More on Heisenberg's model for high energy nucleon-nucleon scattering

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We revisit Heisenberg's model for nucleon-nucleon scattering which admits a saturation of the Froissart bound. We examine its uniqueness, and find that up to certain natural generalizations, it is the only action that saturates the bound. We find that we can extract also subleading behavior for $\sigma_{\rm tot}(s)$ from it, though that requires a knowledge of the wave function solution that is hard to obtain, and a black-disk model allows the calculation of $\sigma_{\rm elastic}(s)$ as well. The wave-function solution is analyzed perturbatively, and its source is interpreted. Generalizations to several mesons, the addition of vector mesons, and curved space regimes are also found. We discuss the relations between Heisenberg's model and holographic models that are dual to QCD-like theories.

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I. INTRODUCTION

In quantum field theory unitarity constrains the asymptotic dependence of the total cross section of any scattering process to be bounded by the well-known Froissart bound [1,2],

$$\sigma_{\rm tot}(s) \le C \ln^2 \frac{s}{s_0}; \qquad C \le \frac{\pi}{m^2}, \tag{1.1}$$

where s is Mandelstam's dynamical variable and m is the mass of the lightest particle that can be exchanged by the scattering projectiles. In the case of QCD $m = m_\pi$ is the pion mass, and the bound is supposed to be saturated in the $s \to \infty$ limit.[1] However, being that the saturation of the bound is governed by nonperturbative, IR, physics, attempts to describe the saturation of the bound in QCD have not been successful. Strikingly, nine years before the discovery of the bound, and in fact even before the birth of QCD, Heisenberg proposed a simple effective model for the maximal behavior of $\sigma_{\rm tot}(s)$, in terms of a Dirac-Born-Infeld (DBI) action for the pion field, that gives an (almost) saturation of the Froissart bound in the case of QCD [8].

The paper of Heisenberg [8] included two revolutionary ideas: (i) the extraction of the dependence of the total cross section on the Mandelstam s variable from the average energy per pion determined from the classical energy density of the scalar field; (ii) describing the dynamics of the scalar field using the DBI action. The first idea is obviously very different from the way one usually determines the cross section in perturbation theory. Instead of computing Feynman diagrams of scattering amplitudes and then from the amplitudes determining the cross section, Heisenberg's proposition was to derive the cross section in a very simple manner from the following relation:

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[1]Note that experimentally, it is extremely difficult to distinguish between a $\log^2 s$ behavior and a small power law. In 2001, the Particle Data Group (PDG) still presented the data for $\sigma_{\rm tot}(s)$ above $\sqrt{s} = 9$ GeV as a fit to a small power law $X_{AB} s^\epsilon$, with $\epsilon \simeq 0.093$ and $X_{AB} \sim 10$–35 mb (Table 38.2 in the 2001 PDG; see also the 2000 PDG [3]). In the 2004 PDG (Ref. [4] Table 40.2), the fit was instead $Z_{AB} + B\log^2(s/s_0)$, now extending down to $\sqrt{s} = 5$ GeV, which however (as already explained in Refs. [5,6]), given that the fit was only 3 percent better (χ^2/d.o.f of 0.971 instead of 1), can be seen as simply the coincidence of fitting instead of $As^\epsilon \simeq A + A\epsilon \log s + A\epsilon^2/2\log^2 s + \cdots$, just the first and the third terms, with the ratio $\simeq \epsilon^2/2$. Of course, then one finds $B \sim 0.3$ mb instead of $\pi/m_\pi^2 \simeq 60$ mb, but nevertheless $Z_{AB} \sim 18$–65 mb, which now just seems like a big coincidence. The improvement in the latest PDG from 2014 (Ref. [7], Table 50) is minimal (χ^2/d.o.f. of 0.96 for the same range), and one finds $B \equiv \pi/M^2 \sim 0.27$ mb implying $M \sim 2.1$ GeV. The justification behind this new fit was that in various models for nonperturbative QCD, which of course include lots of assumptions, one can find a coefficient $B \simeq \pi/M_g^2$, with $M_g \sim \mathcal{O}(\rm GeV)$ a glueball-type mass. Yet in the Froissart derivation m was really necessarily the pion mass, and represented the *truly asymptotic* form of the cross section. It is perfectly plausible that an approximately valid model is written in terms of an effective Froissart bound with a gluon-type mass M_g, but the real behavior in the measured energy range is $\propto s^\epsilon$, which would be achieved if $M_g = M_g(s)$ varies *extremely* slowly with the scale. So we assume that the truly asymptotic regime with coefficient π/m_π^2 has not been achieved yet [from the experimental data cited above it would be achieved just over 1 TeV, such that (a few TeV/9 GeV)$^{0.093} \sim 60$ mb$/X_{AB}$]. This interpretation is more in line with Froissart's original derivation, so we will adopt it in this paper.

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$$\langle k_0\rangle = \sqrt{s}e^{-m_\pi b_{\max}}, \to \sigma_{\rm tot} = \pi b_{\max}^2 = \frac{\pi}{m_\pi^2}\log^2\frac{s}{\langle k_0\rangle^2}, \tag{1.2}$$

where $\langle k_0\rangle$ is the energy per pion, m_π is its mass, and $b_{\max}$ is the maximal impact parameter for which there is still an interaction between the two nucleon projectiles. The assumption of the model is that there is an "effective action" for the scalar field that mediates the interaction from which one can compute $\langle k_0\rangle = \frac{\mathcal{E}}{n}$ where $\mathcal{E}$ is the total energy and n is the number of the pions. Thus, the dependence of $\sigma_{\rm tot}$ on s follows from the dependence of $\langle k_0\rangle$ on s. A physical system for which $\langle k_0\rangle$ does not depend on s saturates Froissart's bound. The second original idea is to use a nonstandard action to describe the dynamics of the pion field. In his paper Heisenberg found that using an action of the scalar that is based on the ordinary kinetic term (regardless of its potential) cannot saturate the Froissart bound. In fact, it will yield a constant cross section. However, using a DBI action yields $\langle k_0\rangle \sim \log\frac{s}{m_\pi^2}$. This mild dependence of $\langle k_0\rangle$ on s means that the total cross section of the model is close to that of the bound.

Experimental data of the total cross section of proton-proton (and proton-antiproton) collisions is well established in a very wide range of energies starting from sub-GeV energies all the way to $\sqrt{s} = 7$ in the TOTEM experiment at the LHC and in fact even higher up to $\sqrt{s} = 57$ TeV from cosmic-ray observations. Figure 1 shows the data points together with a fit based on Eq. (1.2) but with a mass $m \simeq 1$ GeV and not the pion mass [9]. Thus, regardless of the Froissart bound, one would like to have a theoretical model that resembles the behavior of Eq. (1.2) since it seems to fit the experimental data quite well. Needless to say there is no direct derivation from QCD that can reproduce such a fit.

The goals of this paper are fourfold. (i) Since part of Ref. [8] was written in a concise form, we elaborate the discussion, perform several additional calculations and provide some further evidence for the claims of the paper. In particular we analyze the pion field including the passage from the $1+1$-dimensional solution to a full four-dimensional one. The ratio of the elastic to total cross section is derived using a black disk model. (ii) We examine the uniqueness of the DBI action as an action that can (almost) saturate the bound. We prove that only an action with an infinite tower of higher powers of the derivative term, as the DBI action admits, can do the job. (iii) We propose and analyze several generalizations of Heisenberg's model. We add a general potential instead of only a mass term, and we analyze a sigma model with several scalars. We examine the "highly effective action" of Ref. [10] for the case of a single scalar in AdS_5. For that case the ordinary kinetic term in the square-root Lagrangian density undergoes the following transformation: $\partial_\mu\phi\partial^\mu\phi \to \frac{1}{\phi^4}\partial_\mu\phi\partial^\mu\phi$. (iv) The last goal is to relate Heisenberg's model to the DBI action used in gauge/gravity duality and furthermore to two different holographic approaches to the nucleon-nucleon scattering.

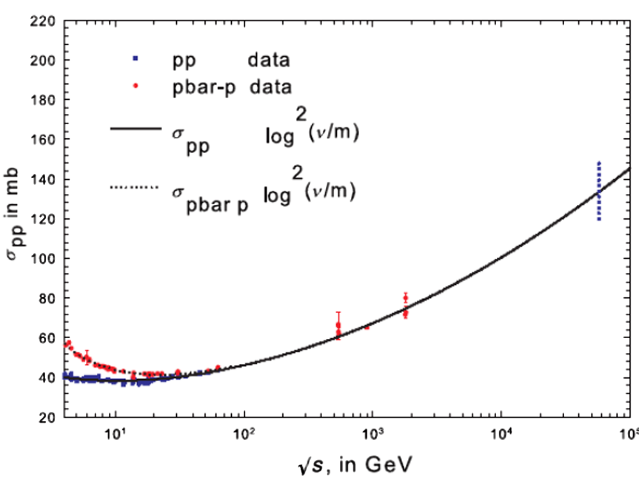

FIG. 1 (color online). The total cross section as a function of $\sqrt{s}$ for pp and $p\bar{p}$ scattering.

One approach to the latter is based on a simple effective model for QCD scattering at high energies: the Polchinski-Strassler model [11] was developed in terms of a metric in a cutoff AdS_5 background, dual to glueball fields, and the fluctuation of an IR brane (IR cutoff), dual to a pion field [the model was extended to the phenomenological "hard-wall" model with the addition (by hand) of extra fields in the bulk in Ref. [12]]. Based on earlier work in Ref. [13], in Ref. [14] it was shown that the saturation of the Froissart bound can arise through scattering of gravitational shock waves located on (or close to) the IR brane, with the formation of a black hole on the IR brane. Then in Ref. [5] it was shown that one can map exactly the description of the saturation of the bound in the dual, through gravitational shock-wave collision with black hole formation, with the saturation of the bound in the Heisenberg model, through pion field shock-wave collisions. This picture was used further in Refs. [15–17] (see Ref. [18] for a review) to describe the strongly-coupled Quark-Gluon Plasma fireball obtained in heavy-ion collisions as the object dual to the black hole formed on the IR brane.

In the second approach we relate Heisenberg's DBI action to the DBI that describes the fluctuation of the flavor branes in confining the gravitational background and in particular in the generalized Sakai-Sugimoto model [19,20]. We show that the source of the scalar field in that model is the flavor instanton density [21], that corresponds to the proton density.

The paper is organized as follows. In the next section we review the model of Ref. [8] and elaborate on certain issues by performing additional calculations. In Sec. III we discuss the uniqueness of the DBI action used in Ref. [8] for the saturation of the Froissart bound. We show that to have a solution of the form $\phi(s) = A\sqrt{s}$, which is what is needed to saturate the Froissart bound, one cannot use an action with a finite series of higher derivatives and only the infinite series that follows from the DBI action does the job (though we are not able to show that some other action with an infinite number of higher-derivative terms could not do the job also). In Sec. IV we consider various generalizations of the action used in Ref. [8]. First we add a potential term in addition to the mass term. We then analyze a DBI sigma model for several scalar fields. For the simple case of replacing one kinetic term in the square root with a sum of kinetic terms for several scalars the behavior is similar to the original model. Next we discuss the case of a DBI action associated with the $AdS_5 \times S^5$ case. We show that for the case of a single scalar field performing the determinant in the DBI action yields a close cousin of the action used in Ref. [8]. Another generalization discussed is the DBI for vector mesons. Assuming here again a dependence only on the coordinate s defined in Eq. (2.3) we show that the behavior of the vector mesons is similar to that of the scalar meson. In Sec. V we discuss the "wave function" of the pion. First we elevate the solution $\phi(s)$ to $\phi(s,r)$. We argue that for Heisenberg's solution we have a shock-like behavior, where T_{++} blows up at $x^+ = 0$, even though we do not have a $\delta(x^+)$ behavior. Next we consider a perturbative expansion around $r = 0$ and then a perturbative solution around the asymptotics $r \to \infty$. Section VI is devoted to analyzing the sources of the pion field. We first consider sources for the nonlinear Born-Infeld theory of electrodynamics. We then in Sec. VI B discuss in a similar manner the source of a scalar DBI theory. Section VII deals with the original question behind this paper, namely, the cross section of the nucleon-nucleon scattering process. We discuss corrections away from the Froissart bound. We then describe the model of the black disk and the corresponding ratio between the elastic and total cross sections. Section VIII is devoted to an examination of the relation between Heisenberg' model and the holographic description of nucleon-nucleon scattering. We show that the nucleon-nucleon scattering process takes the form of the scattering of instantons of the flavored gauge fields that reside on the flavor branes. We end this paper with a summary and open questions.

II. HEISENBERG'S MODEL: A REVIEW AND ELABORATIONS

In this section, we first review the work of Heisenberg in modern language, and we perform some additional computations that clarify some aspects of the model.

With a remarkable insight, Heisenberg considered a nonlinear higher-derivative action for the pion, the DBI action with a mass term inside the square root,

$$\mathcal{L} = l^{-4}[1 - \sqrt{1 + l^4[(\partial_\mu \phi)^2 + m^2\phi^2]}]. \tag{2.1}$$

The reasoning is that in the high-energy limit, many pions (lowest-mass particles) will be created, so we need to consider a pion field as the effective one, but this process is both nonperturbative and high energy, and hence one needs a nonlinear action. As we will soon see, a polynomial interaction does not have the required properties, so the DBI action is the natural one to consider.

In the high-energy limit, colliding hadrons will look like pancakes due to Lorentz contraction, but moreover we need to consider them as just sources for the pion field surrounding them, that will also get Lorentz contracted and look like a shock wave. Therefore the process considered in the asymptotic regime is a collision of pion field shock waves with the action (2.1). We look for (classical) shock-wave solutions to the action (2.1). The equations of motion are

$$-\Box\phi + m^2\phi + l^4 \frac{[(\partial_\mu\partial_\nu\phi)(\partial_\mu\phi)\partial_\nu\phi + (\partial_\mu\phi)^2 m^2\phi]}{1 + l^4[(\partial_\mu\phi)^2 + m^2\phi^2]} = 0. \tag{2.2}$$

The crucial simplification that allowed Heisenberg to do exact calculations is to consider that for a shock-wave solution, only the physics near the shock is relevant, and by focusing near that, we can ignore the dependence on the two transverse dimensions y, z (with $r = \sqrt{y^2 + z^2}$), and consider the $1+1$-dimensional problem for time t and the longitudinal direction x (along the direction of propagation).

Then from Lorentz invariance, he considers only solutions that depend on

$$s = t^2 - x^2. \tag{2.3}$$

(Note that from now on, we will use s to denote this variable only, and not the Mandelstam invariant, which will be called $\tilde{s}$.) This requires some explanation. The first point is that $\phi = \phi(s)$ is boost invariant for boosts in x: under a boost, we have $x^+ \to e^\beta x^+$, $x^- \to e^{-b} x^-$. But why do we need a boost-invariant solution? The fact that ϕ is a scalar means that $\phi'(x'^+, x'^-) = \phi(x^+, x^-)$, where $x^\pm = t \pm x$. We could say that we find the solution $\phi(x^+, x^-)$ in a reference system and then define the one in another reference system by $\phi'(x'^+, x'^-) = \phi(x^+, x^-)$, so any solution would work.

However, the essential point is that we use the ultra-relativistic approximation, in which even though the pion is massive, we consider that the source moves on a light cone, $x^+ = 0$ or $x^- = 0$. As a result, we impose that $\phi(x^+ = 0) = 0$ or $\phi(x^- = 0) = 0$. This in turn implies a power-law behavior near the light cone, i.e. (for $x^- = 0$), $\phi \propto (x^-)^q$, $q > 0$ for $x^- \sim 0$. But if we have an arbitrary dependence on x^+, then in the boosted system ϕ' would have a power of e^β in front, unless we have the same power law for x^+, i.e. unless $\phi(x^+, x^-) = \phi(x^+ x^-) = \phi(s)$.

