_B)(\Delta(\mathfrak{C})) &= (\pi_{-p} \otimes \bar{\pi}_B)(\Delta(\mathfrak{C})),\end{aligned}$$

for all $\mathfrak{C} \in \{\mathfrak{H}_a, \mathfrak{P}, \mathfrak{P}^\dagger\}$. This implies that the total number of left or right states is a conserved quantum number as a result of the central elements in \mathcal{B}_T . Scattering off a vector boundary, which intertwines the representations $\pi_p \otimes \bar{\pi}_B$ and $\bar{\pi}_{-p} \otimes \pi_B$ or $\bar{\pi}_p \otimes \pi_B$ and $\pi_{-p} \otimes \bar{\pi}_B$, is forbidden rather by the intertwining equations for the supercharges. Thus, the K -matrix $\mathbb{K}_{\mathcal{B}_T}(p)$ decomposes into the four sectors (left-from-left, right-from-right, left-from-right and right-from-left) with $\mathbb{K}_{\mathcal{B}_T}^{\text{ab}}(p)$ describing the chiral reflection of a magnon of chirality \mathfrak{a} from a boundary of chirality \mathfrak{b} .

1. Complete and partial K -matrices

The complete K -matrix decomposes into the direct sum

$$\begin{aligned}\mathbb{K}_{\mathcal{B}_T}(p, \mathbf{B}) &= \mathbb{K}_{\mathcal{B}_T}^{\text{LL}}(p, \mathbf{B}) \oplus \mathbb{K}_{\mathcal{B}_T}^{\text{RR}}(p, \mathbf{B}) \\ &\oplus \mathbb{K}_{\mathcal{B}_T}^{\text{LR}}(p, \mathbf{B}) \oplus \mathbb{K}_{\mathcal{B}_T}^{\text{RL}}(p, \mathbf{B})\end{aligned}\quad (4.38)$$

of partial K -matrices in the LL, RR, LR and RL decoupled sectors. The partial K -matrices are solutions of the boundary intertwining and reflection equations, and the unitarity condition.

Left-left and right-right sectors.—We write the partial K -matrices in the LL and RR sectors as

$$\begin{aligned}\mathbb{K}_{\mathcal{B}_T}^{\text{LL}}(p, \mathbf{B}) &= \sum_{i,j,k,l=1,2} (K^{\text{LL}}(p, \mathbf{B}))_{jl}^{ik} (\mathbb{E}_{ij} \otimes \mathbb{E}_{kl}), \\ \mathbb{K}_{\mathcal{B}_T}^{\text{RR}}(p, \mathbf{B}) &= \sum_{i,j,k,l=\bar{1},\bar{2}} (K^{\text{RR}}(p, \mathbf{B}))_{jl}^{ik} (\mathbb{E}_{ij} \otimes \mathbb{E}_{kl}),\end{aligned}$$

which depend on the magnon momentum p and its mass m_p through the Zhukovski variables x_p^\pm , and the boundary mass parameter m_B through x_B . The boundary intertwining equations are

$$\begin{aligned}(\pi_{-p} \otimes \pi_B)(\Delta(\mathfrak{b})) \mathbb{K}_{\mathcal{B}_T}^{\text{LL}}(p, \mathbf{B}) \\ &= \mathbb{K}_{\mathcal{B}_T}^{\text{LL}}(p, \mathbf{B})(\pi_p \otimes \pi_B)(\Delta(\mathfrak{b})), \\ (\bar{\pi}_{-p} \otimes \bar{\pi}_B)(\Delta(\mathfrak{b})) \mathbb{K}_{\mathcal{B}_T}^{\text{RR}}(p, \mathbf{B}) \\ &= \mathbb{K}_{\mathcal{B}_T}^{\text{RR}}(p, \mathbf{B})(\bar{\pi}_p \otimes \bar{\pi}_B)(\Delta(\mathfrak{b}))\end{aligned}$$

for all $\mathfrak{b} \in \mathcal{B}_T$. The unitarity condition implies $\mathbb{K}_{\mathcal{B}_T}^{\text{LL}}(-p, \mathbf{B}) \mathbb{K}_{\mathcal{B}_T}^{\text{LL}}(p, \mathbf{B}) = \mathbb{1}$ and $\mathbb{K}_{\mathcal{B}_T}^{\text{RR}}(-p, \mathbf{B}) \mathbb{K}_{\mathcal{B}_T}^{\text{RR}}(p, \mathbf{B}) = \mathbb{1}$. We find the solutions to these boundary intertwining equations to be

$$\begin{aligned}\mathbb{K}_{\mathcal{B}_T}^{\text{LL}}(p, \mathbf{B}) \\ &= k_{\mathcal{B}_T}^{\text{LL}}(p, \mathbf{B}) \left[\mathbb{E}_{11} \otimes \mathbb{E}_{11} + \frac{(x_p^+ - u_p^{-2} x_B)}{(x_p^- - x_B)} \mathbb{E}_{11} \otimes \mathbb{E}_{22} \right. \\ &\quad + \frac{(x_p^- + u_p^2 x_B)}{(x_p^- - x_B)} \mathbb{E}_{22} \otimes \mathbb{E}_{11} + \frac{(x_p^+ + x_B)}{(x_p^- - x_B)} \mathbb{E}_{22} \otimes \mathbb{E}_{22} \\ &\quad + \frac{i(u_p + u_p^{-1}) \eta_p \eta_B}{(x_p^- - x_B)} \mathbb{E}_{12} \otimes \mathbb{E}_{21} \\ &\quad \left. - \frac{i(u_p + u_p^{-1}) \eta_p \eta_B}{(x_p^- - x_B)} \mathbb{E}_{21} \otimes \mathbb{E}_{12} \right],\end{aligned}\quad (4.39)$$

$$\begin{aligned}\mathbb{K}_{\mathcal{B}_T}^{\text{RR}}(p, \mathbf{B}) \\ &= k_{\mathcal{B}_T}^{\text{RR}}(p, \mathbf{B}) \left[\mathbb{E}_{\bar{1}\bar{1}} \otimes \mathbb{E}_{\bar{1}\bar{1}} + \frac{(x_p^+ - u_p^{-2} x_B)}{(x_p^- - x_B)} \mathbb{E}_{\bar{1}\bar{1}} \otimes \mathbb{E}_{\bar{2}\bar{2}} \right. \\ &\quad + \frac{(x_p^- + u_p^2 x_B)}{(x_p^- - x_B)} \mathbb{E}_{\bar{2}\bar{2}} \otimes \mathbb{E}_{\bar{1}\bar{1}} + \frac{(x_p^+ + x_B)}{(x_p^- - x_B)} \mathbb{E}_{\bar{2}\bar{2}} \otimes \mathbb{E}_{\bar{2}\bar{2}} \\ &\quad + \frac{i(u_p + u_p^{-1}) \eta_p \eta_B}{(x_p^- - x_B)} \mathbb{E}_{\bar{1}\bar{2}} \otimes \mathbb{E}_{\bar{2}\bar{1}} \\ &\quad \left. - \frac{i(u_p + u_p^{-1}) \eta_p \eta_B}{(x_p^- - x_B)} \mathbb{E}_{\bar{2}\bar{1}} \otimes \mathbb{E}_{\bar{1}\bar{2}} \right],\end{aligned}\quad (4.40)$$

where $k_{\mathcal{B}_T}^{\text{LL}}(-p, \mathbf{B}) k_{\mathcal{B}_T}^{\text{LL}}(p, \mathbf{B}) = 1$ and $k_{\mathcal{B}_T}^{\text{RR}}(-p, \mathbf{B}) k_{\mathcal{B}_T}^{\text{RR}}(p, \mathbf{B}) = 1$ for unitarity.

Left-right and right-left sectors.—We write the partial K -matrices in the LR and RL sectors as

$$\begin{aligned}\mathbb{K}_{\mathcal{B}_T}^{\text{LR}}(p, \mathbf{B}) &= \sum_{\substack{i,j=1,2 \\ k,l=\bar{1},\bar{2}}} (K^{\text{LR}}(p, \mathbf{B}))_{jl}^{ik} (\mathbb{E}_{ij} \otimes \mathbb{E}_{kl}), \\ \mathbb{K}_{\mathcal{B}_T}^{\text{RL}}(p, \mathbf{B}) &= \sum_{\substack{i,j=\bar{1},\bar{2} \\ k,l=1,2}} (K^{\text{RL}}(p, \mathbf{B}))_{jl}^{ik} (\mathbb{E}_{ij} \otimes \mathbb{E}_{kl}).\end{aligned}$$

The left-right K -matrix is a solution of the boundary intertwining equations

$$\begin{aligned}(\pi_{-p} \otimes \bar{\pi}_B)(\Delta(\mathfrak{b})) \mathbb{K}_{\mathcal{B}_T}^{\text{LR}}(p, \mathbf{B}) \\ &= \mathbb{K}_{\mathcal{B}_T}^{\text{LR}}(p, \mathbf{B})(\pi_p \otimes \bar{\pi}_B)(\Delta(\mathfrak{b})), \\ (\bar{\pi}_{-p} \otimes \pi_B)(\Delta(\mathfrak{b})) \mathbb{K}_{\mathcal{B}_T}^{\text{RL}}(p, \mathbf{B}) \\ &= \mathbb{K}_{\mathcal{B}_T}^{\text{RL}}(p, \mathbf{B})(\bar{\pi}_p \otimes \pi_B)(\Delta(\mathfrak{b}))\end{aligned}$$

for all $\mathfrak{b} \in \mathcal{B}_T$. The unitarity condition implies $\mathbb{K}_{\mathcal{B}_T}^{\text{LR}}(-p, \mathbf{B}) \mathbb{K}_{\mathcal{B}_T}^{\text{LR}}(p, \mathbf{B}) = \mathbb{1}$ and $\mathbb{K}_{\mathcal{B}_T}^{\text{RL}}(-p, \mathbf{B}) \mathbb{K}_{\mathcal{B}_T}^{\text{RL}}(p, \mathbf{B}) = \mathbb{1}$. The solutions to the boundary intertwining equations take the form

$$\begin{aligned}\mathbb{K}_{\mathcal{B}_T}^{\text{LR}}(p, \mathbf{B}) &= k_{\mathcal{B}_T}^{\text{LR}}(p, \mathbf{B}) [(1 - x_p^+ x_B)(1 + x_p^- x_B)]^{-\frac{1}{2}} [(x_p^+ x_B + u_p^{-2}) \mathbb{E}_{11} \otimes \mathbb{E}_{\bar{1}\bar{1}} + (x_p^- x_B + 1) \mathbb{E}_{11} \otimes \mathbb{E}_{\bar{2}\bar{2}} + (x_p^+ x_B - 1) \mathbb{E}_{22} \otimes \mathbb{E}_{\bar{1}\bar{1}} \\ &\quad + (x_p^- x_B - u_p^2) \mathbb{E}_{22} \otimes \mathbb{E}_{\bar{2}\bar{2}} - (u_p + u_p^{-1}) \eta_p \eta_B \mathbb{E}_{12} \otimes \mathbb{E}_{\bar{1}\bar{2}} + (u_p + u_p^{-1}) \eta_p \eta_B \mathbb{E}_{21} \otimes \mathbb{E}_{\bar{2}\bar{1}}],\end{aligned}\quad (4.41)$$

$$\begin{aligned} \mathbb{K}_{B\tau}^{\text{RL}}(p, \mathbf{B}) &= k_{B\tau}^{\text{LR}}(p, \mathbf{B})[(1 - x_p^+ x_B)(1 + x_p^- x_B)]^{-\frac{1}{2}}[(x_p^+ x_B + u_p^{-2})\mathbb{E}_{\bar{1}\bar{1}} \otimes \mathbb{E}_{11} + (x_p^- x_B + 1)\mathbb{E}_{\bar{1}\bar{1}} \otimes \mathbb{E}_{22} + (x_p^+ x_B - 1)\mathbb{E}_{\bar{2}\bar{2}} \otimes \mathbb{E}_{11} \\ &\quad + (x_p^- x_B - u_p^2)\mathbb{E}_{\bar{2}\bar{2}} \otimes \mathbb{E}_{22} - (u_p + u_p^{-1})\eta_p \eta_B \mathbb{E}_{\bar{1}\bar{2}} \otimes \mathbb{E}_{12} + (u_p + u_p^{-1})\eta_p \eta_B \mathbb{E}_{\bar{2}\bar{1}} \otimes \mathbb{E}_{21}], \end{aligned} \quad (4.42)$$

where $k_{B\tau}^{\text{LR}}(-p, \mathbf{B})k_{B\tau}^{\text{LR}}(p, \mathbf{B})=1$ and $k_{B\tau}^{\text{RL}}(-p, \mathbf{B})k_{B\tau}^{\text{RL}}(p, \mathbf{B})=1$ for unitarity.

We note that it is necessary to impose *boundary crossing symmetry conditions* on all our K -matrices, which constrain the scale factors $k^a(p)$ and $a^a(p)$. We anticipate that these will be related to the dressing phases of Ref. [38] in the bulk R -matrix. We leave this for future research.

2. Reflection equation

The reflection equation for the complete K -matrix is

$$\begin{aligned} \mathbb{K}_{23}(q, \mathbf{B})\mathbb{R}_{21}(q, -p)\mathbb{K}_{13}(p, \mathbf{B})\mathbb{R}_{12}(p, q) \\ = \mathbb{R}_{21}(-q, -p)\mathbb{K}_{13}(p, \mathbf{B})\mathbb{R}_{12}(p, -q)\mathbb{K}_{23}(q, \mathbf{B}), \end{aligned} \quad (4.43)$$

which is equivalent, for the partial R -matrices and K -matrices, to

$$\begin{aligned} \mathbb{K}_{23}^{\text{bc}}(q, \mathbf{B})\mathbb{R}_{21}^{\text{ba}}(q, -p)\mathbb{K}_{13}^{\text{ac}}(p, \mathbf{B})\mathbb{R}_{12}^{\text{ab}}(p, q) \\ = \mathbb{R}_{21}^{\text{ba}}(-q, -p)\mathbb{K}_{13}^{\text{ac}}(p, \mathbf{B})\mathbb{R}_{12}^{\text{ab}}(p, -q)\mathbb{K}_{23}^{\text{bc}}(q, \mathbf{B}), \end{aligned} \quad (4.44)$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \{\text{L}, \text{R}\}$ (see Fig. 6). Here we define

$$\begin{aligned} \mathbb{R}_{12}^{\text{ab}}(p, q) &= \mathbb{R}^{\text{ab}}(p, q) \otimes \mathbb{1}, \\ \mathbb{R}_{21}^{\text{ab}}(p, q) &= (\mathbb{P}\mathbb{R}^{\text{ab}}(p, q)\mathbb{P}) \otimes \mathbb{1}, \\ \mathbb{K}_{13}^{\text{ab}}(p, \mathbf{B}) &= (\mathbb{1} \otimes \mathbb{P})(\mathbb{K}^{\text{ab}}(p, \mathbf{B}) \otimes \mathbb{1})(\mathbb{1} \otimes \mathbb{P}), \\ \mathbb{K}_{23}^{\text{ab}}(p, \mathbf{B}) &= \mathbb{1} \otimes \mathbb{K}^{\text{ab}}(p, \mathbf{B}), \end{aligned}$$

and similarly for $\mathbb{R}_{12}(p, q)$, $\mathbb{R}_{21}(q, p)$, $\mathbb{K}_{13}(p, \mathbf{B})$ and $\mathbb{K}_{23}(q, \mathbf{B})$ in terms of complete R - and K -matrices.

A direct computation shows that the R -matrices and K -matrices for the vector boundary satisfy (4.44) for (\mathbf{abc}) in the homogeneous (LLL, RRR) and mixed (LLR, LRL,

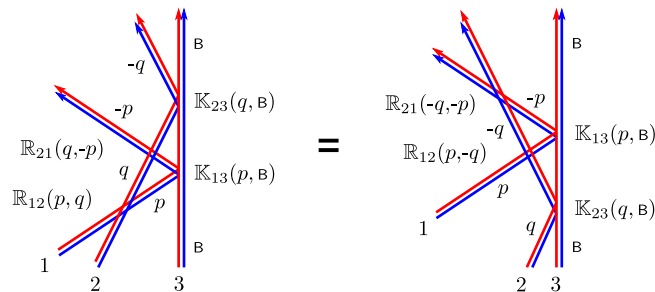


FIG. 6 (color online). A two-magnon reflection off a vector boundary factorizes into a succession of single-magnon reflections and two-magnon scattering events. The double red and blue line again indicates a *direct sum of left and right magnons or boundary vector states*.

RLL, LRR, RLR, RRL) sectors. This ensures that the complete R -matrix and K -matrix satisfy (4.43).

V. INTEGRABLE CLOSED $\mathfrak{psu}(1,1|2)^2$ SPIN-CHAIN AND SCATTERING MATRICES IN AdS₃ \times S³ \times T⁴

The bosonic isometry group of the AdS₃ \times S³ \times T⁴ supergravity background is

$$SO(2, 2) \times SO(4) \times U(1)^4,$$

with the Lie algebra splitting into left- and right-movers

$$\begin{aligned} \mathfrak{so}(2, 2) &\sim \mathfrak{su}(1, 1)_{\text{L}} \oplus \mathfrak{su}(1, 1)_{\text{R}}, \\ \mathfrak{so}(4) &\sim \mathfrak{su}(2)_{\text{L}} \oplus \mathfrak{su}(2)_{\text{R}}. \end{aligned}$$

The bosonic isometries can thus be rearranged into

$$[\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2)]_{\text{L}} \oplus [\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2)]_{\text{R}} \oplus \mathfrak{u}(1)^4,$$

which is the bosonic part of the superisometry algebra

$$\mathfrak{psu}(1, 1|2)_{\text{L}} \oplus \mathfrak{psu}(1, 1|2)_{\text{R}} \oplus \mathfrak{u}(1)^4.$$

Massive excitations of the world sheet of a closed superstring on AdS₃ \times S³ \times T⁴ can be identified with the magnon excitations of a homogeneous double-row $\mathfrak{psu}(1, 1|2)^2$ closed spin-chain. These magnons transform under a centrally extended $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$ algebra¹⁸ [two copies of $\mathfrak{su}(1|1)_{\mathbb{C}}^2$ with the Cartan and central elements identified, respectively]. The left- and right-moving excitations decouple in the weak coupling limit. This section contains a review based on Ref. [21] of this integrable closed $\mathfrak{psu}(1, 1|2)^2$ spin-chain and the S -matrix describing the scattering of magnon excitations.

