$A_k\bar{F}$ chiral gauge theories

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We study asymptotically free chiral gauge theories with an SU(N) gauge group and chiral fermions transforming according to the antisymmetric rank-k tensor representation, $A_k \equiv [k]_N$, and the requisite number, $n_{\bar{F}}$, of copies of fermions in the conjugate fundamental representation, $\bar{F} \equiv \overline{[1]}_N$, to render the theories anomaly-free. We denote these as $A_k\bar{F}$ theories. We take $N \geq 2k+1$ so that $n_{\bar{F}} \geq 1$. The $A_2\bar{F}$ theories form an infinite family with $N \geq 5$, but we show that the $A_3\bar{F}$ and $A_4\bar{F}$ theories are only asymptotically free for N in the respective ranges $7 \leq N \leq 17$ and $9 \leq N \leq 11$, and that there are no asymptotically free $A_k\bar{F}$ theories with $k \geq 5$. We investigate the types of ultraviolet to infrared evolution for these $A_k\bar{F}$ theories and find that, depending on k and N, they may lead to a non-Abelian Coulomb phase, or may involve confinement with massless gauge-singlet composite fermions, bilinear fermion condensation with dynamical gauge and global symmetry breaking, or formation of multifermion condensates that preserve the gauge symmetry. We also show that there are no asymptotically free, anomaly-free SU(N) $S_k\bar{F}$ chiral gauge theories with $k \geq 3$, where S_k denotes the rank-k symmetric representation.

I. INTRODUCTION

The properties of chiral gauge theories, especially in the strong-coupling regime, remain a challenge for theoretical understanding. One requires that such a theory must be free of any triangle anomaly in gauged currents, since such an anomaly would spoil the renormalizability of the theory. Imposing the additional requirement that such a theory must be asymptotically free guarantees that there is at least one region, namely the deep ultraviolet (UV) at large Euclidean energy/momentum scales μ, where the running gauge coupling $g(\mu)$ is small, so that the properties of the theory are reliably calculable using perturbative methods. A chiral gauge theory is said to be irreducibly chiral if it does not contain any vectorlike subsector. In this case, the chiral gauge symmetry precludes any fermion mass terms in the underlying Lagrangian. We shall focus on irreducibly chiral theories here. In an asymptotically free gauge theory, as the reference scale μ decreases toward the infrared (IR) from the ultraviolet, the running gauge coupling grows. One possibility is that the beta function has an infrared zero at a small value of the coupling, which constitutes an infrared fixed point of the renormalization group (RG). In this case, in the full renormalization-group evolution of the theory from the deep UV to the IR, the gauge interaction remains weakly coupled, and one expects that the IR behavior is that of a (deconfined) non-Abelian Coulomb phase. In contrast, if the beta function has an IR zero at a sufficiently large value of the coupling, or if the beta function does not have any IR zero, then the theory becomes strongly coupled in the infrared. In this case, several types of behavior can occur. Since the fermions are massless, the theory is invariant under a global flavor symmetry. If one can construct gauge-singlet fermionic operator products that

match the global anomalies of the fundamental fermions, a condition known as 't Hooft global anomaly matching, then the strongly coupled chiral gauge interaction may confine and produce massless gauge-singlet composite spin-1/2 fermions [1–13]. Alternatively, the strong gauge interaction may produce fermion condensate(s) that spontaneously break gauge and global chiral symmetries [8,9,14–22]. This latter type of behavior can occur in several stages at different energy scales, resulting in a hierarchy of symmetry-breaking scales. In addition to their intrinsic field-theoretic interest, strongly coupled chiral gauge theories have been applied in efforts to construct (preon) models of composite quarks and leptons and models explaining electroweak symmetry breaking and the structure of fermion generations and masses in the Standard Model. Some work along these lines includes [1–21].

In this paper we shall study asymptotically free chiral gauge theories in four spacetime dimensions (at zero temperature) with an SU(N) gauge group and chiral fermions transforming according to the antisymmetric rank-k tensor representation, denoted $A_k \equiv [k]_N$, and the requisite number, $n_{\bar{F}}$, of copies of fermions in the conjugate fundamental representation, $\bar{F} \equiv \overline{[1]}_N$, to render the theories anomaly-free [23]. We denote these as $A_k\bar{F}$ theories. We take $N \geq 2k+1$ so that the theory is chiral and $n_{\bar{F}} \geq 1$. We extend previous studies on the $A_2\bar{F}$ theories [2,8–12,18] with further analysis of fermion condensation channels and sequential symmetry breaking, while the $A_k\bar{F}$ theories with $k \geq 3$ are, to our knowledge, new here. These $A_k\bar{F}$ theories with $k \geq 3$ form a natural generalization of the $A_2\bar{F}$ theories and are of particular interest because of the fact that each $A_k\bar{F}$ theory is irreducibly chiral and is minimal in the sense that it only

contains two types of fermion representations. The $A_2\bar{F}$ theories were of interest both as preon models [2,8–12] and as reasonably ultraviolet-complete models of dynamical electroweak symmetry breaking and quark and lepton mass generation [18]. The $A_2\bar{F}$ theories form an infinite family with $N \geq 5$, but we show that the $A_3\bar{F}$ and $A_4\bar{F}$ theories are only asymptotically free for N in the respective ranges $7 \leq N \leq 17$ and $9 \leq N \leq 11$, and there are no asymptotically free $A_k\bar{F}$ theories with $k \geq 5$. We investigate the types of ultraviolet to infrared evolution for these $A_k\bar{F}$ theories and find that, depending on k and N, they may lead to a non-Abelian Coulomb phase, or may involve confinement with massless gauge-singlet composite fermions, or fermion condensation with dynamical gauge and global symmetry breaking. One of the methods that we use for our analysis is the most-attractive-channel criterion for bilinear fermion condensation [2]. We find two cases in each of which two channels are equally attractive, so the most-attractive channel criterion cannot determine which is more likely to occur; for these we show how vacuum alignment arguments prefer one channel over the other. We also discuss the possibility that the strongly coupled gauge interaction can produce multifermion condensates that preserve the gauge symmetry. Finally, we show that there are no asymptotically free, anomaly-free $SU(N)$ $S_k\bar{F}$ chiral gauge theories with $k \geq 3$, where S_k denotes the rank-k symmetric representation. We restrict our consideration here to chiral gauge theories with only gauge and fermion fields, but without any scalar fields; the nonperturbative behavior of systems with interacting gauge, fermion, and scalar fields has been studied, e.g., in [24], and some recent work on RG flows in chiral theories with scalar fields includes [25].

This paper is organized as follows. In Sec. II we briefly review the theoretical methods that we use for our work. In Sec. III we discuss the construction of the $A_k\bar{F}$ chiral gauge theories and determine the constraints from anomaly cancellation and asymptotic freedom. In Sec. IV we ascertain whether or not the maximal scheme-independent information from the beta function indicates that the theory has an infrared zero, and, if so, we calculate the value of α at this zero in the beta function. Sections V and VI contain general discussions of the global flavor symmetry group and the most attractive channel for bilinear fermion condensation formation in an $A_k\bar{F}$ theory. In Secs. VII, VIII, and IX we present our results on the specific $A_2\bar{F}$, $A_3\bar{F}$, and $A_4\bar{F}$ classes of theories, respectively. In Sec. X we discuss multifermion condensates that can preserve chiral gauge symmetry. In Sec. XI we prove that there are no asymptotically free $S_k\bar{F}$ chiral gauge theories with $k \geq 3$, where S_k denotes the symmetric rank-k tensor representation of $SU(N)$. Our conclusions are given in Sec. XII and some auxiliary formulas are included in two appendices.

II. METHODS OF ANALYSIS

In this section we briefly discuss the methods of analysis that we use for our work. We refer the reader to [12,13,22] for more detailed discussions of these methods. To determine the constraints due to the requirement of asymptotic freedom and, for asymptotically free theories, to study the UV to IR evolution, we calculate the beta function to its maximal scheme-independent order, namely two-loops. We denote $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) \equiv g(\mu)^2/16\pi^2$. The beta function is $\beta_\alpha = d\alpha/dt$, where $dt = d\ln\mu$, with the series expansion

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \alpha^\ell. \qquad (2.1)$$

In Eq. (2.1) we have extracted an overall minus sign, b_ℓ is the ℓ-loop coefficient, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$ is the reduced ℓ-loop coefficient. The n-loop beta function, denoted $\beta_{\alpha,n\ell}$, is given by Eq. (2.1) with the upper limit on the ℓ-loop summation equal to n instead of ∞. The requirement of asymptotic freedom means that $\beta_\alpha < 0$ for small α, which holds if $b_1 > 0$. General expressions for b_1 [26] and b_2 [27] are given in Appendix A.

Given that $b_1 > 0$, it follows that if $b_2 < 0$, then the two-loop beta function, $\beta_{\alpha,2\ell}$, has an IR zero at $a_{\mathrm{IR},2\ell} = -b_1/b_2$, or equivalently,

$$\alpha_{\mathrm{IR},2\ell} = -\frac{\bar{b}_1}{\bar{b}_2} = -\frac{4\pi b_1}{b_2}. \qquad (2.2)$$

For sufficiently large fermion content in an asymptotically free theory, b_2 may be negative, so that the beta function exhibits such an infrared zero. This was discussed for vectorial gauge theories in [27,28] and is also important for the chiral gauge theories under consideration here. As the fermion content is reduced, $\alpha_{\mathrm{IR},2\ell}$ increases, and for sufficiently small fermion content, the beta function may not exhibit any IR zero. In both the case where the IR zero occurs at a substantial value of $\alpha_{\mathrm{IR},2\ell}$ and the case where the beta function has no IR zero, the theory becomes strongly coupled in the infrared. Higher-loop calculations of the IR zero for various fermion representations, including rank-2 tensor representations, were presented in [29,30]. Since these higher-loop calculations are scheme-dependent, it was necessary to assess the sensitivity of the IR zero to scheme transformations. This was done in [31–33]. For our purposes here, it will suffice to use the maximal scheme-independent information available from the beta function, as encoded in the coefficients up to the two-loop level.

In these situations of strong coupling in the infrared, one can apply various methods to analyze the resultant nonperturbative behavior of the chiral gauge theory. First, one may investigate whether the fermion content of the theory satisfies the 't Hooft global anomaly matching conditions [1].

These are necessary but not sufficient conditions for the gauge interaction to confine and produce massless gauge-singlet composite spin-1/2 fermions. If such massless spin-1/2 fermions are actually produced, they would saturate the massless-fermion sector of the theory, since any composite fermions with spins $J \geq 3/2$ would be massive [3].