For $\phi = \phi(s)$, we have

$$(\partial_\mu \phi)^2 = -4s\left(\frac{d\phi}{ds}\right)^2, \tag{2.4}$$

the DBI action becomes

$$\mathcal{L} = l^{-4}\left[1 - \sqrt{1 + l^4\left(-4s\left(\frac{d\phi}{ds}\right)^2 + m^2\phi^2\right)}\right], \tag{2.5}$$

and its equation of motion becomes

$$4\frac{d}{ds}\left(s\frac{d\phi}{ds}\right) + m^2\phi + \frac{8l^4 s(\frac{d\phi}{ds})^2}{1 + l^4[-4s(\frac{d\phi}{ds})^2 + m^2\phi^2]} \times \left[\frac{d\phi}{ds} - \frac{m^2\phi}{2} + 2s\frac{d^2\phi}{ds^2}\right] = 0. \tag{2.6}$$

However, by multiplying with the denominator (assuming that it does not vanish), canceling and rewriting the terms we are led to the form

$$4\frac{d}{ds}\left(s\frac{d\phi}{ds}\right) + m^2\phi = 8sl^4\left(\frac{d\phi}{ds}\right)^2 \frac{[\frac{d\phi}{ds} + m^2\phi]}{1 + l^4 m^2\phi^2}. \tag{2.7}$$

When $m = 0$, one can find an exact solution depending on an arbitrary parameter a,

$$\phi = \frac{1}{a}\log\left(1 + \frac{a^2}{2l^4}s + \frac{a}{2l^4}\sqrt{4l^4 s + a^2 s^2}\right), \quad s \geq 0, \tag{2.8}$$

and $\phi = 0$ for $s < 0$.

When $m \neq 0$, one can find a perturbative solution at small s,

$$\phi = \frac{\sqrt{s}}{l^2}(1 + a\,s\,m^2 + \cdots), \quad 0 \leq s \ll 1/m^2, \tag{2.9}$$

and $\phi = 0$ for $s < 0$, as well as a solution at large s,

$$\phi \simeq \gamma s^{-1/4} m^{-1/2}\cos(m\sqrt{s} + \delta), \quad s \gg 1/m^2. \tag{2.10}$$

At this point, Heisenberg notes that for the model to be reasonable, we need $(\partial_\mu\phi)^2$ to be a finite constant at the position of the shock, $s = 0$, since we need the nonlinearities to play a role there. But for the free Klein-Gordon (KG) equation, the result is infinite, which is also unphysical.[2] Thus the only possibility to correctly describe the shock at $s = 0$ is to have $(\partial_\mu\phi)^2$ be a finite constant, which leads to $\phi \sim A\sqrt{s}$ for $s \to 0$. This, as we will show below, is incompatible with an action with a canonical kinetic term and a polynomial potential.

A. From the pion field to the nucleon-nucleon cross section

The energy (Hamiltonian) density of the pion field is

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{l^{-4} + (\nabla\phi)^2 + m^2\phi^2}{\sqrt{1 + l^4[(\partial_\mu\phi)^2 + m^2\phi^2]}} - l^{-4}, \tag{2.11}$$

and similarly the momentum density is

$$\mathcal{P} = \pi\nabla\phi = \frac{\dot{\phi}\nabla\phi}{\sqrt{1 + l^4[(\partial_\mu\phi)^2 + m^2\phi^2]}}. \tag{2.12}$$

Both densities have a denominator which is the square-root term of the Lagrangian density. In the massless case, by substituting the solution (2.8) into Eq. (2.11) we find that the energy density diverges at $s = 0$ due to the denominator going to zero as $a\sqrt{s}/(2l^2)$. Similarly, in the massive case, by substituting the solution (2.9) into Eq. (2.11) we find the same divergence due to the denominator going to zero at $s = 0$ as $m\sqrt{s(1 - 6a)}$.

Now following Ref. [8] we assume that one can introduce a small perturbation so that the denominator can be taken as a nonvanishing constant. In this case we can use the standard method of Fourier transforming Eq. (2.9) over x to k, as

$$\phi(k, t) = l^{-2}\int_0^t dx e^{ikx}\sqrt{t^2 - x^2}(1 + am^2(t^2 - x^2) + \cdots), \tag{2.13}$$

which for $a = 0$ (only the leading term) gives

$$\phi(k, t) \simeq l^{-2}\frac{\pi}{2}\frac{|t|}{|k|}(J_1(|k||t|) + i\mathbf{H_1}(|\mathbf{k}||\mathbf{t}|)), \tag{2.14}$$

where J_1 is a Bessel function and $\mathbf{H_1}$ is a Struve function. When expanded at large k, we obtain

$$\phi - l^{-2}i\frac{|t|}{|k|} \simeq \sqrt{-i}l^{-2}\sqrt{\frac{\pi}{2}}|t|^{1/2}|k|^{-3/2}e^{-i|k||t|} \times \left(1 + \frac{3}{8|k||t|}e^{2i|k||t|}\right). \tag{2.15}$$

Note that the nonoscillatory part of ϕ is not a radiative piece, and hence is dropped.

However, as discussed above, in reality the shock wave should have a finite thickness in $\sqrt{s}$ of the order of the Lorentz-contracted $1/m$, i.e. $\sqrt{s}_{0m} \equiv \sqrt{s}_{\rm min} = \sqrt{1 - v^2}/m$, which means that at sufficiently large t, $\phi(k, t)$ should be cut

[2] The KG equation for $\phi = \phi(s)$ is just $d/ds(sd\phi/ds) = 0$, with the solution $\phi = A\log(s/s_0)$, which means $(\partial_\mu\phi)^2 = -4s(d\phi/ds)^2 = -4A^2/s \to \infty$.

off at $k_{0m} = 1/r_{0m} = \gamma m$, the relativistic mass of the pion. With the assumption of a constant denominator we get

$$\frac{dE}{dk} \propto k^2 \phi(k)^2 \sim \frac{\text{const}}{k}, \tag{2.16}$$

where in the last equality we have substituted Eq. (2.15). But this is valid only for $k \leq k_{0m}$.

Finally, the momentum k is identified with the momentum of a pion k_0, and moreover the classical field close to the shock is identified with the classical limit of the field of radiated pions in a hadron collision. Thus the radiated energy $\mathcal{E}$ (identified through canonical quantization with the pion field energy E) per unit frequency of radiated pions is given by (denoting the constant by B)

$$\frac{d\mathcal{E}}{dk_0} = \frac{B}{k_0}, \quad m \leq k \leq k_{0m}. \tag{2.17}$$

This integrates to

$$\mathcal{E} = B \ln \frac{k_{0m}}{m} = B \ln \gamma, \tag{2.18}$$

and leads to a relation for the number of pions emitted for a given energy, since $dE = k_0 dn$, giving

$$\frac{dn}{dk_0} = \frac{B}{k_0^2}, \quad m \leq k_0 \leq k_{0m}, \tag{2.19}$$

which integrates to

$$n = \frac{B}{m}\left(1 - \frac{m}{k_{0m}}\right). \tag{2.20}$$

Then the average emitted energy per pion is

$$\langle k_0 \rangle \equiv \frac{\mathcal{E}}{n} = m \frac{\ln(k_{0m}/m)}{1 - m/k_{0m}} = m \frac{\ln \gamma}{1 - \frac{1}{\gamma}} \simeq m \ln \gamma, \tag{2.21}$$

which is approximately constant (only logarithmic dependence on the energy).

The last step in the Heisenberg model is to assume that the emitted energy is proportional to the total energy of the system, $\sqrt{\tilde{s}}$ (here $\tilde{s}$ is the Mandelstam variable), with the constant of proportionality (ratio of emitted energy) being approximately given by the pion wave-function overlap. Since at large transverse distance $r\,(= \sqrt{y^2 + z^2})$, the wave function is small $\phi l \ll 1$, and thus it satisfies the free massive KG equation, with the solution $\phi(r) \sim e^{-mr}$, the wave-function overlap is $\sim e^{-mb}$, where b is the impact parameter, i.e. the transverse separation between the colliding hadrons at the impact point $x = 0$. Then we have approximately

$$\mathcal{E} \sim \sqrt{\tilde{s}} e^{-mb}. \tag{2.22}$$

The maximum impact parameter for which we have an interaction, $b_{\max}$, arises when the emitted energy equals the average emitted energy per pion $\langle k_0 \rangle$, so that it corresponds to emitting just one pion. Then we have

$$\sqrt{\tilde{s}} e^{-mb_{\max}} = \langle k_0 \rangle \Rightarrow b_{\max} = \frac{1}{m} \ln \frac{\sqrt{\tilde{s}}}{\langle k_0 \rangle} \Rightarrow$$
$$\sigma_{\text{tot}} = \frac{\pi}{m^2} \ln^2 \frac{\sqrt{\tilde{s}}}{\langle k_0 \rangle}. \tag{2.23}$$

We see then that the saturation of the Froissart bound arises only if $\langle k_0 \rangle$ is approximately constant as a function of energy.

Next we would like to compare this result with what one gets for an "ordinary field theory" with a canonical kinetic term and a polynomial potential of the form

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \lambda \phi^n. \tag{2.24}$$

The corresponding equation of motion resulting from it for the $\phi = \phi(s)$ ansatz,

$$4\left(s \frac{d^2\phi}{ds^2} + \frac{d\phi}{ds}\right) + m^2 \phi + n\lambda \phi^{n-1} = 0, \tag{2.25}$$

does not have $\phi \sim A\sqrt{s}$ as a solution, since the first bracket is divergent, as it equals $-A/2\sqrt{s}$, and the other terms give zero, as they are positive powers of s. In fact, we can see that the only way to satisfy the equation of motion at leading order in s with a canonical kinetic term plus a potential is for a potential that includes the logarithmic term $\Lambda \ln(\phi/\phi_0)$, since then in the equation of motion we have Λ/ϕ, and we can solve the equation with $A^2 = 2\Lambda$. But it is unclear how such a term could arise in the potential (especially since it is unbounded from below at $\phi = 0$).

Now let us study for this class of theories the energy per emitted pion. In a way similar to the one described above we can prove that

$$\frac{d\mathcal{E}}{dk_0} = B, \quad m \leq k_0 \leq k_{0m}; \Rightarrow \frac{dn}{dk_0} = \frac{B}{k_0}, \quad m \leq k_0 \leq k_{0m} \Rightarrow$$
$$\langle k_0 \rangle = \frac{\mathcal{E}}{n} \simeq \frac{k_{0m}}{\ln \frac{k_{0m}}{m}} = m\gamma \frac{1}{\ln \gamma} \propto \frac{\sqrt{\tilde{s}}}{\ln \sqrt{\tilde{s}}}. \tag{2.26}$$

That means that we do not get the saturation of the Froissart bound, but rather we get a constant $\sigma_{\text{tot}}(\sqrt{\tilde{s}})$.

In fact, we can check that the saturation of the bound is obtained only for $d\mathcal{E}/dk_0 \propto 1/k_0^n$, with $n \geq 1$, whereas for actions with polynomial potentials this is not satisfied. In fact, as we saw, $d\mathcal{E}/dk_0 \propto 1/k_0$ was obtained from the behavior $\phi \propto \sqrt{s}$ of the field near $s = 0$, which was due to the DBI form of the action.

In conclusion, we have two physical ways to restrict the form of the action. As Heisenberg argued, we need $(\partial_\mu\phi)^2$ to be a finite constant in order to describe the correct physics, which restricts us to $\phi(s) \propto \sqrt{s}$, arising only in DBI. On the other hand, if we are to be able to saturate the Froissart bound (which should happen, as Froissart argued), we again need $d\mathcal{E}/dk_0 \propto 1/k_0$, which again requires the DBI action.

III. UNIQUENESS OF THE HEISENBERG MODEL ACTION

In his paper [8], Heisenberg showed that an action based on an ordinary kinetic term with any kind of a potential term does not saturate the Froissart bound. We now seek to check the uniqueness of Heisenberg's choice of having the DBI action as the action of the pion field. We have seen that we need an action with higher derivatives. The question is then can we have other higher-derivative actions? In particular we want to examine whether one needs an infinite series of any power of the derivative term or if it is enough to have certain finite series. And furthermore if one needs an infinite series is the DBI action used by Heisenberg unique?

We now examine this question, by considering Lagrangians of the type $\mathcal{L}(\phi, X)$, where $X = (\partial_\mu\phi)^2$.

A. DBI truncated to first term

We will start by truncating the DBI Lagrangian from the previous section (with $m^2\phi^2$ promoted to $2V$ for more generality) to the first interaction term, i.e.

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + \frac{l^4}{8}[(\partial_\mu\phi)^2 + 2V(\phi)]^2 \\ &= -\frac{1}{2}(\partial_\mu\phi)^2 - \tilde{V}(\phi) + \frac{l^4}{8}[(\partial_\mu\phi)^2]^2 + \frac{l^4}{2}(\partial_\mu\phi)^2 V(\phi),\end{aligned} \tag{3.1}$$

where $\tilde{V}(\phi) = V(\phi) - l^4 V^2(\phi)/2$, but for generality we will consider arbitrary $\tilde{V}$.

Then the equation of motion is

$$\begin{aligned}&-\Box\phi + \tilde{V}'(\phi) + \frac{l^4}{2}(\partial_\mu\phi)^2\Box\phi + l^4(\partial_\mu\phi)(\partial_\nu\phi)(\partial_\mu\partial_\nu\phi) \\ &\quad + l^4(\partial^2\phi)V(\phi) + \frac{l^4}{2}(\partial_\mu\phi)^2 V'(\phi) = 0,\end{aligned} \tag{3.2}$$

and for $\phi = \phi(s)$ we get the equation of motion

$$\begin{aligned}&4\frac{d}{ds}\left(s\frac{d\phi}{ds}\right)(1 - l^4 V(\phi)) + \tilde{V}'(\phi) + 8sl^4\left(\frac{d\phi}{ds}\right)^2 \\ &\quad \times\left(2\frac{d\phi}{ds} + 3s\frac{d^2\phi}{ds^2} - \frac{V'(\phi)}{4}\right) = 0.\end{aligned} \tag{3.3}$$

We want to see whether a solution of the type $\phi = A\sqrt{s}$ near $s = 0$ is possible. First we note that in this case, the terms with V and V' are irrelevant (they are subleading), so we will drop them for simplicity (we can add them for free at the end). Then, substituting, we get the equation of motion for the leading term

$$\frac{A}{\sqrt{s}}\left(1 + \frac{A^2 l^4}{2}\right) = 0, \tag{3.4}$$

so we see that for a real scalar field (as we want), when $A^2 > 0$, there is no solution.

B. Generalization with first derivative interaction

Next, we drop the irrelevant V terms and generalize by writing an arbitrary coefficient for the interaction term,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 + C[(\partial_\mu\phi)^2]^2, \tag{3.5}$$

giving the equation of motion

$$-\Box\phi + 4C(\partial_\mu\phi)^2\Box\phi + 8C(\partial_\mu\phi)(\partial_\nu\phi)(\partial_\mu\partial_\nu\phi) = 0, \tag{3.6}$$

and on $\phi = \phi(s)$, we get

$$\begin{aligned}&4\frac{d}{ds}\left(s\frac{d\phi}{ds}\right) + 64Cs\left(\frac{d\phi}{ds}\right)^2\frac{d}{ds}\left(s\frac{d\phi}{ds}\right) \\ &\quad + 64Cs\left(\frac{d\phi}{ds}\right)^2\left[\frac{d\phi}{ds} + 2s\frac{d^2\phi}{ds^2}\right] = 0.\end{aligned} \tag{3.7}$$

Substituting the ansatz $\phi \simeq A\sqrt{s}$, we get

$$\frac{A}{\sqrt{s}}(1 + 4CA^2) + 16CA^2\left(\frac{A}{2\sqrt{s}} - \frac{A}{2\sqrt{s}}\right) = 0. \tag{3.8}$$

Note that we have kept the last term, which is zero, for reasons to be explained later.