A. $\mathfrak{psu}(1,1|2)^2$ closed spin-chain with $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]^2$ excitations

1. Single-row $\mathfrak{psu}(1,1|2)$ spin-chain with $\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)$ excitations

Symmetry generators.—The $\mathfrak{psu}(1, 1|2)$ superalgebra shown in Appendix B has bosonic generators

$$\mathfrak{F}_0, \mathfrak{F}_b \in \mathfrak{su}(1, 1), \quad \mathfrak{Q}_\beta, \mathfrak{Q}_5 \in \mathfrak{su}(2)$$

¹⁸Note that here we use $\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)$ to denote the direct sum of Lie algebras as vector spaces.

of $\mathfrak{su}(1,1) \oplus \mathfrak{su}(2)$, and fermionic generators $\mathfrak{Q}_{b\beta\dot{\beta}}$ labeled by $b, \beta, \dot{\beta} = \pm$ indices. There is also a $\mathfrak{u}(1)$ automorphism \mathfrak{R}_8 .

Sites.—A site in this homogeneous single-row $\mathfrak{psu}(1,1|2)$ spin-chain is the module

$$M \equiv M\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \text{span}_{\mathbb{C}}\{|\phi_{\dot{\beta}}^{(n)}\rangle, |\psi_{\beta}^{(n)}\rangle\}.$$

A vector at this site transforms in the half-BPS representation of the $\mathfrak{psu}(1,1|2)$ superalgebra shown in Appendix B. The vacuum state is

$$|\mathcal{Z}\rangle = |\phi_+^{(0)}\rangle, \quad (5.1)$$

and the four fundamental excitations $\varphi^{\beta\dot{\beta}}$ are

$$\begin{aligned} |\varphi^{+\dot{+}}\rangle &= |\phi_+^{(0)}\rangle, & |\varphi^{-\dot{-}}\rangle &= |\phi_+^{(1)}\rangle, \\ |\varphi^{+\dot{-}}\rangle &= -|\psi_-^{(0)}\rangle, & |\varphi^{-\dot{+}}\rangle &= |\psi_+^{(0)}\rangle, \end{aligned} \quad (5.2)$$

which transform under a $\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)$ algebra with fermionic $\mathfrak{psu}(1|1)^2$ generators

$$\begin{aligned} \mathfrak{Q}_1 &\equiv \mathfrak{Q}_{-++}, & \mathfrak{Q}_2 &\equiv -\mathfrak{Q}_{--+}, \\ \mathfrak{S}_1 &\equiv \mathfrak{Q}_{+--}, & \mathfrak{S}_2 &\equiv \mathfrak{Q}_{-++} \end{aligned}$$

satisfying $\{\mathfrak{Q}_i, \mathfrak{S}_j\} = \mathfrak{H}\delta_{ij}$, for $i, j \in \{1, 2\}$. The additional bosonic $\mathfrak{u}(1)$ generator $\mathfrak{H} = -\mathfrak{F}_0 - \mathfrak{L}_5$ is the magnon Hamiltonian. We notice that this $\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)$ algebra can, alternatively, be viewed as two copies of $\mathfrak{su}(1|1)^2$ with the Cartan elements identified. We can extend this algebra to $\mathfrak{u}(1)^2 \ltimes [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]$ by introducing $\mathfrak{X}_1 = -\frac{1}{2}\mathfrak{L}_5 - \frac{1}{2}\mathfrak{R}_8$ and $\mathfrak{X}_2 = -\frac{1}{2}\mathfrak{L}_5 + \frac{1}{2}\mathfrak{R}_8$, which satisfy

$$\begin{aligned} \{\mathfrak{Q}_i, \mathfrak{S}_j\} &= \mathfrak{H}\delta_{ij}, & [\mathfrak{X}_i, \mathfrak{Q}_j] &= -\frac{1}{2}\delta_{ij}\mathfrak{Q}_i, \\ [\mathfrak{X}_i, \mathfrak{S}_j] &= \frac{1}{2}\delta_{ij}\mathfrak{S}_i, \end{aligned} \quad (5.3)$$

but do not annihilate the vacuum state.

Spin-chain.—The homogeneous single-row spin-chain with J sites is now identified with the module $M^{\otimes J}$. The spin-chain vacuum and fundamental excitations are

$$|0\rangle = |\mathcal{Z}^J\rangle, \quad |\varphi_{(n)}^{\beta\dot{\beta}}\rangle = |\mathcal{Z}^{n-1}\varphi^{\beta\dot{\beta}}\mathcal{Z}^{J-n}\rangle. \quad (5.4)$$

Single-magnon excitations are constructed as vectors in momentum space:

$$|\varphi_p^{\beta\dot{\beta}}\rangle = \sum_{n=1}^J e^{ipn} |\varphi_{(n)}^{\beta\dot{\beta}}\rangle. \quad (5.5)$$

The action of the fermionic $\mathfrak{psu}(1|1)^2$ generators on a magnon state is

$$\begin{aligned} \mathfrak{Q}_1|\varphi_p^{+\dot{\beta}}\rangle &= |\varphi_p^{-\dot{\beta}}\rangle, & \mathfrak{S}_1|\varphi_p^{+\dot{\beta}}\rangle &= 0, \\ \mathfrak{Q}_1|\varphi_p^{-\dot{\beta}}\rangle &= 0, & \mathfrak{S}_1|\varphi_p^{-\dot{\beta}}\rangle &= |\varphi_p^{+\dot{\beta}}\rangle, \\ \mathfrak{Q}_2|\varphi_p^{\beta+}\rangle &= (-1)^{\delta_{\beta-}}|\varphi_p^{\beta-}\rangle, & \mathfrak{S}_2|\varphi_p^{\beta+}\rangle &= 0, \\ \mathfrak{Q}_2|\varphi_p^{\beta-}\rangle &= 0, & \mathfrak{S}_2|\varphi_p^{\beta-}\rangle &= (-1)^{\delta_{\beta-}}|\varphi_p^{\beta+}\rangle. \end{aligned} \quad (5.6)$$

These low-lying magnon excitations have energy 1 with respect to the magnon Hamiltonian \mathfrak{H} . Here also

$$\begin{aligned} \mathfrak{X}_i|0\rangle &= -\frac{J}{2}|0\rangle, & \mathfrak{X}_i|\varphi_p^{+\dot{+}}\rangle &= \left(-\frac{J}{2} + \frac{1}{2}\right)|\varphi_p^{+\dot{+}}\rangle, \\ \mathfrak{X}_i|\varphi_p^{-\dot{-}}\rangle &= -\frac{J}{2}|\varphi_p^{-\dot{-}}\rangle, \\ \mathfrak{X}_1|\varphi_p^{+\dot{-}}\rangle &= \left(-\frac{J}{2} + \frac{1}{2}\right)|\varphi_p^{+\dot{-}}\rangle, & \mathfrak{X}_2|\varphi_p^{+\dot{-}}\rangle &= -\frac{J}{2}|\varphi_p^{+\dot{-}}\rangle, \\ \mathfrak{X}_1|\varphi_p^{-\dot{+}}\rangle &= -\frac{J}{2}|\varphi_p^{-\dot{+}}\rangle, & \mathfrak{X}_2|\varphi_p^{-\dot{+}}\rangle &= \left(-\frac{J}{2} + \frac{1}{2}\right)|\varphi_p^{-\dot{+}}\rangle. \end{aligned}$$

The standard procedure by which single-magnon excitations may be generalized to multimagnon excitations was described in Sec. III A 1.

2. Double-row $\mathfrak{psu}(1,1|2)^2$ closed spin-chain with $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]^2$ excitations

The homogeneous double-row $\mathfrak{psu}(1,1|2)^2$ spin-chain consists of left- and right-moving $\mathfrak{psu}(1,1|2)_{\text{L}}$ and $\mathfrak{psu}(1,1|2)_{\text{R}}$ spin-chains which decouple at weak coupling.

Sites.—Sites of the left- and right-moving spin-chains form the module $M_{\text{L}} \otimes M_{\text{R}}$, with M_{L} and M_{R} the left and right copies of the module M . The vacuum state and fundamental excitations are

$$\begin{aligned} |\mathcal{Z}\rangle &= \left| \begin{pmatrix} \mathcal{Z}_{\text{L}} \\ \mathcal{Z}_{\text{R}} \end{pmatrix} \right\rangle, & |\varphi^{\beta\dot{\beta}}\rangle &= \left| \begin{pmatrix} \varphi_{\text{L}}^{\beta\dot{\beta}} \\ \mathcal{Z}_{\text{R}} \end{pmatrix} \right\rangle, \\ |\bar{\varphi}^{\beta\dot{\beta}}\rangle &= \left| \begin{pmatrix} \mathcal{Z}_{\text{L}} \\ \varphi_{\text{R}}^{\beta\dot{\beta}} \end{pmatrix} \right\rangle, \end{aligned} \quad (5.7)$$

which transform under the $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]^2$ algebra

$$\begin{aligned} \{\mathfrak{Q}_{\text{a}i}, \mathfrak{S}_{\text{b}j}\} &= \mathfrak{H}_{\text{a}}\delta_{\text{ab}}\delta_{ij}, & [\mathfrak{X}_{\text{a}i}, \mathfrak{Q}_{\text{b}j}] &= -\frac{1}{2}\mathfrak{Q}_{\text{a}i}\delta_{\text{ab}}\delta_{ij}, \\ [\mathfrak{X}_{\text{a}i}, \mathfrak{S}_{\text{b}j}] &= \frac{1}{2}\mathfrak{S}_{\text{a}i}\delta_{\text{ab}}\delta_{ij} \end{aligned}$$

with $\text{a}, \text{b} \in \{\text{L}, \text{R}\}$ and $i, j \in \{1, 2\}$. Notice that $\mathfrak{X}_i = \mathfrak{X}_{\text{L}i} - \mathfrak{X}_{\text{R}i}$ do annihilate the vacuum state, although $\mathfrak{X}_{\text{L}i}$ and $\mathfrak{X}_{\text{R}i}$ individually do not. We define $\mathfrak{H} = \mathfrak{H}_{\text{L}} + \mathfrak{H}_{\text{R}}$ and

$\mathfrak{M} = \mathfrak{H}_L - \mathfrak{H}_R$, with \mathfrak{H} the magnon Hamiltonian. Again, we will focus on well-separated magnon excitations in the $J \rightarrow \infty$ limit so that the left- and right-moving excitations $\varphi_L^{\beta\dot{\beta}}$ and $\varphi_R^{\beta\dot{\beta}}$ do not coincide.

Spin-chain.—The homogeneous double-row spin-chain is identified with the module $(M_L \otimes M_R)^{\otimes J}$. The ground state is

$$|0\rangle = |\mathcal{Z}^J\rangle = \left| \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^J \right\rangle, \quad (5.8)$$

and left- and right-moving fundamental excitations are

$$\begin{aligned} |\varphi_{(n)}^{\beta\dot{\beta}}\rangle &= |\mathcal{Z}^{n-1} \varphi^{\beta\dot{\beta}} \mathcal{Z}^{J-n}\rangle = \left| \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^{n-1} \begin{pmatrix} \varphi_L^{\beta\dot{\beta}} \\ \mathcal{Z}_R \end{pmatrix} \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^{J-n} \right\rangle, \\ |\bar{\varphi}_{(n)}^{\beta\dot{\beta}}\rangle &= |\mathcal{Z}^{n-1} \bar{\varphi}^{\beta\dot{\beta}} \mathcal{Z}^{J-n}\rangle = \left| \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^{n-1} \begin{pmatrix} \mathcal{Z}_L \\ \varphi_R^{\beta\dot{\beta}} \end{pmatrix} \begin{pmatrix} \mathcal{Z}_L \\ \mathcal{Z}_R \end{pmatrix}^{J-n} \right\rangle, \end{aligned} \quad (5.9)$$

with low-lying left- and right-moving single-magnon excitations given by

$$|\varphi_p^{\beta\dot{\beta}}\rangle = \sum_{n=1}^J e^{ipn} |\varphi_{(n)}^{\beta\dot{\beta}}\rangle, \quad |\bar{\varphi}_p^{\beta\dot{\beta}}\rangle = \sum_{n=1}^J e^{ipn} |\bar{\varphi}_{(n)}^{\beta\dot{\beta}}\rangle. \quad (5.10)$$

The left- and right-moving magnon excitations $|\varphi_p^{\beta\dot{\beta}}\rangle$ and $|\bar{\varphi}_p^{\beta\dot{\beta}}\rangle$ have energy eigenvalue 1 of the magnon Hamiltonian \mathfrak{H} , whereas the left/right-movers have mass eigenvalues ± 1 , with thus a particle/antiparticle interpretation. The nontrivial action of the fermionic generators of the $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]^2$ algebra on these magnon states is

$$\begin{aligned} \mathfrak{Q}_{L1} |\varphi_p^{+\dot{\beta}}\rangle &= |\varphi_p^{-\dot{\beta}}\rangle, & \mathfrak{E}_{L1} |\varphi_p^{-\dot{\beta}}\rangle &= |\varphi_p^{+\dot{\beta}}\rangle, \\ \mathfrak{Q}_{L2} |\varphi_p^{\beta+}\rangle &= (-1)^{\delta_{\beta-}} |\varphi_p^{\beta-}\rangle, & \mathfrak{E}_{L2} |\varphi_p^{\beta-}\rangle &= (-1)^{\delta_{\beta-}} |\varphi_p^{\beta+}\rangle, \\ \mathfrak{Q}_{R1} |\bar{\varphi}_p^{+\dot{\beta}}\rangle &= |\bar{\varphi}_p^{-\dot{\beta}}\rangle, & \mathfrak{E}_{R1} |\bar{\varphi}_p^{-\dot{\beta}}\rangle &= |\bar{\varphi}_p^{+\dot{\beta}}\rangle, \\ \mathfrak{Q}_{R2} |\bar{\varphi}_p^{\beta+}\rangle &= (-1)^{\delta_{\beta-}} |\bar{\varphi}_p^{\beta-}\rangle, & \mathfrak{E}_{R2} |\bar{\varphi}_p^{\beta-}\rangle &= (-1)^{\delta_{\beta-}} |\bar{\varphi}_p^{\beta+}\rangle, \end{aligned} \quad (5.11)$$

with the nontrivial action of the $\mathfrak{u}(1)$ generators \mathfrak{X}_i of the $\mathfrak{u}(1)^2 \ltimes [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]^2$ algebra, which annihilate the ground state, given by

$$\begin{aligned} \mathfrak{X}_i |\varphi_p^{++}\rangle &= \frac{1}{2} |\varphi_p^{++}\rangle, & \mathfrak{X}_1 |\varphi_p^{+-}\rangle &= \frac{1}{2} |\varphi_p^{+-}\rangle, \\ \mathfrak{X}_2 |\varphi_p^{-+}\rangle &= \frac{1}{2} |\varphi_p^{-+}\rangle, & \mathfrak{X}_i |\bar{\varphi}_p^{++}\rangle &= -\frac{1}{2} |\bar{\varphi}_p^{++}\rangle, \\ \mathfrak{X}_1 |\bar{\varphi}_p^{-+}\rangle &= -\frac{1}{2} |\bar{\varphi}_p^{-+}\rangle, & \mathfrak{X}_2 |\bar{\varphi}_p^{-+}\rangle &= -\frac{1}{2} |\bar{\varphi}_p^{-+}\rangle. \end{aligned}$$

We again make use of the standard generalization to multimagnon excitations.

B. $\mathfrak{psu}(1,1|2)^2$ spin-chain with centrally extended $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_c^2$ excitations

Beyond the weak-coupling limit of the $\mathfrak{psu}(1,1|2)^2$ spin-chain in which the left- and right-moving excitations decouple, we must centrally extend the subalgebra of the massive magnon excitations to $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_c^2$. This centrally extended algebra has fermionic generators \mathfrak{Q}_{ai} , \mathfrak{E}_{ai} and bosonic generators \mathfrak{H}_a , \mathfrak{P} , \mathfrak{P}^\dagger , with $a \in \{L, R\}$ and $i \in \{1, 2\}$. These generators satisfy

$$\begin{aligned} \{\mathfrak{Q}_{ai}, \mathfrak{E}_{bj}\} &= \mathfrak{H}_a \delta_{ab} \delta_{ij}, & \{\mathfrak{Q}_{Li}, \mathfrak{Q}_{Rj}\} &= \mathfrak{P} \delta_{ij}, \\ \{\mathfrak{E}_{Li}, \mathfrak{E}_{Rj}\} &= \mathfrak{P}^\dagger \delta_{ij}. \end{aligned} \quad (5.12)$$

Here \mathfrak{P} and \mathfrak{P}^\dagger are the new central elements. The dynamic and nondynamic $\mathfrak{psu}(1,1|2)^2$ spin-chains were studied in detail in Ref. [21]. Let us briefly review these constructions.

1. Finite spin-chain with length-changing effects

The bosonic central elements \mathfrak{P} and \mathfrak{P}^\dagger have length-changing effects on the closed finite spin-chain above. Here \mathcal{Z}^+ and \mathcal{Z}^- insert or remove a vacuum state, as described in Sec. III B 1.