An alternative possibility is that the gauge interaction can produce bilinear fermion condensates. In an irreducibly chiral theory (without a vectorlike subsector), these condensates break the gauge symmetry, as well as global flavor symmetries. A common method to identify the most likely channel in which this condensation occurs is the most-attractive-channel (MAC) criterion [2]. Let us consider a fermion condensation channel in which fermions in the representations R_1 and R_2 of the gauge group G form a (Lorentz-invariant) bilinear fermion condensate that transforms according to the representation R_c of G, denoted as

$$R_1 \times R_2 \to R_c, \qquad (2.3)$$

where the subscript c stands for "condensate channel." An approximate measure of the attractiveness of this condensation channel is

$$\Delta C_2 \equiv C_2(R_1) + C_2(R_2) - C_2(R_c), \qquad (2.4)$$

where $C_2(R)$ is the quadratic Casimir invariant for the representation R (see Appendix B). In this approach, the most attractive channel for bilinear fermion condensation is the one with the largest (positive) value of ΔC_2, and this is thus the most likely to occur. Here one bears in mind that the MAC is based on the one-gluon exchange and hence is only a rough guide to the nonperturbative phenomenon of fermion condensation. If two or more such channels have the same value of ΔC_2, then we make use of a vacuum alignment argument [14,15], as follows. Let us consider the case where two channels have the same ΔC_2 and produce condensates in the representations R_{c_1} and R_{c_2}. Assume that these condensates break the initial gauge group G to the respective subgroups $H_{c_1} \subset G$ and $H_{c_2} \subset G$. The vacuum alignment argument favors the condensation channel that yields the larger residual subgroup, H_{c_1} or H_{c_2}, namely the one with the larger order, $o(H_{c_1})$ or $o(H_{c_2})$ [where the order, $o(H)$, of a Lie group H is the number of generators of the associated Lie algebra]. This is based on an energy minimization argument, since the channel that respects the largest residual gauge symmetry will minimize the number of gauge bosons that pick up masses.

A rough estimate of the minimal critical strength of the coupling for fermion condensation in a given channel has been obtained from an analysis of the Schwinger-Dyson equation for the fermion propagator and is [34]

$$\alpha_{cr,c} \sim \frac{2\pi}{3\Delta C_2(R_c)}. \qquad (2.5)$$

Owing to the uncertainties inherent in the strong-coupling physics describing this condensation phenomenon, Eq. (2.5) is only a rough estimate. For our purposes, it will be convenient to define the ratio for the channel (2.3):

$$\rho_c \equiv \frac{\alpha_{\mathrm{IR},2\ell}}{\alpha_{cr,c}}. \qquad (2.6)$$

If ρ_c is considerably larger (smaller) than unity, then condensation in the given channel (2.3) is likely (unlikely). We note that fermion condensation in a strongly coupled gauge theory may, in principle, involve a product of an even number of fermion operators larger than just two [35,36]. A conjecture for a thermally motivated inequality concerning a measure of field degrees of freedom, as evaluated in the UV and in the IR, was proposed and studied for vectorial and chiral gauge theories in [10,37] and investigated further in several works, including [11–13,38]. Here our main methods will consist of analyses of beta functions, various channels for fermion condensation, and construction of low-energy effective field theories resulting from self-breaking of chiral gauge theories.

III. $A_k\bar{F}$ THEORIES AND CONSTRAINTS FROM ANOMALY CANCELLATION AND ASYMPTOTIC FREEDOM

The chiral gauge theories that we study here have an SU(N) gauge group and chiral fermions transforming according to a rank-k antisymmetric tensor representation $A_k \equiv [k]_N$ of this group, and the requisite number of chiral fermions in the conjugate fundamental representation, $\bar{F} \equiv \overline{[1]}_N$, to render the theories free of any anomaly in gauged currents [23]. Here we determine the constraints on these theories from anomaly cancellation and asymptotic freedom. These theories are irreducibly chiral, i.e., they do not contain any vectorlike subsector. Consequently, the chiral gauge symmetry forbids any fermion mass terms in the underlying Lagrangian. We denote the number of copies (flavors) of $\bar{F}$ fermions as $n_{\bar{F}}$. The contribution to the triangle anomaly in gauged currents of a chiral fermion in the A_k representation is [39] (see Appendix B)

$$\mathcal{A}([k]_N) = \frac{(N-3)!(N-2k)}{(N-k-1)!(k-1)!}. \qquad (3.1)$$

The total anomaly in the theory is

$$\begin{aligned} \mathcal{A} &= \mathcal{A}([k]_N) + n_{\bar{F}}\mathcal{A}(\overline{[1]}_N) \\ &= \mathcal{A}([k]_N) - n_{\bar{F}}\mathcal{A}([1]_N), \end{aligned} \qquad (3.2)$$

so $\mathcal{A} = 0$, i.e., the theory is free of anomalies in gauged currents, if and only if

$$n_{\bar{F}} = \mathcal{A}([k]_N). \qquad (3.3)$$

If N is even and $k = N/2$, the $[k]_N = [k]_{2k}$ representation is self-conjugate, with zero anomaly, so Eq. (3.3) yields $n_{\bar{F}} = 0$ and a nonchiral theory. In order to get a chiral theory, with positive $n_{\bar{F}}$, it is necessary and sufficient that

$$N \geq N_{\min} = 2k + 1, \qquad (3.4)$$

so, for a given k, we will restrict N to this range. For N in this range, the anomaly $\mathcal{A}([k]_N)$ is an integer greater than unity. A member of this set of chiral gauge theories is thus determined by its values of k and N and has the form

$$G = SU(N), \qquad \text{fermions}: A_k + n_{\bar{F}}\bar{F}, \qquad (3.5)$$

i.e., $[k]_N + n_{\bar{F}}\overline{[1]}_N$, where N is bounded below by (3.4). The $A_k\bar{F}$ theories with $k = 3$ and $k = 4$ have respective upper bounds on N imposed by the requirement of asymptotic freedom.

To determine the upper bounds on N for these values of k, we calculate the one-loop coefficient in the beta function, b_1. To indicate explicitly the dependence of the b_ℓ coefficients with $\ell = 1, 2$ on k, we shall write them as $b_\ell^{(k)}$. For a general $SU(N)$ $A_k\bar{F}$ theory, we have

$$b_1^{(k)} = \frac{1}{3}[11N - 2\{T(A_k) + n_{\bar{F}}T(\bar{F})\}]$$

$$= \frac{1}{3}\left[11N - \frac{1}{(k-1)!}\left\{\left[\prod_{j=2}^{k}(N-j)\right]\right.\right.$$

$$\left.\left.+ \frac{(N-3)!(N-2k)}{(N-k-1)!}\right\}\right]. \qquad (3.6)$$

In Eq. (3.6), both $T(A_k)$ and $n_{\bar{F}} = \mathcal{A}(A_k)$ are polynomials of degree $\max(1, k-1)$ in N and hence, $b_1^{(k)}$ is a polynomial of degree $\max(1, k-1)$ in N. Specifically, we find

$$b_1^{(2)} = 3N + 2, \qquad (3.7)$$

$$b_1^{(3)} = \frac{1}{3}(-N^2 + 18N - 12), \qquad (3.8)$$

and

$$b_1^{(4)} = \frac{1}{9}(-N^3 + 12N^2 - 14N + 60). \qquad (3.9)$$

The $A_2\bar{F}$ theories are thus asymptotically free without any upper bound on N. For the $A_3\bar{F}$ theories, the asymptotic-freedom requirement that $b_1^{(3)}$ must be positive yields the upper bound $N \leq 17$. (If one were to generalize N from positive integer values to real values, $b_1^{(3)}$ is positive for N in the range $9 - \sqrt{69} < N < 9 + \sqrt{69}$, i.e., $0.6934 < N < 17.3066$ to the indicated floating-point accuracy.) In the

$A_4\bar{F}$ theories, $b_1^{(4)}$ is positive only for the integer values $N = 9$, 10, 11. (With N generalized to a positive real number, $b_1^{(4)} > 0$ for $N < 11.2291$.) Denoting $N_{\max}$ as the maximal value of N, for a given k, for which an $SU(N)$ $A_k\bar{F}$ theory is asymptotically free, we summarize these results as

$$N_{\max} = \begin{cases} \infty & \text{for } k = 2 \\ 17 & \text{for } k = 3 \\ 11 & \text{for } k = 4 \end{cases}. \qquad (3.10)$$

Combining these results, we explicitly exhibit the asymptotically free, anomaly-free chiral gauge theories of this type with $2 \leq k \leq 4$ together with the respective allowed ranges of N, $N_{\min} \leq N \leq N_{\max}$:

$$k = 2 \Longrightarrow N \geq 5, \qquad (3.11)$$

$$k = 3 \Longrightarrow 7 \leq N \leq 17, \qquad (3.12)$$

$$k = 4 \Longrightarrow 9 \leq N \leq 11. \qquad (3.13)$$

The $SU(N)$ $A_2\bar{F}$ theories have been studied in several works [2,4,8,10–12] and have fermion content given by

$$k = 2: \quad \text{fermions}: A_2 + (N-4)\bar{F}. \qquad (3.14)$$

The $SU(N)$ $A_k\bar{F}$ theories with $k = 3$ and $k = 4$ are, to our knowledge, new here. These have the fermion contents

$$k = 3: \quad \text{fermions}: A_3 + \frac{(N-3)(N-6)}{2}\bar{F} \qquad (3.15)$$

and

$$k = 4: \quad \text{fermions}: A_4 + \frac{(N-3)(N-4)(N-8)}{6}\bar{F}. \qquad (3.16)$$

For the $A_k\bar{F}$ theories with $k = 2$, 3, 4, we denote the fermion field in the $A_k = [k]_N$ representation as ψ_L^{ab}, ψ_L^{abd}, and ψ_L^{abde}, respectively, where a, b, d, e are $SU(N)$ gauge indices (the symbol c is reserved to mean charge conjugation) where N is in the respective intervals (3.14)–(3.16), and we denote the $\bar{F}$ fermions as $\chi_{a,i,L}$, where i is a copy (flavor) index taking values in the respective ranges $1 \leq i \leq n_{\bar{F}} = \mathcal{A}([k]_N)$.