C. Generalization to arbitrary powers

Next we consider on top of the previous interaction, an arbitrary nth order interaction,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 + C_2[(\partial_\mu\phi)^2]^2 + C_n[(\partial_\mu\phi)^2]^n, \tag{3.9}$$

with the equation of motion

$$\begin{aligned}&-\Box\phi + 4C_2(\partial_\mu\phi)^2\Box\phi\left[1 + \frac{nC_n}{2C_2}[(\partial_\mu\phi)^2]^{n-2}\right] \\ &\quad + 8C_2(\partial_\mu\phi)(\partial_\nu\phi)(\partial_\mu\partial_\nu\phi)\left[1 + \frac{n(n-1)C_n}{2C_2}[(\partial_\mu\phi)^2]^{n-2}\right] \\ &\quad = 0,\end{aligned} \tag{3.10}$$

and on the ansatz $\phi = \phi(s) = A\sqrt{s}$, we have

$$\frac{A}{\sqrt{s}} + 4C_2A^2\frac{A}{\sqrt{s}}\left(1 + \frac{nC_n}{2C_2}(-A^2)^{n-2}\right) + 16C_2A^2\left(\frac{A}{2\sqrt{s}} - \frac{A}{2\sqrt{s}}\right)\left(1 + \frac{n(n-1)C_n}{2C_2}(-A^2)^{n-2}\right) = 0. \tag{3.11}$$

We can finally generalize to a sum of arbitrary powers,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 + \sum_{n\geq 2} C_n[(\partial_\mu\phi)^2]^n, \tag{3.12}$$

with the equation of motion

$$-\Box\phi + \Box\phi\sum_{n\geq 2}2nC_n[(\partial_\mu\phi)^2]^{n-1} + (\partial_\mu\phi)(\partial_\nu\phi)(\partial_\mu\partial_\nu\phi) \times\sum_{n\geq 2}4n(n-1)[(\partial_\mu\phi)^2]^{n-1} = 0. \tag{3.13}$$

On the solution $\phi = \phi(s) = A\sqrt{s}$, we get

$$\frac{A}{\sqrt{s}}\left(1 + \sum_{n\geq 2}2nC_n(-1)^nA^{2(n-1)}\right) + \left(\frac{A}{2\sqrt{s}} - \frac{A}{2\sqrt{s}}\right)\sum_{n\geq 2}8n(n-1)C_n(-1)^nA^{2(n-1)} = 0. \tag{3.14}$$

For the DBI action, the coefficients, coming from the expansion of

$$-(1+x)^{1/2} = -1 - \frac{x}{2} - \frac{1/2(1/2-1)\ldots(1/2-n+1)}{1\cdot 2\cdot\ldots\cdot(n)}x^n, \tag{3.15}$$

give therefore $\mathrm{sgn}(C_n) = (-1)^n$, meaning that the coefficients inside the two brackets in Eq. (3.14) are all positive. Moreover, from the arguments in Ref. [22], the signs coming from the DBI action are the ones needed for causality and locality of an action: any other sign for C_n in the expansion of $\mathcal{L} = \sum_n C_n[(\partial\phi)^2]^n$ was argued to lead to violations of causality and locality (though note that the metric convention in Ref. [22] is mostly minus, so all coefficients C_n there are positive, but that corresponds to alternating signs for us). Note also that in principle we could imagine also having $(\partial^m\phi)^n$ terms in the effective action, though in our analysis we restrict to the case where they are absent. We will just mention that Ref. [22] found that having only those types of terms and $C_n = 0$ is problematic, as it leads to causality violation, but there was no argument against having both kinds of terms. It will be interesting to further analyze this possibility.

That means that for a general action with arbitrary C_n, at any finite order in the terms, $\phi = A\sqrt{s}$ is not a solution.

But then the question is, how is it possible that the DBI action has this as a solution? To answer that, we look at Eq. (2.6), which is the equivalent of what we have here. On the ansatz $\phi = A\sqrt{s}$, we get from it

$$\frac{A}{\sqrt{s}} + \frac{2l^4A^2}{1-A^4l^2}\left(\frac{A}{2\sqrt{s}} - \frac{A}{2\sqrt{s}}\right) = 0, \tag{3.16}$$

which at first seems not to have a solution, just like our finite-order truncations, but looking more closely we see that $A^2 = l^{-4}$ is a solution, since then the second term is $0/0$, and there is a solution, as seen by going to the form (2.7). The essential fact is the existence of the factor

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \tag{3.17}$$

for $x = 1$, multiplying the $(A/2\sqrt{s} - A/2\sqrt{s}) = 0$ term, but not the nonzero term. Thus in the case of the finite truncation, we have the ratio of the zero and nonzero terms being

$$\frac{\sum_{n\geq 2}8n(n-1)C_n(-1)^nA^{2(n-1)}}{1 + \sum_{n\geq 2}2nC_n(-1)^nA^{2(n-1)}} \to \infty, \tag{3.18}$$

which goes to infinity for an infinite number of terms, allowing the solution.

In conclusion, the DBI action is the unique one satisfying the physical requirement $\phi(s) \simeq A\sqrt{s}$ near $s = 0$ (it could be that there are other derivative actions, with an infinite number of terms, and the same signs as DBI for the coefficients, but it is unlikely); however we can add a potential inside or outside the square root without modifying the result.

IV. GENERALIZATIONS

We first consider a simple generalization: instead of just a mass term inside the square root, we consider a general potential V, with Lagrangian

$$\mathcal{L} = l^{-4}\left[1 - \sqrt{1 + l^4[(\partial_\mu\phi)^2 + 2V(\phi)]}\right]. \tag{4.1}$$

Its equation of motion is

$$-\Box\phi + \partial_\phi V(\phi) + l^4\frac{[(\partial_\mu\partial_\nu\phi)(\partial_\mu\phi)\partial_\nu\phi + (\partial_\mu\phi)^2\partial_\phi V(\phi)]}{1 + l^4[(\partial_\mu\phi)^2 + 2V(\phi)]} = 0, \tag{4.2}$$

and for a solution $\phi = \phi(s)$, we obtain (after the same manipulations as in the Heisenberg case)

$$4\frac{d}{ds}\left(s\frac{d\phi}{ds}\right)+V'(\phi)=8sl^4\left(\frac{d\phi}{ds}\right)^2\frac{[\frac{d\phi}{ds}+V'(\phi)]}{1+2l^4V(\phi)}. \quad (4.3)$$

It is easy to check that this equation has again the same small s solution (2.9) for $a=0$, i.e. the leading term, since the terms with V in the equation of motion are actually subleading with respect to the others. This in turn leads to the same analysis of Heisenberg, so this generalization is allowed.

We can also consider adding V outside the square root,

$$\mathcal{L}=l^{-4}\left[1-\sqrt{1+l^4[(\partial_\mu\phi)^2]}\right]-V(\phi), \quad (4.4)$$

and we can again check that the same thing happens: the solution $\phi(s)=l^{-2}\sqrt{s}+\cdots$ is still valid, since again the terms with V in the equation of motion are subleading on the solution.

A. Several mesons and sigma model

We can also consider N scalar fields, corresponding to having several scalar mesons, ϕ^i, $i=1,\ldots,N$ and for generality consider it in $d+1$ dimensions. A generalized DBI model would be

$$\mathcal{L}=l^{-(d+1)}\left[h(\phi^i)-f(\phi)\sqrt{1+l^{d+1}[g_{ij}(\phi^k)(\partial_\mu\phi^i)(\partial_\mu\phi^j)+2V(\phi^i)]}\right]. \quad (4.5)$$

Then when $h(\phi^i)=g(\phi^i)=1$, for small fields we keep only the leading term in the expansion of the square root, and obtain the usual sigma model with a potential,

$$\mathcal{L}_2\approx-\frac{1}{2}g_{ij}(\phi^k)(\partial_\mu\phi^i)(\partial_\mu\phi^j)-V(\phi^i). \quad (4.6)$$

The equations of motion of the action (4.5) are

$$\begin{aligned}&l^{-(d+1)}\left[\partial_{\phi^i}h(\phi)-\partial_{\phi^i}f(\phi)\sqrt{1-l^{d+1}[g_{ij}(\phi^k)(\partial_\mu\phi^i)(\partial_\mu\phi^j)-2V(\phi^i)]}\right]\sqrt{1-l^{d+1}[g_{ij}(\phi^k)(\partial_\mu\phi^i)(\partial_\mu\phi^j)-2V(\phi^i)]}\\&+f(\phi)\left[\frac{1}{2}\partial^{\phi^i}g_{jk}(\phi)(\partial_\mu\phi^j)\partial^\mu\phi^k-\partial_{\phi^i}V(\phi)-\partial_\mu[g_{ij}(\phi)\partial_\mu\phi^j]\right]-\partial_\mu f(\phi)g_{ij}(\phi)\partial_\mu\phi^j\\&-\frac{1}{2}l^{(d+1)}\frac{f(\phi)g^{ij}(\phi)\partial_\mu\phi^j}{1-l^{d+1}[g_{ij}(\phi^k)(\partial_\mu\phi^i)(\partial_\mu\phi^j)-2V(\phi^i)]}(\partial_\mu[g_{ij}(\phi)(\partial_\mu\phi^i)(\partial_\mu\phi^j)-2V(\phi^i)])=0.\end{aligned} \quad (4.7)$$

To analyze this case, first note that Heisenberg already considered the case of several mesons with the DBI action, but that meant that the sum was outside the square root,

$$\mathcal{L}=l^{-4}\sum_a\left[1-\sqrt{1+l^4[(\partial_\mu\phi^a)^2+m_a^2\phi_a^2]}\right]. \quad (4.8)$$

That case worked in the same way as for a single meson. We now consider the generalization with the sum inside the square root, and a sigma model metric,

$$\mathcal{L}=l^{-4}\left[1-\sqrt{1+l^4\left[\sum_{ab}G_{ab}(\phi_c)(\partial_\mu\phi^a)(\partial_\mu\phi^b)+\sum_a m_a^2\phi_a^2\right]}\right], \quad (4.9)$$

where we can replace everywhere the mass terms with a general potential, since as we already saw that does not change anything.

Now in terms of the asymptotic value of the cross section (the Froissart behavior), nothing changes, since the maximum cross section is governed by the pion of smallest mass, that has the largest wave function at large distances, according to the mechanism reviewed below. What does change is the value of the cross section at intermediate energies, where now we have cross sections for emissions of various scalar mesons.

The equations of motion coming from the action (4.9) are

$$-\left[4\frac{d}{ds}\left(G_{ab}(\phi)s\frac{d\phi^b}{ds}\right)+m_a^2\phi_a^2\right]\left[1+l^4\left(-4sG_{ef}\frac{d\phi^e}{ds}\frac{d\phi^f}{ds}+m_e^2\phi_e^2\right)\right]$$
$$-l^4m_e^2G_{ab}\left(-4s\frac{d\phi^b}{ds}\frac{d\phi^e}{ds}\right)\phi^e-8sl^4G_{ab}\frac{d\phi^b}{ds}\frac{d}{ds}\left[sG_{ef}\frac{d\phi^e}{ds}\frac{d\phi^f}{ds}\right]=0. \quad (4.10)$$

The simplifications that occurred when $G_{ab}=\delta_{ab}$ do not occur anymore. However, the fields ϕ^a will also have in general the interpretation of some brane coordinates in the gravity dual descriptions of Sec. VIII. In the Heisenberg case, we had a single field, corresponding to a single coordinate transverse to the brane, but in general we can have many. Then the origin of the coordinate, corresponding to the position of the brane, must be a stable point. Around it, we can expand the metric as $G_{ab}=\delta_{ab}+\mathcal{O}(|\phi|)$, and write an ansatz for the fields as

$$\phi^a=A^a\sqrt{s}, \quad (4.11)$$

for $s\to 0$. At $s=0$, the fields are at 0, i.e. the stable point (the "IR brane" or IR cutoff of the gravity dual), and with the metric expanded as above, we can check that the ansatz is a solution of the equations of motion if

$$\sum_a (A^a)^2=1/l^4. \quad (4.12)$$

In order to understand the asymptotic cross sections, we consider the behavior of the wave functions for large transverse r. The large-r behavior of the cross section is governed by the lightest meson, the pion. Indeed, the wave functions go like $\phi^a\propto e^{-m_a r}$ at $r\to\infty$. Therefore, if we are in the asymptotic regime for the pion, $\phi^\pi\propto e^{-m_\pi r}$, which implies also $|\phi^a|\ll|\phi^\pi|\ll l^{-1}$, then from the equations of motion we can check that we also have $\phi^a\propto e^{-m_a r}$, which by the usual Heisenberg argument implies that

$$e^{-m_\pi b_\pi}\sim\frac{\langle k_0\rangle}{\sqrt{s}}\Rightarrow\sigma_{\rm tot}\simeq\sigma_\pi\simeq\frac{\pi}{m_\pi^2}\ln^2\left(\frac{\sqrt{s}}{\langle k_0\rangle}\right),$$
$$e^{-m_a b_a}\sim\frac{\langle k_0\rangle}{\sqrt{s}}\Rightarrow\sigma_a\simeq\frac{\pi}{m_a^2}\ln^2\left(\frac{\sqrt{s}}{\langle k_0\rangle}\right)\ll\sigma_{\rm tot}, \quad (4.13)$$

where σ_a is the cross section for the production of mesons a. Therefore in the asymptotic regime, all the σ_a should behave like Froissart saturation, with corresponding coefficients π/m_a^2.

B. Anti–de Sitter case and curved-space generalizations; solutions

In Sec. VIII we will discuss possible relations between the Heisenberg model and a holographic description of the nucleon-nucleon scattering. We have seen in the previous subsection that a natural generalization of Heisenberg's model includes several scalars corresponding to several mesons, and a sigma model for them. Here we discuss a particular example of such a generalization which is a DBI sigma model in anti–de Sitter (AdS) spacetime. This also arises naturally in the context of gauge/gravity duality.

The DBI action on the flat worldvolume in $d+1$ dimensions, i.e. for a Dd-brane, takes the form

$$S_{\rm DBI}=T_d\int d^{d+1}\sigma e^{-\tilde{\phi}}\sqrt{-\det[\partial_\mu X^i\partial_\nu X^j g_{ij}(X)+2\pi\alpha' F_{\mu\nu}]}, \quad (4.14)$$

where T_d is the Dd-brane tension, $\tilde{\phi}$ is the dilaton, σ_μ are the worldvolume coordinates, X^i $i=1,\ldots D$ are the target-space coordinates and $g_{ij}(X^k)$ is the metric on that target space. The $d+1$-dimensional DBI action describes in particular the physics of Dd-branes. The D-brane action in fact also includes a Chern-Simons (CS) term, but for our purposes, the effect of that will be just to subtract $\int d^{d+1}\sigma T_d$ from the above action.

Imposing $d+1$-dimensional Lorentz invariance, switching off the gauge fields, writing $T_d=l^{-(d+1)}$, using the static gauge $\sigma_\mu=\delta^I_\mu X^I$ for $I=0,1,2,d$, and defining the vector $\vec{\phi}\equiv X^i/l^{(d+1)/2}\equiv iv^i$ with $i=d+1,\ldots,D$, the DBI action reduces to

$$S_{Dd}=l^{-(d+1)}\int d^{d+1}xe^{-\tilde{\phi}}\left[\sqrt{-\det\left(\eta_{\mu\nu}\tilde{g}(\phi)+l^{d+1}\partial_\mu\phi^i\partial_\nu\phi^j g_{ij}(\phi)\right)}-1\right]. \quad (4.15)$$

For the special case of the D3-brane moving in an $AdS_5\times S^5$ space (the space generated by a large number N of other D3-branes), this action is the "highly effective action" for the $\mathcal{N}=4$ Super Yang-Mills (SYM) theory on D3-branes written recently in Ref. [10], which takes the form

$$S_{D3}\sim\int d^4x\phi^4\left[\sqrt{-\det\left(\eta_{\mu\nu}+\frac{\partial_\mu\vec{\phi}\cdot\partial_\nu\vec{\phi}}{\phi^4}\right)}-1\right], \quad (4.16)$$

where now the dilaton is a constant, the target-space coordinates are $\vec{\phi} \equiv v^i$ with $I = 4, \ldots, 9$, and the metric on the target space was taken to be

$$ds^2 = R^2\left[\phi^2 \eta_{IJ} dx^I dx^J + \frac{1}{\phi^2} d\vec{\phi} \cdot d\vec{\phi}\right]. \qquad (4.17)$$

For the special case of a single scalar with $\tilde{\phi} = 0$, $\tilde{g} = 1$, using the identity

$$-\det\left[\eta_{\mu\nu} + g(\phi)\partial_\mu\phi\partial_\nu\phi\right] = 1 + g(\phi)\partial_\mu\phi\partial^\mu\phi, \qquad (4.18)$$

the action for a D-brane [Eq. (4.15)] reduces to the Heisenberg-type action with a $g(\phi)$, or a one-dimensional sigma model action of the type (4.9).