Single-magnon excitations.—The action of the fermionic generators of the $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_c^2$ algebra on the left-moving magnon excitations is [21]

$$\begin{aligned} \mathfrak{Q}_{L1} |\varphi_p^{+\dot{\beta}}\rangle &= a_p |\varphi_p^{-\dot{\beta}}\rangle, & \mathfrak{E}_{L1} |\varphi_p^{-\dot{\beta}}\rangle &= b_p |\varphi_p^{+\dot{\beta}}\rangle, \\ \mathfrak{Q}_{L2} |\varphi_p^{\beta+}\rangle &= (-1)^{\delta_{\beta-}} a_p |\varphi_p^{\beta-}\rangle, & \mathfrak{E}_{L2} |\varphi_p^{\beta-}\rangle &= (-1)^{\delta_{\beta-}} b_p |\varphi_p^{\beta+}\rangle, \\ \mathfrak{Q}_{R1} |\varphi_p^{-\dot{\beta}}\rangle &= c_p |\mathcal{Z}^+ \varphi_p^{+\dot{\beta}}\rangle, & \mathfrak{E}_{R1} |\varphi_p^{+\dot{\beta}}\rangle &= d_p |\mathcal{Z}^- \varphi_p^{-\dot{\beta}}\rangle, \\ \mathfrak{Q}_{R2} |\varphi_p^{\beta-}\rangle &= (-1)^{\delta_{\beta-}} c_p |\mathcal{Z}^+ \varphi_p^{\beta+}\rangle, \\ \mathfrak{E}_{R2} |\varphi_p^{\beta+}\rangle &= (-1)^{\delta_{\beta-}} d_p |\mathcal{Z}^- \varphi_p^{\beta-}\rangle, \end{aligned} \quad (5.13)$$

and, similarly, on the right-moving magnon excitations,

$$\begin{aligned} \mathfrak{Q}_{R1} |\bar{\varphi}_p^{+\dot{\beta}}\rangle &= \bar{a}_p |\bar{\varphi}_p^{-\dot{\beta}}\rangle, & \mathfrak{E}_{R1} |\bar{\varphi}_p^{-\dot{\beta}}\rangle &= \bar{b}_p |\bar{\varphi}_p^{+\dot{\beta}}\rangle, \\ \mathfrak{Q}_{R2} |\bar{\varphi}_p^{\beta+}\rangle &= (-1)^{\delta_{\beta-}} \bar{a}_p |\bar{\varphi}_p^{\beta-}\rangle, \\ \mathfrak{E}_{R2} |\bar{\varphi}_p^{\beta-}\rangle &= (-1)^{\delta_{\beta-}} \bar{b}_p |\bar{\varphi}_p^{\beta+}\rangle, \\ \mathfrak{Q}_{L1} |\bar{\varphi}_p^{-\dot{\beta}}\rangle &= \bar{c}_p |\mathcal{Z}^+ \bar{\varphi}_p^{+\dot{\beta}}\rangle, & \mathfrak{E}_{L1} |\bar{\varphi}_p^{+\dot{\beta}}\rangle &= \bar{d}_p |\mathcal{Z}^- \bar{\varphi}_p^{-\dot{\beta}}\rangle, \\ \mathfrak{Q}_{L2} |\bar{\varphi}_p^{\beta-}\rangle &= (-1)^{\delta_{\beta-}} \bar{c}_p |\mathcal{Z}^+ \bar{\varphi}_p^{\beta+}\rangle, \\ \mathfrak{E}_{L2} |\bar{\varphi}_p^{\beta+}\rangle &= (-1)^{\delta_{\beta-}} \bar{d}_p |\mathcal{Z}^- \bar{\varphi}_p^{\beta-}\rangle. \end{aligned} \quad (5.14)$$

The energy eigenvalues of $\mathfrak{H} = \mathfrak{H}_L + \mathfrak{H}_R$ on the left- and right-moving magnons are

$$E_p = a_p b_p + c_p d_p, \quad \bar{E}_p = \bar{a}_p \bar{b}_p + \bar{c}_p \bar{d}_p,$$

and the eigenvalues of the mass operator $\mathfrak{M} = \mathfrak{H}_L - \mathfrak{H}_R$ are

$$m = a_p b_p - c_p d_p = 1, \quad -\bar{m} = -(\bar{a}_p \bar{b}_p - \bar{c}_p \bar{d}_p) = -1.$$

Physical single-magnon states should again be annihilated by the central elements \mathfrak{P} and \mathfrak{P}^\dagger , which would imply $c_p = d_p = \bar{c}_p = \bar{d}_p = 0$, so we revert to a magnon state of the decoupled spin-chain.

Two-magnon excitations. We can write similar excitations for two left-moving magnons of momenta p and q to those in Sect. III B 1, as well as for two right-moving magnons, and for left- and right-moving magnons $|\varphi_p^{\beta\beta} \varphi_q^{\gamma\gamma}\rangle$, $|\bar{\varphi}_p^{\beta\beta} \bar{\varphi}_q^{\gamma\gamma}\rangle$, $|\varphi_p^{\beta\beta} \bar{\varphi}_q^{\gamma\gamma}\rangle$, $|\bar{\varphi}_p^{\beta\beta} \varphi_q^{\gamma\gamma}\rangle$ in terms of these length-changing effects. We find that, for these two-magnon states to be annihilated by the central elements \mathfrak{P} and \mathfrak{P}^\dagger , we must make use of the same parametrization (3.17) for both a_p, b_p, c_p, d_p and $\bar{a}_p, \bar{b}_p, \bar{c}_p, \bar{d}_p$, satisfying the same constraints (3.18), with the unit mass now $m = \bar{m} = 1$.

2. Infinite spin-chain with Hopf algebra structure

Now, in the $J \rightarrow \infty$ infinite spin-chain limit, we can encode the length-changing effects rather in a \mathfrak{U} -braided Hopf algebra structure for the $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$ superalgebra, as in Appendix B of Ref. [21].

Single-magnon representations.—We will write the actions (5.13) and (5.14) in terms of matrix representations of $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$. We must first introduce the vector spaces

$$\begin{aligned} \mathcal{V}_p &= \text{span}_{\mathbb{C}}\{|\varphi_p^{++}\rangle, |\varphi_p^{+-}\rangle, |\varphi_p^{-+}\rangle, |\varphi_p^{--}\rangle\}, \\ \bar{\mathcal{V}}_p &= \text{span}_{\mathbb{C}}\{|\bar{\varphi}_p^{++}\rangle, |\bar{\varphi}_p^{+-}\rangle, |\bar{\varphi}_p^{-+}\rangle, |\bar{\varphi}_p^{--}\rangle\} \end{aligned}$$

for left- and right-moving magnons. We can identify these vector spaces with $\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1}$. The action (5.13) can be encoded in the left-moving representation $\pi_p: [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2 \rightarrow \text{End}(\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1})$:

$$\begin{aligned} \pi_p(\mathfrak{Q}_{L1}) &= a_p \mathbb{E}_{21} \otimes \mathbb{1}, & \pi_p(\mathfrak{Q}_{L2}) &= a_p \mathbb{1} \otimes \mathbb{E}_{21}, \\ \pi_p(\mathfrak{Q}_{R1}) &= c_p \mathbb{E}_{12} \otimes \mathbb{1}, & \pi_p(\mathfrak{Q}_{R2}) &= c_p \mathbb{1} \otimes \mathbb{E}_{12}, \\ \pi_p(\mathfrak{C}_{L1}) &= b_p \mathbb{E}_{12} \otimes \mathbb{1}, & \pi_p(\mathfrak{C}_{L2}) &= b_p \mathbb{1} \otimes \mathbb{E}_{12}, \\ \pi_p(\mathfrak{C}_{R1}) &= d_p \mathbb{E}_{21} \otimes \mathbb{1}, & \pi_p(\mathfrak{C}_{R2}) &= d_p \mathbb{1} \otimes \mathbb{E}_{21}, \\ \pi_p(\mathfrak{H}_L) &= a_p b_p \mathbb{1} \otimes \mathbb{1}, & \pi_p(\mathfrak{H}_R) &= c_p d_p \mathbb{1} \otimes \mathbb{1}, \\ \pi_p(\mathfrak{P}) &= a_p c_p \mathbb{1} \otimes \mathbb{1}, & \pi_p(\mathfrak{P}^\dagger) &= b_p d_p \mathbb{1} \otimes \mathbb{1}, \end{aligned} \quad (5.15)$$

and the action (5.14) can be encoded in the right-moving magnon representation $\bar{\pi}_p: [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2 \rightarrow \text{End}(\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1})$:

$$\begin{aligned} \pi_p(\mathfrak{Q}_{R1}) &= a_p \mathbb{E}_{\bar{2}\bar{1}} \otimes \mathbb{1}, & \pi_p(\mathfrak{Q}_{R2}) &= a_p \mathbb{1} \otimes \mathbb{E}_{\bar{2}\bar{1}}, \\ \pi_p(\mathfrak{Q}_{L1}) &= c_p \mathbb{E}_{\bar{1}\bar{2}} \otimes \mathbb{1}, & \pi_p(\mathfrak{Q}_{L2}) &= c_p \mathbb{1} \otimes \mathbb{E}_{\bar{1}\bar{2}}, \\ \pi_p(\mathfrak{C}_{R1}) &= b_p \mathbb{E}_{\bar{1}\bar{2}} \otimes \mathbb{1}, & \pi_p(\mathfrak{C}_{R2}) &= b_p \mathbb{1} \otimes \mathbb{E}_{\bar{1}\bar{2}}, \\ \pi_p(\mathfrak{C}_{L1}) &= d_p \mathbb{E}_{\bar{2}\bar{1}} \otimes \mathbb{1}, & \pi_p(\mathfrak{C}_{L2}) &= d_p \mathbb{1} \otimes \mathbb{E}_{\bar{2}\bar{1}}, \\ \pi_p(\mathfrak{H}_R) &= a_p b_p \mathbb{1} \otimes \mathbb{1}, & \pi_p(\mathfrak{H}_L) &= c_p d_p \mathbb{1} \otimes \mathbb{1}, \\ \pi_p(\mathfrak{P}) &= a_p c_p \mathbb{1} \otimes \mathbb{1}, & \pi_p(\mathfrak{P}^\dagger) &= b_p d_p \mathbb{1} \otimes \mathbb{1}, \end{aligned} \quad (5.16)$$

both with parameters (3.17).

Hopf algebra.—Again we introduce an additional generator \mathfrak{U} , which is central with respect to the $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$ superalgebra. The action on any single-magnon excitation is

$$\mathfrak{U}|\varphi_p^{\beta\beta}\rangle = e^{i\frac{\pi}{2}}|\varphi_p^{\beta\beta}\rangle, \quad \mathfrak{U}|\bar{\varphi}_p^{\beta\beta}\rangle = e^{i\frac{\pi}{2}}|\bar{\varphi}_p^{\beta\beta}\rangle, \quad (5.17)$$

and hence

$$\begin{aligned} \pi_p(\mathfrak{U}) &= -c_p d_p^{-1} \mathbb{1} \otimes \mathbb{1} = e^{i\frac{\pi}{2}} \mathbb{1} \otimes \mathbb{1}, \\ \bar{\pi}_p(\mathfrak{U}) &= -\bar{c}_p \bar{d}_p^{-1} \mathbb{1} \otimes \mathbb{1} = e^{i\frac{\pi}{2}} \mathbb{1} \otimes \mathbb{1} \end{aligned} \quad (5.18)$$

in the left- and right-moving single-magnon representations. As in Sec. III B 2, we define a Hopf algebra structure on $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$, denoting this Hopf superalgebra by \mathcal{A} throughout Secs. V and VI.

$$L(\mathcal{A}) = [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$$

is the associated Lie superalgebra.

We again choose a symmetric frame in which the coproduct takes a form similar to that of [21]

$$\begin{aligned} \Delta(\mathfrak{Q}_{ai}) &= \mathfrak{Q}_{ai} \otimes \mathbf{1} + \mathfrak{U} \otimes \mathfrak{Q}_{ai}, \\ \Delta(\mathfrak{P}) &= \mathfrak{P} \otimes \mathbf{1} + \mathfrak{U}^2 \otimes \mathfrak{P}, \\ \Delta(\mathfrak{C}_{ai}) &= \mathfrak{C}_{ai} \otimes \mathbf{1} + \mathfrak{U}^{-1} \otimes \mathfrak{C}_{ai}, \\ \Delta(\mathfrak{P}^\dagger) &= \mathfrak{P}^\dagger \otimes \mathbf{1} + \mathfrak{U}^{-2} \otimes \mathfrak{P}^\dagger, \\ \Delta(\mathfrak{H}_a) &= \mathfrak{H}_a \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{H}_a, \\ \Delta(\mathfrak{U}^{\pm 1}) &= \mathfrak{U}^{\pm 1} \otimes \mathfrak{U}^{\pm 1}, \end{aligned} \quad (5.19)$$

with opposite coproduct $\Delta^{\text{op}}(\mathfrak{a}) = \mathcal{P}\Delta(\mathfrak{a})$. The central elements co-commute, $\Delta(\mathfrak{C}) = \Delta^{\text{op}}(\mathfrak{C})$ for $\mathfrak{C} \in \{\mathfrak{H}_a, \mathfrak{P}, \mathfrak{P}^\dagger\}$, which implies $\mathfrak{P} = \nu_1(1 - \mathfrak{U}^2)$ and $\mathfrak{P}^\dagger = \nu_2(1 - \mathfrak{U}^{-2})$, again using $\nu_1 = \nu_2 = h$ to obtain our representations π_p and $\bar{\pi}_p$. All the other Hopf algebra structures of Sec. III B 2 generalize in the obvious manner. In particular, the antipode $S: \mathcal{A} \rightarrow \mathcal{A}$ is

$$\begin{aligned}
 \mathcal{S}(\mathfrak{Q}_{ai}) &= -\mathcal{U}^{-1}\mathfrak{Q}_{ai}, & \mathcal{S}(\mathfrak{S}_{ai}) &= -\mathcal{U}\mathfrak{S}_{ai}, \\
 \mathcal{S}(\mathfrak{H}_a) &= -\mathfrak{H}_a, & \mathcal{S}(\mathfrak{P}) &= -\mathcal{U}^{-2}\mathfrak{P}, \\
 \mathcal{S}(\mathfrak{P}^\dagger) &= -\mathcal{U}^2\mathfrak{P}^\dagger, & \mathcal{S}(\mathcal{U}) &= \mathcal{U}^{-1},
 \end{aligned} \tag{5.20}$$

which relates left- and right-movers in the representations π_p and $\pi_{\bar{p}}$ through

$$\pi_p(\mathcal{S}(\mathfrak{a})) = (\pi_{\bar{p}}(\bar{\mathfrak{a}}))^{\text{str}}, \tag{5.21}$$

with the charge conjugation matrix trivial. Here $\mathfrak{a} \in \mathcal{A}$ with $\bar{\mathfrak{a}} \in \mathcal{A}$ defined by

$$\begin{aligned}
 \bar{\mathfrak{Q}}_{ai} &= \delta_{aL}\mathfrak{Q}_{Ri} + \delta_{aR}\mathfrak{Q}_{Li}, & \bar{\mathfrak{P}} &= \mathfrak{P}, \\
 \bar{\mathfrak{H}}_a &= \delta_{aL}\mathfrak{H}_R + \delta_{aR}\mathfrak{H}_L, \\
 \bar{\mathfrak{S}}_{ai} &= \delta_{aL}\mathfrak{S}_{Ri} + \delta_{aR}\mathfrak{S}_{Li}, & \bar{\mathfrak{P}}^\dagger &= \mathfrak{P}^\dagger, & \bar{\mathcal{U}}^{\pm 1} &= \mathcal{U}^{\pm 1}.
 \end{aligned} \tag{5.22}$$

C. Two-magnon scattering and R-matrices

The two-magnon scattering matrix is

$$\mathcal{S}(p, q) = PR(p, q), \quad \text{with } R(p, q) \in \text{End}(\mathcal{W}_p \otimes \mathcal{W}_q), \tag{5.23}$$

where now

$$\begin{aligned}
 \mathcal{W}_p &= \mathcal{V}_p \oplus \bar{\mathcal{V}}_p \\
 &= \text{span}_{\mathbb{C}}\{|\varphi_p^{++}\rangle, |\varphi_p^{+-}\rangle, |\varphi_p^{-+}\rangle, |\varphi_p^{--}\rangle, \\
 &\quad |\bar{\varphi}_p^{++}\rangle, |\bar{\varphi}_p^{+-}\rangle, |\bar{\varphi}_p^{-+}\rangle, |\bar{\varphi}_p^{--}\rangle\}.
 \end{aligned}$$

Again, $R(p, q)$ can be decomposed into a direct sum of four sectors $R^{\text{ab}}(p, q)$ corresponding to the partial R -matrices $\mathbb{R}^{\text{ab}}(p, q) \in \text{End}((\mathbb{C}^{1|1} \otimes \dot{\mathbb{C}}^{1|1}) \otimes (\mathbb{C}^{1|1} \otimes \dot{\mathbb{C}}^{1|1}))$. Here the complete R -matrix

$$\begin{aligned}
 \mathbb{R}(p, q) &= \mathbb{R}^{\text{LL}}(p, q) \oplus \mathbb{R}^{\text{LR}}(p, q) \oplus \mathbb{R}^{\text{RL}}(p, q) \\
 &\quad \oplus \mathbb{R}^{\text{RR}}(p, q)
 \end{aligned} \tag{5.24}$$

satisfies the intertwining equations (3.34), with π_p and $\bar{\pi}_p$ now the representations (5.15) and (5.16), and a similar unitarity condition.