We next show that there are no asymptotically free $A_k\bar{F}$ theories with $k \geq 5$. Consider first the $k = 5$ theory, for which

$$b_1^{(5)} = \frac{1}{36}(-N^4 + 18N^3 - 119N^2 + 474N - 360). \qquad (3.17)$$

TABLE I. Some properties of SU(N) $A_k\bar{F}$ chiral gauge theories. The quantities listed are k, N, $n_{\bar{F}}$, $\bar{b}_1$, $\bar{b}_2$, and, for negative $\bar{b}_2$, $\alpha_{\mathrm{IR},2\ell} = -\bar{b}_1/\bar{b}_2$, α_{cr} for the most attractive bilinear fermion condensation channel (2.3) in the SU(N) theory, and the ratio ρ_c. The dash notation − means that the two-loop beta function has no IR zero. The likely IR behavior is indicated in the last column, with the abbreviations SC, MC, WC for the type coupling in the IR (SC = strong, MC = moderate, WC = weak coupling). In the WC case, the UV to IR evolution is to a non-Abelian Coulomb phase (NACP). The various possibilities for the evolution involving strong and moderately strong coupling are discussed in the text. For $k = 2$, we include illustrative results covering the interval $5 \leq N \leq 10$; for $k = 3, 4$ we list results for all (asymptotically free) $A_k\bar{F}$ theories.

k	N	$n_{\bar{F}}$	$\bar{b}_1$	$\bar{b}_2$	$\alpha_{\mathrm{IR},2\ell}$	α_{cr}	ρ_c	IR coupling
2	5	1	1.3528	1.4996	−	0.44	−	SC
2	6	2	1.59155	2.0486	−	0.45	−	SC
2	7	3	1.8303	2.6796	−	0.37	−	SC
2	8	4	2.0690	3.3927	−	0.31	−	SC
2	9	5	2.3077	4.1879	−	0.27	−	SC
2	10	6	2.5465	5.0654	−	0.24	−	SC
3	7	2	1.7242	2.1525	−	0.20	−	SC
3	8	5	1.8038	1.9784	−	0.21	−	SC
3	9	9	1.8303	1.3805	−	0.21	−	SC
3	10	14	1.8038	0.2573	−	0.21	−	SC
3	11	20	1.7242	−1.4926	1.155	0.21	5.4	SC
3	12	27	1.59155	−3.9705	0.4008	0.21	1.9	MC
3	13	35	1.4059	−7.2779	0.1932	0.19	0.99	MC
3	14	44	1.1671	−11.5161	0.1013	0.18	0.57	MC
3	15	54	0.8753	−16.7864	0.05215	0.16	0.32	WC, NACP
3	16	65	0.5305	−23.1901	0.02288	0.15	0.15	WC, NACP
3	17	77	0.1326	−30.8287	0.00430	0.14	0.03	WC, NACP
4	9	5	1.5650	−0.5896	2.6542	0.12	22.5	SC
4	10	14	1.0610	−5.3310	0.1990	0.12	1.7	MC
4	11	28	0.2387	−13.410	0.0178	0.12	0.15	WC, NACP

With N generalized to a real variable, $b_1^{(5)}$ is positive only for N in the range $0.9585 < N < 10.7379$. But for an $A_k\bar{F}$ theory, N is bounded below by $2k + 1$, which has the value 11 here, so for this $k = 5$ theory there is no value of N that simultaneously satisfies both the lower bound (3.4) and the requirement of asymptotic freedom. We reach the same conclusion in the $k = 6$ case, for which

$$b_1^{(6)} = \frac{1}{180}(-N^5 + 25N^4 - 245N^3 + 1175N^2 - 2094N + 2520). \tag{3.18}$$

With N extended from physical values to real numbers, $b_1^{(6)}$ is positive if $N < 11.098$, but N is required to satisfy $N \geq 13$, which again means that for $k = 6$ there is no value of N that satisfies the lower bound (3.4) and the requirement of asymptotic freedom. Similarly, we find that for the $k = 7$ case, $b_1^{(7)}$ is only positive for the range $1.094 < N < 11.742$, while N must be in the range $N \geq 15$ by (3.4), and so forth for higher k. The underlying reason for the nonexistence of asymptotically free $A_k\bar{F}$ chiral gauge theories with these higher values of k is that, as noted above, both $T([k]_N)$ and $\mathcal{A}([k]_N)$ are polynomials of degree $\max(1, k - 1)$ in N, and they both contribute negatively to $b_1^{(k)}$ for the relevant range $N \geq 2k + 1$. Their negative contributions eventually outweigh the positive contribution of the $(11/3)N$ term from the gauge fields.

In passing, we remark that there are two possible ways that one could expand the fermion content of the $A_k\bar{F}$ models considered here for certain k and N values, as restricted by the constraint of asyamptotic freedom, namely (i) to have n_{cp} replications of the chiral fermion content and (ii) to add vectorlike subsectors. For example, in category (i), the following $k = 3$ theories are asymptotically free: $n_{cp} = 2$ and $7 \leq N \leq 11$; $n_{cp} = 3$ and $7 \leq N \leq 9$; $n_{cp} = 4$ and $N = 7$, 8; and $n_{cp} = 5$ and $N = 7$. We have studied different chiral gauge theories with this sort of n_{cp} replication of a minimal irreducible chiral fermion content in [22]. We shall not pursue these expansions here but instead focus on studying the minimal $A_k\bar{F}$ theories.

IV. BETA FUNCTION ANALYSIS OF $A_k\bar{F}$ THEORIES

In this section we give a general analysis of the beta function applicable to all of the (anomaly-free)

asymptotically free $A_k\bar{F}$ theories, with N in the respective ranges $N \geq 5$ for $k = 2$ and the finite intervals $7 \leq N \leq 17$ for $k = 3$ and $9 \leq N \leq 11$ for $k = 4$ as given in (3.14)–(3.16). In Sec. III we gave the one-loop coefficient for the $A_k\bar{F}$ theories, which we used to determine the upper bound on N for a given k. Here we proceed to give the two-loop coefficient, $b_2^{(k)}$, and use it to analyze the UV to IR evolution. We have (again with $A_k \equiv [k]_N$, and $F \equiv [1]_N$)

$$b_2^{(k)} = \frac{1}{3}[34N^2 - 2\{(5C_2(G) + 3C_2(A_k))T(A_k)$$
$$+ n_{\bar{F}}(5C_2(G) + 3C_2(\bar{F}))T(\bar{F})\}], \qquad (4.1)$$

where the various group invariants are listed in Appendix B. For the three relevant cases, $k = 2, 3, 4$, the explicit expressions are

$$b_2^{(2)} = \frac{13N^3 + 30N^2 + N - 12}{2N}, \qquad (4.2)$$

$$b_2^{(3)} = \frac{-16N^4 + 183N^3 - 204N^2 - 27N + 108}{6N}, \qquad (4.3)$$

and

$$b_2^{(4)} = (36N)^{-1}[-35N^5 + 429N^4 - 1321N^3 + 2235N^2$$
$$+ 588N - 1440]. \qquad (4.4)$$

In Table I we list values of the reduced coefficients $\bar{b}_1$ and $\bar{b}_2$ for an illustrative set of the $A_2\bar{F}$ theories and for all of the (asymptotically free) $A_3\bar{F}$ and $A_4\bar{F}$ theories. In the cases where $\bar{b}_2 < 0$ so that the two-loop beta function has a physical IR zero, we have also listed the value of $\alpha_{IR,2\ell}$. The value of the resultant ratio ρ_c for condensation in the most attractive channel for bilinear fermion condensation (discussed further below) gives an estimate of whether the theories are weakly or strongly coupled in the infrared. This is indicated by the abbreviations WC, MC, and SC (weak coupling, moderate coupling, and strong coupling) in Table I.

V. GLOBAL SYMMETRY OF $A_k\bar{F}$ THEORIES

Because the $A_k\bar{F}$ theories are irreducibly chiral, so that the chiral gauge symmetry requires the fermions to be massless, each such theory has a classical global flavor symmetry

$$G_{fl,cl}^{(k)} = U(n_{\bar{F}})_{\bar{F}} \otimes U(1)_{A_k}, \qquad (5.1)$$

where $n_{\bar{F}} = \mathcal{A}([k]_N)$ as given in Eq. (3.3). Equivalently,

$$G_{fl,cl}^{(k)} = \begin{cases} U(1)_{\bar{F}} \otimes U(1)_{A_k} & \text{if } n_{\bar{F}} = 1 \\ SU(n_{\bar{F}})_{\bar{F}} \otimes U(1)_{\bar{F}} \otimes U(1)_{A_k} & \text{if } n_{\bar{F}} \geq 2 \end{cases}. \qquad (5.2)$$

For $n_{\bar{F}} \geq 2$, the multiplet $(\chi_{a,1,L}, \ldots, \chi_{a,n_{\bar{F}},L})$ may be taken to transform as the conjugate fundamental, $\bar{\square}$, representation of the global flavor group, $SU(n_{\bar{F}})$. The $U(1)_{\bar{F}}$ and $U(1)_{A_k}$ symmetries in (5.2) are both broken by $SU(N)$ instantons [40]. As in [12], we define a vector whose components are comprised of the instanton-generated contributions to the breaking of these symmetries. In the basis $(A_k, \bar{F})$, this vector is

$$\vec{v}^{(k)} = (T([k]_N), n_{\bar{F}}T(\bar{F}))$$
$$= \lambda_{N,k}(N - 2, N - 2k), \qquad (5.3)$$

where

$$\lambda_{N,k} = \frac{(N - 3)!}{2(k - 1)!(N - k - 1)!}. \qquad (5.4)$$

We can construct one linear combination of the two original currents that is conserved in the presence of $SU(N)$ instantons. We denote the corresponding global $U(1)$ flavor symmetry as $U(1)'$ and the fermion charges under this $U(1)'$ as

$$\vec{Q}^{(k)'} = (Q'_{A_k}, Q'_{\bar{F}}). \qquad (5.5)$$

The $U(1)'$ current is conserved if and only if

$$\sum_f n_f T(R_f) Q_f^{(k)'} = \vec{v} \cdot \vec{Q}^{(k)'} = 0. \qquad (5.6)$$

This condition only determines the vector $\vec{Q}^{(k)'}$ up to an overall multiplicative constant. A solution is

$$\vec{Q}^{(k)'} = (N - 2k, -(N - 2)). \qquad (5.7)$$

The actual global chiral flavor symmetry group (preserved in the presence of instantons) is then

$$G_{fl}^{(k)} = \begin{cases} U(1)' & \text{if } n_{\bar{F}} = 1 \\ SU(n_{\bar{F}}) \otimes U(1)' & \text{if } n_{\bar{F}} \geq 2 \end{cases}. \qquad (5.8)$$

For the three k values relevant here, this is

$$G_{fl}^{(2)} = \begin{cases} U(1)' & \text{if } N = 5 \\ SU(N - 4)_{\bar{F}} \otimes U(1)' & \text{if } N \geq 6 \end{cases} \qquad (5.9)$$

with $U(1)'$ charges

$$\vec{Q}^{(2)\prime} = (N-4, -(N-2)), \qquad (5.10)$$

$$G_{fl}^{(3)} = \text{SU}\left(\frac{(N-3)(N-6)}{2}\right)_{\bar{F}} \otimes \text{U}(1)' \qquad (5.11)$$

with U(1)′ charges

$$\vec{Q}^{(3)\prime} = (N-6, -(N-2)), \qquad (5.12)$$

and

$$G_{fl}^{(4)} = \text{SU}\left(\frac{(N-3)(N-4)(N-8)}{6}\right)_{\bar{F}} \otimes \text{U}(1)' \qquad (5.13)$$

with U(1)′ charges

$$\vec{Q}^{(4)\prime} = (N-8, -(N-2)). \qquad (5.14)$$

VI. MOST ATTRACTIVE CHANNEL FOR BILINEAR FERMION CONDENSATION IN $A_k\bar{F}$ THEORIES

A. General analysis

The ultraviolet to infrared evolution of a particular SU(N) $A_k\bar{F}$ theory is determined by the values of N and k. In the cases where it can lead to the formation of a bilinear fermion condensate, one should then determine the most attractive channel in which this condensate can form. We present this analysis here. Since the $A_k\bar{F}$ theories that we consider here are irreducibly chiral, a bilinear condensate breaks the gauge symmetry. In Sec. X below, we will discuss the possible formation of multifermion condensates involving more than just two fermions, which can preserve the chiral gauge symmetry.