For the case of several scalars, but with the metric trivialized around the position of the brane, i.e. $g_{ij}(\phi) \simeq g(\phi)\delta_{ij}$, we have

$$\begin{aligned}
-\det(\eta_{\mu\nu} + g(\phi)\partial_\mu\phi^i\partial_\nu\phi^i) &= -\frac{1}{d!}\epsilon^{\mu_1\ldots\mu_d}\epsilon^{\nu_1\ldots\nu_d}(\eta_{\mu_1\nu_1} + g(\phi)\partial_{\mu_1}\phi^i\partial_{\nu_1}\phi^i)\ldots \\
(\eta_{\mu_d\nu_d} + g(\phi)\partial_{\mu_d}\phi^i\partial_{\nu_d}\phi^i) &= \frac{1}{d!}[\epsilon_{\mu_1\ldots\mu_d}\epsilon^{\mu_1\ldots\mu_d} + dg(\phi)\epsilon^{\mu}{}_{\nu_2\ldots\nu_d}\epsilon^{\nu\nu_2\ldots\nu_d}\partial_\mu\phi^i\partial_\nu\phi^i + \cdots] \\
&= 1 + g(\phi)\partial_\mu\phi^i\partial^\mu\phi^i + g^2(\phi)(\partial_{\mu_1}\phi^i\partial_{\nu_1}\phi^i)(\partial_{\mu_2}\phi^j\partial_{\nu_2}\phi^j)\delta^{\nu_1\nu_2}_{\mu_1\mu_2} \\
&\quad + g^3(\phi)(\partial_{\mu_1}\phi^{i_1}\partial_{\nu_1}\phi^{i_1})(\partial_{\mu_2}\phi^{i_2}\partial_{\nu_2}\phi^{i_2})(\partial_{\mu_3}\phi^{i_3}\partial_{\nu_3}\phi^{i_3})\delta^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3} + \cdots \qquad (4.19)
\end{aligned}$$

Then *on the solution* $\phi = \phi(s)$, we have

$$\begin{aligned}
2(\partial_{\mu_1}\phi^i\partial_{\nu_1}\phi^i)(\partial_{\mu^2}\phi^j\partial_{\nu_2}\phi^j)\delta^{\nu_1\nu_2}_{\mu_1\mu_2} &= \partial_{\mu_1}\phi^i\partial^{\mu_1}\phi^i\partial_{\mu^2}\phi^j\partial^{\mu_2}\phi^j - \partial_{\mu_1}\phi^i\partial^{\mu_1}\phi^j\partial_{\mu^2}\phi^i\partial^{\mu_2}\phi^j \\
&= 16s^2\frac{d\phi^i}{ds}\frac{d\phi^i}{ds}\frac{d\phi^j}{ds}\frac{d\phi^j}{ds} - 16^2\frac{d\phi^i}{ds}\frac{d\phi^j}{ds}\frac{d\phi^i}{ds}\frac{d\phi^j}{ds} = 0 \qquad (4.20)
\end{aligned}$$

and we can easily see that for the higher terms the same happens. Therefore *on the solution* $\phi = \phi(s)$, the presence of higher-order terms inside the square root in the D-brane DBI action (4.15) is not relevant, and we still have a sigma model action like Eq. (4.9). That means that the Heisenberg analysis is still valid in the case of the general DBI D-brane action.

1. Shock waves for a D-brane in curved space

Consider the action of a D3-brane moving in AdS_5, i.e. the "highly effective action" of Eq. (4.16) for a single scalar ϕ and metric $g(\phi) = \phi^{-4}$,

$$\mathcal{L} = l^{-4}\left[1 - \sqrt{1 + \frac{(\partial_\mu\phi)(\partial_\mu\phi)}{\phi^4}}\right]. \qquad (4.21)$$

Its equation of motion on the ansatz $\phi = \phi(s)$ is

$$s\phi'' + \phi' - 2\frac{s}{\phi}(\phi')^2 - 2\frac{s}{\phi^4}(\phi')^3 = 0, \qquad (4.22)$$

which is a special case of the more general form with an arbitrary metric $g(\phi)$,

$$s\phi'' + \phi' + \frac{1}{2}\frac{g'(\phi)}{g(\phi)}s(\phi')^2 - 2sg(\phi)(\phi')^3 = 0, \qquad (4.23)$$

for $g(\phi) = \frac{1}{\phi^4}$.

It is easy to check that an exact solution for this nonlinear equation is

$$\phi(s) = \frac{1}{\sqrt{s}} = \frac{1}{\sqrt{t^2 - x^2}}. \qquad (4.24)$$

Substituting the solution into the Lagrangian density (4.21) we find that the square root vanishes and $\mathcal{L} = l^{-4}$. [The same holds for the solution of the massless Heisenberg model where $g(\phi) = 1$.]

In fact one can use this property to find solutions for other target-space metrics. For $g(\phi) = l^{4-n}(\phi)^{-n}$, we get

$$\begin{aligned}
\mathcal{L} = l^{-4} \to 4sg(\phi)(\phi')^2 = 1 \to \phi(s) \\
= l^{-1}(l^{-2}s)^{\frac{1}{2-n}}\left[\frac{2-n}{4}\right]^{\frac{1}{2-n}}. \qquad (4.25)
\end{aligned}$$

Furthermore, for a general metric $g(\phi)$ we find that the solution for $\phi(s)$ is

$$\int d\phi\sqrt{g(\phi)} = \sqrt{s}. \qquad (4.26)$$

It is easy to check that this solution indeed solves the equation of motion (4.23).

C. Introducing vector mesons

As we saw in Eq. (4.14), the DBI action on the worldvolume of a Dd-brane has vector fields. In fact, in

AdS/QCD approaches with probe branes, like for instance the Sakai-Sugimoto model or the models of Sec. VIII, these vectors on the gravitational side give rise on the dual field theory side to towers of vector-meson states. We are considering mainly a flat metric for the scalar fields (trivial sigma model), which arises as an approximation in the IR of the gravity dual, as we discussed. Then we must consider the effect of the gravity dual metric (that drives the brane to the stable point around which the metric is flat) to be to give masses to the fields.

Therefore we consider the DBI action with a mass for the vector inside the square root,

$$\mathcal{L} = l^{-4}\left[1 - \sqrt{\det(\eta_{ab} + l^4\partial_a\phi\partial_b\phi + l^2F_{ab}) + m^2\phi^2 + M_V^2A_a^2}\right]$$
$$= l^{-4}\left[1 - \sqrt{1 + l^4[(\partial_\mu\phi)^2 + m^2\phi^2] + \frac{l^4}{2}F_{ab}F^{ab} - l^8\left(\frac{1}{4}\tilde{F}_{ab}F^{ab}\right)^2 + M_V^2A_a^2 + \cdots}\right] \quad (4.27)$$

where $\tilde{F}_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$.

For the vector wave functions $A_a(r)$, like for the pion field $\phi(r)$, we need to give some initial data (boundary condition), and then the wave function is determined from the equation of motion of the above action.

Let us consider first the case with no pions, just vector mesons, i.e. $\phi = 0$. At sufficiently large r we have again the usual free field decay

$$A_a(r) = A_a e^{-M_V r}, \quad (4.28)$$

and again, with the *additional* assumption that σ_V, the cross section for the emission of V vector mesons, is obtained when the emitted vector-meson energy (which we should calculate) equals the average per vector-meson emitted energy, i.e.

$$\frac{\langle k_0\rangle}{\sqrt{s}} = e^{-M_V b_{\max}}, \quad (4.29)$$

we obtain

$$\sigma_V = \pi b_{\max}^2. \quad (4.30)$$

Of course, the correct calculation would be the one where we have *both* the pions and the vector-meson wave functions, and then we can calculate σ_V as above.

The action for only vector mesons with mass M_V and no pions is

$$\mathcal{L} = l^{-4}\left[1 - \sqrt{1 + \frac{l^4}{2}F_{ab}F^{ab} - l^8\left(\frac{\tilde{F}_{ab}F^{ab}}{4}\right)^2 + l^4M_V^2A_a^2}\right]. \quad (4.31)$$

We would like to restrict again the dependence of the gauge fields to a dependence on (t,x) and furthermore to only an s dependence, but now with all four vector fields A_a, with $a = 0,1,2,3$, since there is no gauge invariance due to the fact that the vector mesons are massive ones.

Substituting $A_a = A_a(s)$ in F_{ab}, we find

$$F_{ab}F^{ab} = 2[-(F_{01})^2 - (F_{02})^2 - (F_{03})^2 + F_{12}^2 + F_{13}^2]$$
$$= -8\left[s\left(\frac{dA_2}{ds}\right)^2 + s\left(\frac{dA_3}{ds}\right)^2 + \left(t\frac{dA_1}{ds} + x\frac{dA_0}{ds}\right)^2\right],$$
$$\tilde{F}_{ab}F^{ab} = \frac{1}{2}\epsilon^{abcd}F_{ab}F_{cd}$$
$$= \frac{1}{2}\epsilon^{0123}(8F_{01}F_{23} - 8F_{02}F_{13} + 8F_{03}F_{12})$$
$$= 8\left(-8t\frac{dA_2}{ds}x\frac{dA_3}{ds} + t\frac{dA_3}{ds}x\frac{dA_2}{ds}\right) = 0, \quad (4.32)$$

so that the action for the ansatz $A_a(s)$ is

$$\mathcal{L} = l^{-4}\left[1 - \left(1 - 4l^4\left[s\left(\frac{dA_2}{ds}\right)^2 + s\left(\frac{dA_3}{ds}\right)^2 + \left(t\frac{dA_1}{ds} + x\frac{dA_0}{ds}\right)^2\right] + l^4M_V^2(A_0^2 + A_1^2 + A_2^2 + A_3^2)\right)^{1/2}\right]. \quad (4.33)$$

Note that in this action we can consistently truncate $A_0 = A_1 = 0$, and then the DBI action for A_2 and A_3 are the same as for two DBI pions of Heisenberg, for which we already saw that we need the full nonlinear DBI action.

V. THE PION WAVE FUNCTION

The pion wave function should be a solution of the equations of motion coming from the pion action. Following Heisenberg, we have considered only the $1+1$-dimensional case of $\phi(s)$ that describes the physics near the shock, at $s \sim 0$, and the weak-field case $\phi(r)$, spherically symmetric in the transverse coordinates, so a function of only $r = \sqrt{y^2 + z^2}$.

Note that in general, we do not even need to have an ansatz depending on both r and s, i.e. $\phi(s,r)$, but rather one depending independently on all four coordinates; however considering $\phi(s,r)$ is a simple way to start the analysis.

A. Possible generalizations to $\phi(r)$ and $\phi(s,r)$

1. Static spherically symmetric solutions

Consider first spherically symmetric solutions depending on all three coordinates, i.e. on $r=\sqrt{x^2+y^2+z^2}$. Moreover, we generalize to n space dimensions. The Lagrangian (2.1) becomes

$$\mathcal{L}=l^{-4}r^{n-1}\left[1-\sqrt{1+l^4(\phi'^2+m^2\phi^2)}\right], \tag{5.1}$$

and its equation of motion is

$$\left(\phi''+\frac{n-1}{r}\phi'-m^2\phi\right)[1+l^4(\phi'^2+m^2\phi^2)] - l^4\phi'^2(\phi''+m^2\phi)=0 \tag{5.2}$$

where $'$ denotes differentiation with respect to r. After simplifications, it is rewritten as

$$\phi''+\frac{n-1}{r}\phi'-m^2\phi=l^4\frac{\phi'^2}{1+l^4m^2\phi}\left(2m^2\phi-\frac{n-1}{r}\phi'\right). \tag{5.3}$$

a. One-dimensional solution.—In $n=1$ space dimension, the ansatz

$$\phi(r)=\frac{A}{1+\beta r} \tag{5.4}$$

is an approximate solution. Indeed upon substituting this ansatz into the equation of motion, we obtain

$$2\beta^2-m^2(1+\beta r)^2=\frac{2\beta^2}{1+\frac{(1+\beta r)^2}{l^4m^2A^2}}, \tag{5.5}$$

after simplifying by a common factor $A/(1+\beta r)^3$. The ansatz satisfies the equation of motion, if $\beta\gg m$ and

$$\frac{2\beta^2}{l^4m^2A^2}=m^2, \tag{5.6}$$

and then it is valid even in the $\beta r\sim\mathcal{O}(1)$ regime, since then we can approximate

$$\frac{2\beta^2}{1+\frac{(1+\beta r)^2}{l^4m^2A^2}}\simeq 2\beta^2\left(1-\frac{(1+\beta r)^2}{l^4m^2A^2}\right)\simeq 2\beta^2-m^2(1+\beta r)^2. \tag{5.7}$$

In conclusion, the solution is

$$\phi(r)\simeq\frac{A}{1+\frac{Ar}{\sqrt{2}m^2l^2}}, \tag{5.8}$$

and as we can see, it is parametrized by A, and is valid for $A\gg m^3l^2$.

However, this solution is only valid in $n=1$ space dimension.

At very large distances, the wave function in any dimension becomes

$$\phi(r)=Be^{-mr}, \tag{5.9}$$

which is what Heisenberg considered as well.

2. Solutions of the form $\phi(s,r)$

A possible generalization that would include both the $\phi(s)$ near $s=0$ and the $\phi(r)$ near $r\to\infty$ is $\phi(s,r)$. Substituting this ansatz into the DBI action, we obtain first

$$(\partial_\mu\phi)^2=-4s\left(\frac{d\phi}{ds}\right)^2+\phi'^2, \tag{5.10}$$

and then for the Lagrangian

$$\mathcal{L}=l^{-4}r^{n-1}\left[1-\sqrt{1+l^4\left(\phi'^2-4s\left(\frac{d\phi}{ds}\right)^2+m^2\phi^2\right)}\right], \tag{5.11}$$

where n is now the number of transverse space dimensions ($n=2$ in the physical case). As before, we find the equation of motion

$$\begin{aligned}&\left(\phi''+\frac{n-1}{r}\phi'-m^2\phi-4\frac{d}{ds}\left[s\frac{d\phi}{ds}\right]\right)[1+l^4m^2\phi^2]\\&+8l^4s\left(\frac{d\phi}{ds}\right)^2\left[\frac{d\phi}{ds}+m^2\phi\right]-2m^2l^4\phi'^2\phi\\&+\frac{n-1}{r}l^4\phi'\left[\phi'^2-4\left(s\frac{d\phi}{ds}\right)^2\right]\\&-4sl^4\left(\frac{d\phi}{ds}\right)^2\phi''-4l^4\phi'^2\frac{d}{ds}\left[s\frac{d\phi}{ds}\right]+8l^4s\frac{d\phi}{ds}\phi'\frac{d\phi'}{ds}=0,\end{aligned} \tag{5.12}$$

where the third line contains terms with mixed derivatives.

We can check that again at $s\simeq0$, $\phi=A\sqrt{s}$ is a solution, but $\phi=A\sqrt{s}f(r)$ is *not* a solution at nonzero r, since the leading terms in the equations of motion for such an ansatz are

$$\begin{aligned}0&\simeq-4\frac{d}{ds}\left[s\frac{d\phi}{ds}\right]+8l^4s\left(\frac{d\phi}{ds}\right)^3\\&=-\frac{A}{\sqrt{s}}f(r)+\frac{A}{\sqrt{s}}A^2l^4f^3(r),\end{aligned} \tag{5.13}$$

and this equation has as its only solution $f(r) = \pm 1$. Therefore the solution at nonzero r and $s \simeq 0$ must be of the type

$$\phi \simeq A\sqrt{s} + s^n f(r), \tag{5.14}$$

where $n \geq 1$.

B. Delta-function shock wave?

Before we continue with $\phi(s, r)$, we want to address the issue of a possible delta function shockwave. In the gravity dual theory, the gravitational shock waves that scatter are delta-function shock waves [5,14], so one can ask whether the same happens also in the field-theory picture.