The intertwining equations of \mathfrak{Q}_{a1} and \mathfrak{S}_{a1} intertwine only the $\mathbb{C}^{1|1}$ space nontrivially, and similarly those of \mathfrak{Q}_{a2} and \mathfrak{S}_{a2} intertwine only $\dot{\mathbb{C}}^{1|1}$. The intertwining equations of \mathfrak{H}_a may be thought of as acting nontrivially on either space and trivially on the other, and just ensure the decomposition of the R -matrix into block diagonal form. Thus, two copies of the partial R -matrices of the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain given in Sec. III C 1,

$$\begin{aligned}
 \mathbb{R}^{\text{ab}}(p, q) &= \sum_{i,j,k,l} (R^{\text{ab}}(p, q))_{jl}^{ik} \mathbb{E}_{ij} \otimes \mathbb{E}_{kl}, \\
 \dot{\mathbb{R}}^{\text{ab}}(p, q) &= \sum_{i,j,k,l} (R^{\text{ab}}(p, q))_{jl}^{ik} \mathbb{E}_{ij} \otimes \mathbb{E}_{kl},
 \end{aligned}$$

can be used to build an R -matrix $\mathbb{R}(p, q)$ for the $\mathfrak{psu}(1, 1|2)^2$ spin-chain [21]. This decomposes into the partial R -matrices given by

$$\mathbb{R}^{\text{ab}}(p, q) = \sum_{\substack{i,j,j', \\ k,k',l}} (R^{\text{ab}}(p, q))_{j',\ell\ell'}^{ii,kk'} (\mathbb{E}_{ij} \otimes \mathbb{E}_{i'j'}) \otimes (\mathbb{E}_{k\ell} \otimes \mathbb{E}_{k'\ell'}), \tag{5.25}$$

with¹⁹

$$(R^{\text{ab}}(p, q))_{j',\ell\ell'}^{ii,kk'} = (-1)^{(k+\ell')(i+j)} (R^{\text{ab}}(p, q))_{j\ell}^{ik} (R^{\text{ab}}(p, q))_{j'\ell'}^{i'k'}, \tag{5.26}$$

where $i, i', j, j' \in \{1, 2\}$ and $\{\bar{1}, \bar{2}\}$ for $\mathfrak{a} = \text{L}$ and $\mathfrak{a} = \text{R}$, respectively, and $k, k', \ell, \ell' \in \{1, 2\}$ and $\{\bar{1}, \bar{2}\}$ for $\mathfrak{b} = \text{L}$ and $\mathfrak{b} = \text{R}$. Now the Zhukovski variables satisfy a mass shell constraint (3.18) with unit mass $m = 1$. This R -matrix satisfies the Yang-Baxter equation (3.40) and a unitarity condition.

VI. INTEGRABLE OPEN $\mathfrak{psu}(1, 1|2)^2$ SPIN-CHAIN AND REFLECTION MATRICES IN $\text{AdS}_3 \times S^3 \times T^4$

Let us now consider the boundary scattering of magnon excitations of a $\mathfrak{psu}(1, 1|2)^2$ open spin-chain off an integrable boundary. These correspond to massive excitations of an open superstring ending on D-branes in $\text{AdS}_3 \times S^3 \times T^4$, such as one of the maximal giant gravitons discussed in Sec. II B.

A. Open spin-chains and boundary scattering

1. Double-row $\mathfrak{psu}(1, 1|2)^2$ open spin-chain with $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$ excitations

Semi-infinite open spin-chain.—Again we consider a semi-infinite open spin-chain with $J \rightarrow \infty$ which has a boundary site on the right side. The ground state is

$$|0\rangle = |\mathcal{Z}^J \mathcal{F}_B\rangle, \tag{6.1}$$

with \mathcal{F}_B an infinitely heavy boundary field. Fundamental excitations now take the form

¹⁹We make use of the isomorphism $\mathbb{1} \otimes \mathbb{P} \otimes \mathbb{1}$ between $\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1} \otimes \dot{\mathbb{C}}^{1|1} \otimes \dot{\mathbb{C}}^{1|1}$ and $\mathbb{C}^{1|1} \otimes \dot{\mathbb{C}}^{1|1} \otimes \mathbb{C}^{1|1} \otimes \dot{\mathbb{C}}^{1|1}$, which maps the graded tensor product $R^{\text{ab}}(p, q) \otimes \dot{R}^{\text{ab}}(p, q)$ of the R -matrices in Sec. III C 1 to the R -matrix $R^{\text{ab}}(p, q)$ here.

$$\begin{aligned} |\varphi_{(n)}^{\beta\dot{\beta}}\rangle_{\mathbf{B}} &= |\mathcal{Z}^{J-n} \varphi^{\beta\dot{\beta}} \mathcal{Z}^{n-1} \mathcal{F}_{\mathbf{B}}\rangle, \\ |\bar{\varphi}_{(n)}^{\beta\dot{\beta}}\rangle_{\mathbf{B}} &= |\mathcal{Z}^{n-1} \bar{\varphi}^{\beta\dot{\beta}} \mathcal{Z}^{J-n} \mathcal{F}_{\mathbf{B}}\rangle \end{aligned} \quad (6.2)$$

in terms of the left- and right-moving excitations $\varphi^{\beta\dot{\beta}}$ and $\bar{\varphi}^{\beta\dot{\beta}}$ of Sec. VA 2. The low-lying left and right single-magnon excitations of the double-row open spin-chain are thus

$$\begin{aligned} |\varphi_p^{\beta\dot{\beta}}\rangle_{\mathbf{B}} &= \sum_{n=1}^J e^{-ipn} |\varphi_{(n)}^{\beta\dot{\beta}}\rangle_{\mathbf{B}}, \\ |\bar{\varphi}_p^{\beta\dot{\beta}}\rangle_{\mathbf{B}} &= \sum_{n=1}^J e^{-ipn} |\bar{\varphi}_{(n)}^{\beta\dot{\beta}}\rangle_{\mathbf{B}}. \end{aligned} \quad (6.3)$$

In the $J \rightarrow \infty$ limit, magnon states of the semi-infinite open spin-chain can again be identified with magnon states of the closed spin-chain, with an additional boundary state,

$$|\varphi_p^{\beta\dot{\beta}}\rangle_{\mathbf{B}} = |\varphi_p^{\beta\dot{\beta}}\rangle \otimes |0\rangle_{\mathbf{B}}, \quad |\bar{\varphi}_p^{\beta\dot{\beta}}\rangle_{\mathbf{B}} = |\bar{\varphi}_p^{\beta\dot{\beta}}\rangle \otimes |0\rangle_{\mathbf{B}}, \quad (6.4)$$

with $|\varphi_p^{\beta\dot{\beta}}\rangle$ and $|\bar{\varphi}_p^{\beta\dot{\beta}}\rangle$ bulk magnon excitations in \mathcal{V}_p and $\bar{\mathcal{V}}_p$. The length-changing effects of the dynamic spin-chain are encoded in the Hopf algebra \mathcal{A} of Sec. VB 2. The boundary field $\mathcal{F}_{\mathbf{B}}$ is represented by the boundary vacuum state $|0\rangle_{\mathbf{B}}$. We can generalize to multimagnon states.

Boundary algebra.—The boundary algebra $\mathcal{B} \subset \mathcal{A}$ must be a coideal subalgebra of the bulk Hopf superalgebra \mathcal{A} . The singlet boundary state $|0\rangle_{\mathbf{B}}$ transforms in the trivial representation of \mathcal{B} defined by the counit map ϵ . A vector state $|\Phi\rangle_{\mathbf{B}}$ at the boundary $|0\rangle_{\mathbf{B}}$ is also possible.

2. Boundary scattering and K-matrices

Outgoing single-magnon representations π_{-p} and $\bar{\pi}_{-p}$.—Incoming magnons are states in \mathcal{V}_p and $\bar{\mathcal{V}}_p$, whereas outgoing magnons are states in the vector spaces \mathcal{V}_{-p} and $\bar{\mathcal{V}}_{-p}$. The incoming single-magnon representations π_p and $\bar{\pi}_p$ are shown in (5.15) and (5.16). The outgoing single-magnon representations $\pi_{-p}: \mathcal{A} \rightarrow \text{End}(\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1})$ and $\bar{\pi}_{-p}: \mathcal{A} \rightarrow \text{End}(\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1})$ are obtained by replacing the parameters a_p, b_p, c_p, d_p with the parameters $a_{-p}, b_{-p}, c_{-p}, d_{-p}$ given in (4.8).

Singlet boundary scattering.—The boundary scattering matrix is

$$S_{\text{boundary}}(p) = \kappa K(p), \quad \text{with } K(p) \in \text{End}(\mathcal{W}_p), \quad (6.5)$$

with κ the reflection map. The scattering of magnons off a singlet boundary $|0\rangle_{\mathbf{B}}$ was described in Sec. IVA 2. The vector space $\mathcal{W}_p = \mathcal{V}_p \oplus \bar{\mathcal{V}}_p$ with $\mathcal{V}_p, \bar{\mathcal{V}}_p \cong \mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1}$ is discussed in Sec. VC.

Vector boundary scattering.—The boundary scattering matrix is now

$$S_{\text{boundary}}(p, \mathbf{B}) = \kappa K(p, \mathbf{B}), \quad \text{with } K(p, \mathbf{B}) \in \text{End}(\mathcal{W}_p \otimes \mathcal{W}_{\mathbf{B}}). \quad (6.6)$$

Here \mathcal{W}_p and $\mathcal{W}_{\mathbf{B}}$ are both modules of $\mathcal{B}_{(\mathbf{T}, \mathbf{T})} = [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbf{C}}^2$. The scattering of magnon excitations of an open spin-chain off a vector boundary $|\Phi_{\mathbf{B}}\rangle \otimes |0\rangle_{\mathbf{B}}$ was described in Sec. IVA 2.

We will discuss singlet boundaries in Sec. VIB and the vector boundary in Sec. VIC.

B. Singlet boundaries

Boundary algebras.—There are now more possible coideal boundary subalgebras \mathcal{B} of the bulk Hopf superalgebra \mathcal{A} describing the scattering of magnons off singlet boundaries than in Sec. IV B. The associated Lie algebras $L(\mathcal{B})$ can be compared with the boundary algebras shown in Table III.

- (1) *Left, right and mixed half-supersymmetric boundary algebras $\mathcal{B}_{(\mathbf{L}, \mathbf{L})}, \mathcal{B}_{(\mathbf{R}, \mathbf{R})}, \mathcal{B}_{(\mathbf{L}, \mathbf{R})}, \mathcal{B}_{(\mathbf{R}, \mathbf{L})}$,* corresponding to D-branes preserving half the bulk supersymmetries, and the magnon Hamiltonian \mathfrak{H} and \mathfrak{M} , implying chiral boundary scattering. The boundary Lie algebras associated with these coideal boundary subalgebras are $\mathfrak{psu}(1|1)_{\mathbf{L}}^2 \oplus \mathfrak{u}(1)_{\mathbf{L}} \oplus \mathfrak{u}(1)_{\mathbf{R}}, \mathfrak{u}(1)_{\mathbf{L}} \oplus \mathfrak{psu}(1|1)_{\mathbf{R}}^2 \oplus \mathfrak{u}(1)_{\mathbf{R}},$ and $\mathfrak{su}(1|1)_{\mathbf{L}} \oplus \mathfrak{su}(1|1)_{\mathbf{R}}.$
- (2) *Non-supersymmetric chiral boundary algebra $\mathcal{B}_{(\mathbf{NC}, \mathbf{NC})}$,* corresponding to D-branes which preserve none of the bulk supersymmetries, but do preserve \mathfrak{H} and \mathfrak{M} . The associated boundary Lie algebra is $\mathfrak{u}(1)_{\mathbf{L}} \oplus \mathfrak{u}(1)_{\mathbf{R}}.$ We will show that scattering off this boundary has a *hidden symmetry*, denoted $\mathcal{B}_{(\mathbf{D}, \mathbf{D})}$, contained in the bulk Hopf superalgebra.
- (3) *Left and right quarter-supersymmetric boundary algebras $\mathcal{B}_{(\mathbf{L}, \mathbf{NC})}, \mathcal{B}_{(\mathbf{NC}, \mathbf{L})}, \mathcal{B}_{(\mathbf{R}, \mathbf{NC})}, \mathcal{B}_{(\mathbf{NC}, \mathbf{R})}$,* corresponding to D-branes preserving a quarter of the bulk supersymmetries, and \mathfrak{H} and \mathfrak{M} . The associated boundary Lie algebras are $\mathfrak{su}(1|1)_{\mathbf{L}} \oplus \mathfrak{u}(1)_{\mathbf{R}}$ and $\mathfrak{u}(1)_{\mathbf{L}} \oplus \mathfrak{su}(1|1)_{\mathbf{R}}.$ We will show that these boundary scattering processes preserve *hidden symmetries*, denoted $\mathcal{B}_{(\mathbf{L}, \mathbf{D})}, \mathcal{B}_{(\mathbf{D}, \mathbf{L})}, \mathcal{B}_{(\mathbf{R}, \mathbf{D})}, \mathcal{B}_{(\mathbf{D}, \mathbf{R})}$, at the level of the Hopf superalgebra.
- (4) *Non-supersymmetric achiral boundary algebra $\mathcal{B}_{(\mathbf{NA}, \mathbf{NA})}$,* corresponding to D-branes preserving \mathfrak{H} , but no bulk supersymmetries. The associated boundary Lie algebra is $\mathfrak{u}(1)_{+}.$

Boundary intertwining equations.—The K -matrix $K(p) \in \text{End}(\mathcal{W}_p)$ again has four sectors: $\mathbb{K}^{\mathbf{a}}(p)$ are chiral reflections, and $\mathbb{A}^{\mathbf{a}}(p)$ are achiral reflections. The partial K -matrices $\mathbb{K}^{\mathbf{a}}(p), \mathbb{A}^{\mathbf{a}}(p) \in \text{End}(\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1})$. The complete K -matrix $\mathbb{K}(p)$ takes the form shown in (4.13). The boundary intertwining equations (4.11) for all $\mathfrak{b} \in \mathcal{B}$

simplify to (4.12) for those $\mathfrak{b} \in [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$. Here π_p and $\bar{\pi}_p$ are now the representations (5.15) and (5.16), and similarly for the reflected representations.

Integrable K -matrices satisfying the reflection equation (4.43) can be built from two of the K -matrices $\mathbb{K}_{\mathcal{B}_1}(p)$ and $\mathbb{K}_{\mathcal{B}_2}(p)$ given in Sec. IV B 1—which have partial K -matrices

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_1}^{\mathfrak{a}}(p) &= \sum_{i,j} (K_{\mathcal{B}_1}^{\mathfrak{a}}(p))_j^i E_{ij}, & \mathbb{A}_{\mathcal{B}_1}^{\mathfrak{a}}(p) &= \sum_{i,j} (A_{\mathcal{B}_1}^{\mathfrak{a}}(p))_j^i E_{ij}, \\ \mathbb{K}_{\mathcal{B}_2}^{\mathfrak{a}}(p) &= \sum_{i,j} (K_{\mathcal{B}_2}^{\mathfrak{a}}(p))_j^i E_{ij}, & \mathbb{A}_{\mathcal{B}_2}^{\mathfrak{a}}(p) &= \sum_{i,j} (A_{\mathcal{B}_2}^{\mathfrak{a}}(p))_j^i E_{ij}. \end{aligned} \quad (6.7)$$

The result is a complete K -matrix solution of the boundary intertwining equations associated with a boundary coideal subalgebra, denoted $\mathcal{B}_{(1,2)}$. The partial K -matrices are given by

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_{(1,2)}}^{\mathfrak{a}}(p) &= \mathbb{K}_{\mathcal{B}_1}^{\mathfrak{a}}(p) \otimes \mathbb{K}_{\mathcal{B}_2}^{\mathfrak{a}}(p) = \sum_{i,j,i',j'} (K_{\mathcal{B}_{(1,2)}}^{\mathfrak{a}}(p))_{j'j}^{ii'} E_{ij} \otimes E_{i'j'}, \\ \mathbb{A}_{\mathcal{B}_{(1,2)}}^{\mathfrak{a}}(p) &= \mathbb{A}_{\mathcal{B}_1}^{\mathfrak{a}}(p) \otimes \mathbb{A}_{\mathcal{B}_2}^{\mathfrak{a}}(p) = \sum_{i,j,i',j'} (A_{\mathcal{B}_{(1,2)}}^{\mathfrak{a}}(p))_{j'j}^{ii'} E_{ij} \otimes E_{i'j'}, \end{aligned} \quad (6.8)$$

with

$$\begin{aligned} (K_{\mathcal{B}_{(1,2)}}^{\mathfrak{a}}(p))_{j'j}^{ii'} &= (K_{\mathcal{B}_1}^{\mathfrak{a}}(p))_j^i (K_{\mathcal{B}_2}^{\mathfrak{a}}(p))_{j'}^{i'}, \\ (A_{\mathcal{B}_{(1,2)}}^{\mathfrak{a}}(p))_{j'j}^{ii'} &= (A_{\mathcal{B}_1}^{\mathfrak{a}}(p))_j^i (A_{\mathcal{B}_2}^{\mathfrak{a}}(p))_{j'}^{i'}. \end{aligned}$$

We notice that $\mathbb{K}_{\mathcal{B}_1}(p)$ and $\mathbb{K}_{\mathcal{B}_2}(p)$ must both be chiral or must both be achiral for a nonzero reflection matrix $\mathbb{K}_{\mathcal{B}_{(1,2)}}(p)$ built in this way.

Constraints from the central elements.—Again, boundary subalgebras associated with singlet boundaries may not contain the central elements \mathfrak{P} and \mathfrak{P}^\dagger . The inclusion of \mathfrak{H}_L or \mathfrak{H}_R in the boundary algebra \mathcal{B} once more implies a chiral K -matrix, although achiral K -matrices are allowed if only $\mathfrak{H} = \mathfrak{H}_L + \mathfrak{H}_R$ is contained in \mathcal{B} .