For the theories that we are discussing here, there are two relevant bilinear fermion condensation channels. First, there is a channel with a condensate that involves the contraction of $2k$ gauge indices of the antisymmetric tensor density $\epsilon_{a_1,\ldots,a_N}$ with the bilinear fermion product $A_k \times A_k$, which transforms like $\bar{A}_{N-2k}$. This channel can thus be written as

$$A_k \times A_k \to \bar{A}_{N-2k}. \qquad (6.1)$$

This channel has attractiveness measure

$$\Delta C_2 = \frac{k^2(N+1)}{N} \quad \text{for } A_k \times A_k \to \bar{A}_{N-2k}. \qquad (6.2)$$

For a given k, this ΔC_2 is a monotonically decreasing function of N, decreasing gradually from its value at $N = 2k+1$,

$$\Delta C_2 = \frac{2k^2(k+1)}{2k+1} = \frac{(N-1)^2(N+1)}{4N} \quad \text{at } N = 2k+1$$
$$\text{for } A_k \times A_k \to \bar{A}_{N-2k} = \bar{A}_1 = \bar{F} \qquad (6.3)$$

and approaching the limit k^2 for $N \gg k$.

Second, there is the channel

$$A_k \times \bar{F} \to A_{k-1}, \qquad (6.4)$$

with

$$\Delta C_2 = \frac{(N+1)(N-k)}{N} \quad \text{for } A_k \times \bar{F} \to A_{k-1}. \qquad (6.5)$$

For a given k, this ΔC_2 is a monotonically increasing function of N, increasing from the value

$$\Delta C_2 = \frac{2(k+1)^2}{2k+1} = \frac{(N+1)^2}{2N} \quad \text{at } N = 2k+1$$
$$\text{for } A_k \times \bar{F} \to A_{k-1}, \qquad (6.6)$$

and approaching a linear growth with N for $N \gg k$. In Table II we list the value of ΔC_2 in Eq. (6.2) for the $A_k \times A_k \to \bar{A}_{N-2k}$ channel and the value of ΔC_2 in Eq. (6.5) for the $A_k \times \bar{F} \to A_{k-1}$ channel for an illustrative set of $A_2\bar{F}$ theories and for the full set of (asymptotically free) $A_3\bar{F}$ and $A_4\bar{F}$ theories.

The most attractive channel for bilinear fermion condensation is the one among these two channels with the larger value of ΔC_2 (assuming that these two values are unequal; we discuss the cases where they are equal below). For a given value of k, we thus determine the MAC as a function of N in its allowed range $N_{\min} \le N \le N_{\max}$ by examining the difference,

$$\Delta C_2(A_k \times A_k \to \bar{A}_{N-2k}) - \Delta C_2(A_k \times \bar{F} \to A_{k-1})$$
$$= \left(\frac{N+1}{N}\right)[k(k+1) - N]. \qquad (6.7)$$

For $N = N_{\min} = 2k+1$, ΔC_2 is larger for the first channel, $A_k \times A_k \to \bar{A}_{N-2k} = \bar{A}_1 = \bar{F}$, than for the second channel, $A_k \times \bar{F} \to A_{k-1}$. This is evident analytically from the fact that with $N = 2k+1$, the difference (6.7) is

$$\frac{2(k+1)(k^2-k-1)}{2k+1} = \frac{(N+1)(N^2-4N-1)}{4N}, \qquad (6.8)$$

which is positive for the relevant range $k \ge 2$ considered here. Since ΔC_2 for the first channel decreases monotonically as a function of N, while the ΔC_2 for the second channel increases monotonically as a function of N, it follows that at some value of N, which we denote N_e (where e stands for "equal"), these values are equal, and for $N > N_e$, the ΔC_2 for the second channel is larger than that

TABLE II. ΔC_2 values for the SU(N) $A_k\bar{F}$ chiral gauge theories and most attractive channels for bilinear fermion condensation. The quantities listed are k, N, and the respective ΔC_2 values for the $A_k \times A_k \to \bar{A}_{N-2k}$ and $A_k \times \bar{F} \to A_{k-1}$ channels. In the last column, we list the most attractive channel for bilinear fermion condensation in the strongly coupled and moderately strongly coupled (SC,MC) cases. If the UV to IR evolution remains weakly coupled (WC), it flows to a non-Abelian Coulomb phase (NACP). For $k = 2$, we include illustrative results including the interval $5 \le N \le 10$; for $k = 3, 4$ we list results for all (asymptotically free) $A_k\bar{F}$ theories. See text for further discussion of the $k = 2$, $N = 6$ and $k = 3$, $N = 12$ cases where the ΔC_2 values are equal. The $A_2\bar{F}$ theories could confine, yielding massless composite fermions. Possible multifermion condensates are also discussed in the text.

k	N	$\Delta C_2(A_k \times A_k \to \bar{A}_{N-2k})$	$\Delta C_2(A_k \times \bar{F} \to A_{k-1})$	MAC for (S,M)C
2	5	4.800	3.600	$A_2 \times A_2 \to \bar{F}$
2	6	4.667	4.667	$A_2 \times \bar{F} \to F$
2	7	4.571	5.714	$A_2 \times \bar{F} \to F$
2	8	4.500	6.750	$A_2 \times \bar{F} \to F$
2	9	4.444	7.778	$A_2 \times \bar{F} \to F$
2	10	4.400	8.800	$A_2 \times \bar{F} \to F$
3	7	10.29	4.571	$A_3 \times A_3 \to \bar{F}$
3	8	10.125	5.625	$A_3 \times A_3 \to \bar{A}_2$
3	9	10.000	6.667	$A_3 \times A_3 \to \bar{A}_3$
3	10	9.900	7.700	$A_3 \times A_3 \to \bar{A}_4$
3	11	9.818	8.727	$A_3 \times A_3 \to \bar{A}_5$
3	12	9.750	9.750	$A_3 \times \bar{F} \to A_2$ or NACP
3	13	9.692	10.769	$A_3 \times \bar{F} \to A_2$ or NACP
3	14	9.643	11.786	$A_3 \times \bar{F} \to A_2$ or NACP
3	15	9.600	12.800	NACP
3	16	9.5625	13.8125	NACP
3	17	9.529	14.824	NACP
4	9	17.78	5.556	$A_4 \times A_4 \to \bar{F}$
4	10	17.60	6.600	$A_4 \times A_4 \to \bar{A}_2$ or NACP
4	11	17.45	7.636	NACP

for the first channel. Setting the two ΔC_2 values equal and solving for $N = N_e$, we find

$$N_e = k(k + 1). \tag{6.9}$$

Evaluating Eq. (6.9) for the three relevant values of k, we have

$$N_e = \begin{cases} 6 & \text{for } k = 2 \\ 12 & \text{for } k = 3. \\ 20 & \text{for } k = 4 \end{cases} \tag{6.10}$$

The first two of these values are within the respective allowed ranges for N, while the value for $k = 4$ is larger than the upper bound $N_{\max} = 11$ for $k = 4$.

Consequently, with $N_{\min} = 2k + 1$ and $N_{\max}$ as given in Eqs. (3.4) and (3.10), we find that, for a given k,

If $2k + 1 \le N < k(k + 1)$ then

MAC $= A_k \times A_k \to \bar{A}_{N-2k}$

If $k(k + 1) < N \le N_{\max}$ then

MAC $= A_k \times \bar{F} \to A_{k-1}$, (6.11)

with the proviso that the second possibility only applies if $k(k + 1) < N_{\max}$, and hence only for $k = 2$ and $k = 3$. Thus, in particular, if $N = N_{\min} = 2k + 1$, then the MAC is the special case of (6.1):

If $N = 2k + 1$ then MAC $= A_k \times A_k \to \bar{F}$. (6.12)

In addition to breaking the original SU(N) gauge symmetry, these condensates also break both the non-Abelian factor group SU($n_{\bar{F}}$) (which is present if $n_{\bar{F}} \ge 2$) and the U(1)′ factor group in the global flavor symmetry (5.8). In particular, the breaking of the U(1)′ symmetry is evident from the fact that the respective condensates in these channels have the nonzero U(1)′ charges

$$Q'^{(k)} = 2Q'_{A_k} = 2(N - 2k) \quad \text{for } A_k \times A_k \to \bar{A}_{N-2k} \tag{6.13}$$

and

$$Q'^{(k)} = Q'_{A_k} + Q'_{\bar{F}} = 2(1 - k) \quad \text{for } A_k \times \bar{F} \to A_{k-1}. \tag{6.14}$$

The marginal case $N = N_e = k(k+1)$ requires further analysis, since the ΔC_2 values for the $A_k \times A_k \to \bar{A}_{N-2k}$ and $A_k \times \bar{F} \to A_{k-1}$ channels are equal, so the procedure of picking the channel with the largest ΔC_2 cannot determine which is more likely to occur. To deal with this marginal case, we use a vacuum alignment argument, which, as applied to possible bilinear fermion condensation channels, favors the one whose condensate respects the larger residual gauge symmetry. To apply the vacuum alignment argument, we must thus determine the residual gauge symmetry group respected by the condensates that occur in these two channels. The resultant bilinear fermion condensate transforms like an n-fold antisymmetric tensor representation of SU(N), where $n = N - 2k$ for the $A_k \times A_k \to \bar{A}_{N-2k}$ channel and $n = k - 1$ for the $A_k \times \bar{F} \to A_{k-1}$ channel. [The fact that in the first case the condensate transforms like $\bar{A}_{N-2k}$ rather than A_{N-2k} does not affect how this breaks SU(N).] From the point of view of the group theory, the problem of determining the residual gauge symmetry is effectively the same as the problem of determining the residual gauge symmetry that results when one has a Higgs field transforming according to the antisymmetric rank-n representation of SU(N). An analysis of this, within the context of Higgs-induced symmetry breaking, was given in [41], and the results depend, in that context, on the parameters in the Higgs potential, which one has the freedom to choose, subject to the overall constraint that the energy must be bounded below. As emphasized in Ref. [20], the situation is different in dynamical gauge symmetry breaking; in principle, given an initial gauge group and set of fermions, there is a unique answer for how the symmetry breaks; this breaking does not depend on any parameters in a Higgs potential. Despite this basic difference between dynamical and Higgs-induced gauge symmetry breaking, we can make use of the general group-theoretic analysis performed for the Higgs case. The result is that there are, *a priori*, three possibilities for the gauge symmetries respected by a condensate or Higgs vacuum expectation value transforming as the rank-n antisymmetric tensor representation of SU(N), $[n]_N$. Denoting the integral part of a real number r as $[r]$ and setting

$$\kappa \equiv [N/n], \qquad (6.15)$$

these are [41]

$$\mathrm{SU}(N-n) \otimes \mathrm{SU}(n) \quad \text{with} \quad 2 \leq n < [N/2], \quad (6.16)$$

$$[\mathrm{SU}(n)]^\kappa \quad \text{with} \quad 3 \leq n \leq [N/2] \quad \text{if} \quad N - [N/n]n = 0 \text{ or } 1, \qquad (6.17)$$

and the symplectic group

$$\mathrm{Sp}(2\kappa) \quad \text{if } n = 2. \qquad (6.18)$$

We analyze the respective cases $k = 2, 3, 4$ next.