We want then to try a delta-function ansatz for a $\phi = \phi(x^-, r)$, where $x^- = (x - t)/\sqrt{2}$,

$$\phi(x^-, r) = \delta(x^-)\Phi(r). \tag{5.15}$$

The equations of motion for $\phi(x^-, r)$ in $n = 2$ transverse dimensions are

$$\phi'' + \frac{1}{r}\phi' - m^2\phi = l^4 \frac{\phi'^2}{1 + l^4 m^2 \phi}\left(2m^2\phi - \frac{1}{r}\phi'\right). \tag{5.16}$$

Then for the delta-function ansatz, in the denominator on the right-hand side of Eq. (5.16) the 1 is negligible with respect to the ϕ^2 term (since the whole term is proportional to a delta function, so the denominator is relevant only on the delta function, when the value is infinite), and the equation becomes (after simplifying the common delta function on both sides)

$$\Phi'' + \frac{1}{r}\Phi' - m^2\Phi = 2\frac{\Phi'^2}{\Phi}\left(1 - \frac{1}{2m^2 r}\frac{\Phi'}{\Phi}\right). \tag{5.17}$$

But we can easily verify that this equation has no solutions of the form Ae^{-mr} at large distances, nor of Ar^{-p} type, and if we use $Ae^{-\alpha\rho^p}$ we find that the only solution is $p = 2$, $\alpha = -m^2/2$, i.e. $Ae^{m^2 r^2/2}$, which is clearly nonphysical.

We do have in fact the solution Ae^{imr}, but it is a complex solution for a real scalar, and $A\cos(mr)$ is not a solution (the equation is nonlinear, so we do not have a superposition principle). So the conclusion seems to be that this case $\phi = \delta(x^-)\Phi(r)$ is unphysical.

In fact, there are some ways around that. We can consider a case when the ϕ resembles a delta function, but it has a finite thickness, and the height of the delta function is not only finite, but such that the 1 in the denominator of Eq. (5.16) actually dominates, so we get the equation of motion

$$\Phi'' + \frac{1}{r}\Phi' - m^2\Phi = l^4\Phi'^2\left(2m\Phi - \frac{1}{r}\Phi'\right). \tag{5.18}$$

Another possibility is to add by hand a source $\phi\delta(x^-)f(r)$ to the action, leading to the modified equation of motion

$$\begin{aligned}\Phi'' &+ \frac{1}{r}\Phi' - m^2\Phi \\ &= 2\frac{\Phi'^2}{\Phi}\left(1 - \frac{1}{2m^2 r}\frac{\Phi'}{\Phi}\right) + l^2 m\Phi\left(1 + \frac{\Phi'^2}{m^2\Phi^2}\right)^{3/2} f(r),\end{aligned} \tag{5.19}$$

but we are still left with the issue of understanding the source-free shock waves like Heisenberg's.

Instead, we can notice that we do not really need a delta-function shockwave in x^+; we only need T_{++} to become infinite at $x^+ = 0$. Normally that happens because of a $\delta(x^+)$ in T_{++} which implies also a $\delta(x^+)$ in the field (in the case of the gravity dual, a delta function in the metric). But in the case of the solution of Heisenberg, we just have an energy density that blows up slowly near $x^+ = 0$.

Indeed, near $s = 0$, we have [see Eq. (2.11)]

$$\mathcal{H} \simeq \frac{\phi'^2}{m\sqrt{s}} \sim \frac{l^{-4}x^2}{ms^{3/2}}, \tag{5.20}$$

which blows up at $s = 0$. Moreover, we can calculate T_{++}, which turns the x^2 in the numerator into s, implying $T_{++} \simeq l^{-4}/m\sqrt{s} \to 0$. That means that there is a source at $s = 0$, since T_{++} becomes infinite there, i.e. at $x^+ = 0$ and $x^- = 0$ (two plane waves, travelling in opposite directions). In the next section we will study the source of the pion field in more detail.

C. Perturbative solution near $r = 0$

We now return to $\phi(s, r)$ and consider the expansion near $r = 0$ of $\phi(s, r)$. We have found the equation of motion (5.12), and the ansatz (5.14). We first plug this ansatz into the equation of motion for $n = 1$, but we find that while at zeroth order we get zero for $A = l^{-2}$, at first order we do not have a cancellation.

It means that we need to consider the next order in $\sqrt{s}$, namely $n = 3/2$. Then we can check that the relevant terms are only

$$\begin{aligned}-4\frac{d}{ds}\left[s\frac{d}{ds}\right] &= -\frac{A}{\sqrt{s}} - 9f\sqrt{s}, \\ +8l^4 s\left(\frac{d\phi}{ds}\right)^3 &= \frac{A^3 l^4}{\sqrt{s}} + 9A^2 l^4 f\sqrt{s} + \cdots,\end{aligned} \tag{5.21}$$

but now we see that with $A = l^{-2}$ we cancel both zeroth-order and first-order terms. Moreover, now we have other terms in the equation of motion that contribute, in particular

$$-m^2\phi - 4\frac{d}{ds}\left[s\frac{d}{ds}\right]l^4m^2\phi^2 + 8l^4sm^2\phi\left(\frac{d\phi}{ds}\right)^2 = -m^2A\sqrt{s} - A^3l^4m^2\sqrt{s} + 2A^3l^4m^2\sqrt{s} = 0, \quad (5.22)$$

but these also cancel!

That means that we need to consider also the second subleading term in the expansion in s of $\phi(s,r)$,

$$\phi = A\sqrt{s} + s^{3/2}f(r) + s^{5/2}g(r), \quad (5.23)$$

and check the terms of order $s^{3/2}$ in the equation of motion as well. Incidentally, we can check that considering a power s^α smaller than $5/2$ does not work either, since then again only the two terms above contribute to the second subleading order, the first giving $-4\alpha^2 g(r)s^{\alpha-1}$, and the second giving $+6\alpha g(r)s^{\alpha-1}$, so they only cancel for $\alpha = 3/2$, which is excluded (it is the first subleading term).

Then we obtain

$$-4\frac{d}{ds}\left[s\frac{d}{ds}\right] = -\frac{A}{\sqrt{s}} - 9f\sqrt{s} - 25gs^{3/2},$$
$$+8l^4s\left(\frac{d\phi}{ds}\right)^3 = \frac{A^3l^4}{\sqrt{s}} + 9A^2l^4f\sqrt{s} + 27Al^4f^2s^{3/2} + 15A^2l^4gs^{3/2} + \cdots \quad (5.24)$$

The other terms in the first line of Eq. (5.12) give to order $s^{3/2}$

$$s^{3/2}\left[f'' + \frac{n-1}{r}f' - m^2f - m^4A + 47m^2f\right], \quad (5.25)$$

the terms in the second line do not contribute to this order, and the terms in the third line give $-A^2l^4f''s^{3/2}$. Summing up all the contributions, and using the zeroth-order condition $A = l^{-2}$, we obtain

$$s^{3/2}\left[27l^2f^2 - 10g + \frac{n-1}{r}f' + 46m^2f - m^4l^{-2}\right], \quad (5.26)$$

and equating this to zero fixes $g(r)$ to be

$$g(r) = \frac{1}{10}\left[27l^2f^2(r) + \frac{n-1}{r}f'(r) + 46m^2f(r) - m^4l^{-2}\right]. \quad (5.27)$$

The interpretation is that we can specify arbitrarily the function $f(r)$, or in another way specify the function

$$\left[s^{-3/2}\frac{d\phi}{dr}\right]\Big|_{s=0}, \quad (5.28)$$

which is an initial datum on the Cauchy surface $s = 0$. Once this is given, the rest of the function ϕ should be fixed by the equation of motion.

D. Perturbative solution near $r = \infty$

However, we are interested instead in the behavior at large, but finite r, needed for the calculation of $\sigma_{\rm tot}(s)$ through $b_{\rm max}$.

We know that at small field and derivatives, the DBI action reduces to the free massive scalar action. Indeed, viewed as an expansion in l^4, or in nonlinearities of the field, the equations of motion reduce at zeroth order to the free equation

$$\phi'' + \frac{n-1}{r}\phi' - m^2\phi - 4\frac{d}{ds}\left[s\frac{d\phi}{ds}\right] = 0, \quad (5.29)$$

and so, under the assumption that the s dependence is subleading, and we can ignore the last term involving only d/ds, we obtain at $mr \gg 1$ the solution

$$\phi \simeq Ae^{-mr}, \quad (5.30)$$

as expected. Note that the ϕ' term in the equations of motion is subleading in mr and it does not contribute to this order.

But we can be more precise, since the exact solution to the free equation of motion is known. If we had $n = 3$, the exact solution would be the Yukawa potential,

$$\phi(r) = \frac{Ae^{-mr}}{r}. \quad (5.31)$$

For $n = 2$, the exact solution is a bit more complicated. We can write the equation of motion at nonzero mass m as

$$\frac{d^2\phi}{d(imr)^2} + \frac{1}{imr}\frac{d}{d(imr)}\phi + \phi = 0, \quad (5.32)$$

which matches the defining differential equations of the Bessel functions at index $\nu = 0$,

$$\frac{d^2Z_0}{dz^2} + \frac{1}{z}\frac{dZ_0}{dz} + Z_0 = 0, \quad (5.33)$$

and therefore we have

$$\phi = Z_0(imr). \quad (5.34)$$

We want to choose the Bessel function of imaginary argument that decays exponentially at infinity. This is

$$K_0(mr) = \frac{\pi i}{2}H_0^{(1)}(imr), \quad (5.35)$$

giving for the scalar

$$\phi(r) = AK_0(mr). \tag{5.36}$$

The asymptotics at $mr \to \infty$ give

$$\phi(r) \simeq A\sqrt{\frac{\pi}{2mr}}e^{-mr}, \tag{5.37}$$

but we should also note the asymptotics at $mr \to 0$, where

$$K_0(z) \simeq -\ln\frac{z}{2}I_0(z) \simeq -\ln\frac{z}{2}. \tag{5.38}$$

To find corrections to this free solution, we could think of expanding the equations of motion in l^4, or equivalently the mass dimension of the remaining expression (once l^4 is removed), but besides the free terms above, all the other terms are linear in l^4, that is, of mass dimension seven with respect to the rest.

We can instead take the ansatz that ϕ depends only on r, and not on s, with a coefficient that is of the order of l^4, i.e.

$$\phi = AK_0(mr) + Bg(r), \tag{5.39}$$

and $B \propto l^4$. Then the full equation of motion reduces to

$$B\left(g''(r) + \frac{1}{r}g'(r) - m^2 g(r)\right) = l^4\phi'^2\left[2m^2\phi - \frac{1}{r}\phi'\right]. \tag{5.40}$$

With the assumption that $B \propto l^4$ we can consider on the right-hand side only the order-zero term with A, to obtain

$$\begin{aligned} &B\left(g''(r) + \frac{1}{r}g'(r) - m^2 g(r)\right) \\ &\quad = (ml)^4 A^3\left(\frac{d}{d(mr)}K_0(mr)\right)^2 \\ &\qquad \times\left[2K_0(mr) - \frac{1}{mr}\frac{d}{d(mr)}K_0(mr)\right]. \end{aligned} \tag{5.41}$$

However, even for that, we can only find the leading-order solution at $mr \to \infty$. Then $g(r) \simeq e^{-3mr}/(mr)^{3/2}$ solves the equation to leading order, and we find the solution

$$\phi \simeq AK_0(mr) + \left(\frac{\pi}{2}\right)^{3/2}\frac{m^2 l^4 A^3}{4}\frac{e^{-3mr}}{(mr)^{3/2}} \tag{5.42}$$

at $mr \to \infty$.

VI. THE SOURCE FOR THE PION FIELD

In this section we would like to understand what is the source of the pion field, which is supposed to represent the nucleons. To do so, we first look at the original Born-Infeld action for nonlinear electrodynamics, in terms of a field strength $F_{\mu\nu}$, and apply the lessons learned to our DBI case, first for a static solution, then for the shock wave.

A. The vector Born-Infeld case

In the original paper of Born and Infeld on nonlinear electrodynamics [23], the issue of the source for solutions of the nonlinear Maxwell field was explored, and in fact it was the crucial motivation for the work: to obtain a smooth "electron" solution to the equations of motion, free of singularities.

The BI Lagrangian can be written as

$$\mathcal{L} = \sqrt{1 + F - G^2} - 1, \tag{6.1}$$

where we defined

$$F \equiv \frac{1}{b^2}(\vec{B}^2 - \vec{E}^2); \qquad G \equiv \frac{1}{b^2}(\vec{B}\cdot\vec{E}). \tag{6.2}$$

Using these definitions we can define quantities analogous to the quantities defined for electromagnetism in a medium, namely

$$\begin{aligned} \vec{H} &\equiv b^2\frac{\partial\mathcal{L}}{\partial\vec{B}} = \frac{\vec{B} - G\vec{E}}{\sqrt{1 + F - G^2}}, \\ \vec{D} &\equiv b^2\frac{\partial\mathcal{L}}{\partial\vec{E}} = \frac{\vec{E} - G\vec{B}}{\sqrt{1 + F - G^2}}. \end{aligned} \tag{6.3}$$

The equations of motion and Bianchi identities of the BI Lagrangian, that correspond to Maxwell's equations of the linear theory, are

$$\begin{aligned} \vec{\nabla}\times\vec{E} + \partial_0\vec{B} = 0; &\qquad \vec{\nabla}\cdot\vec{B} = 0, \\ \vec{\nabla}\times\vec{H} - \partial_0\vec{D} = 0; &\qquad \vec{\nabla}\cdot\vec{D} = 0. \end{aligned} \tag{6.4}$$

The Hamiltonian density can be written as

$$\mathcal{H} = \sqrt{1 + P - Q^2} - 1, \tag{6.5}$$

where

$$P = \frac{1}{b^2}(\vec{D}^2 - \vec{H}^2); \qquad Q = \frac{1}{b^2}(\vec{D}\cdot\vec{H}). \tag{6.6}$$

The inverse relations for the fields are obtained from the Hamiltonian as

$$\begin{aligned} \vec{B} &= b^2\frac{\partial\mathcal{H}}{\partial\vec{H}} = \frac{\vec{H} + Q\vec{D}}{\sqrt{1 + P - Q^2}}, \\ \vec{E} &= b^2\frac{\partial\mathcal{H}}{\partial\vec{D}} = \frac{\vec{D} + Q\vec{H}}{\sqrt{1 + P - Q^2}}. \end{aligned} \tag{6.7}$$

At zero magnetic field, $\vec{B} = \vec{H} = 0$, the equations of motion and Bianchi identities reduce to

$$\vec{\nabla}\times\vec{E}=0;\qquad \vec{\nabla}\cdot\vec{D}=0, \tag{6.8}$$

and we also find $Q=0$, $P=\vec{D}^2/b^2$, $G=0$, $F=-\vec{E}^2/b^2$. Then $\vec{\nabla}\cdot\vec{D}=0$ reduces to

$$\frac{d}{dr}(r^2 D_r)=0, \tag{6.9}$$

which admits a nontrivial solution of the form

$$D_e=\frac{e}{r^2}, \tag{6.10}$$

the same as in Maxwell theory. More precisely, the solution is at $r\neq 0$, which means that we have actually

$$\vec{\nabla}\cdot\vec{D}=4\pi e\delta^3(r), \tag{6.11}$$

which gives the integral formula (from Gauss's law)

$$4\pi e=\int_{\Sigma_r}D_r\sigma=\int_{\Sigma_r}d\vec{S}\cdot\vec{D}. \tag{6.12}$$

So from the point of view of $\vec{D}$ (the field in the medium in electromagnetism, where $\vec{D}=\epsilon_0\vec{E}+\vec{P}$ and $\vec{\nabla}\cdot\vec{D}=0$ in the absence of external sources, and otherwise just includes the charges external to the medium) the sources are point-like as in Maxwell theory.