1. Boundary subalgebras and K -matrices

Left, right and mixed half-supersymmetric boundary algebras.—The left, right and mixed half-supersymmetric boundary superalgebras, $\mathcal{B}_{(L,L)}$, $\mathcal{B}_{(R,R)}$, $\mathcal{B}_{(L,R)}$ and $\mathcal{B}_{(R,L)}$, are defined to be coideal subalgebras of \mathcal{A} generated as follows:

$$\begin{aligned} \mathcal{B}_{(L,L)} &= \langle \mathfrak{Q}_{L1}, \mathfrak{E}_{L1}, \mathfrak{Q}_{L2}, \mathfrak{E}_{L2}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\ \mathcal{B}_{(R,R)} &= \langle \mathfrak{Q}_{R1}, \mathfrak{E}_{R1}, \mathfrak{Q}_{R2}, \mathfrak{E}_{R2}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\ \mathcal{B}_{(L,R)} &= \langle \mathfrak{Q}_{L1}, \mathfrak{E}_{L1}, \mathfrak{Q}_{R2}, \mathfrak{E}_{R2}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\ \mathcal{B}_{(R,L)} &= \langle \mathfrak{Q}_{R1}, \mathfrak{E}_{R1}, \mathfrak{Q}_{L2}, \mathfrak{E}_{L2}, \mathfrak{H}_L, \mathfrak{H}_R \rangle. \end{aligned} \quad (6.9)$$

That $\mathcal{B}_{(a,b)}$ is an integrable boundary algebra follows from the fact that $(\mathfrak{g}, \mathfrak{h}_{(a,b)})$ forms a symmetric pair of Lie algebras, if we define

$$\begin{aligned} \mathfrak{g} &\equiv L(\mathcal{A}) = [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2, \\ \mathfrak{h}_{(L,L)} &\equiv L(\mathcal{B}_{(L,L)}) = [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_L \oplus \mathfrak{u}(1)_R, \\ \mathfrak{h}_{(R,R)} &\equiv L(\mathcal{B}_{(R,R)}) = \mathfrak{u}(1)_L \oplus [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_R, \\ \mathfrak{h}_{(L,R)} &\equiv L(\mathcal{B}_{(L,R)}) = \mathfrak{su}(1|1)_L \oplus \mathfrak{su}(1|1)_R, \\ \mathfrak{h}_{(R,L)} &\equiv L(\mathcal{B}_{(R,L)}) = \mathfrak{su}(1|1)_L \oplus \mathfrak{su}(1|1)_R. \end{aligned} \quad (6.10)$$

Here $\mathfrak{g} = \mathfrak{h}_{(a,b)} \oplus \mathfrak{m}_{(a,b)}$, with $\mathfrak{m}_{(L,L)}$, $\mathfrak{m}_{(R,R)}$, $\mathfrak{m}_{(L,R)}$ and $\mathfrak{m}_{(R,L)}$ generated by $\{\mathfrak{Q}_{R1}, \mathfrak{E}_{R1}, \mathfrak{Q}_{R2}, \mathfrak{E}_{R2}, \mathfrak{P}, \mathfrak{P}^\dagger\}$, $\{\mathfrak{Q}_{L1}, \mathfrak{E}_{L1}, \mathfrak{Q}_{L2}, \mathfrak{E}_{L2}, \mathfrak{P}, \mathfrak{P}^\dagger\}$, $\{\mathfrak{Q}_{R1}, \mathfrak{E}_{R1}, \mathfrak{Q}_{L2}, \mathfrak{E}_{L2}, \mathfrak{P}, \mathfrak{P}^\dagger\}$ and $\{\mathfrak{Q}_{L1}, \mathfrak{E}_{L1}, \mathfrak{Q}_{R2}, \mathfrak{E}_{R2}, \mathfrak{P}, \mathfrak{P}^\dagger\}$. The boundary Lie superalgebras $\mathfrak{h}_{(L,L)}$ and $\mathfrak{h}_{(R,R)}$ are associated with open superstrings on AdS₃ × S³ × T⁴ ending on $Y = 0$ and $\bar{Y} = 0$ D1- and D5-brane maximal giant gravitons, respectively.

The K -matrix solutions of the relevant boundary intertwining equations are given by

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_{(a,b)}}(p) &= \mathbb{K}_{\mathcal{B}_{(a,b)}}^L(p) \oplus \mathbb{K}_{\mathcal{B}_{(a,b)}}^R(p), \\ \text{with } \mathbb{K}_{\mathcal{B}_{(a,b)}}^c(p) &= \mathbb{K}_{\mathcal{B}_a}^c(p) \otimes \mathbb{K}_{\mathcal{B}_b}^c(p), \end{aligned} \quad (6.11)$$

for $a, b, c \in \{L, R\}$, in terms of solutions in Sec. IV B 1. This satisfies the reflection equation (4.44).

Non-supersymmetric chiral boundary algebra.—A boundary coideal subalgebra which contains no supercharges, but leads to boundary scattering processes which preserve chirality, is

$$\mathcal{B}_{(NC,NC)} = \langle \mathfrak{H}_L, \mathfrak{H}_R \rangle. \quad (6.12)$$

The boundary Lie algebra $L(\mathcal{B}_{(NC,NC)}) = \mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$ is that of an open superstring on AdS₃ × S³ × T⁴ ending on the $\bar{Z} = 0$ D1- or D5-brane maximal giant graviton. A K -matrix solution of the boundary intertwining equation for $\mathcal{B}_{(NC,NC)}$ and the reflection equation is

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_{(NC,NC)}}(p) &= \mathbb{K}_{\mathcal{B}_{(NC,NC)}}^L(p) \oplus \mathbb{K}_{\mathcal{B}_{(NC,NC)}}^R(p), \\ \text{with } \mathbb{K}_{\mathcal{B}_{(NC,NC)}}^a(p) &= \mathbb{K}_{\mathcal{B}_{NC}}^a(p) \otimes \mathbb{K}_{\mathcal{B}_{NC}}^a(p). \end{aligned} \quad (6.13)$$

Let us now show that there is a *hidden symmetry*. A diagonally supersymmetric boundary algebra is defined by

$$\mathcal{B}_{(D,D)} = \langle \mathfrak{q}_{+1}, \mathfrak{q}_{-1}, \mathfrak{q}_{+2}, \mathfrak{q}_{-2}, \mathfrak{d}, \tilde{\mathfrak{d}} \rangle, \quad (6.14)$$

which is a coideal subalgebra of \mathcal{A} . The K -matrix (6.13) intertwines representations of these hidden boundary symmetries $\mathcal{B}_{(D,D)}$. Here

$$\begin{aligned}
 \mathfrak{q}_{+j} &= \mathfrak{P}^\dagger \mathfrak{Q}_{Lj} + ic\mathfrak{P}\mathfrak{E}_{Rj}, & \mathfrak{q}_{-j} &= \mathfrak{P}\mathfrak{E}_{Lj} + ic\mathfrak{P}^\dagger \mathfrak{Q}_{Rj}, \\
 \mathfrak{d} &= (\mathfrak{H}_L - c^2\mathfrak{H}_R + ic(\mathfrak{P} + \mathfrak{P}^\dagger))\mathfrak{K}, \\
 \tilde{\mathfrak{d}} &= (\mathfrak{H}_L - c^2\mathfrak{H}_R - ic(\mathfrak{P} + \mathfrak{P}^\dagger))\mathfrak{K},
 \end{aligned} \tag{6.15}$$

with $\mathfrak{K} = \mathfrak{P}\mathfrak{P}^\dagger$. The space $\mathcal{M}_{(D,D)}$ is generated by $\{\mathfrak{s}_{+1}, \mathfrak{s}_{-1}, \mathfrak{s}_{+2}, \mathfrak{s}_{-2}, \mathfrak{n}, \tilde{\mathfrak{n}}\}$, where we define

$$\begin{aligned}
 \mathfrak{s}_{+j} &= \mathfrak{P}^\dagger \mathfrak{Q}_{Lj} - ic\mathfrak{P}\mathfrak{E}_{Rj}, \\
 \mathfrak{s}_{-j} &= \mathfrak{P}\mathfrak{E}_{Lj} - ic\mathfrak{P}^\dagger \mathfrak{Q}_{Rj}, \\
 \mathfrak{n} &= (\mathfrak{H}_L + c^2\mathfrak{H}_R + ic(\mathfrak{P} - \mathfrak{P}^\dagger))\mathfrak{K}, \\
 \tilde{\mathfrak{n}} &= (\mathfrak{H}_L + c^2\mathfrak{H}_R - ic(\mathfrak{P} - \mathfrak{P}^\dagger))\mathfrak{K}.
 \end{aligned} \tag{6.16}$$

The generators of $\mathcal{B}_{(D,D)}$ and $\mathcal{M}_{(D,D)}$ satisfy

$$\begin{aligned}
 \{\mathfrak{q}_{+i}, \mathfrak{q}_{-j}\} &= \mathfrak{d}\delta_{ij}, & \{\mathfrak{q}_{+i}, \mathfrak{s}_{-j}\} &= \mathfrak{n}\delta_{ij}, & \{\mathfrak{q}_{+i}, \mathfrak{s}_{+j}\} &= 0, \\
 \{\mathfrak{s}_{+i}, \mathfrak{s}_{-j}\} &= \tilde{\mathfrak{d}}\delta_{ij}, & \{\mathfrak{q}_{-i}, \mathfrak{s}_{+j}\} &= \tilde{\mathfrak{n}}\delta_{ij}, & \{\mathfrak{q}_{-i}, \mathfrak{s}_{-j}\} &= 0,
 \end{aligned} \tag{6.17}$$

with $\mathfrak{d}, \tilde{\mathfrak{d}}$ and $\mathfrak{n}, \tilde{\mathfrak{n}}$ central elements. These relations are identical to (5.12). The associated Lie superalgebra $L(\mathcal{B}_{(D,D)} \oplus \mathcal{M}_{(D,D)})$ is isomorphic to $L(\mathcal{A})$, while here $L(\mathcal{B}_{(D,D)})$ is generated by $\{\mathfrak{q}_{+1}, \mathfrak{q}_{-1}, \mathfrak{q}_{+2}, \mathfrak{q}_{-2}, \mathfrak{d}, \tilde{\mathfrak{d}}\}$. Now $(\mathfrak{g}, \mathfrak{h}_{(D,D)})$ defines a symmetric pair, with

$$\begin{aligned}
 \mathfrak{g} &\equiv L(\mathcal{B}_{(D,D)} \oplus \mathcal{M}_{(D,D)}) \cong L(\mathcal{A}) \\
 &= [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2, \\
 \mathfrak{h}_{(D,D)} &\equiv L(\mathcal{B}_{(D,D)}) \\
 &= [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_D \oplus \mathfrak{u}(1)_D.
 \end{aligned} \tag{6.18}$$

Left and right quarter-supersymmetric boundary algebras.—The following left and right quarter-supersymmetric boundary superalgebras can also be defined:

$$\begin{aligned}
 \mathcal{B}_{(L,NC)} &= \langle \mathfrak{Q}_{L1}, \mathfrak{E}_{L1}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\
 \mathcal{B}_{(NC,L)} &= \langle \mathfrak{Q}_{L2}, \mathfrak{E}_{L2}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\
 \mathcal{B}_{(R,NC)} &= \langle \mathfrak{Q}_{R1}, \mathfrak{E}_{R1}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\
 \mathcal{B}_{(NC,R)} &= \langle \mathfrak{Q}_{R2}, \mathfrak{E}_{R2}, \mathfrak{H}_L, \mathfrak{H}_R \rangle.
 \end{aligned} \tag{6.19}$$

Here the associated boundary Lie superalgebras are $L(\mathcal{B}_{(L,NC)}) = L(\mathcal{B}_{(NC,L)}) = \mathfrak{su}(1|1)_L \oplus \mathfrak{u}(1)_R$ and $L(\mathcal{B}_{(R,NC)}) = L(\mathcal{B}_{(NC,R)}) = \mathfrak{u}(1)_L \oplus \mathfrak{su}(1|1)_R$. The K -matrices which satisfy the boundary intertwining equations of $\mathcal{B}_{(a,NC)}$ and $\mathcal{B}_{(NC,a)}$, respectively, and the reflection equation are

$$\mathbb{K}_{\mathcal{B}_{(a,NC)}}(p) = \mathbb{K}_{\mathcal{B}_{(a,NC)}}^L(p) \oplus \mathbb{K}_{\mathcal{B}_{(a,NC)}}^R(p),$$

$$\text{with } \mathbb{K}_{\mathcal{B}_{(a,NC)}}^b(p) = \mathbb{K}_{\mathcal{B}_a}^b(p) \otimes \mathbb{K}_{\mathcal{B}_{NC}}^b(p),$$

$$\mathbb{K}_{\mathcal{B}_{(NC,a)}}(p) = \mathbb{K}_{\mathcal{B}_{(NC,a)}}^L(p) \oplus \mathbb{K}_{\mathcal{B}_{(NC,a)}}^R(p),$$

$$\text{with } \mathbb{K}_{\mathcal{B}_{(NC,a)}}^b(p) = \mathbb{K}_{\mathcal{B}_{NC}}^b(p) \otimes \mathbb{K}_{\mathcal{B}_a}^b(p), \tag{6.20}$$

for $a, b \in \{L, R\}$.

There are again *hidden symmetries* in the Hopf superalgebra. We define coideal subalgebras of \mathcal{A} which take the form

$$\begin{aligned}
 \mathcal{B}_{(L,D)} &= \langle \mathfrak{Q}_{L1}, \mathfrak{E}_{L1}, \mathfrak{q}_{+2}, \mathfrak{q}_{-2}, \mathfrak{d}, \tilde{\mathfrak{d}}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\
 \mathcal{B}_{(D,L)} &= \langle \mathfrak{q}_{+1}, \mathfrak{q}_{-1}, \mathfrak{Q}_{L2}, \mathfrak{E}_{L2}, \mathfrak{d}, \tilde{\mathfrak{d}}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\
 \mathcal{B}_{(R,D)} &= \langle \mathfrak{Q}_{R1}, \mathfrak{E}_{R1}, \mathfrak{q}_{+2}, \mathfrak{q}_{-2}, \mathfrak{d}, \tilde{\mathfrak{d}}, \mathfrak{H}_L, \mathfrak{H}_R \rangle, \\
 \mathcal{B}_{(D,R)} &= \langle \mathfrak{q}_{+1}, \mathfrak{q}_{-1}, \mathfrak{Q}_{R2}, \mathfrak{E}_{R2}, \mathfrak{d}, \tilde{\mathfrak{d}}, \mathfrak{H}_L, \mathfrak{H}_R \rangle.
 \end{aligned} \tag{6.21}$$

We define $\mathcal{M}_{(L,D)}$, $\mathcal{M}_{(D,L)}$, $\mathcal{M}_{(R,D)}$ and $\mathcal{M}_{(D,R)}$ to be spaces generated by $\{\mathfrak{Q}_{R1}, \mathfrak{E}_{R1}, \mathfrak{s}_{+2}, \mathfrak{s}_{-2}, \mathfrak{n}, \tilde{\mathfrak{n}}\}$, $\{\mathfrak{s}_{+1}, \mathfrak{s}_{-1}, \mathfrak{Q}_{R2}, \mathfrak{E}_{R2}, \mathfrak{n}, \tilde{\mathfrak{n}}\}$, $\{\mathfrak{Q}_{L1}, \mathfrak{E}_{L1}, \mathfrak{s}_{+2}, \mathfrak{s}_{-2}, \mathfrak{n}, \tilde{\mathfrak{n}}\}$ and $\{\mathfrak{s}_{+1}, \mathfrak{s}_{-1}, \mathfrak{Q}_{L2}, \mathfrak{E}_{L2}, \mathfrak{n}, \tilde{\mathfrak{n}}\}$. Here we can construct symmetric pairs $(\mathfrak{g}, \mathfrak{h}_{(a,NC)})$ and $(\mathfrak{g}, \mathfrak{h}_{(NC,a)})$ of Lie algebras:

$$\begin{aligned}
 \mathfrak{g} &\equiv L(\mathcal{B}_{(a,D)} \oplus \mathcal{M}_{(a,D)}) = L(\mathcal{B}_{(D,a)} \oplus \mathcal{M}_{(D,a)}) \\
 &\cong L(\mathcal{A}) = [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2, \\
 \mathfrak{h}_{(a,D)} &\equiv L(\mathcal{B}_{(a,D)}) = \mathfrak{su}(1|1)_a \oplus \mathfrak{su}(1|1)_D, \\
 \mathfrak{h}_{(D,a)} &\equiv L(\mathcal{B}_{(D,a)}) = \mathfrak{su}(1|1)_D \oplus \mathfrak{su}(1|1)_a.
 \end{aligned} \tag{6.22}$$

Non-supersymmetric achiral boundary algebra.—A boundary algebra which contains no supercharges and breaks chiral symmetry is

$$\mathcal{B}_{(NA,NA)} = \langle \mathfrak{H} \rangle. \tag{6.23}$$

The boundary Lie algebra is $L(\mathcal{B}_{(NA,NA)}) = \mathfrak{u}(1)_+$. A K -matrix solution of the boundary intertwining equation of $\mathcal{B}_{(NA,NA)}$ and reflection equation is

$$\begin{aligned}
 \mathbb{K}_{(NA,NA)}(p) &= \begin{pmatrix} 0 & \mathbb{A}_{(NA,NA)}^R(p) \\ \mathbb{A}_{(NA,NA)}^L(p) & 0 \end{pmatrix}, \\
 \text{with } \mathbb{A}_{(NA,NA)}^a(p) &= \mathbb{A}_{NA}^a(p) \otimes \mathbb{A}_{NA}^a(p).
 \end{aligned} \tag{6.24}$$

It is not clear if there is a hidden symmetry in this case.