B. Case $k = 2$

From the special case for $k = 2$ of our general result (6.11) above, we infer that the $A_2 \times A_2 \to \bar{A}_1 = \bar{F}$ channel is the most attractive channel for bilinear fermion condensation in the $A_2\bar{F}$ theories for the lowest value of N, namely $N = 5$, while the $A_2 \times \bar{F} \to F$ channel is the MAC for the infinite interval $N \geq 7$. For the marginal case $k = 2$, $N = 6$, the $A_2 \times A_2 \to \bar{A}_{N-2k} = \bar{A}_2$ and $A_2 \times \bar{F} \to F$ channels have the same value of ΔC_2, namely $\Delta C_2 = 14/3 = 4.667$ (see Table II), so the ΔC_2 attractiveness criterion cannot be used to decide which is more likely to occur. Now the condensate in the $A_2 \times \bar{F} \to F$ channel leaves invariant an SU(5) subgroup of SU(6), with order 24. To analyze the possible invariance groups of a condensate in the $A_2 \times A_2 \to \bar{A}_2$ channel, we apply our discussion above with $N = 6$, $n = 2$, and hence $\kappa = [6/2] = 3$, so the *a priori* possible invariance groups of the condensate are SU(4) $\otimes$ SU(2) with order 18 and Sp(6) with order 21. Neither of these groups has an order as large as that of SU(5), so the vacuum alignment argument predicts that, if a bilinear fermion condensate forms, then this condensate will form in the $A_2 \times \bar{F} \to F$ channel. Summarizing our results for $k = 2$ and all N, we thus find that if bilinear fermion condensation occurs, then

$$k = 2 \Rightarrow \mathrm{MAC} = \begin{cases} A_2 \times A_2 \to \bar{A}_1 & \text{for } N = 5 \\ A_2 \times \bar{F} \to F & \text{for } N \geq 6 \end{cases}. \qquad (6.19)$$

As noted above, since this class of (asymptotically free) $A_2\bar{F}$ chiral gauge theories satisfies the 't Hooft global anomaly matching conditions, there is also the possibility of confinement, yielding massless composite fermions. There is also the possibility of multifermion condensate formation, which we will discuss below. Since the early works such as [1,2,4], for a class of asymptotically free chiral gauge theories such as the $A_2\bar{F}$ class discussed here, for which the UV to IR evolution leads to strong coupling and hence could lead to confinement with massless composite fermions or to fermion condensation, there has not, to our knowledge, been a rigorous argument presented that actually determines the type of UV to IR evolution in an asymptotically free chiral gauge theory.

C. Case $k = 3$

From the special case for $k = 3$ of our general result (6.11) above, we infer that the $A_3 \times A_3 \to \bar{A}_{N-2k} = \bar{A}_{N-6}$ channel is the most attractive channel for bilinear fermion condensation not only for the minimal value of N, namely $N = 7$, but also for the interval of N values up to $N = 11$. We discuss the marginal case of $N = 12$ last. Again

substituting $k = 3$ into (6.11), it follows formally that the MAC for $12 \leq N \leq 17$ is the $A_3 \times \bar{F} \to A_2$ channel. However, for $N = 13, 14$, the respective values of the IR zero in the beta function are sufficiently close to the rough estimate of the minimal critical value of α for condensate formation in the $A_3 \times \bar{F} \to A_2$ channel (see Tables I and II) that it is possible that the system could evolve from the UV to a deconfined, non-Abelian Coulomb phase in the IR with no fermion condensate formation or associated spontaneous chiral symmetry breaking.

The $k = 3$, $N = 12$ case is again marginal; the $A_3 \times A_3 \to \bar{A}_6$ and $A_3 \times \bar{F} \to A_2$ channels have the same value of ΔC_2, namely $\Delta C_2 = 39/4 = 9.750$. Hence, we use a vacuum alignment argument to decide on which of these channels is more likely to occur. For the $A_3 \times A_3 \to \bar{A}_{N-6} = \bar{A}_6$ channel, we apply our discussion above with $N = 12$, $n = 6$, and hence $\kappa = [12/6] = 2$, so the invariance group of the $\bar{A}_2$ condensate is $[\mathrm{SU}(6)]^2$, with order 70. For the $A_3 \times \bar{F} \to A_2$ channel, we have $N = 12$, $n = 2$ and hence $\kappa = [12/2] = 6$, so the *a priori* possible invariance groups of the A_2 condensate are $\mathrm{SU}(10) \otimes \mathrm{SU}(2)$ with order 102 and $\mathrm{Sp}(12)$ with order 78. The vacuum alignment argument thus favors condensation in the $A_3 \times \bar{F} \to \bar{A}_2$ channel for this $N = 12$ case. Summarizing these results, we have

$$k = 3 \Rightarrow \mathrm{MAC} = \begin{cases} A_3 \times A_3 \to \bar{A}_{N-6} & \text{for } 7 \leq N \leq 11 \\ A_3 \times \bar{F} \to A_2 & \text{for } 12 \leq N \leq 17 \end{cases}.$$

(6.20)

However, as mentioned above, for $N = 13, 14$ (and also for $N = 12$), the respective values of ρ_c are sufficiently close to unity that, in view of the intrinsic theoretical uncertainties in the analysis of the strong-coupling physics, it is possible that the UV to IR evolution could lead either to the formation of a fermion condensate or to a non-Abelian Coulomb phase without spontaneous chiral symmetry breaking.

If N is in the higher interval $15 \leq N \leq 17$, then ρ_c is sufficiently small that we definitely expect the evolution to lead to a chirally symmetric non-Abelian Coulomb phase in the IR. Hence, in these cases, the MAC is not directly relevant to the dynamics of the theory.

D. Case $k = 4$

Finally, we discuss the theories with $k = 4$, for which the interval of values of N is $9 \leq N \leq 11$. Since the value of N_e, namely $N_e = 20$, is larger than $N_{\max}$, the most attractive channel for bilinear fermion condensation in all of these theories is $A_4 \times A_4 \to \bar{A}_{N-8}$, i.e., $A_4 \times A_4 \to \bar{F}$ for $N = 9$, $A_4 \times A_4 \to \bar{A}_2$ for $N = 10$, and $A_4 \times A_4 \to \bar{A}_3$ for $N = 11$. In the SU(9) $A_4\bar{F}$ theory, the IR zero in the two-loop beta function is much larger than α_{cr} for this channel, so it is likely that the SU(9) gauge interaction

would produce a condensate in this channel, thereby breaking SU(9) to SU(8). For $N = 10$, $\alpha_{\mathrm{IR},2\ell}/\alpha_{cr} = 1.7$, which is sufficiently close to unity that, taking account of the uncertainties in the strong-coupling estimates, the UV to IR evolution might produce a condensate in the respective most attractive bilinear fermion channel or might lead to a non-Abelian Coulomb phase. For $N = 11$, the IR zero in the two-loop beta function is small compared with the estimated α_{cr} for the $A_4 \times A_4 \to \bar{A}_3$ condensation channel, so we definitely expect the system to evolve from the UV to a non-Abelian Coulomb phase in the IR.

VII. $A_2\bar{F}$ THEORIES

A. General

In this section we analyze the UV to IR evolution of some $A_2\bar{F}$ theories in detail. Recall that the explicit fermion fields are A_2: ψ_L^{ab} and $\bar{F}$: $\chi_{a,i,L}$, where a, b are the SU(N) gauge indices and $i = 1, .., N - 4$ is a copy (flavor) index. The one-loop and two-loop coefficients were given in Eqs. (3.7) and (4.2). We find that for all $N \geq N_{\min} = 5$, the coefficient $b_2^{(2)}$ is positive, so the two-loop beta function of the $A_2\bar{F}$ theory has no IR zero. Hence, as the Euclidean reference scale μ decreases from the UV to the IR, the gauge coupling increases until it eventually exceeds the region where it is perturbatively calculable. This IR behavior is thus marked as SC, for strong coupling, in Table I.

The global flavor symmetry group for this theory is given in Eq. (5.9) with the U(1)$'$ charge assignments in (5.10). This theory satisfies the 't Hooft global anomaly matching conditions [1,10], so, as it becomes strongly coupled in the infrared, it could confine and produce massless gauge-singlet composite spin-1/2 fermions as well as massive gauge-singlet mesons and also primarily gluonic states. If this happens, then it is a complete description of the UV to IR evolution. The three-fermion operator for the composite gauge-singlet fermion can be written as

$$f_{ij} \propto [\chi_{a,i,L}^T C \psi_L^{ab}] \chi_{b,j,L} + (i \leftrightarrow j).$$

(7.1)

From Eq. (5.10), the U(1)$'$ charge of this composite fermion is

$$Q_{f_{ij}} = Q_{\bar{A}} + 2Q_{\bar{F}} = -N.$$

(7.2)

If $N \geq 6$, f_{ij} transforms as the conjugate symmetric rank-2 tensor representation, $\overline{\square\square}$, of the SU($N - 4$)$_{\bar{F}}$ factor group in the global flavor symmetry group $G_{fl}^{(2)} = \mathrm{SU}(N - 4)_{\bar{F}} \otimes \mathrm{U}(1)'$ of the theory.

Another possibility is that the SU(N) gauge interaction could produce bilinear fermion condensates, thereby breaking both gauge and global symmetries. The most attractive channel for this fermion condensation was determined, as a

function of N, in Eq. (6.19). It can also be possible to form multifermion condensates involving more than two fermion fields, which preserve the chiral gauge symmetry. We will discuss this latter possibility in Sec. X. Here we proceed to analyze bilinear fermion condensate formation for various specific theories.

B. SU(5) $A_2 \bar{F}$ theory

The simplest chiral gauge theory in the $A_2 \bar{F}$ family of theories has the gauge group SU(5), with fermion content given by the $N = 5$ special case of Eq. (3.14), namely $A_2 + \bar{F} = [2]_5 + \overline{[1]}_5$. Like the other $A_2 \bar{F}$ theories considered here that become strongly coupled in the infrared, this one could confine and produce a massless composite fermion. Alternatively, it could produce fermion condensates. The most attractive channel for bilinear fermion condensation in this theory is $A_2 \times A_2 \to \bar{A}_1$. If the dynamics is such that this condensate does, indeed, form, then we denote the mass scale at which it is produced as Λ_5. This condensate breaks the SU(5) gauge symmetry to SU(4). Without loss of generality, we take the gauge index corresponding to the breaking direction to be $a = 5$. The condensate then has the form

$$\langle \epsilon_{abde5} \psi_L^{abT} C \psi_L^{de} \rangle \propto [\langle \psi_L^{12T} C \psi_L^{34} \rangle - \langle \psi_L^{13T} C \psi_L^{24} \rangle + \langle \psi_L^{14T} C \psi_L^{23} \rangle]. \quad (7.3)$$

The fermions involved in this condensate gain dynamical masses of order Λ_5, as do the nine gauge bosons in the coset SU(5)/SU(4). In addition to breaking the SU(5) gauge symmetry, the condensate has the nonzero value of the U(1)′ charge $Q'^{(2)} = -2$ given by the $k = 2$ special case of Eq. (6.14) and hence breaks the global U(1)′ symmetry. Since this symmetry is not gauged, this breaking yields one Nambu-Goldstone boson (NGB).