For a static system we have $\vec{E}=-\vec{\nabla}A_0$, where A_0 is the zero component of the gauge field vector potential, and it is related to $\vec{D}$ by

$$\vec{D}=\frac{\vec{E}}{\sqrt{1-\frac{\vec{E}^2}{b^2}}}\Rightarrow\frac{e}{r^2}=D_r=\frac{E_r}{\sqrt{1-\frac{E_r^2}{b^2}}}=\frac{-A_0'(r)}{\sqrt{1-\frac{A_0'^2}{b^2}}}, \tag{6.13}$$

which implies for the electric field E

$$-E_r=A_0'=\pm\frac{e/r_0^2}{\sqrt{1+r^4/r_0^4}}, \tag{6.14}$$

where

$$r_0=\sqrt{\frac{e}{b}} \tag{6.15}$$

is a radius related to the radius of the electron. We then also obtain the electric potential

$$A_0(r)=\frac{e}{r_0}f\left(\frac{r}{r_0}\right), \tag{6.16}$$

where

$$f(x)=\int_x^\infty\frac{dy}{\sqrt{1+y^4}}, \tag{6.17}$$

and we obtain that $f(0)\simeq 1.8541$ and $A_0\simeq 1.8541e/r_0$.

The finite maximum of the electric field $\vec{E}$ is obtained at $r=0$, and equals $e/r_0^2=b$, as expected, since $\mathcal{L}=\sqrt{1-\vec{E}^2/b^2}$.

As in electromagnetism, we can regard the total charge Q in the material as either $\int dV\vec{\nabla}\cdot\vec{D}$ or $\int dV\vec{\nabla}\cdot\vec{E}$ with the difference that the former expression counts only the outside charge introduced, whereas the latter expression counts all the charge, including the polarization response of the material, which tends to spread out the charge density.

Also now, we can define $4\pi\rho=\vec{\nabla}\cdot\vec{E}$ and find after an easy calculation that

$$\rho=\frac{e}{2\pi r_0^3}\frac{1}{(r/r_0)(1+(r/r_0)^4)^{3/2}} \tag{6.18}$$

and we see that this charge density is spread out, going as $1/r^7$ at $r\to\infty$, but only as $1/r$ at $r\to 0$. We can also verify the fact that its integral gives the same result as the integral of D_r, namely e.

B. The scalar DBI action and its source

1. Static scalar DBI results

A similar thing happens for the scalar DBI action. We start by reviewing the construction of the *static* scalar solutions paralleling the nonlinear electrodynamics solutions, as presented in Ref. [24].

On static solutions, $\partial_t\phi=0$, the scalar DBI action reduces to

$$\mathcal{L}=\sqrt{1+\vec{F}^2}, \tag{6.19}$$

where

$$\vec{F}\equiv\vec{\nabla}\phi, \tag{6.20}$$

where ϕ is the DBI scalar. Note then that this action is the same as the vector BI action above for the case $\vec{B}=0$, just with a different sign inside the square root. Therefore we can follow the same analysis, and first define

$$\vec{C}=\frac{\partial\mathcal{L}}{\partial\vec{F}}=\frac{\vec{F}}{\sqrt{1+\vec{F}^2}}. \tag{6.21}$$

In terms of it, the equation of motion is

$$\vec{\nabla}\cdot\vec{C}=0, \tag{6.22}$$

which is solved by

$$C_r=\frac{e}{r^2}, \tag{6.23}$$

so that really we have $\vec{\nabla}\cdot\vec{C}=4\pi e$ in three spatial dimensions. Therefore the solution for the scalar is given by

$$F_r = \partial_r\phi = \frac{e/r^2}{\sqrt{1-e^2/r^4}}, \tag{6.24}$$

which is called the "catenoid." The solution has a horizon-like structure at $r=\sqrt{e}\equiv r_0$, due to the fact that it has the interpretation (in the case it is the action of a D-brane) of one half of a D-brane–anti-D-brane solution connected by a throat.

2. The DBI scalar shock wave

In the spirit of the model of Ref. [8] we now consider a four-dimensional scalar field $\phi(r,s)$. In particular $\phi(s)$ can be recast from a $1+1$-dimensional action which for the massless case reads

$$\mathcal{L} = l^{-4}\left[1-\sqrt{1-4l^4 s\left(\frac{d\phi}{ds}\right)^2}\right]. \tag{6.25}$$

We define first the analog of the electric field from the Born-Infeld paper,

$$E_s = 2\sqrt{s}\frac{d\phi}{ds}. \tag{6.26}$$

In terms of it, the Lagrangian becomes

$$\mathcal{L} = l^{-4}\left[1-\sqrt{1-E_s^2}\right], \tag{6.27}$$

just like the BI vector case. Then we also define the analog of the electric induction,

$$D_s = \frac{\partial\mathcal{L}}{\partial E_s}, \tag{6.28}$$

which gives

$$D_s = \frac{E_s}{\sqrt{1-l^4E_s^2}}. \tag{6.29}$$

The equation of motion (the analog of Maxwell's equation) is

$$\frac{d}{ds}(\sqrt{s}D_s) = 0, \tag{6.30}$$

which is solved by

$$D_s = \frac{A}{\sqrt{s}}, \quad s>0. \tag{6.31}$$

Causality then requires that we have $D_s=0$ for $s<0$. Then by inverting $D_s(E_s)$ we get

$$E_s = \frac{D_s}{\sqrt{1+l^4D_s^2}} = \frac{A/\sqrt{s}}{\sqrt{1+l^4A^2/s}}. \tag{6.32}$$

But since $D_s=0$ for $s<0$, we also have $E_s=0$ for $s<0$, which means that really,

$$D_s = \frac{A}{\sqrt{s}}\theta(s); \qquad E_s = \frac{A\theta(s)}{\sqrt{s+l^4A^2}}. \tag{6.33}$$

We can also integrate the above to find that ϕ is given by

$$\phi = \int ds\frac{E_s}{2\sqrt{s}} = \frac{A}{2}\int ds\frac{1}{\sqrt{s(s+l^4A^2)}} = A\log\left[\frac{\sqrt{s}+\sqrt{s+l^4A^2}}{l^2A}\right], \tag{6.34}$$

at $s>0$ and 0 at $s<0$ which has the same structure as Eq. (2.8).

This reduces at small s to

$$\phi(s)\simeq l^{-2}\sqrt{s}\theta(s), \tag{6.35}$$

which is the same solution as Heisenberg's. Note that the constant A determining D_s is arbitrary, even though $\phi(s)$ near $s=0$ is completely determined.

Then the electric field is a step function,

$$E_s = 2\sqrt{s}\frac{d\phi}{ds}\simeq l^{-2}\theta(s), \tag{6.36}$$

and the electric induction is

$$D_s = \frac{A}{\sqrt{s}}\theta(s). \tag{6.37}$$

Plugging this back into the equation of motion for D_s, for the analog of $\rho_{\rm ext}=\vec{\nabla}\cdot\vec{D}$ we really have,

$$\frac{d}{ds}(\sqrt{s}D_s) = \frac{d}{ds}(A\theta(s)) = A\delta(s). \tag{6.38}$$

So as in the BI case, there is a source term, which is a delta function when viewed from the point of view of the induction D_s (i.e., it is an "external source" to the medium). The value of the charge, A, is arbitrary, even though $\phi(s)$ near $s=0$ is completely determined.

We can also define the equivalent of the $\vec{\nabla}\cdot\vec{E}=\rho$, the total charge (including the one due to the "polarization of the medium"), which is spread out. We define the density

$$\rho = \frac{d}{ds}(\sqrt{s}E_s) = A\frac{d}{ds}\left(\frac{\sqrt{s}\theta(s)}{\sqrt{s+l^4A^2}}\right) = \frac{l^4A^2}{2\sqrt{s}(s+Al^4A^2)^{3/2}}\theta(s). \tag{6.39}$$

Note that we dropped a term coming from the derivative of $\theta(s)$, proportional to $\sqrt{s}\delta(s)$, since this is zero. We see that this charge drops at infinity as $1/s^2$, and at 0 only as $1/\sqrt{s}$, and integrates to the same total value as the one defined via D_s,

$$A\frac{\sqrt{s}\theta(s)}{\sqrt{s+l^4A^2}}\Big|_0^\infty = A. \qquad (6.40)$$

In conclusion, there is an "external source" located at $s=0$ (the shock's position), with an arbitrary charge, but the "in-medium" source is spread out, over an s of the order of l^4A^2.

VII. THE CROSS SECTION

We can now finally consider the calculation of cross sections arising from the Heisenberg model.

A. Corrections away from the Froissart limit

The first issue to address is of a systematic expansion away from the limit of Froissart bound saturation. It is clear that by considering a $\phi(r)$ that is not yet completely dominated by the e^{-mr} term, we can find corrections to the Froissart behavior of the cross section. If we have an exact wave function, we can obtain a $\sigma_{\rm tot}(\tilde{s})$ that would be different in the leading behavior, like a power law $\sigma_{\rm tot}(\tilde{s})\propto\tilde{s}^\alpha$, appearing before the onset of Froissart saturation.

1. Corrections to leading behavior

We first consider corrections to the e^{-mr} behavior of $\phi(r)$, which were found in Eq. (5.42), with the free part being asymptotically (5.38). The e^{-3mr} behavior is subleading with respect to the $1/\sqrt{r}$ in the first factor, so we consider

$$\phi(r)\propto\frac{e^{-mr}}{\sqrt{mr}}. \qquad (7.1)$$

Then as usual, the emitted energy is proportional to $\phi(b)\sqrt{\tilde{s}}$, and when it gets down to $\langle k_{0,\pi}\rangle$ (the average per pion emitted energy), we reach $b_{\rm max}$. Thus

$$\sqrt{\tilde{s}}\frac{e^{-m_\pi b_{\rm max}}}{\sqrt{b_{\rm max}m_\pi}}\simeq\langle k_{0,\pi}\rangle, \qquad (7.2)$$

giving

$$b_{\rm max}\simeq\frac{1}{m_\pi}\ln\frac{\sqrt{\tilde{s}}}{\langle E_\pi\rangle\sqrt{\ln(\sqrt{\tilde{s}}/\langle k_{0,\pi}\rangle)}}$$
$$\simeq\frac{1}{m_\pi}\ln\frac{\sqrt{\tilde{s}}}{\langle k_{0,\pi}\rangle}-\frac{1}{2m_\pi}\ln\left[\ln\frac{\sqrt{\tilde{s}}}{\langle k_{0,\pi}\rangle}\right], \qquad (7.3)$$

and $\sigma_{\rm tot}(\tilde{s})=\pi b_{\rm max}(\tilde{s})^2$.

Here we should make a comment on the gravity dual calculation. In Ref. [14], the leading behavior for $\sigma_{\rm tot}=\pi b_{\rm max}^2$ was obtained, and it is easy to find from the result in that paper (see Eq. 4.16 of that reference and below it) that since one obtains an Aichelburg-Sexl shockwave profile $\Phi(r)\propto e^{-mr}/\sqrt{mr}$ that exactly matches the Heisenberg model profile found here [Eq. (5.37)], the subleading correction for the gravity dual cross section also exactly matches Eq. (7.3), with the correct coefficient. In Ref. [25], the same result was found from the gravity dual, with a $1/4$ coefficient instead of our $1/2$, but that is due to their naive starting point adopted from Ref. [13], of considering static point masses, and then making them traceless by hand, instead of considering the more rigorous AS shock-wave scattering that are correctly generated by ultrarelativistic photons and are full nonlinear solutions (as needed for the calculation of the position of the dual black hole horizon), as was explained in Ref. [14]. Taking these correct shock waves does not affect the leading cross section, but it affects the coefficient of the subleading term.

2. Possible new regime

But besides the small corrections to the Froissart saturation regime above, we can in principle have also a situation where a new regime for $\sigma_{\rm tot}(\tilde{s})$ appears.

To avoid the leading Froissart behavior, we must avoid the exponential e^{-mr} for $r=b_{\rm max}$, so we need to have $m_\pi b_{\rm max}(\tilde{s})<1$. This can indeed exist in some energy regime $\tilde{s}$, for small mass $m=m_\pi\ll l^{-1}$.

Since the scale l in Heisenberg's DBI action can presumably be identified with $\Lambda_{\rm QCD}$, and $\Lambda_{\rm QCD}\sim 2m_\pi$, the corrections of order $(ml)^2\sim(m_\pi/\Lambda_{\rm QCD})^2=1/4$ are small, so it could be a good approximation.

But if $ml\ll 1$, there is a regime where the wave function is linear, and when solving for $\phi(r)$ from the equation of motion we never get into the nonlinear regime. That means that the full solution to the free equation, $\phi=AK_0(mr)$, is exact. At distances $r\ll m^{-1}$, we obtain

$$\phi(r)\simeq -A\ln\frac{mr}{2}. \qquad (7.4)$$

Then the condition for $b_{\rm max}$ at energies $\tilde{s}$ for which the above $\phi(r)$ are still in the linear regime is

$$\sqrt{\tilde{s}}\left[-\frac{1}{2}\ln(m_\pi b_{\rm max}(\tilde{s}))\right]=\langle k_{0,\pi}\rangle, \qquad (7.5)$$

giving

$$b_{\rm max}(\tilde{s})=\frac{1}{m_\pi}e^{-2\frac{\langle k_{0,\pi}\rangle}{\sqrt{\tilde{s}}}}\Rightarrow$$
$$\sigma_{\rm tot}(\tilde{s})=\pi b_{\rm max}(\tilde{s})^2=\frac{\pi}{m_\pi^2}e^{-4\frac{\langle k_{0,\pi}\rangle}{\sqrt{\tilde{s}}}}, \qquad (7.6)$$

for $\sqrt{\tilde{s}} > \langle k_{0,\pi}\rangle$, which gives a mildly increasing dependence, that could be easily mistaken for a small power law or the $\log^2$ behavior of Froissart saturation.

In conclusion, such a new energy regime could appear in QCD just before the onset of Froissart saturation, but it would be hard to distinguish experimentally from the small power-law ("soft Pomeron") behavior, or from the Froissart saturation behavior.

B. Black disk model and ratio of elastic to total cross section

Until now we have discussed the total cross section, or in the case of several mesons, also individual meson cross sections. But we want now to discuss also the elastic cross section. For that however, we need a quantum amplitude, whose forward part gives the total cross section, and whose absolute value squared gives the elastic cross section. Note that there are QCD models that try to describe the Froissart behavior and perhaps the total elastic cross section, like black disk eikonal ones [26], dynamical gluon mass ones [27], QCD minijet ones [28] (also emphasizing the effect of soft gluons and the IR dynamics, as done here), or Ref. [29]. But in these QCD cases, it was crucial that one could say something about the quantum amplitude itself.

Since we do not have a quantum amplitude, only a total cross section, we can engineer an amplitude that gives this total cross section, and from it calculate the elastic amplitude. The simplest model is a black disk eikonal amplitude, with S matrix $S = e^{i\delta}$ and $\mathrm{Im}(\delta) = \infty$ for $b \leq b_{\max}(\tilde{s})$ and with $\delta = 0$ for $b > b_{\max}(\tilde{s})$. This reproduces the cross section $\pi b_{\max}(\tilde{s})^2$.

For the scattering of massless states, we have in general

$$\frac{1}{\tilde{s}}\mathcal{A}(\tilde{s},t) = -i\int d^2b e^{i\vec{q}\cdot\vec{b}}(e^{i\delta(b,\tilde{s})}-1)$$
$$= i\int_0^{b_{\max}(\tilde{s})} bdb\int_0^{2\pi} d\theta e^{iqb\cos\theta}(e^{i\delta}-1), \quad (7.7)$$

where $\vec{b}$ is the impact parameter (transversal), and its Fourier conjugate is $\vec{q}$, with $\vec{q}^{\,2} = t$.

For the black disk eikonal,

$$\frac{1}{\tilde{s}}\mathcal{A}(\tilde{s},t) = 2\pi i\frac{b_{\max}(\tilde{s})}{\sqrt{t}}J_1(\sqrt{t}b_{\max}(\tilde{s})). \quad (7.8)$$

The total cross section is found from

$$\frac{1}{\tilde{s}}\mathrm{Im}\mathcal{A}_{\mathrm{elastic}}(\tilde{s},t=0) = \sigma_{\mathrm{tot}}(k_1,k_2\to\text{anything}), \quad (7.9)$$

and it is easy to calculate that for the black disk eikonal we get $\sigma_{\mathrm{tot}} = \pi b_{\max}(\tilde{s})^2$.