C. Vector boundaries

The totally supersymmetric boundary superalgebra $\mathcal{B}_{(\mathcal{T},\mathcal{T})}$ is the coideal subalgebra of \mathcal{A} generated as

$$\mathcal{B}_{(\mathcal{T},\mathcal{T})} = \langle \mathcal{Q}_{\mathcal{L}1}, \mathcal{E}_{\mathcal{L}1}, \mathcal{Q}_{\mathcal{L}2}, \mathcal{E}_{\mathcal{L}2}, \mathcal{Q}_{\mathcal{R}1}, \mathcal{E}_{\mathcal{R}1}, \mathcal{Q}_{\mathcal{R}2}, \mathcal{E}_{\mathcal{R}2}, \mathcal{H}_{\mathcal{R}}, \mathcal{H}_{\mathcal{L}}, \mathfrak{P}, \mathfrak{P}^\dagger \rangle, \quad (6.25)$$

with $\mathcal{U}, \mathcal{U}^{-1} \notin \mathcal{B}_{(\mathcal{T},\mathcal{T})}$.

The vector boundary state transforms in the left- or right-moving representation, $\pi_{\mathcal{B}}$ or $\bar{\pi}_{\mathcal{B}}$, of the boundary Lie superalgebra $L(\mathcal{B}_{(\mathcal{T},\mathcal{T})}) = [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$. Table III shows that this is the boundary superalgebra of an open superstring on AdS₃ × S³ × T⁴ attached to the Z = 0 D1- or D5-brane maximal giant graviton.

Boundary representations $\pi_{\mathcal{B}}$ and $\bar{\pi}_{\mathcal{B}}$.—We define the vector spaces associated with left- and right-moving boundary vector states in the same way as for the magnons in the bulk:

$$\begin{aligned} \mathcal{V}_{\mathcal{B}} &= \text{span}_{\mathbb{C}}\{|\varphi_{\mathcal{B}}^{++}\rangle, |\varphi_{\mathcal{B}}^{+-}\rangle, |\varphi_{\mathcal{B}}^{-+}\rangle, |\varphi_{\mathcal{B}}^{--}\rangle\}, \\ \bar{\mathcal{V}}_{\mathcal{B}} &= \text{span}_{\mathbb{C}}\{|\bar{\varphi}_{\mathcal{B}}^{++}\rangle, |\bar{\varphi}_{\mathcal{B}}^{+-}\rangle, |\bar{\varphi}_{\mathcal{B}}^{-+}\rangle, |\bar{\varphi}_{\mathcal{B}}^{--}\rangle\}, \end{aligned}$$

both isomorphic to $\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1}$. We set $\mathcal{W}_{\mathcal{B}} = \mathcal{V}_{\mathcal{B}} \oplus \bar{\mathcal{V}}_{\mathcal{B}}$. The left and right boundary representations $\pi_{\mathcal{B}}: \mathcal{B}_{(\mathcal{T},\mathcal{T})} \rightarrow \text{End}(\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1})$ and $\bar{\pi}_{\mathcal{B}}: \mathcal{B}_{(\mathcal{T},\mathcal{T})} \rightarrow \text{End}(\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1})$ are given by (5.15) and (5.16), with the subindex p replaced by \mathcal{B} . We choose the parametrization (4.35) for $a_{\mathcal{B}}, b_{\mathcal{B}}, c_{\mathcal{B}}$ and $d_{\mathcal{B}}$, with $m_{\mathcal{B}}$ the boundary mass parameter.

Boundary intertwining equations and K -matrix.—The K -matrix $K(p, \mathbf{B}) \in \text{End}(\mathcal{W}_p \otimes \mathcal{W}_{\mathcal{B}})$ decomposes into a direct sum of K -matrices in four sectors $K_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{ab}}(p, \mathbf{B})$, which correspond to the partial K -matrices $\mathbb{K}_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{ab}}(p, \mathbf{B}) \in \text{End}((\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1}) \otimes (\mathbf{C}^{1|1} \otimes \dot{\mathbf{C}}^{1|1}))$. Here the complete K -matrix

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}(p, \mathbf{B}) &= \mathbb{K}_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{LL}}(p, \mathbf{B}) \oplus \mathbb{K}_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{LR}}(p, \mathbf{B}) \\ &\quad \times \oplus \mathbb{K}_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{RL}}(p, \mathbf{B}) \oplus \mathbb{K}_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{RR}}(p, \mathbf{B}) \end{aligned} \quad (6.26)$$

satisfies the boundary intertwining equations (4.37) for all $\mathfrak{b} \in [\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$, with π_p and $\bar{\pi}_p$ the representations (5.15) and (5.16), and the reflected representations π_{-p} and $\bar{\pi}_{-p}$ and boundary representations $\pi_{\mathcal{B}}$ and $\bar{\pi}_{\mathcal{B}}$ similarly defined. The complete K -matrix is required to be unitary.

This K -matrix of a $\mathfrak{psu}(1, 1|2)^2$ spin-chain with a vector boundary can be built from two copies of the K -matrix of a $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain with a similar boundary given in Sec. IV C—which have partial K -matrices of the form

$$\mathbb{K}_{\mathcal{B}_{\mathcal{T}}}^{\text{ab}}(p, \mathbf{B}) = \sum_{i,j,k,l} (K_{\mathcal{B}_{\mathcal{T}}}^{\text{ab}}(p, \mathbf{B}))_{jl}^{ik} \mathbb{E}_{ij} \otimes \mathbb{E}_{kl},$$

$$\mathbb{K}_{\mathcal{B}_{\mathcal{T}}}^{\text{ab}}(p, \mathbf{B}) = \sum_{i,j,k,l} (K_{\mathcal{B}_{\mathcal{T}}}^{\text{ab}}(p, \mathbf{B}))_{ji}^{lk} \mathbb{E}_{ij} \otimes \mathbb{E}_{kl}.$$

The K -matrix $\mathbb{K}_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}(p, \mathbf{B})$ then has partial K -matrices

$$\begin{aligned} \mathbb{K}_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{ab}}(p, \mathbf{B}) &= \sum_{\substack{i,j,j', \\ k,k',l}} (K_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{ab}}(p, \mathbf{B}))_{j'j}^{ii,kk'} (\mathbb{E}_{ij} \otimes \mathbb{E}_{j'i}) \\ &\quad \otimes (\mathbb{E}_{k\ell} \otimes \mathbb{E}_{k'\ell'}), \end{aligned} \quad (6.27)$$

where

$$\begin{aligned} (K_{\mathcal{B}_{(\mathcal{T},\mathcal{T})}}^{\text{ab}}(p, \mathbf{B}))_{j'j}^{ii,kk'} &= (-1)^{(k+\ell)(i+j)} (K_{\mathcal{B}_{\mathcal{T}}}^{\text{ab}}(p, \mathbf{B}))_{j\ell}^{ik} \\ &\quad \times (K_{\mathcal{B}_{\mathcal{T}}}^{\text{ab}}(p, \mathbf{B}))_{j'\ell'}^{i'k'} \end{aligned} \quad (6.28)$$

and satisfies the reflection equation (4.43).

VII. DISCUSSION

We have derived integrable boundary S -matrices which describe magnon scattering off vector and singlet boundaries for $\mathfrak{d}(2, 1; \alpha)^2$ and $\mathfrak{psu}(1, 1|2)^2$ open spin-chains in AdS₃/CFT₂. These massive magnon excitations have $\mathfrak{su}(1|1)_{\mathbb{C}}^2$ and $[\mathfrak{psu}(1|1)^2 \oplus \mathfrak{u}(1)]_{\mathbb{C}}^2$ bulk symmetries, which are the level-0 Lie superalgebras of bulk Hopf superalgebras. The matrix parts of these boundary S -matrices are reflection K -matrices which are solutions of the boundary Yang-Baxter equation and the boundary intertwining equations associated with a coideal subalgebra \mathcal{B} of the bulk superalgebra \mathcal{A} .

In the case of the $\mathfrak{d}(2, 1; \alpha)^2$ open spin-chain, we find *chiral integrable reflections* associated with

- (1) A totally supersymmetric boundary algebra, $\mathcal{B}_{\mathcal{T}}$.
- (2) Left and right half-supersymmetric boundary algebras, $\mathcal{B}_{\mathcal{L}}$ and $\mathcal{B}_{\mathcal{R}}$.
- (3) A non-supersymmetric boundary algebra, \mathcal{B}_{NC} .

We also derive an *achiral integrable reflection* corresponding to a non-supersymmetric boundary algebra \mathcal{B}_{NA} generated by the magnon Hamiltonian only. These all match to D1- and D5-brane maximal giant graviton boundaries in AdS₃ × S³ × S¹ × S¹. We uncover a *hidden symmetry* which enhances the non-supersymmetric chiral boundary \mathcal{B}_{NC} to a diagonally supersymmetric coideal subalgebra $\mathcal{B}_{\mathcal{D}}$ of \mathcal{A} . This hidden symmetry $\mathcal{B}_{\mathcal{D}}$ has no known analogue in AdS₅/CFT₄.

In the case of the $\mathfrak{psu}(1, 1|2)^2$ open spin-chain, the integrable bulk S -matrix was found in Ref. [21] to be essentially two copies of the bulk S -matrix of the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain [20]. The same is true for the integrable boundary S -matrices. We can put together two K -matrices for the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain, which are associated with

boundary coideal subalgebras \mathcal{B}_1 and \mathcal{B}_2 , to form a K -matrix for a $\mathfrak{psu}(1,1|2)^2$ spin-chain corresponding to a boundary algebra denoted $\mathcal{B}_{(1,2)}$. In this way, we derived *chiral integrable reflections* associated with

- (1) A totally supersymmetric boundary algebra, $\mathcal{B}_{(T,T)}$.
 - (2) Left, right and mixed half-supersymmetric boundary algebras, $\mathcal{B}_{(L,L)}$, $\mathcal{B}_{(R,R)}$, $\mathcal{B}_{(L,R)}$ and $\mathcal{B}_{(R,L)}$.
 - (3) Left and right quarter-supersymmetric boundary algebras, $\mathcal{B}_{(L,NC)}$, $\mathcal{B}_{(NC,L)}$, $\mathcal{B}_{(R,NC)}$ and $\mathcal{B}_{(NC,R)}$.
 - (4) A non-supersymmetric boundary algebra, $\mathcal{B}_{(NC,NC)}$.
- Now only $\mathcal{B}_{(T,T)}$, $\mathcal{B}_{(L,L)}$, $\mathcal{B}_{(R,R)}$ and $\mathcal{B}_{(NC,NC)}$ have obvious interpretations as D1- and D5-brane maximal giant gravitons in $\text{AdS}_3 \times S^3 \times T^4$. There are *hidden symmetries* enhancing the quarter supersymmetric and non-supersymmetric chiral boundary algebras to $\mathcal{B}_{(L,D)}$, $\mathcal{B}_{(D,L)}$, $\mathcal{B}_{(R,D)}$, $\mathcal{B}_{(D,R)}$ and $\mathcal{B}_{(D,D)}$. There is also an *achiral integrable reflection* with a non-supersymmetric boundary algebra $\mathcal{B}_{(NA,NA)}$, which now has no clear D-brane interpretation.

It is well known that $\mathfrak{su}(1|1)_c^2$ R -matrices can be identified with certain subsectors of the $\mathfrak{su}(2|2)_c$ R -matrix [6,70,71]. We find that our K -matrices for the totally supersymmetric boundary can be identified with the corresponding subsectors of the K -matrix associated with the $Z = 0$ giant graviton [8] and the right factor of the K -matrix associated with the D7-brane [9]. The K -matrices for the half-supersymmetric boundaries can be identified with the corresponding subsectors of the K -matrix associated with the $Y = 0$ giant graviton [8] and its dual [57]. The K -matrix for the non-supersymmetric chiral boundary is essentially two copies of the reflection matrix of Ref. [10], with its free parameter a identified with our c and $-\frac{1}{c}$ in these copies. For certain values of the parameter c , this K -matrix can also be identified with the corresponding subsectors of the left factor of the K -matrix associated with the D7-brane [9]. The remaining K -matrices have no such analogues.

This work takes the first steps in an exploration of boundary integrability in $\text{AdS}_3/\text{CFT}_2$. There are many important questions which may now be addressed. We expect to present the boundary crossing symmetry relations and an analysis of boundary bound states in future research. The underlying boundary Yangian symmetries and the boundary Bethe equations in $\text{AdS}_3/\text{CFT}_2$ still remain to be studied. Recently, the authors of Ref. [72] were able to incorporate massless magnons into the $\text{AdS}_3/\text{CFT}_2$ bulk scattering picture. It would be interesting to extend our analysis to include the scattering of massless excitations off integrable boundaries. The Wilson loop computations of Refs. [73,74] relied upon results [11] from $\text{AdS}_5/\text{CFT}_4$ boundary scattering, and our work may prove useful should similar computations be undertaken in $\text{AdS}_3/\text{CFT}_2$ dualities. Finally, it would be interesting to study integrable boundaries for open superstrings on AdS_3 supergravity backgrounds with mixed NS-NS and R-R flux [72,75–82].

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APPENDIX A: SPINOR CONVENTIONS

Here we use the following 10D gamma matrices²⁰:

$$\begin{aligned}\Gamma^\mu &= \sigma^1 \otimes \sigma^2 \otimes \gamma^\mu \otimes \mathbb{1} \otimes \mathbb{1}, & \Gamma^n &= \sigma^1 \otimes \sigma^1 \otimes \mathbb{1} \otimes \gamma^n \otimes \mathbb{1}, \\ \Gamma^{\dot{n}} &= \sigma^1 \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma^{\dot{n}}, & \Gamma^9 &= \sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},\end{aligned}\tag{A1}$$

where we now choose $\gamma^\mu = (i\sigma^3, \sigma^2, \sigma^1)$, $\gamma^n = (\sigma^1, \sigma^3, \sigma^2)$ and $\gamma^{\dot{n}} = (\sigma^1, \sigma^3, \sigma^2)$. Hence,

$$\begin{aligned}\Gamma^{012} &= \sigma^1 \otimes \sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\ \Gamma^{345} &= -i\sigma^1 \otimes \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\ \Gamma^{678} &= -i\sigma^1 \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}.\end{aligned}\tag{A2}$$

The Weyl condition $\Gamma\varepsilon = \varepsilon$ is written in terms of the chirality matrix as

$$\Gamma \equiv \Gamma_{0123456789} = \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}.\tag{A3}$$

The Majorana condition on left- and right-moving spinors is $(B\varepsilon^L)^* = \varepsilon^L$ and $(B\varepsilon^R)^* = -\varepsilon^R$, with

$$B \equiv \Gamma^2\Gamma^5\Gamma^8\Gamma^9 = -\sigma^3 \otimes \mathbb{1} \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^2,\tag{A4}$$

which satisfies $B\Gamma^M B^{-1} = (\Gamma^M)^*$, with $B^{-1} = B$. The charge conjugation matrix is defined to be

$$C \equiv B\Gamma^0 = i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2,\tag{A5}$$

²⁰We make use of the conventions of Ref. [13] with a rearrangement of the Pauli matrices.

satisfying $C\Gamma^M C^{-1} = -(\Gamma_M)^*$, with $C^{-1} = -C$. We can compute the bilinears

$$\begin{aligned}\Gamma_{12} &= -i\mathbb{1} \otimes \mathbb{1} \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1}, \\ \Gamma_{35} &= i\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma^3 \otimes \mathbb{1}, \\ \Gamma_{68} &= i\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma^3.\end{aligned}\quad (\text{A6})$$

We define kappa symmetry projection operators for the AdS₃ × S³ × S³ × S¹ background as

$$\begin{aligned}K^\pm(\alpha) &\equiv \frac{1}{2} [1 \pm (\sqrt{\alpha}\Gamma^{012}\Gamma^{345} + \sqrt{1-\alpha}\Gamma^{012}\Gamma^{678})] \\ &= \frac{1}{2} \mathbb{1} \otimes [\mathbb{1} \mp (\sqrt{\alpha}\sigma^3 - \sqrt{1-\alpha}\sigma^1)] \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},\end{aligned}\quad (\text{A7})$$

which are dependent on the parameter α , which controls the relative size of the 3-spheres, and appears also in the superconformal algebra $\mathfrak{d}(2, 1; \alpha)_L \oplus \mathfrak{d}(2, 1; \alpha)_R \oplus \mathfrak{u}(1)$. In the limit as $\alpha \rightarrow 1$, we obtain the kappa symmetry projectors for the AdS₃ × S³ × T⁴ background

$$\begin{aligned}K^\pm &\equiv \frac{1}{2} (1 \pm \Gamma^{012}\Gamma^{345}) \\ &= \frac{1}{2} \mathbb{1} \otimes (\mathbb{1} \mp \sigma^3) \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},\end{aligned}\quad (\text{A8})$$

with the superconformal algebra now $\mathfrak{psu}(1, 1|2)_L \oplus \mathfrak{psu}(1, 1|2)_R \oplus \mathfrak{u}(1)^4$.