To construct the low-energy effective field theory with SU(4) chiral gauge invariance that describes the physics as the scale μ decreases below Λ_5, we decompose the fermion representations of SU(5) with respect to the unbroken SU(4) subgroup. It will be useful to give this decomposition more generally for SU(N) relative to an SU($N-1$) subgroup in our usual notation and also in terms of the corresponding Young tableaux:

$$[2]_N = \{[2]_{N-1} + [1]_{N-1}\} \, , \ i.e.,$$
$$\square\!\square\, _{\text{SU}(N)} = \left[\square\!\square + \square \right]_{\text{SU}(N-1)} . \quad (7.4)$$

The $[2]_4$ field is comprised of ψ_L^{ab} fermions with $1 \le a$, $b \le 4$ that gained dynamical masses of order Λ_5 and were integrated out of the low-energy theory. The other massless SU(4)-nonsinglet fermions are the $[1]_4 = F$ fermion ψ_L^{5b} with $1 \le b \le 4$ and the $\overline{[1]}_4 = \bar{F}$ fermion $\chi_{a,1,L}$ with $1 \le a \le 4$. Hence, the massless SU(4)-nonsinglet fermion

content of this theory consists of $F + \bar{F}$, so this theory is vectorial. This SU(4) theory also contains the SU(4)-singlet fermion $\chi_{5,1,L}$. The one-loop and two-loop coefficients of the SU(4) beta function have the same sign, so again, this function has no IR zero, and therefore the SU(4) gauge coupling inherited from the SU(5) UV theory continues to increase as the reference scale μ decreases. Rewriting the left-handed $\bar{F}$ as a right-handed F, one sees that this is a vectorial SU(4) gauge theory with a massless $N_f = 1$ Dirac fermion in the fundamental representation. It therefore has a classical global chiral flavor symmetry group $U(1)_F \otimes U(1)_{\bar{F}}$, or equivalently, $U(1)_V \otimes U(1)_A$ in standard notation. The $U(1)_A$ is broken by SU(4) instantons, so the nonanomalous global flavor symmetry is $U(1)_V$. At a scale $\Lambda_4 \lesssim \Lambda_5$, one expects that the SU(4) gauge interaction produces a bilinear fermion condensate in the most attractive channel, which is $F \times \bar{F} \to 1$, thus preserving the SU(4) gauge symmetry. The condensate is

$$\left\langle \sum_{b=1}^{4} \psi_L^{5bT} C \chi_{b,1,L} \right\rangle. \quad (7.5)$$

This condensate respects the $U(1)_V$ global symmetry, and hence does not produce any Nambu-Goldstone bosons. Thus, this SU(4) theory confines and produces gauge-singlet hadrons (with the baryons being bosonic). In the infrared limit, the only remaining massless particles are the SU(4)-singlet fermion $\chi_{a,1,L}$ and the one Nambu-Goldstone boson resulting from the breaking of the U(1)′ global flavor symmetry by the condensate (7.3).

C. SU(6) $A_2 \bar{F}$ theory

We next consider an SU(6) $A_2 \bar{F}$ theory. The fermion content of this theory is the $N = 6$ special case of (3.14), namely $A_2 + 2\bar{F} = [2]_6 + 2\overline{[1]}_6$. The A_2 fermion is denoted $\psi_L^{ab} = -\psi_L^{ba}$, and the two copies of the $\bar{F}$ fermion are denoted $\chi_{a,i,L}$, where $1 \le a$, $b \le 6$ are gauge indices and $i = 1$, 2 is the copy index. We consider possible bilinear fermion condensates for this theory. As discussed above, although the bilinear fermion condensation channels $A_2 \times A_2 \to \bar{A}_2$ and $A_2 \times \bar{F} \to F$ have the same ΔC_2, a vacuum alignment argument favors the $A_2 \times \bar{F} \to F$ channel because it leaves a larger residual gauge symmetry, namely SU(5). Assuming that a condensate in this channel does form, we denote the scale at which it is produced as Λ_6. Again, by convention we take the breaking direction as $a = 6$ and the copy index as $i = 2$ on the $\bar{F}$ fermion in the condensate, which can thus be written as

$$\left\langle \sum_{b=1}^{5} \psi_L^{6bT} C \chi_{b,2,L} \right\rangle. \quad (7.6)$$

This condensate also breaks the $SU(2)_{\bar{F}} \otimes U(1)'$ global flavor symmetry. The ψ_L^{6b} and $\chi_{b,2,L}$ fermions with

$1 \leq b \leq 5$ involved in the condensate (7.6) get dynamical masses of order Λ_6, as do the 11 gauge bosons in the coset SU(6)/SU(5). These are integrated out of the low-energy effective SU(5)-invariant theory that describes the physics as the scale μ decreases below Λ_6.

From the $N = 6$ special case of the general decomposition (7.4) in conjunction with the form of the condensate (7.6), it follows that the massless SU(5)-nonsinglet fermion content of the descendant SU(5) theory is $A_2 + \bar{F}$, together with the (massless) SU(5)-singlet fermions $\chi_{6,1,L}$ and $\chi_{6,2,L}$. Thus, the SU(5)-nonsinglet fermion content of this theory is the same as that of the SU(5) theory discussed above, and our analysis there applies here. Since this SU(5) theory satisfies the 't Hooft global anomaly matching conditions, when it becomes strongly coupled, it could confine and produce massless SU(5)-singlet composite fermions, as well as massive mesons and primarily gluonic states, or it could self-break via fermion condensate formation. We also discuss below a possible SU(5)-preserving four-fermion condensate that might form.

D. SU(N) $A_2\bar{F}$ theories with $N \geq 7$

For $N \geq 7$, the most attractive channel for bilinear fermion condensation is $A_2 \times \bar{F} \to F$, with ΔC_2 given by the $k = 2$ special case of (6.5),

$$\Delta C_2 = C_2([2]_N)$$
$$= \frac{(N - 2)(N + 1)}{N} \quad \text{for } A_2 \times \bar{F} \to F. \quad (7.7)$$

The UV to IR evolution of these theories is similar to that of the SU(6) theory. At each stage, owing to the fact that the SU(N) theory and the various descendant theories satisfy 't Hooft global anomaly matching conditions, as the coupling gets strong in the IR, the gauge interaction may confine and produce massless composite fermions or may produce various fermion condensates. The most attractive channel for bilinear fermion condensation at a given stage is $A_2 \times \bar{F} \to F$, breaking the theory down to the next descendant low-energy theory. If the theory follows the first type of UV to IR flow, namely confinement with massless composite fermions, this extends all the way to the IR limit, while if the theory follows the second type of flow with condensate formation, then there is, in general, a resultant sequence of low-energy effective theories that describe the physics of the massless dynamical degrees of freedom at lower scales. If all of the stages involve gauge (and global) symmetry breaking by fermion condensates, then the gauge symmetry breaking is of the form

$$\text{SU}(N) \to \text{SU}(N - 1) \to \ldots \to \text{SU}(4). \quad (7.8)$$

Here, the last theory, namely the SU(4) theory, is vectorial, while all of the higher-lying theories are chiral gauge theories.

VIII. $A_3\bar{F}$ THEORIES

The fermion content of the $A_3\bar{F}$ theories was displayed in Eq. (3.15). The one-loop and two-loop coefficients in the beta function were given in Eqs. (3.8) and (4.3), with numerical results for $\bar{b}_1$ and $\bar{b}_2$ displayed in Table I. As is evident in Table I, for $7 \leq N \leq 10$, the coefficient $\bar{b}_2$ is positive, so the two-loop beta function has no IR zero, and hence, as the reference scale μ decreases from large values in the UV toward the IR, the gauge coupling increases until it exceeds the region where it is perturbatively calculable. These theories are thus strongly coupled in the infrared (marked as SC in Table I).

The next step in the analysis of the UV to IR flow in these theories is to determine if one or more of them might satisfy the 't Hooft global anomaly matching conditions. If this were to be the case, then, as in the $A_2\bar{F}$ theories, one would have a two-fold possibility for the strongly coupled IR physics, namely confinement with gauge-singlet composite fermions but no spontaneous chiral symmetry breaking or formation of bilinear fermion condensates with associated breaking of gauge and global symmetries. For this purpose, we have examined possible SU(N) gauge-singlet fermionic operator products to determine if any of them could satisfy these global anomaly matching conditions. The global flavor symmetry group was given in Eq. (5.11) with (5.12). We have not found any such fermionic operator products. As an illustration of our analysis, let us consider the case $N = 7$, which contains a $[3]_7$ fermion ψ_L^{abd} and two fermions, $\chi_{a,i,L}$ with $i = 1, 2$ comprising two copies of the $\overline{[1]}_7$ representation. For this case,

$$G_{fl,N=7}^{(3)} = \text{SU}(2)_{\bar{F}} \otimes \text{U}(1)', \quad (8.1)$$

with $\vec{Q}' = (1, -5)$. A fermionic operator product that is an SU(7) singlet is of the form

$$f_{i,R} = \epsilon_{abdefgh}[\psi_L^{abdT} C \psi_L^{efg}](\chi^c)_{i,R}^h, \quad (8.2)$$

where the c superscript denotes the charge conjugate fermion field. However, this vanishes identically. This can be seen as follows: an interchange (transposition) of ψ_L^{abd} and ψ_L^{efg} entails a minus sign from the switching of an odd number of indices in the antisymmetric SU(7) tensor density, a second minus sign from Fermi statistics, and a third minus sign from the fact that $C^T = -C$ for the Dirac charge conjugation matrix, so the operator is equal to minus itself and hence is zero.

Therefore, when theory becomes strongly coupled in the infrared, we will focus on the type of UV to IR evolution that leads to fermion condensates, and we consider bilinear fermion condensates here. The most attractive channel for these condensates, as a function of N, was given in Eq. (6.20).

As an explicit example of the $A_3\bar{F}$ class of chiral gauge theories, let us consider the SU(7) theory, which has chiral fermion content given by the $N = 7$ special case of Eq. (3.15), namely

$$A_3 + 2\bar{F} = [3]_7 + 2\overline{[1]}_7. \tag{8.3}$$

The most attractive channel for this theory is $A_3 \times A_3 \to \bar{F}$, which breaks the gauge symmetry SU(7) to SU(6) and also breaks the global flavor symmetry group $\text{SU}(2)_{\bar{F}} \otimes \text{U}(1)'$. We denote the scale at which this condensate forms as Λ_7. Without loss of generality, we label the gauge index for the broken direction to be $a = 7$. The condensate then has the form

$$\langle \epsilon_{abdefg7} \psi_L^{abdT} C \psi_L^{efg} \rangle. \tag{8.4}$$

Of the $\binom{7}{3} = 35$ components of the A_3 fermion, denoted generically as ψ_L^{abd}, the $\binom{7}{3} - \binom{6}{2} = 20$ components with $1 \le a, b, d \le 6$ that are involved in this condensate gain dynamical masses of order Λ_7, as do the 13 gauge bosons in the coset SU(7)/SU(6). These are integrated out of the low-energy effective theory SU(6) chiral gauge theory that describes the physics as the scale decreases below Λ_7.