We should note here that most of the time, like for instance in Ref. [26], the black disk eikonal model starts with a partial amplitude $a_l(k) = (e^{2i\delta_l(k)}-1)/(2i)$, suggested by the partial-wave expansion, which is a factor of 2 smaller than Eq. (7.7). After the normalization of the cross section is properly taken into account, this leads to $\sigma_{\mathrm{tot}} = 2\pi b_{\max}^2$ and, since $\sigma_{\mathrm{tot}} \sim \mathrm{Im}a$, but $\sigma_{\mathrm{el}} \sim |a|^2$, so a rescaling of a leads to a rescaling of $\sigma_{\mathrm{el}}/\sigma_{\mathrm{tot}}$, to $\sigma_{el} = \pi b_{\max}^2$. But our model, also used for instance in Ref. [14], is physically different, since we considered simply, as usual, the amplitude as the Fourier transform of the T-matrix, and $S = 1 + iT = e^{i\delta}$. This leads to $\sigma_{\mathrm{tot}} = \pi b_{\max}^2(\tilde{s})$, which we believe is a model more deserving of the name black disk, as the total cross section equals the classical one. Then, as we shall see, we obtain $\sigma_{\mathrm{el}}/\sigma_{\mathrm{tot}} \simeq 1/4$, instead of $1/2$.

In the case that the particles are massive with mass m instead, the $1/\tilde{s}$ is replaced by $1/(2p_{\mathrm{CM}}E_{\mathrm{CM}})$. But if $m_1 = m_2 = m$, $E_{\mathrm{CM}} = 2\sqrt{p_{\mathrm{CM}}^2 + m^2} = \sqrt{\tilde{s}}$, so we have

$$2E_{\mathrm{CM}}p_{\mathrm{CM}} = \sqrt{\tilde{s}(\tilde{s}-4m^2)}. \quad (7.10)$$

On the other hand, for the differential cross section, we have the center-of-mass formula

$$\left.\frac{d\sigma_{el}}{d\Omega}\right|_{\mathrm{CM}} = \frac{|\mathcal{A}|^2}{64\pi^2E_{\mathrm{CM}}^2}, \quad (7.11)$$

and the relativistically invariant differential cross section is

$$\frac{d\sigma}{dt} = \frac{|\mathcal{A}(\tilde{s},t)|^2}{16\pi\tilde{s}(\tilde{s}-4m^2)}. \quad (7.12)$$

For the black disk eikonal, we obtain

$$\sigma_{\mathrm{elastic}} = \frac{4\pi^2b_{\max}^2(\tilde{s})\tilde{s}(\tilde{s}-4m^2)}{16\pi\tilde{s}(\tilde{s}-4m^2)}\int\frac{dt}{t}[J_1(\sqrt{t}b_{\max}(\tilde{s}))]^2, \quad (7.13)$$

and since $\sigma_{\mathrm{tot}} = \pi b_{\max}^2(\tilde{s})$, we get

$$\frac{\sigma_{\mathrm{elastic}}}{\sigma_{\mathrm{tot}}} = \frac{1}{4}\int\frac{dt}{t}[J_1(\sqrt{t}b_{\max}(s))]^2. \quad (7.14)$$

It remains to define the range of integration for t, given $\tilde{s}$. In the center-of-mass system,

$$\tilde{s} = E_{\mathrm{CM}}^2;$$
$$t = (\vec{p}_{\mathrm{CM}} - \vec{k}_{\mathrm{CM}})^2 = k_{\mathrm{CM}}^2 + p_{\mathrm{CM}}^2 - 2k_{\mathrm{CM}}p_{\mathrm{CM}}\cos\theta, \quad (7.15)$$

where $\vec{p}_{\mathrm{CM}}$ and $\vec{k}_{\mathrm{CM}}$ are momenta of the same particle, before and after the collision in the center of mass. Then the range of integration for t, given $\tilde{s}$, which fixes p_{CM} and k_{CM}, is

$$t \in [(p_{\rm CM} - k_{\rm CM})^2, (p_{\rm CM} + k_{\rm CM})^2]. \tag{7.16}$$

But $p_{\rm CM} = k_{\rm CM}$ and $\sqrt{\tilde{s}}/2 = E_{\rm CM}/2 = E = \sqrt{p_{\rm CM}^2 + m^2}$, meaning that

$$p_{\rm CM} = k_{\rm CM} = \sqrt{\frac{\tilde{s}}{4} - m^2}, \tag{7.17}$$

and then the range of integration of t is

$$t \in [0, \tilde{s} - 4m^2], \tag{7.18}$$

so that finally

$$\frac{\sigma_{\rm elastic}}{\sigma_{\rm tot}} = \frac{1}{4}\int_0^{\tilde{s}-4m^2} \frac{dt}{t}[J_1(\sqrt{t}b_{\rm max}(\tilde{s}))]^2. \tag{7.19}$$

By using the recurrence relations for $J_\nu(x)$, we do the integral and obtain

$$\frac{\sigma_{\rm elastic}}{\sigma_{\rm tot}} = \frac{1}{4}[1 - (J_0(|b_{\rm max}(\tilde{s})|\sqrt{\tilde{s} - 4m^2}))^2 - (J_1(|b_{\rm max}(\tilde{s})|\sqrt{\tilde{s} - 4m^2}))^2]. \tag{7.20}$$

At large z,

$$J_0(z) \simeq \sqrt{\frac{2}{\pi z}}\cos(z - \pi/4),$$
$$J_1(z) \simeq \sqrt{\frac{2}{\pi z}}\cos(z - 3\pi/4) = \sqrt{\frac{2}{\pi z}}\sin(z - \pi/4), \tag{7.21}$$

so that finally we obtain

$$\frac{\sigma_{\rm elastic}}{\sigma_{\rm tot}} \simeq \frac{1}{4}\left[1 - \frac{2}{\pi b_{\rm max}(\tilde{s})\sqrt{\tilde{s} - 4m_N^2}}\right] \tag{7.22}$$

where we put m_N for a nucleon or nucleus mass, corresponding to the case when we collide nucleons or nuclei. Then from the Heisenberg model $b_{\rm max}(s) \simeq 1/m_\pi \ln(s/s_0)$, so that the sought-for ratio is

$$\frac{\sigma_{\rm elastic}}{\sigma_{\rm tot}} \simeq \frac{1}{4}\left[1 - \frac{2m_\pi}{\pi \ln(\tilde{s}/s_0)\sqrt{\tilde{s} - 4m_N^2}}\right], \tag{7.23}$$

asymptoting very fast to $1/4$.

This compares very well with the experimental results from the TOTEM experiment [30].

VIII. HEISENBERG MODEL AND HOLOGRAPHY

In Sec. III C we described a sigma model in AdS space. This can be directly related to another holographic model, the "hard-wall" model, which is an AdS background chopped off at a certain value of the radial coordinate. This scenario is addressed in the following subsection. We then present an alternative approach that includes a systematical analysis of the relations between Heisenberg's model and the holographic description of a proton-proton scattering. This in fact involves two steps. In the first we will establish the relations between the DBI action used in Heisenberg's model and the DBI action that emerges as the action of flavor branes in confining backgrounds. The second step is to lay out the holographic dual of scattering of baryons and to relate it to the extraction of the cross section from Heisenberg's model. The two steps are described in the second and third subsections of this section.

A. The relation to the holographic "hard-wall" model

The remarkable fact is that, even though the Heisenberg model was proposed before string theory was discovered, the DBI action used by Heisenberg emerges naturally in holographic models of QCD since it relates to the effective action of open strings. In the simplest model for high-energy QCD scattering introduced by Polchinski and Strassler, one considers an AdS_5 space,

$$ds^2 = \frac{r^2}{R^2}d\vec{x}^2 + R^2\frac{dr^2}{r^2}, \tag{8.1}$$

cut off at an $r_{\rm min} = R^2\Lambda$, with Λ identified with the (pure) QCD scale (glueball scale). It was soon realized that one can think of the IR cutoff as a dynamical IR brane (like in the Randall-Sundrum model), and the appropriately normalized fluctuation in the position $r_{\rm min}$, the scalar $\sim\phi$, can be identified with the pion in QCD. But the action for the fluctuation in the position of a brane is exactly the DBI action!

The only nontrivial part of the action is the potential for the brane position, which can appear, depending on the mechanism, either inside or outside the square root.

The picture of high-energy scattering is also similar in the gravity dual [5,14–18]. In a purely gravitational theory, we have gravitational shock-wave collision, happening near the IR cutoff, creating a black hole on the IR cutoff, being mapped to the pion field shock-wave collisions creating a fireball. But more precisely, when we consider also the fluctuation of the IR cutoff giving the pion, we have the same picture, of pion field shock waves colliding and creating a fireball.

Note that the presence of an infinite number of higher-derivative terms in the DBI action is, from the QCD point of view, related to the chiral symmetry (chiral perturbation theory), whereas in the gravity dual this symmetry becomes geometrized, and the action is determined by the motion of a D-brane in a symmetric background. In the hard-wall model, this is somewhat obscured, as we only have the translational symmetry broken by the position of the D-brane.

B. The DBI action of flavor branes in confining backgrounds

Heisenberg's model assumes that the scalar fields that are in charge of the interaction between nucleons are governed by a DBI action in flat spacetime. Holography provides dual string descriptions to certain strongly coupled gauge dynamical systems. As was mentioned above, the DBI action is a basic tool in the toolkit of string theories. Thus, an obvious question to ask is whether one can relate Heisenberg's model to a holographic description of proton-proton scattering, and in what way. To answer this question one has to address first the issue of what is the holographic laboratory dual of QCD in its confining phase.

The basic $\mathrm{AdS}_5 \times S^5$ string theory, the dual of $\mathcal{N} = 4$ SYM is clearly not the right setup. It is both conformal and maximally supersymmetric. One has to deform the geometrical background in such a way that the isometry group is not $SO(4,2) \times SO(6)$ but rather only the four-dimensional Poincaré symmetry. Obviously the desired background should be equipped with a scale which breaks scale invariance. To check whether a given background corresponds to a boundary confining field theory, one should investigate the stringy dual of the Wilson line. A necessary condition for a "confining background" is that any rectangular Wilson line along one of the space directions and the time direction should admit a confining area-law behavior. In Ref. [31] it was shown that this is achieved provided that either $g_{tt}g_{xx}(u)$ has a nonvanishing minimum value or that it does not vanish at the value of the radial coordinate u where $g_{tt}g_{uu}(u) \to \infty$, and where g_{tt}, g_{xx} and g_{uu} are the metric components along time, the space direction of the Wilson line, and the radial direction respectively. Not surprisingly the $\mathrm{AdS}_5 \times S^5$ background does not obey this requirement.

A close cousin of this background that does admit confinement is the "hard-wall model" discussed in the previous subsection where one, by hand, chops off the radial direction to be $u \geq u_\Lambda$ where u_Λ is a scale in the bulk that corresponds to Λ_{QCD} of the boundary confining gauge theory. This however is not a solution of the equations of motion.

A prototype confining background that is a solution is the AdS_5 background with one spatial coordinate compactified [32] on a circle in such a way that the two-dimensional manifold spanned by the radial direction and the circle has a cigar-like geometry. It is easy to check that upon imposing antiperiodic boundary conditions for fermions, the only massless fields of the dual large-N_c gauge theory are only the gauge fields and all their supersymmetric partners become massive. In that way supersymmetry is broken and the dual field theory is that of pure large-N_c gauge theory in three spacetime dimensions. To get a gravity model dual of four-dimensional confining large-N_c gauge theory, one can compactify the near-horizon background of a large number of $D4$ branes [33] rather than the $\mathrm{AdS}_5 \times S^5$ model which is the background of a large number of $D3$ branes. In fact the dual gauge theory is an effective confining theory with energies smaller than $\frac{1}{R}$ where R is the radius of the compact circle which maps into the mass of the dual glueballs. There are several other solutions of the ten-dimensional supergravity equations of motion that admit confinement but with no loss of generality we will discuss here only this model.

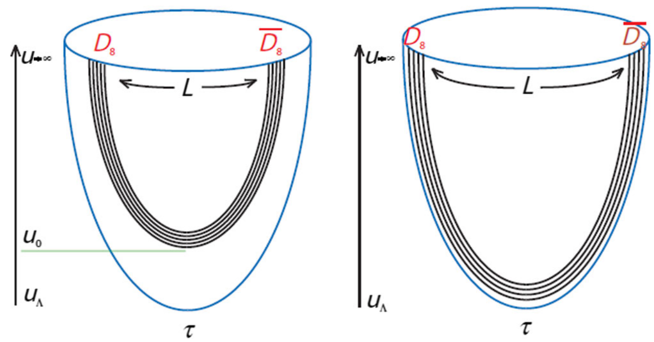

FIG. 2 (color online). On the right side we have the antipodal setup of the Sakai-Sugimoto model where $u_0 = u_\Lambda$. On the left side we have the generalized nonantipodal setup.

To incorporate in the gravity side the quark degrees of freedom one introduces N_f flavor D-branes. For $N_f \ll N_c$ one can neglect the backreaction of the flavor brane on the bulk and hence treat them as probe branes. In the Sakai-Sugimoto model [19] a stack of N_f D8 and a stack of N_f anti-D8 branes are placed so that asymptotically at large radial direction the transverse direction to their worldvolumes is along the compact circle x_4.

In the IR in the region of the tip of the cigar the two stacks of branes have to merge into each other hence breaking the original $U_L(N_f) \times U_R(N_f)$ chiral symmetry into a diagonal subgroup of $U_D(N_f)$. In the original model, the U-shaped branes were in an antipodal setup $u_0 = u_\Lambda$; see the right panel of Fig. 2. This was generalized (see the left panel) to incorporate an additional parameter $u_0 \neq u_\Lambda$ [20] which, as will be shown below, is crucial for coupling the protons to pions in the holographic picture. The physics of the degrees of freedom that resides on the flavor branes, namely the $U(N_f)$ gauge fields and the scalars in the adjoint of the $U(N_f)$ group, is described by a DBI action. In fact the action also includes, on top of the DBI action, a CS term. That is obviously where Heisenberg's model and holography meet. The action on the flavor branes in the Sakai-Sugimoto model reads

$$S_{\mathrm{DBI}} = T_8 \int d^9\sigma e^{-\tilde{\phi}} \sqrt{-\det[\partial_\mu X^i \partial_\nu X^j g_{ij}(X) + 2\pi\alpha' F_{\mu\nu}]}, \tag{8.2}$$

where the dilaton ϕ, the metric g_{ij} and the Ramond-Ramond (RR) four-form are given by [20]

$$\begin{aligned} ds^2 &= \left(\frac{u}{R_{D4}}\right)^{3/2} [-dt^2 + \delta_{ij} dx^i dx^j + f(u) dx_4^2] \\ &\quad + \left(\frac{R_{D4}}{u}\right)^{3/2} \left[\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right] \\ F_4 &= \frac{2\pi N_c}{V_4} \epsilon_4, \qquad e^\phi = g_s \left(\frac{u}{R_{D4}}\right)^{3/4}, \\ R_{D4}^3 &= \pi g_s N_c l_s^3, \qquad f(u) = 1 - \left(\frac{u_\Lambda}{u}\right)^3, \end{aligned} \tag{8.3}$$

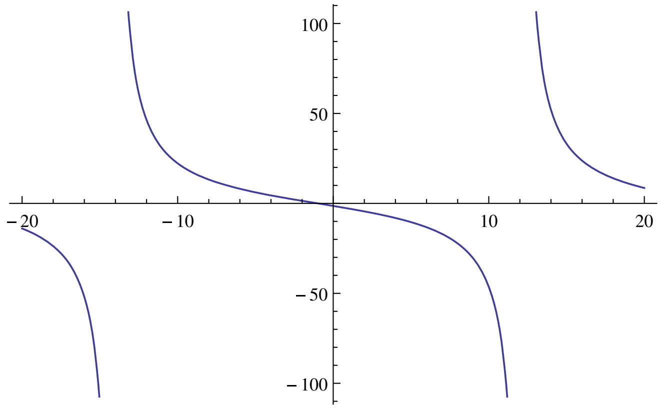

FIG. 3 (color online). $\phi(t)$ as a function of t for $\frac{m}{l} = 0.1$.