APPENDIX B: REPRESENTATIONS OF $\mathfrak{d}(2, 1; \alpha)$ AND $\mathfrak{psu}(1, 1|2)$

1. $\mathfrak{d}(2, 1; \alpha)$ superalgebra and BPS representations

Let us briefly review the representation theory of the exceptional Lie superalgebra $\mathfrak{d}(2, 1; \alpha)$ based on Refs. [13,20,83]. The bosonic subalgebra of $\mathfrak{d}(2, 1; \alpha)$ is $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)'$. The bosonic generators are denoted

$$\mathfrak{F}_\mu \in \mathfrak{su}(1, 1), \quad \mathfrak{L}_m \in \mathfrak{su}(2), \quad \mathfrak{R}_{\dot{m}} \in \mathfrak{su}(2)',$$

with $\mu \in \{0, \pm\}$, $m \in \{5, \pm\}$, $\dot{m} \in \{8, \pm\}$, and the fermionic generators $\mathfrak{Q}_{b\dot{\beta}\dot{\beta}}$, with \pm indices. The full $\mathfrak{d}(2, 1; \alpha)$ superalgebra is given by

$$\begin{aligned}[\mathfrak{F}_0, \mathfrak{F}_\pm] &= \pm \mathfrak{F}_\pm, & [\mathfrak{F}_+, \mathfrak{F}_-] &= 2\mathfrak{F}_0, \\ [\mathfrak{F}_0, \mathfrak{Q}_{\pm\dot{\beta}\dot{\beta}}] &= \pm \frac{1}{2} \mathfrak{Q}_{\pm\dot{\beta}\dot{\beta}}, & [\mathfrak{F}_\pm, \mathfrak{Q}_{\mp\dot{\beta}\dot{\beta}}] &= \mathfrak{Q}_{\pm\dot{\beta}\dot{\beta}}, \\ [\mathfrak{L}_5, \mathfrak{L}_\pm] &= \pm \mathfrak{L}_\pm, & [\mathfrak{L}_+, \mathfrak{L}_-] &= 2\mathfrak{L}_5, \\ [\mathfrak{L}_5, \mathfrak{Q}_{b\pm\dot{\beta}}] &= \pm \frac{1}{2} \mathfrak{Q}_{b\pm\dot{\beta}}, & [\mathfrak{L}_\pm, \mathfrak{Q}_{b\mp\dot{\beta}}] &= \mathfrak{Q}_{b\pm\dot{\beta}}, \\ [\mathfrak{R}_8, \mathfrak{R}_\pm] &= \pm \mathfrak{R}_\pm, & [\mathfrak{R}_+, \mathfrak{R}_-] &= \mathfrak{R}_8, \\ [\mathfrak{R}_8, \mathfrak{Q}_{b\dot{\beta}\pm}] &= \pm \frac{1}{2} \mathfrak{Q}_{b\dot{\beta}\pm}, & [\mathfrak{R}_\pm, \mathfrak{Q}_{b\dot{\beta}\pm}] &= \mathfrak{Q}_{b\dot{\beta}\pm}, \\ \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \pm \mathfrak{F}_\pm, & \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \mp \mathfrak{F}_\pm, \\ \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \mp \alpha \mathfrak{L}_\pm, & \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \pm \alpha \mathfrak{L}_\pm, \\ \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \mp (1-\alpha) \mathfrak{R}_\pm, \\ \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \pm (1-\alpha) \mathfrak{R}_\pm, \\ \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\mp\mp\mp}\} &= -\mathfrak{F}_0 \pm \alpha \mathfrak{L}_5 \pm (1-\alpha) \mathfrak{R}_8, \\ \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\mp\mp\mp}\} &= \mathfrak{F}_0 \mp \alpha \mathfrak{L}_5 \pm (1-\alpha) \mathfrak{R}_8.\end{aligned}\quad (\text{B1})$$

The bosonic and fermionic generators $\{\mathfrak{F}_\pm, \mathfrak{L}_\pm, \mathfrak{R}_\pm\}$ and $\{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\}$ are raising operators. The Cartan subalgebra is generated by $\{\mathfrak{H}, \mathfrak{L}_5, \mathfrak{R}_8\}$, with $\mathfrak{H} = -\mathfrak{F}_0 - \alpha \mathfrak{L}_5 - (1-\alpha) \mathfrak{R}_8$.

Half-BPS representation $(-\frac{\alpha}{2}, \frac{1}{2}, 0)$.—The module $M^\alpha(-\frac{\alpha}{2}, \frac{1}{2}, 0)$ on which this representation of $\mathfrak{d}(2, 1; \alpha)$ acts is spanned by bosons $|\phi_\pm^{(n)}\rangle$, which transform in the $\mathbf{2}$ of $\mathfrak{su}(2)$, and fermions $|\psi_\pm^{(n)}\rangle$, which transform in the $\mathbf{2}$ of $\mathfrak{su}(2)'$, both transforming nontrivially also under $\mathfrak{su}(1, 1)$. The nontrivial action of $\mathfrak{d}(2, 1; \alpha)$ on this module is given by

$$\begin{aligned}\mathfrak{L}_5 |\phi_\pm^{(n)}\rangle &= \pm \frac{1}{2} |\phi_\pm^{(n)}\rangle, & \mathfrak{L}_\pm |\phi_\mp^{(n)}\rangle &= |\phi_\pm^{(n)}\rangle, \\ \mathfrak{R}_8 |\psi_\pm^{(n)}\rangle &= \pm \frac{1}{2} |\psi_\pm^{(n)}\rangle, & \mathfrak{R}_\pm |\psi_\mp^{(n)}\rangle &= |\psi_\pm^{(n)}\rangle, \\ \mathfrak{F}_0 |\phi_\beta^{(n)}\rangle &= -\left(\frac{\alpha}{2} + n\right) |\phi_\beta^{(n)}\rangle, \\ \mathfrak{F}_0 |\psi_\beta^{(n)}\rangle &= -\left(\frac{\alpha}{2} + \frac{1}{2} + n\right) |\psi_\beta^{(n)}\rangle, \\ \mathfrak{F}_\pm |\phi_\beta^{(n)}\rangle &= \pm \sqrt{n\left(n - \frac{1}{2} \mp \frac{1}{2} + \alpha\right)} |\phi_\beta^{(n\mp 1)}\rangle, \\ \mathfrak{F}_\pm |\psi_\beta^{(n)}\rangle &= \pm \sqrt{n\left(n + \frac{1}{2} \mp \frac{1}{2} + \alpha\right)} |\psi_\beta^{(n\mp 1)}\rangle, \\ \mathfrak{Q}_{-\pm\dot{\beta}} |\phi_\mp^{(n)}\rangle &= \pm \sqrt{n + \alpha} |\psi_\dot{\beta}^{(n)}\rangle, \\ \mathfrak{Q}_{+\pm\dot{\beta}} |\phi_\mp^{(n)}\rangle &= \pm \sqrt{n} |\psi_\dot{\beta}^{(n-1)}\rangle, \\ \mathfrak{Q}_{-\dot{\beta}\pm} |\psi_\mp^{(n)}\rangle &= \mp \sqrt{n + 1} |\phi_\dot{\beta}^{(n+1)}\rangle, \\ \mathfrak{Q}_{+\dot{\beta}\pm} |\psi_\mp^{(n)}\rangle &= \mp \sqrt{n + \alpha} |\phi_\dot{\beta}^{(n)}\rangle.\end{aligned}\quad (\text{B2})$$

The highest weight state $|\phi_+^{(0)}\rangle$ is annihilated by all the raising operators, as well as by \mathfrak{R}_- and by two of the four fermionic lowering operators, \mathfrak{Q}_{-++} and \mathfrak{Q}_{-+-} . This representation is therefore half-BPS with the shortening condition

$$\begin{aligned} & \{\mathfrak{Q}_{+-\mp}, \mathfrak{Q}_{-+\pm}\}|\phi_+^{(0)}\rangle \\ &= (\mp \mathfrak{F}_0 \mp \alpha \mathfrak{L}_5 + (1-\alpha)\mathfrak{R}_8)|\phi_+^{(0)}\rangle = 0, \end{aligned} \quad (\text{B3})$$

which implies $\mathfrak{H}|\phi_+^{(0)}\rangle = 0$. The lowering operators act nontrivially on this highest weight state as

$$\begin{aligned} \mathfrak{L}_-|\phi_+^{(0)}\rangle &= |\phi_-^{(0)}\rangle, & \mathfrak{F}_-|\phi_+^{(0)}\rangle &= -|\phi_+^{(1)}\rangle, \\ \mathfrak{Q}_{--+}\phi_+^{(0)}\rangle &= -\sqrt{\alpha}|\psi_+^{(0)}\rangle, & \mathfrak{Q}_{---}\phi_+^{(0)}\rangle &= -\sqrt{\alpha}|\psi_-^{(0)}\rangle. \end{aligned} \quad (\text{B4})$$

Half-BPS representation $(-\frac{1-\alpha}{2}, 0, \frac{1}{2})$.—The module $M^{(1-\alpha)}(-\frac{1-\alpha}{2}, \frac{1}{2}, 0)$ on which this representation of $\mathfrak{d}(2, 1; \alpha)$ acts is spanned by bosons $|\phi_{\pm}^{(n)}\rangle$, which transform in the $\mathbf{2}$ of $\mathfrak{su}(2)'$, and fermions $|\psi_{\pm}^{(n)}\rangle$, which transform in the $\mathbf{2}$ of $\mathfrak{su}(2)$, both transforming nontrivially under $\mathfrak{su}(1, 1)$. The nontrivial action of $\mathfrak{d}(2, 1; \alpha)$ on this module is given by

$$\begin{aligned} \mathfrak{R}_8|\phi_{\pm}^{(n)}\rangle &= \pm \frac{1}{2}|\phi_{\pm}^{(n)}\rangle, & \mathfrak{R}_{\pm}|\phi_{\mp}^{(n)}\rangle &= |\phi_{\pm}^{(n)}\rangle, \\ \mathfrak{L}_5|\psi_{\pm}^{(n)}\rangle &= \pm \frac{1}{2}|\psi_{\pm}^{(n)}\rangle, & \mathfrak{L}_{\pm}|\psi_{\mp}^{(n)}\rangle &= |\psi_{\pm}^{(n)}\rangle, \\ \mathfrak{F}_0|\phi_{\beta}^{(n)}\rangle &= -\left(\frac{1-\alpha}{2} + n\right)|\phi_{\beta}^{(n)}\rangle, \\ \mathfrak{F}_0|\psi_{\beta}^{(n)}\rangle &= -\left(\frac{1-\alpha}{2} + \frac{1}{2} + n\right)|\psi_{\beta}^{(n)}\rangle, \\ \mathfrak{F}_{\pm}|\phi_{\beta}^{(n)}\rangle &= \pm \sqrt{n\left(n + \frac{1}{2} \mp \frac{1}{2} - \alpha\right)}|\phi_{\beta}^{(n\mp 1)}\rangle, \\ \mathfrak{F}_{\pm}|\psi_{\beta}^{(n)}\rangle &= \pm \sqrt{n\left(n + \frac{3}{2} \mp \frac{1}{2} - \alpha\right)}|\psi_{\beta}^{(n\mp 1)}\rangle, \\ \mathfrak{Q}_{-\beta\pm}|\phi_{\mp}^{(n)}\rangle &= \pm \sqrt{n+1-\alpha}|\psi_{\beta}^{(n)}\rangle, \\ \mathfrak{Q}_{+\beta\pm}|\phi_{\mp}^{(n)}\rangle &= \pm \sqrt{n}|\psi_{\beta}^{(n-1)}\rangle, \\ \mathfrak{Q}_{-\pm\beta}|\psi_{\mp}^{(n)}\rangle &= \mp \sqrt{n+1}|\phi_{\beta}^{(n+1)}\rangle, \\ \mathfrak{Q}_{+\pm\beta}|\psi_{\mp}^{(n)}\rangle &= \mp \sqrt{n+1-\alpha}|\phi_{\beta}^{(n)}\rangle. \end{aligned} \quad (\text{B5})$$

The highest weight state $|\phi_+^{(0)}\rangle$ is annihilated by all the raising operators, as well as by \mathfrak{L}_- and by two of the four

fermionic lowering operators, \mathfrak{Q}_{-++} and \mathfrak{Q}_{-+-} . This representation is therefore half-BPS with the shortening condition

$$\begin{aligned} & \{\mathfrak{Q}_{+\mp-}, \mathfrak{Q}_{-+\pm}\}|\phi_+^{(0)}\rangle \\ &= (\mp \mathfrak{F}_0 - \alpha \mathfrak{L}_5 \mp (1-\alpha)\mathfrak{R}_8)|\phi_+^{(0)}\rangle = 0, \end{aligned} \quad (\text{B6})$$

which implies $\mathfrak{H}|\phi_+^{(0)}\rangle = 0$. The lowering operators act nontrivially on this highest weight state as

$$\begin{aligned} \mathfrak{L}_-|\phi_+^{(0)}\rangle &= |\phi_-^{(0)}\rangle, & \mathfrak{F}_-|\phi_+^{(0)}\rangle &= -|\phi_+^{(1)}\rangle, \\ \mathfrak{Q}_{--+}\phi_+^{(0)}\rangle &= -\sqrt{1-\alpha}|\psi_+^{(0)}\rangle, \\ \mathfrak{Q}_{---}\phi_+^{(0)}\rangle &= -\sqrt{1-\alpha}|\psi_-^{(0)}\rangle. \end{aligned} \quad (\text{B7})$$

Quarter-BPS representation $(-\frac{\alpha}{2}, \frac{1}{2}, 0) \otimes (-\frac{1-\alpha}{2}, 0, \frac{1}{2})$.—The tensor product representation of the two half-BPS representations consists of vectors in the module $M = M^{\alpha} \otimes M^{(1-\alpha)}$. The highest weight state is $|\phi_+^{(0)}\phi_+^{(0)}\rangle$, which is annihilated by all the raising operators and one of the four fermionic lowering operators, \mathfrak{Q}_{-++} . This representation is quarter-BPS with the shortening condition

$$\begin{aligned} & \{\mathfrak{Q}_{+--}, \mathfrak{Q}_{-++}\}|\phi_+^{(0)}\phi_+^{(0)}\rangle \\ &= -(\mathfrak{F}_0 + \alpha \mathfrak{L}_5 + (1-\alpha)\mathfrak{R}_8)|\phi_+^{(0)}\phi_+^{(0)}\rangle \\ &= -\mathfrak{H}|\phi_+^{(0)}\phi_+^{(0)}\rangle = 0, \end{aligned} \quad (\text{B8})$$

and the other elements of the Cartan subalgebra act on the highest weight state as

$$\begin{aligned} \mathfrak{L}_5|\phi_+^{(0)}\phi_+^{(0)}\rangle &= \frac{1}{2}|\phi_+^{(0)}\phi_+^{(0)}\rangle, \\ \mathfrak{R}_8|\phi_+^{(0)}\phi_+^{(0)}\rangle &= \frac{1}{2}|\phi_+^{(0)}\phi_+^{(0)}\rangle. \end{aligned} \quad (\text{B9})$$

The nontrivial action of the bosonic lowering operators on this highest weight state is

$$\begin{aligned} \mathfrak{F}_-|\phi_+^{(0)}\phi_+^{(0)}\rangle &= -|\phi_+^{(1)}\phi_+^{(0)}\rangle - |\phi_+^{(0)}\phi_+^{(1)}\rangle, \\ \mathfrak{L}_-|\phi_+^{(0)}\phi_+^{(0)}\rangle &= |\phi_-^{(0)}\phi_+^{(0)}\rangle, & \mathfrak{R}_-|\phi_+^{(0)}\phi_+^{(0)}\rangle &= |\phi_+^{(0)}\phi_-^{(0)}\rangle, \end{aligned} \quad (\text{B10})$$

and the nontrivial action of the fermionic lowering operators is

$$\begin{aligned}
 \mathfrak{Q}_{--+}|\phi_+^{(0)}\phi_+'^{(0)}\rangle &= -\sqrt{\alpha}|\psi_+^{(0)}\phi_+'^{(0)}\rangle, \\
 \mathfrak{Q}_{-+-}|\phi_+^{(0)}\phi_+'^{(0)}\rangle &= -\sqrt{1-\alpha}|\phi_+^{(0)}\psi_+'^{(0)}\rangle, \\
 \mathfrak{Q}_{---}|\phi_+^{(0)}\phi_+'^{(0)}\rangle &= -\sqrt{\alpha}|\psi_-^{(0)}\phi_+'^{(0)}\rangle \\
 &\quad -\sqrt{1-\alpha}|\phi_+^{(0)}\psi_-^{(0)}\rangle. \quad (\text{B11})
 \end{aligned}$$

2. $\mathfrak{psu}(1,1|2)$ superalgebra and BPS representations

Here we review, based on Ref. [21], the representation theory of the Lie superalgebra $\mathfrak{psu}(1,1|2)$, which has the bosonic subalgebra of $\mathfrak{d}(2,1;\alpha)$ is $\mathfrak{su}(1,1) \oplus \mathfrak{su}(2)$. The bosonic generators are

$$\mathfrak{F}_\mu \in \mathfrak{su}(1,1), \quad \mathfrak{L}_m \in \mathfrak{su}(2),$$

with $\mu \in \{0, \pm\}$, $m \in \{5, \pm\}$, and the fermionic generators $\mathfrak{Q}_{b\beta\beta'}$, with \pm indices. The full $\mathfrak{psu}(1,1|2)$ superalgebra is

$$\begin{aligned}
 [\mathfrak{F}_0, \mathfrak{F}_\pm] &= \pm\mathfrak{F}_\pm, & [\mathfrak{F}_+, \mathfrak{F}_-] &= 2\mathfrak{F}_0, \\
 [\mathfrak{F}_0, \mathfrak{Q}_{\pm\beta\beta'}] &= \pm\frac{1}{2}\mathfrak{Q}_{\pm\beta\beta'}, & [\mathfrak{F}_\pm, \mathfrak{Q}_{\mp\beta\beta'}] &= \mathfrak{Q}_{\pm\beta\beta'}, \\
 [\mathfrak{L}_5, \mathfrak{L}_\pm] &= \pm\mathfrak{L}_\pm, & [\mathfrak{L}_+, \mathfrak{L}_-] &= 2\mathfrak{L}_5, \\
 [\mathfrak{L}_5, \mathfrak{Q}_{b\pm\beta}] &= \pm\frac{1}{2}\mathfrak{Q}_{b\pm\beta}, & [\mathfrak{L}_\pm, \mathfrak{Q}_{b\mp\beta}] &= \mathfrak{Q}_{b\pm\beta}, \\
 \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \pm\mathfrak{F}_\pm, & \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \mp\mathfrak{F}_\pm, \\
 \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \mp\mathfrak{L}_\pm, & \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\} &= \pm\mathfrak{L}_\pm, \\
 \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\mp\mp\mp}\} &= -\mathfrak{F}_0 \pm \mathfrak{L}_5, \\
 \{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\mp\mp\mp}\} &= \mathfrak{F}_0 \mp \mathfrak{L}_5. \quad (\text{B12})
 \end{aligned}$$

The bosonic generators $\{\mathfrak{F}_+, \mathfrak{L}_+\}$ and fermionic generators $\{\mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}, \mathfrak{Q}_{\pm\pm\pm}\}$ are the raising operators. The Cartan subalgebra is generated by $\{\mathfrak{F}_0, \mathfrak{L}_5\}$, with $\mathfrak{H} = -(\mathfrak{F}_0 + \mathfrak{L}_5)$. There is also a $\mathfrak{u}(1)$ automorphism \mathfrak{R}_8 which satisfies

$$[\mathfrak{R}_8, \mathfrak{Q}_{b\beta\beta'}] = \pm\frac{1}{2}\mathfrak{Q}_{b\beta\beta'}. \quad (\text{B13})$$

This $\mathfrak{psu}(1,1|2)$ superalgebra can be seen as the $\alpha \rightarrow 1$ limit of the $\mathfrak{d}(2,1;\alpha)$ superalgebra.