The massless SU(6)-nonsinglet fermion content of this SU(6) theory thus consists of $A_2 + 2\bar{F} = [2]_6 + 2\overline{[1]}_6$, comprised by the $\binom{6}{2} = 15$ components ψ_L^{ab7} and the $\chi_{a,i,L}$ with $1 \le a, b \le 6$ and $i = 1, 2$. A theorem proved in [22] states that a low-energy effective theory that arises by dynamical symmetry breaking from an (asymptotically free) anomaly-free chiral gauge theory is also anomaly-free. One sees that the present example is in accord with this general theorem. Indeed, the nonsinglet fermions in this SU(6) descendant theory are precisely those of the SU(6) $A_2\bar{F}$ theory discussed above, and that analysis applies here for the further UV to IR evolution of the theory. In addition to the SU(6)-nonsinglet fermions, this descendant theory also contains the SU(6)-singlet fermions $\chi_{7,i,L}$ with $i = 1, 2$.

IX. $A_4\bar{F}$ THEORIES

The fermion content of the $A_4\bar{F}$ theories was given in Eq. (3.16). The reduced one-loop and two-loop coefficients in the beta function were listed in Eqs. (3.9) and (4.4), with numerical results displayed in Table I. We find that for each of the three relevant values of N, namely $N = 9, 10, 11$, the coefficient $\bar{b}_2$ is negative, so the two-loop beta function has an IR zero. As we noted above, for $N = 11$, this IR zero is at very weak coupling relative to the minimal critical value for bilinear fermion condensation, so we can reliably conclude that the theory evolves from the UV to a (deconfined) non-Abelian Coulomb phase in the infrared. In the $N = 9$ and $N = 10$ theories, the respective IR zeros

in the two-loop beta function occur at strong and moderate coupling, so a full analysis is necessary.

We have examined whether there are SU(N) gauge-singlet composite fermion operators that could satisfy the 't Hooft global anomaly matching conditions, but we have not found any. The global flavor symmetry group was given in Eq. (5.13) with (5.14). As an illustration of our analysis, let us consider the SU(9) $A_4\bar{F}$ theory, which contains a $[3]_9$ fermion ψ_L^{abd} and the fermions, $\chi_{a,i,L}$ with $1 \le i \le 5$ comprising five copies of the $\overline{[1]}_9$ representation of SU(9). The global flavor symmetry group is

$$G_{fl,N=9}^{(4)} = \text{SU}(5)_{\bar{F}} \otimes \text{U}(1)'. \tag{9.1}$$

The $\chi_{a,i,L}$ fermions transform as $\bar{\square}$ of the $\text{SU}(5)_{\bar{F}}$ flavor group, and the vector of U(1)$'$ charges is $\vec{Q}' = (Q'_{A_4}, Q'_{\bar{F}}) = (1, -7)$. A fermionic operator product that is an SU(9) gauge singlet is

$$f_R^i = \epsilon_{abdefghrs}[\psi_L^{abdeT} C \psi_L^{fghr}](\chi^c)_R^{s,i}. \tag{9.2}$$

This transforms as a $\square$ representation of the global $\text{SU}(5)_{\bar{F}}$ symmetry with U(1)$'$ charge $2Q'_{A_4} - Q'_{\bar{F}} = 9$. Since this is a right-handed composite fermion, we actually calculate with the charge conjugate $(f^c)_{i,L}$, which is a left-handed fermion that transforms as a $\bar{\square}$ representation of the global SU(5) with U(1)$'$ charge -9. We find that this composite fermion does not satisfy the global anomaly matching conditions. For example, consider the SU(5)^3 anomaly. The fundamental fields make the following contributions: the A_4 fermion yields zero, while the $\bar{F}$ fermions yield $N\mathcal{A}(\bar{\square}) = 9 \times (-1) = -9$. However, the f_L^c fermion yields $\mathcal{A}(\bar{\square}) = -1$, which does not match. Since we have not found composite fermion operators that satisfy the 't Hooft global anomaly matching conditions, we consider fermion condensation in the cases where the beta function has an IR zero at moderate (for $N = 10$) and strong (for $N = 9$) coupling in the infrared.

As an explicit example, we analyze the SU(9) $A_4\bar{F}$ theory. The fermion content of this theory is given by the $N = 9$ special case of Eq. (3.16), namely

$$A_4 + 5\bar{F} = [4]_9 + 5\overline{[1]}_9. \tag{9.3}$$

The most attractive channel for bilinear fermion condensation is the $N = 9$ special case of (6.12), namely $A_4 \times A_4 \to \bar{F}$. Assuming that this condensate forms, it breaks the gauge symmetry SU(9) to SU(8) and also breaks the global flavor symmetry group $\text{SU}(5)_{\bar{F}} \otimes \text{U}(1)'$. We denote the scale at which this condensate forms as Λ_9. Without loss of generality, we label the gauge index for the broken direction to be $a = 9$. The condensate then has the form

$$\langle \epsilon_{abdefghr9} \psi_L^{abdeT} C \psi_L^{fghr} \rangle. \tag{9.4}$$

Of the $\binom{9}{4} = 126$ components of ψ_L^{abde}, the $\binom{9}{4} - \binom{8}{3} = 70$ components with $1 \le a, b, d, e \le 8$ that are involved in this condensate gain dynamical masses of order Λ_9, as do the 17 gauge bosons in the coset SU(9)/SU(8). These are integrated out of the low-energy effective theory SU(8) chiral gauge theory that describes the physics as the scale decreases below Λ_9.

The massless SU(8)-nonsinglet fermion content of this SU(8) theory thus consists of $A_3 + 5\bar{F} = [3]_8 + 5\overline{[1]}_8$, comprised by the $\binom{8}{3} = 56$ components ψ_L^{abd9} and the $\chi_{a,i,L}$ with $1 \le a, b, d \le 8$ and $1 \le i \le 5$. Again, the theorem proved in [22] guarantees that this SU(8) descendant theory is anomaly-free. Indeed, the nonsinglet fermions in this SU(8) descendant theory are precisely those of the SU(8) $A_3\bar{F}$ theory discussed above, and that analysis applies here for the further UV to IR evolution of the theory. In addition to the SU(8)-nonsinglet fermions, this descendant theory also contains the SU(6)-singlet fermions $\chi_{9,i,L}$ with $1 \le i \le 5$.

X. MULTIFERMION CONDENSATES AND IMPLICATIONS FOR THE PRESERVATION OF CHIRAL GAUGE SYMMETRY

Our discussion above of fermion condensate formation focused on bilinear fermion condensates and resultant dynamical chiral gauge symmetry breaking. However, it is, in principle, possible for a strongly interacting vectorial or chiral gauge theory to produce fermion condensates involving product(s) of more than just two fermion fields [35,36]. Much less attention has been devoted in the literature to such multifermion condensates than to bilinear fermion condensates. This is somewhat analogous to the situation with bound states of (anti)quarks in hadronic physics. For many years the main focus of research was on color-singlet bound states with the minimum number of (anti)quarks, namely qqq, for baryons and $q\bar{q}$ for mesons. (Subsequently, glueballs and mixing between $q\bar{q}$ mesons and glue to form mass eigenstates were also studied.) However, there is increasing experimental evidence that the hadron spectrum also contains bound states with additional quarks, such as $q\bar{q}q\bar{q}$ and $q\bar{q}Q\bar{Q}$, where Q means a heavy quark, c or b, including charged mesons, and possibly $qqqq\bar{q}$ and $qqqQ\bar{Q}$ [42]. In the case of possible condensates involving four or more fermions, we are not aware of a reliable method that can be used to assess the relative likelihood that these would form. The problem of assessing this likelihood is fraught with even more theoretical uncertainty than the uncertainty inherent in the use of the rough MAC criterion to measure the attractiveness of bilinear fermion condensation channels.

Clearly, Lorentz invariance implies that the number of fermion fields in such multifermion condensates must be

even. As usual, we denote the charge conjugate of a generic fermion field χ as $\chi^c \equiv C\bar{\chi}^T$, where C is the Dirac charge conjugation matrix satisfying $C = -C^T$ and $\bar{\chi} \equiv \chi^\dagger \gamma_0$; recall also that for a left-handed fermion χ_L, the charge conjugate is $(\chi_L)^c = (\chi^c)_R$.

As an example, consider the SU(5) $A_2\bar{F}$ theory, with the fields ψ_L^{ab} and $\chi_{a,1,L}$ or equivalently, $\psi_{ab,R}^c$ and $(\chi^c)_R^{a,1}$. When the gauge interaction becomes strong, it could produce several different four-fermion condensates that preserve the SU(5) gauge symmetry. One such condensate that involves all of the fermions is

$$\langle \epsilon_{abdef} [\psi_L^{abT} C \psi_L^{de}] [\psi_L^{fsT} C \chi_{s,1,L}] \rangle, \tag{10.1}$$

where here a, b, d, e, f, s are SU(5) gauge indices. This condensate has U(1)$'$ charge $3Q_{A_2}' + Q_{\bar{F}}'$. Using the results from the $N = 5$ special case of Eq. (5.10), namely, $Q_{A_2}' = 1$, $Q_{\bar{F}}' = -3$, we find that this condensate (10.1) has zero U(1)$'$ charge, so it also preserves the global U(1)$'$ symmetry of the SU(5) theory.

In a similar manner, consider the SU(6) $A_2\bar{F}$ theory, with the fermions ψ_L^{ab} and $\chi_{a,j,L}$ with $j = 1$, 2. As the SU(6) gauge interaction becomes strong in the infrared, it might produce the following four-fermion condensate that is invariant under the SU(6) gauge symmetry:

$$\langle \epsilon_{abdesu} [\psi_L^{abT} C \psi_L^{de}] [(\chi^c)_R^{s,1T} C (\chi^c)_R^{u,2}] \rangle - (1 \leftrightarrow 2)$$
$$= \langle \epsilon_{abdesu} [\psi_L^{abT} C \psi_L^{de}] \epsilon_{ij} [(\chi^c)_R^{s,iT} C (\chi^c)_R^{u,j}] \rangle. \tag{10.2}$$

Note that because of the contraction of the operator product $[(\chi^c)_R^{s,1T} C (\chi^c)_R^{u,2}]$ with the SU(6) ϵ_{abdesu} tensor, the first term in Eq. (10.2) is automatically antisymmetrized in the flavor indices $j = 1$, 2; we have made this explicit by subtracting the term with these indices interchanged. As shown by the second line of Eq. (10.2), this condensate thus preserves the SU(2)$_{\bar{F}}$ factor group in the global flavor symmetry G_{fl} for this theory, namely SU(2)$_{\bar{F}} \otimes$ U(1)$'$. In the $(A_2, \bar{F})$ basis, the U(1)$'$ charges are $(2, -4)$, as given by the $N = 6$ special case of Eq. (5.10). Hence, the U(1)$'$ charge of the condensate (10.2) is -4, so it breaks the U(1)$'$ part of G_{fl}, yielding one Nambu-Goldstone boson.