Half-BPS representation $(-\frac{1}{2}, \frac{1}{2})$.—This representation of $\mathfrak{psu}(1,1|2)$ acts on the module $M(-\frac{1}{2}, \frac{1}{2})$ which is spanned by bosons $|\phi_\pm^{(n)}\rangle$, transforming in the $\mathbf{2}$ of $\mathfrak{su}(2)$, and fermions $|\psi_\pm^{(n)}\rangle$ which are singlets. Both transform nontrivially under $\mathfrak{su}(1,1)$. The superalgebra $\mathfrak{psu}(1,1|2)$ acts as

$$\begin{aligned}
 \mathfrak{L}_5|\phi_\pm^{(n)}\rangle &= \pm\frac{1}{2}|\phi_\pm^{(n)}\rangle, & \mathfrak{L}_\pm|\phi_\mp^{(n)}\rangle &= |\phi_\pm^{(n)}\rangle, \\
 \mathfrak{F}_0|\phi_\beta^{(n)}\rangle &= -\left(\frac{1}{2} + n\right)|\phi_\beta^{(n)}\rangle, \\
 \mathfrak{F}_0|\psi_\beta^{(n)}\rangle &= -(1+n)|\psi_\beta^{(n)}\rangle, \\
 \mathfrak{F}_\pm|\phi_\beta^{(n)}\rangle &= \pm\left(n + \frac{1}{2} \mp \frac{1}{2}\right)|\phi_\beta^{(n\mp 1)}\rangle, \\
 \mathfrak{F}_\pm|\psi_\beta^{(n)}\rangle &= \pm\sqrt{\left(n + \frac{1}{2} \mp \frac{1}{2}\right)\left(n + \frac{3}{2} \mp \frac{1}{2}\right)}|\psi_\beta^{(n\mp 1)}\rangle, \\
 \mathfrak{Q}_{-\pm\beta}|\phi_\mp^{(n)}\rangle &= \pm\sqrt{n+1}|\psi_\beta^{(n)}\rangle, \\
 \mathfrak{Q}_{+\pm\beta}|\phi_\mp^{(n)}\rangle &= \pm\sqrt{n}|\psi_\beta^{(n-1)}\rangle, \\
 \mathfrak{Q}_{-\beta\pm}|\psi_\mp^{(n)}\rangle &= \mp\sqrt{n+1}|\phi_\beta^{(n+1)}\rangle, \\
 \mathfrak{Q}_{+\beta\pm}|\psi_\mp^{(n)}\rangle &= \mp\sqrt{n+1}|\phi_\beta^{(n)}\rangle. \quad (\text{B14})
 \end{aligned}$$

The highest weight state $|\phi_+^{(0)}\rangle$ is annihilated by the raising operators, as well as by two of the four fermionic lowering operators, $\mathfrak{Q}_{\pm\pm\pm}$ and $\mathfrak{Q}_{\mp\mp\mp}$. This representation is half-BPS with the shortening condition

$$\begin{aligned}
 \{\mathfrak{Q}_{\pm\mp\mp}, \mathfrak{Q}_{\mp\pm\pm}\}|\phi_+^{(0)}\rangle &= \mp(\mathfrak{F}_0 + \mathfrak{L}_5)|\phi_+^{(0)}\rangle \\
 &= \mp\mathfrak{H}|\phi_+^{(0)}\rangle = 0. \quad (\text{B15})
 \end{aligned}$$

The lowering operators act nontrivially on this highest weight state as

$$\begin{aligned}
 \mathfrak{L}_-|\phi_+^{(0)}\rangle &= |\phi_-^{(0)}\rangle, & \mathfrak{F}_-|\phi_+^{(0)}\rangle &= -|\phi_+^{(1)}\rangle, \\
 \mathfrak{Q}_{\mp\mp\mp}|\phi_+^{(0)}\rangle &= -|\psi_+^{(0)}\rangle, & \mathfrak{Q}_{\pm\pm\pm}|\phi_+^{(0)}\rangle &= -|\psi_-^{(0)}\rangle. \quad (\text{B16})
 \end{aligned}$$

APPENDIX C: BOSONIC SYMMETRIES

1. $SO(2,2)$ isometry group

$\mathfrak{so}(2,2)$ splitting.—Let us specify the Lie algebra splitting $\mathfrak{so}(2,2) = \mathfrak{su}(1,1)_L \oplus \mathfrak{su}(1,1)_R$. Here $SO(2,2)$ group transformations act on $x^\mu = (x^1, x^2, x^3, x^4) \in \text{AdS}_3 \subset \mathbb{R}^{2+2}$. If we combine the vector components into a quaternion

$$\underline{x} \equiv x^\mu \tau_\mu, \quad \text{with } \tau_\mu = (i\mathbb{1}, i\sigma_3, \sigma_1, \sigma_2) = (i\mathbb{1}, t_k), \quad (\text{C1})$$

then $SO(2,2)$ transformations can be realized as a $SU(1,1)_L \times SU(1,1)_R$ transformation

$$\begin{aligned}
 \underline{x} \rightarrow U_L \underline{x} U_R^{-1} &\approx \underline{x} + \delta \underline{x}, \quad \text{with } U_L = e^{\alpha^k t_k} \in SU(2)_L \\
 &\text{and } U_R = e^{-\bar{\alpha}^k t_k} \in SU(2)_R,
 \end{aligned}$$

where thus

$$\delta \underline{x} = \alpha^k t_k \underline{x} - \bar{\alpha}^k \underline{x} t_k = \delta x^\mu \tau_\mu. \quad (\text{C2})$$

The double-covering nature of this relationship corresponds to the fact that $(\pm U_L, \pm U_R)$ generate the same $SO(2, 2)$ rotation. The rotation angles and boost parameters are

$$\begin{aligned} \theta_{12} = -\theta_{21} = \alpha^3 - \bar{\alpha}^3, \quad \beta_{14} = \beta_{14} = \alpha^2 - \bar{\alpha}^2, \\ \beta_{13} = \beta_{13} = \alpha^1 - \bar{\alpha}^1, \quad \theta_{34} = -\theta_{43} = -(\alpha^3 + \bar{\alpha}^3), \\ \beta_{24} = \beta_{42} = -(\alpha^2 + \bar{\alpha}^2), \quad \beta_{23} = \beta_{32} = \alpha^1 + \bar{\alpha}^1. \end{aligned} \quad (\text{C3})$$

Here $e^{i\alpha^3 \sigma_3}$ and $e^{i\bar{\alpha}^3 \sigma_3}$ are in the Cartan subgroups $U(1)_L \subset SU(1, 1)_L$ and $U(1)_R \subset SU(1, 1)_R$. The $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ generators of rotations by θ_{12} and θ_{34} can be written as $\mathfrak{F}_{L0} - \mathfrak{F}_{R0}$ and $-(\mathfrak{F}_{L0} + \mathfrak{F}_{R0})$ in terms of the left and right generators, \mathfrak{F}_{a0} , of the Cartan subalgebra $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$.

2. $SO(4)$ isometry group

$\mathfrak{so}(4)$ splitting.—Let us specify the Lie algebra splitting $\mathfrak{so}(4) \sim \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. A $SO(4)$ rotation acts on $x^K = (x^1, x^2, x^3, x^4) \in S^3 \subset \mathbb{R}^4$. If we combine the vector components into a quaternion

$$\underline{x} \equiv x^K \tau_K, \quad \text{with } \tau_K = (\mathbb{1}, i\sigma_3, i\sigma_1, i\sigma_2), \quad (\text{C4})$$

then any rotation in $SO(4)$ can be realized (in two ways) as a $SU(2)_L \times SU(2)_R$ transformation

$$\begin{aligned} \underline{x} \rightarrow U_L \underline{x} U_R^{-1} \approx \underline{x} + \delta \underline{x}, \quad \text{with } U_L = e^{i\alpha^k \sigma_k} \in SU(2)_L \\ \text{and } U_R = e^{i\bar{\alpha}^k \sigma_k} \in SU(2)_R, \end{aligned}$$

where

$$\delta \underline{x} = i(\alpha^k \sigma_k \underline{x} - \bar{\alpha}^k \underline{x} \sigma_k) = \delta x^K \tau_K. \quad (\text{C5})$$

The rotation angles are given by

$$\begin{aligned} \theta_{12} = -\theta_{21} = \alpha^3 - \bar{\alpha}^3, \quad \theta_{14} = -\theta_{41} = \alpha^2 - \bar{\alpha}^2, \\ \theta_{13} = -\theta_{42} = \alpha^1 - \bar{\alpha}^1, \quad \theta_{34} = -\theta_{43} = -(\alpha^3 + \bar{\alpha}^3), \\ \theta_{23} = -\theta_{32} = -(\alpha^2 + \bar{\alpha}^2), \quad \theta_{24} = -\theta_{42} = \alpha^1 + \bar{\alpha}^1. \end{aligned} \quad (\text{C6})$$

Here $e^{i\alpha^3 \sigma_3}$ and $e^{i\bar{\alpha}^3 \sigma_3}$ are elements of the Cartan subgroups $U(1)_L \subset SU(2)_L$ and $U(1)_R \subset SU(2)_R$. The $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ generators of rotations by θ_{12} and θ_{34} can be written as $\mathfrak{L}_{L5} - \mathfrak{L}_{R5}$ and $-(\mathfrak{L}_{L5} + \mathfrak{L}_{R5})$ in terms of the left and right generators, \mathfrak{L}_{a5} , of the Cartan subalgebra $\mathfrak{u}(1)_L \oplus \mathfrak{u}(1)_R$.

$\mathfrak{so}(4)'$ splitting.—There is a similar splitting $\mathfrak{so}(4)' \sim \mathfrak{su}(2)'_L \oplus \mathfrak{su}(2)'_R$ with the $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ generators, $\mathfrak{R}_{L5} - \mathfrak{R}_{R5}$ and $-(\mathfrak{R}_{L5} + \mathfrak{R}_{R5})$, of rotations by θ'_{12} and θ'_{34} written in terms of left and right generators, \mathfrak{R}_{a8} , of the Cartan subalgebra $\mathfrak{u}(1)'_L \oplus \mathfrak{u}(1)'_R$.

$\mathfrak{d}(2, 1; \alpha)^2$ spin-chain fields.—This $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain described in Sec. III contains left- and right-moving fields which transform under $\mathfrak{so}(4) \oplus \mathfrak{so}(4)' \sim (\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R) \oplus (\mathfrak{su}(2)'_L \oplus \mathfrak{su}(2)'_R)$.

Let us define $Z = x_1 + ix_2$ and $Y = x_3 + ix_4$, which transform under $SO(4)$. We notice that

$$\underline{x} = \begin{pmatrix} x_1 + ix_2 & i(x_3 - ix_4) \\ i(x_3 + ix_4) & x_1 - ix_2 \end{pmatrix} = \begin{pmatrix} Z & i\bar{Y} \\ iY & \bar{Z} \end{pmatrix}. \quad (\text{C7})$$

We can write a similar relation for $Z' = x'_1 + ix'_2$ and $Y' = x'_3 + ix'_4$ transforming under $SO(4)'$.

Now $\phi_{L\beta}^{(0)} = (\phi_{L+}^{(0)}, \phi_{L-}^{(0)})$ and $\phi_{R\beta}^{(0)} = (\phi_{R+}^{(0)}, \phi_{R-}^{(0)})$ transform nontrivially under $SU(2)_L$ and $SU(2)_R$, respectively. Thus Z and Y (and their complex conjugates \bar{Z} and \bar{Y}) transform under the rotational symmetry $SO(4) \sim SU(2)_L \times SU(2)_R$ as

$$\begin{aligned} Z &\sim \begin{pmatrix} \phi_{L+}^{(0)} \\ \phi_{R+}^{(0)} \end{pmatrix}, & \bar{Z} &\sim \begin{pmatrix} \phi_{L-}^{(0)} \\ \phi_{R-}^{(0)} \end{pmatrix}, \\ Y &\sim \begin{pmatrix} \phi_{L-}^{(0)} \\ \phi_{R+}^{(0)} \end{pmatrix}, & \bar{Y} &\sim \begin{pmatrix} \phi_{L+}^{(0)} \\ \phi_{R-}^{(0)} \end{pmatrix} \end{aligned}$$

in the notation of the fields in our double-row spin-chain. Similarly, for the primed fields associated with the $SO(4)'$ rotational symmetry group,

$$\begin{aligned} Z' &\sim \begin{pmatrix} \phi_{L+}^{\prime(0)} \\ \phi_{R+}^{\prime(0)} \end{pmatrix}, & \bar{Z}' &\sim \begin{pmatrix} \phi_{L-}^{\prime(0)} \\ \phi_{R-}^{\prime(0)} \end{pmatrix}, \\ Y' &\sim \begin{pmatrix} \phi_{L-}^{\prime(0)} \\ \phi_{R+}^{\prime(0)} \end{pmatrix}, & \bar{Y}' &\sim \begin{pmatrix} \phi_{L+}^{\prime(0)} \\ \phi_{R-}^{\prime(0)} \end{pmatrix}. \end{aligned}$$

The vacuum of the $\mathfrak{d}(2, 1; \alpha)^2$ spin-chain therefore transforms as

$$\underline{Z} = \begin{pmatrix} Z_L \\ Z_R \end{pmatrix} = \begin{pmatrix} (\phi_{L+}^{(0)}, \phi_{L+}^{\prime(0)}) \\ (\phi_{R+}^{(0)}, \phi_{R+}^{\prime(0)}) \end{pmatrix} \sim ZZ',$$

while the fundamental bosonic excitations transform as

$$\begin{aligned}\phi &= \begin{pmatrix} \phi_L \\ Z_R \end{pmatrix} = \begin{pmatrix} (\phi_{L-}^{(0)}, \phi_{L+}^{\prime(0)}) \\ (\phi_{R+}^{(0)}, \phi_{R+}^{\prime(0)}) \end{pmatrix} \sim YZ', \\ \phi' &= \begin{pmatrix} \phi'_L \\ Z_R \end{pmatrix} = \begin{pmatrix} (\phi_{L+}^{(0)}, \phi_{L-}^{\prime(0)}) \\ (\phi_{R+}^{(0)}, \phi_{R+}^{\prime(0)}) \end{pmatrix} \sim ZY', \\ \bar{\phi} &= \begin{pmatrix} Z_L \\ \phi_R \end{pmatrix} = \begin{pmatrix} (\phi_{L-}^{(0)}, \phi_{L+}^{\prime(0)}) \\ (\phi_{R+}^{(0)}, \phi_{R+}^{\prime(0)}) \end{pmatrix} \sim \bar{Y}Z', \\ \bar{\phi}' &= \begin{pmatrix} Z_L \\ \phi'_R \end{pmatrix} = \begin{pmatrix} (\phi_{L+}^{(0)}, \phi_{L-}^{\prime(0)}) \\ (\phi_{R+}^{(0)}, \phi_{R+}^{\prime(0)}) \end{pmatrix} \sim \bar{Y}Z'.\end{aligned}$$

Here a composite state of the ϕ and ϕ' excitations would transform as YY' .

$\mathfrak{psu}(1, 1|2)^2$ spin-chain fields.—The $\mathfrak{psu}(1, 1|2)^2$ spin-chain described in Sec. V contains left- and right-moving fields which transform under $\mathfrak{so}(4) \sim \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$.

Defining again $Z=x_1+ix_2$ and $Y=x_3+ix_4$, we find that the vacuum of the $\mathfrak{psu}(1, 1|2)^2$ spin-chain transforms as

$$\mathcal{Z} = \begin{pmatrix} Z_L \\ Z_R \end{pmatrix} = \begin{pmatrix} \phi_{L+}^{(0)} \\ \phi_{R+}^{(0)} \end{pmatrix} \sim Z,$$

while, for the first bosonic excitations,

$$\begin{aligned}\varphi^{+\dagger} &= \begin{pmatrix} \varphi_L^{+\dagger} \\ Z_R \end{pmatrix} = \begin{pmatrix} \phi_{L-}^{(0)} \\ \phi_{R+}^{(0)} \end{pmatrix} \sim Y, \\ \bar{\varphi}^{+\dagger} &= \begin{pmatrix} Z_L \\ \varphi_R^{+\dagger} \end{pmatrix} = \begin{pmatrix} \phi_{L+}^{(0)} \\ \phi_{R-}^{(0)} \end{pmatrix} \sim \bar{Y}.\end{aligned}$$

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