One can give corresponding discussions of gauge-invariant multifermion condensates for other SU(N) $A_k\bar{F}$ theories that become strongly coupled in the infrared. In general, these theories could also produce other types of four-fermion condensates such as

$$\langle [\psi_L^{abT} C \chi_{b,i,L}] [(\psi^c)_{ad,R}^T C (\chi^c)_R^{d,j}] \rangle, \tag{10.3}$$

$$\langle [\bar{\psi}_{ab,L} \gamma_\mu \psi_L^{ab}] [\bar{\psi}_{de,L} \gamma^\mu \psi_L^{de}] \rangle, \tag{10.4}$$

$$\langle [\bar{\psi}_{ab,L} \gamma_\mu \psi_L^{ab}] [(\bar{\chi}_L)^{d,i} \gamma^\mu \chi_{d,j,L}] \rangle, \tag{10.5}$$

and

$$\langle [(\bar{\chi}_L)^{a,i} \gamma_\mu \chi_{a,j,L}][(\bar{\chi}_L)^{b,k} \gamma^\mu \chi_{b,\ell,L}] \rangle, \qquad (10.6)$$

where $1 \le i, j, k, \ell \le n_{\bar{F}}$. There are also multifermion condensates with eight and more fermions that one could consider. Such multifermion condensates merit further study.

XI. NONEXISTENCE OF ASYMPTOTICALLY FREE $S_k\bar{F}$ THEORIES WITH $k \ge 3$

It is natural to carry out an investigation of (anomaly-free) chiral gauge theories with gauge group $SU(N)$ and chiral fermions transforming according to the rank-k symmetric tensor representation with $k \ge 3$ and a requisite number of chiral fermions in the $\bar{F}$ representation so as to render the theories free of an anomaly in gauged currents. We denote such a theory as an $SU(N)$ $S_k\bar{F}$ theory. This investigation would be the analogue of the study that we have performed in this paper for $A_k\bar{F}$ theories with $k \ge 3$ and would generalize the studies that have been carried out in the past on the $S_2\bar{F}$ theory [4,10–13]. As with the $A_k\bar{F}$ theories, we require that the theory must be asymptotically free so that it is perturbatively calculable in at least one regime, namely the deep UV, where the gauge coupling is small.

However, we shall show here that there are no asymptotically free (anomaly-free) $S_k\bar{F}$ chiral gauge theories with $k \ge 3$. As before we denote the number of copies of $\bar{F}$ fermions as $n_{\bar{F}}$. The contribution to the triangle anomaly in gauged currents of a chiral fermion in the S_k representation is (see Appendix B)

$$\mathcal{A}(S_k) = \frac{(N+k)!(N+2k)}{(N+2)!(k-1)!}. \qquad (11.1)$$

The total anomaly in the theory is $\mathcal{A} = \mathcal{A}(S_k) - n_{\bar{F}}\mathcal{A}(F)$, so the condition of anomaly cancellation is that

$$n_{\bar{F}} = \mathcal{A}(S_k). \qquad (11.2)$$

The first few values are of $n_{\bar{F}}$ are

$$n_{\bar{F}} = \begin{cases} N+4 & \text{if } k=2 \\ (1/2)(N+3)(N+6) & \text{if } k=3 \\ (1/6)(N+3)(N+4)(N+8) & \text{if } k=4 \end{cases} \qquad (11.3)$$

and so forth for higher k.

To investigate the restrictions due to the requirement of asymptotic freedom, we calculate the one-loop coefficient of the beta function. We find

$$b_{1,S_k\bar{F}} = \frac{1}{3}[11N - 2\{T(S_k) + \mathcal{A}(S_k)T(\bar{F})\}] = \frac{1}{3}\left[11N - \frac{1}{(k-1)!}\left\{\left[\prod_{j=2}^{k}(N+j)\right] + \frac{(N+k)!(N+2k)}{(N+2)!}\right\}\right]. \qquad (11.4)$$

We exhibit the explicit expressions for $b^{(k)}$ for the first few $k \ge 2$:

$$b_{1,S_2\bar{F}} = 3N + 2, \qquad (11.5)$$

$$b_{1,S_3\bar{F}} = \frac{1}{3}(-N^2 + 4N - 12), \qquad (11.6)$$

$$b_{1,S_4\bar{F}} = -\frac{1}{9}(N^3 + 12N^2 + 14N + 60), \qquad (11.7)$$

$$b_{1,S_5\bar{F}} = -\frac{1}{36}(N^4 + 18N^3 + 119N^2 + 210N + 360),$$
$$\qquad (11.8)$$

and

$$b_{1,S_6\bar{F}} = -\frac{1}{180}(N^5 + 25N^4 + 245N^3 + 1175N^2 + 2094N + 2520). \qquad (11.9)$$

The coefficient $b_{1,S_2\bar{F}}$ is positive for all relevant N, and this property was used in past studies of the $S_2\bar{F}$ theory.

However, the coefficient $b_{1,S_3\bar{F}}$ is negative for relevant $N \ge 3$. [Recall that an SU(2) theory has only real representations and hence is not chiral.] With N generalized from positive integers to real numbers, $b_{1,S_3\bar{F}}$ is negative for all N, reaching its maximum value of $-8/3$ for $N = 2$. We find that the $b_{1,S_k\bar{F}}$ coefficients with $k \ge 4$ are negative-definite for all positive N (either real or integer). This is evident from the illustrative explicit expressions that we have given for $4 \le k \le 6$. This completes our proof that there are no asymptotically free (anomaly-free) $S_k\bar{F}$ chiral gauge theories with $k \ge 3$.

XII. CONCLUSIONS

In summary, in this paper we have constructed and studied asymptotically free chiral gauge theories with an $SU(N)$ gauge group and chiral fermions transforming according to the antisymmetric rank-k representation, A_k, with $k = 2, 3, 4$, and, for each k and N, the requisite number of copies, $n_{\bar{F}}$, of fermions transforming according to the conjugate fundamental representation, $\bar{F}$, of this group to render the theory anomaly-free. For a given k, to

get a theory that is chiral and has $n_{\bar{F}} \geq 1$, we take $N \geq 2k + 1$. We have extended previous studies of the $A_2 \bar{F}$ theories with further analysis of fermion condensation channels and sequential symmetry breaking and have presented a number of new results on the $A_k \bar{F}$ theories with $k \geq 3$. The $A_2 \bar{F}$ theories form an infinite family with $N \geq 5$, but we have shown that the $A_3 \bar{F}$ and $A_4 \bar{F}$ theories are only asymptotically free for N in the respective ranges $7 \leq N \leq 17$ and $9 \leq N \leq 11$, and that there are no asymptotically free $A_k \bar{F}$ theories with $k \geq 5$. We have investigated the types of ultraviolet to infrared evolution for these $A_k \bar{F}$ theories and have found that, depending on k and N, they may lead in the infrared to a non-Abelian Coulomb phase, or may involve confinement with massless gauge-singlet composite fermions, or bilinear fermion condensation with dynamical gauge and global symmetry breaking. In two cases, namely $(k, N) = (2, 6), (3, 12)$, in each of which two bilinear fermion condensation channels are equally attractive, so the MAC criterion does not prefer one over the other, we have applied vacuum alignment arguments to infer which channel is preferred. We have also discussed multifermion condensates. Finally, we have shown that there are no asymptotically free, anomaly-free $SU(N)$ $S_k \bar{F}$ chiral gauge theories with $k \geq 3$, where S_k denotes the rank-k symmetric representation.

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APPENDIX A: BETA FUNCTION COEFFICIENTS AND RELEVANT GROUP INVARIANTS

For reference, we list the one-loop and two-loop coefficients [26,27] in the beta function (2.1) for a non-Abelian chiral gauge theory with gauge group G and a set of chiral fermions comprised of N_i fermions transforming according to the representations R_i:

$$b_1 = \frac{1}{3}\left[11 C_2(G) - 2\sum_{R_i} N_i T(R_i)\right] \qquad (A1)$$

and

$$b_2 = \frac{1}{3}\left[34 C_2(G)^2 - 2\sum_{R_i} N_i \{5 C_2(G) + 3 C_2(R_i)\} T(R_i)\right].$$
$$(A2)$$

APPENDIX B: RELEVANT GROUP INVARIANTS

We list below the group invariants that we use for the relevant case $G = SU(N)$. Recall that the order of the Lie group $SU(N)$ (i.e., the number of infinitesimal generators of the associated Lie algebra) is $o(SU(N)) = N^2 - 1$, and

the order of the symplectic group $Sp(2N)$ is $o(Sp(2N)) = N(2N + 1)$. The antisymmetric rank-k representation of $SU(N)$ is denoted $A_k \equiv [k]_N$. Some group invariants are

$$\sum_{i,j=1}^{\dim(R)} \mathcal{D}_R(T_a)_{ij} \mathcal{D}_R(T_b)_{ji} = T(R)\delta_{ab} \qquad (B1)$$

and

$$\sum_{a=1}^{o(G)} \sum_{j=1}^{\dim(R)} \mathcal{D}_R(T_a)_{ij} \mathcal{D}_R(T_a)_{jk} = C_2(R)\delta_{ik}, \qquad (B2)$$

where T_a are the generators of G, and $\mathcal{D}_R$ is the matrix representation (*Darstellung*) of R.

For the adjoint representation, adj, $C_2(adj) \equiv C_2(G)$, and for $G = SU(N)$, $C_2(G) = N$. For the rank-k antisymmetric representation of $SU(N)$, $A_k \equiv [k]_N$,

$$C_2([k]_N) = \frac{k(N-k)(N+1)}{2N} \qquad (B3)$$

and

$$T([k]_N) = \frac{1}{2}\binom{N-2}{k-1}. \qquad (B4)$$

Thus, for $k = 1$, $T([1]_N) = T(F) = 1/2$ and, for $k \geq 2$,

$$T([k]_N) = \frac{\prod_{j=2}^k (N-j)}{2(k-1)!}. \qquad (B5)$$

For the rank-k symmetric representation, S_k,

$$C_2(S_k) = \frac{k(N+k)(N-1)}{2N} \qquad (B6)$$

and

$$T(S_k) = \frac{\prod_{j=2}^k (N+j)}{2(k-1)!}. \qquad (B7)$$

The anomaly produced by chiral fermions transforming according to the representation R of a group G is defined as

$$\mathrm{Tr}_R(T_a, \{T_b, T_c\}) = \mathcal{A}(R)d_{abc} \qquad (B8)$$

where the d_{abc} are the totally symmetric structure constants of the corresponding Lie algebra. Thus, $\mathcal{A}(\square) = 1$ for $SU(N)$. For the symmetric and antisymmetric rank-k tensor representations of $SU(N)$, the anomaly is, respectively [39],

$$\mathcal{A}(S_k) = \frac{(N+k)!(N+2k)}{(N+2)!(k-1)!} \qquad (B9)$$

and, for $1 \leq k \leq N - 1$,

$$\mathcal{A}(A_k) = \frac{(N-3)!(N-2k)}{(N-k-1)!(k-1)!}. \qquad (B10)$$

[Note that $[N]_N$ is the singlet, so $\mathcal{A}([N]_N) = 0$.] Hence, in particular,

$$\mathcal{A}([2]_N) = N - 4, \qquad (B11)$$

$$\mathcal{A}([3]_N) = \frac{(N-3)(N-6)}{2}, \qquad (B12)$$

and

$$\mathcal{A}([4]_N) = \frac{(N-3)(N-4)(N-8)}{3!}. \qquad (B13)$$

From Eq. (B10), there follows the recursion relation

$$\mathcal{A}([k]_N) + \mathcal{A}([k+1]_N) = \mathcal{A}([k+1]_{N+1}) \quad \text{for } 1 \leq k \leq N-1. \qquad (B14)$$

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