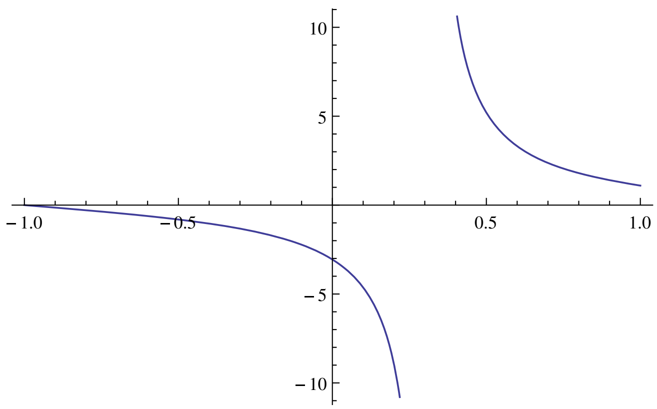

FIG. 4 (color online). $\phi(t)$ as a function of t for $\frac{m}{l} = 1$.

where x_4 is the coordinate of the compactified circle, V_4 is the volume of the unit four-sphere Ω_4 and ϵ_4 is its corresponding volume form. Upon inserting the metric and the dilaton one finds, according to the general analysis in Sec. IV B,

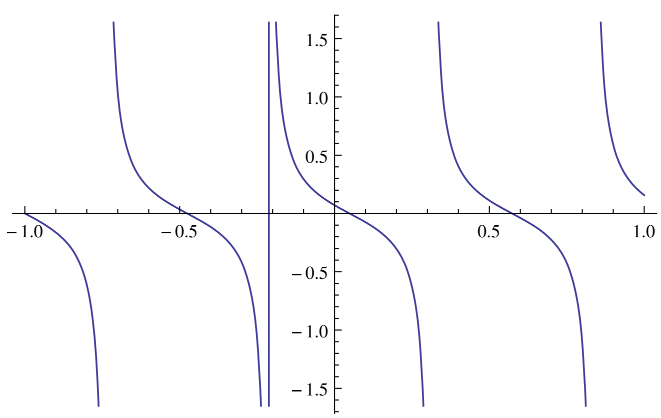

FIG. 5 (color online). $\phi(t)$ as a function of t for $\frac{m}{l} = 5$.

$$S_{\rm DBI} = \tilde{T}_8 \int dtd^3xdx_4\phi^4 \times \sqrt{f(\phi) + \left(\frac{R_{D4}}{\phi}\right)^3 \left[\partial_\mu\phi\partial^\mu\phi + \frac{1}{f(\phi)}(\partial_{x_4}\phi)^2\right]}, \tag{8.4}$$

where $\tilde{T}_8 = T_8\Omega_4/g_s$ and to connect to the rest of the paper we denoted the radial coordinate u by ϕ.

The fluctuations of ϕ translate using the dictionary of holography to scalar mesons. To extract the spectrum of the latter one considers first a profile of the flavor brane given by $\phi_{cl}(x_4)$. One then introduces the fluctuations of ϕ in the following form:

$$\phi(x_4, x^\mu) = \phi_{cl}(x_4) + \sum_n \delta\phi_n(x^\mu)\zeta_n(x_4). \tag{8.5}$$

The lowest mode of the fluctuating field ϕ_0 should be identified with the scalar field $\phi(x^\mu)$ in the Heisenberg model. Next one expands the DBI action to quadratic order in ϕ, integrates over the x_4 direction, and derives a massive spectrum for the $\delta\phi_n(x^\mu)$. Here we do not want to expand the square root but rather maintain the full tower of derivatives of the field. The outcome of the integration of the $\zeta_n(x_4)$ will be mass terms of the form $m_n^2\phi^2$ plus terms higher order in ϕ. We assume here that the truncation to only the mass term in the expansion of the DBI action can be translated to having a mass term in the four-dimensional DBI itself. In that case the action takes the form

$$S_{\rm DBI} = \tilde{T}_8 \int dtd^3x\phi^4 \sqrt{f(\phi) + \left(\frac{R_{D4}}{\phi}\right)^3 [\partial_\mu\phi\partial^\mu\phi + m^2\phi^2]}. \tag{8.6}$$

The equation of motion that is associated with the action (8.6) for the $m = 0$ case can be written in the following form:

$$\begin{aligned}&\left(1 - \left(\frac{u_\Lambda}{\phi}\right)^3\right)\left[8 - 5\left(\frac{u_\Lambda}{\phi}\right)^3 + \left(\frac{R_{D4}}{\phi}\right)^3 \times (\partial_\mu\phi\partial^\mu\phi - 2\phi\partial_\mu\partial^\mu\phi)\right] \\ &\quad + 8\left(\frac{R_{D4}}{\phi}\right)^3 \partial_\mu\phi\partial^\mu\phi - 2\frac{u_\Lambda^3 R_{D4}^3}{\phi^6}(\partial_\mu\phi)^2 - 2\frac{R_{D4}^6}{\phi^6}[(\partial_\mu\phi)^2]^2 \\ &\quad + 2\left(\frac{R_{D4}}{\phi}\right)^6 \phi\partial^\mu\phi[-(\partial_\mu\phi)\partial^\nu\partial_\nu\phi + (\partial^\nu\phi)\partial_\mu\partial_\nu\phi] = 0.\end{aligned} \tag{8.7}$$

We leave the investigation of the relation between the solution of the DBI action given here and the DBI in flat spacetime used in the Heisenberg model to a future investigation.

C. A holographic description of the proton-proton scattering

So far we have discussed the holographic laboratory and its relation to the DBI action applied in Heisenberg's model. Next we would like to see what is the relation between the cross section of a proton-proton scattering in Heisenberg's model and the corresponding cross section in a holographic setup that is associated with a confining theory equipped with flavor degrees of freedom. Here for concreteness we will use the generalized Sakai-Sugimoto model. A stringy realization of a baryon in this model [34] is that of a baryonic vertex made out of a D4 brane that wraps the four-cycle and is connected by N_c strings to the N_f probe flavor branes [35]. *A priori* the baryonic vertex could have been located in the generalized Sakai-Sugimoto model in any place below the flavor brane, but in Ref. [36] it was shown that in fact it must be immersed on the flavor brane. The interaction between two protons in this setup is that of two baryonic vertices each connected to N_c strings that stretch on the flavor branes. The scattering of such two objects is obviously very complicated. Instead it was shown in Ref. [21] that one can view the baryon as a flavored gauge instanton. This follows from the fact that the wrapped D4 brane is a point on the four-dimensional part of the worldvolume of the flavor brane which is spanned by the ordinary three space coordinates and the radial direction. Alternatively it can be shown by expanding the flavor gauge DBI + CS actions, keeping only the leading-order $U(N_f)$ YM + CS action. The five-dimensional action takes the form

$$\begin{aligned} S &= S_{\mathrm{YM}} + S_{\mathrm{CS}}, \\ S_{\mathrm{YM}} &\approx \int d^4x \int dz \frac{1}{2g_{\mathrm{YM}}^2(z)} \mathrm{tr}(\mathcal{F}_{MN}^2), \\ S_{\mathrm{CS}} &= \frac{N_c}{16\pi^2} \int \hat{A} \wedge \mathrm{tr} F^2 + \frac{N_c}{96\pi^2} \int A \wedge \hat{F}^2, \end{aligned} \quad (8.8)$$

where near the bottom of the U-shaped flavor branes we have

$$\frac{1}{2g_{\mathrm{YM}}^2(z)} = \frac{N_c \lambda M_{\mathrm{KK}}}{216\pi^3} \left(\zeta + \frac{8\zeta^3 - 5}{9\zeta} M_{\mathrm{KK}}^2 z^2 + O(M_{\mathrm{KK}}^4 z^4) \right). \quad (8.9)$$

Here $\mathcal{F}$ is the $U(N_f)$ gauge field, and A and $\hat{A}$ denote the gauge one-forms associated with the $SU(N_f)$ and $U(1)$ subgroups respectively. We made a coordinate transformation from (x^μ, x_4) to to a five-dimensional conformal metric $ds^2 = (\frac{u(z)}{R_{D4}})^{3/2}(-dt^2 + dx_i^2 + dz^2)$, $\zeta = \frac{u_0}{u_\Lambda}$. Based on this action it was further shown that the static properties extracted from this model are similar to those derived from the Skyrme model.

Next we would like to examine to what extent Heisenberg's treatment of the scattering of a proton on a proton translates into a scattering process of two instantons in the holographic laboratory. The interaction of the latter can be divided into three zones [37]. In the far zone when the distances between the two instantons is much larger than the inverse of the dual of Λ_{QCD} the interaction is dominated by the exchange of the lightest meson. In the isoscalar channel it was found that the repulsion, due to the exchange of vector mesons, is stronger than the attraction, due to the exchange of scalar mesons, since the lightest meson on the latter type is heavier than the lightest vector meson. In the isovector channel it is obvious that the lightest meson is the pion and the exchange of it yields an attraction. In the near zone, using the solution that carries instanton number two, one finds that there is only a repulsive hard core interaction. In the intermediate zone there is a repulsion due to the interaction of the instanton density with the $U(1)$ of the $U(N_f)$ flavor gauge group. However, as was shown in Ref. [38] there is also an attractive force due to the interaction of the instanton density with the scalar field associated with the fluctuation of the $D8$ branes. The action of this scalar takes the form

$$\begin{aligned} S_\phi = S_{\mathrm{DBI}} + \frac{N_c}{16\pi^2} \int d^4x \int dz C(z) \\ \times \mathrm{tr}(\Phi \mathcal{F}_{MN} \mathcal{F}^{MN}) + \cdots, \end{aligned} \quad (8.10)$$

where $C(z)$ measures the ratio of the attractive to the repulsive forces

$$\frac{F_a}{F_r} = C^2(z) = \frac{1 - \zeta^{-3}}{9} \left(\frac{u_0}{u(z)} \right)^8 \leq \frac{1}{9} < 1. \quad (8.11)$$

Note that for self-dual (instanton) configurations, $Tr[F_{MN}F^{MN}] = Tr[F \wedge F]$ and hence the scalar field that originates from the brane fluctuations couples to the instanton density, namely to the proton density.

Thus, in a holographic description of the interaction between two protons, both in the intermediate as well as in the far zone, the interaction is mediated by a scalar field that is governed by a DBI action. The DBI action (8.4) is not the one Heisenberg used but rather a DBI of a scalar in a curved background. The source of the scalar field and its coupling to the proton given in Eq. (8.10) is different from the source of the scalar field discussed in Sec. V, but a fixed gauge field profile will generate a function $f(r)$ in the action as in Eq. (5.19), or an implicit external source as in Eq. (6.37).

IX. SUMMARY AND OPEN QUESTIONS

As was explained in the Introduction, in this paper we addressed four aspects of Heisenberg's model of the scattering of nucleons.

(i) We elaborated on, and gave further supporting evidence for the model. We made an analysis of the energy of the scalar field, and the conditions under which we obtain the (almost) saturation of the Froissart bound. We have analyzed what happens when we go from a $1+1$-dimensional solution to a $3+1$-dimensional one $\phi(s,r)$. We have understood the implicit source in the Heisenberg solution by analogy with the electromagnetic Born-Infeld action: there is an "external" $\delta(s)$ source that is "spread out" by the medium. One can also consider $\delta(x^-)$ shock-wave solutions by adding an explicit source in the Lagrangian. By using a perturbative $\phi(s,r)$, we have obtained corrections away from the maximal Froissart saturation behavior, as well as a new regime for $\sigma_{\rm tot}(s)$.

(ii) We examined the uniqueness of the DBI action in terms of giving the (almost) saturation of the bound. We have found that, perhaps surprisingly, no action with a potential interaction, or with a finite number of higher-derivative terms can do the job. The DBI action can do the job, though we have not been able to prove that another action with an infinite number of higher-derivative terms cannot do the job as well.

(iii) We proposed and analyzed several generalizations of the Heisenberg model. We added a general potential inside the square root, instead of just the mass term and we considered a sigma model with several scalar mesons. We considered a "curved-space" generalization inspired by holography, in particular the "highly effective action" of Ref. [10] for the case of single scalar in AdS_5, when we replace $\partial_\mu\phi\partial^\mu\phi$ by $\frac{1}{\phi^4}\partial_\mu\phi\partial^\mu\phi$. By considering a "black disk" type of amplitude in the sense of Ref. [14], we have obtained also a value for the ratio of the elastic to the total cross section, $\sigma_{\rm el}/\sigma_{\rm tot}$ that asymptotically goes to $1/4$. We note that the more common model in for instance Ref. [26] would give $1/2$, but the experimental evidence points towards $1/4$.

(iv) We have considered the relation of the Heisenberg model and the DBI action he considered to two holographic approaches to proton-proton (or nucleon-nucleon) scattering: a simple hard-wall model, and a more precise model based on flavor branes in confining backgrounds.

In this paper we have just explored the tip of the iceberg. There are a handful of additional open questions that are awaiting further investigation. Here we list a few of them.

(i) Probably the most interesting topic related to realistic high-energy scattering is performing a precise comparison between the results of Heisenberg's model and experimental data of high-energy scattering of nucleons and of nuclei. One can deduce the scattering total cross section and the ratio between the elastic and total cross sections not only for the asymptotic range of energies as was discussed in Sec. VII A. In Sec. IV we analyzed several possible generalizations of the model, and in Sec. VIII we discussed the relation to certain holographic models. These deviations from the original model can also be confronted with experimental data. One would like to extract the values of the parameters of the various models that admit the best fit to the data, in particular the mass of the scalar particle that mediates the interaction which we referred to as the "pion" in this paper.

(ii) It is well known that there are two approaches for phenomenologically fitting the experimental data. One is based on the Froissart bound, namely $\sigma_{\rm tot}\sim\log^2(s)$ and the other on an exchange of Reggeons and Pomerons between the two scattering nucleons. In this case one uses a relation of the form $\sigma_{\rm tot}\sim as^{-0.47}+bs^{0.08}$. Both approaches yield a reasonable fit (see Ref. [6] for a possible way to connect the gravity dual picture of gravitational shock-wave scattering to the soft Pomeron behavior). Thus, a natural question to ask is what is the relation between the two models? In Sec. VIII we have attempted to relate the model to a holographic model of scattering of nucleons. The latter is an approximated picture of a fully stringy description of the scattering process. The exchange of a Reggeon and a Pomeron seems closely related to an exchange of an open and a closed string. Hence one may be able to find a direct relation between the two approaches.

(iii) One natural generalization of the model that was not discussed here but in fact is quite common in implementing the DBI action in holography is the non-Abelian DBI model. To incorporate the (flavor) non-Abelian nature of the pions is the analog of using N_f probe flavor branes rather than a single one in holographic models. In both cases the non-Abelianization will provide further structure. A first attempt with the non-Abelian model was presented in Ref. [24].

(iv) Describing the scattering of two nucleons as a scattering of two shock waves is clearly only an approximation and one may attempt to introduce a correction beyond the shock-wave limit. Similarly one can introduce corrections to the black disk model.

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APPENDIX A: AN ALTERNATIVE METHOD OF DETERMINING THE SCALAR FIELD ENERGY

The Hamiltonian density was given in Eq. (2.11). It reads

$$\mathcal{H} = p\dot{\phi} - \mathcal{L} = \frac{l^{-4} + (\nabla\phi)^2 + m^2\phi^2}{\sqrt{1 + l^4[(\partial_\mu\phi)^2 + m^2\phi^2]}} - l^{-4}. \tag{A1}$$

To determine the Hamiltonian density in momentum space namely $\mathcal{H}(k)$ is a nontrivial task for the DBI action since we cannot simply, as is done for ordinary free field theories, substitute the Fourier transform of the field into Eq. (A1) since the fields appear also in the denominator. In the case when upon substituting the classical solution $\phi(s)$ into the denominator the latter is a constant one can use the usual method. But as was shown in Sec. II this is not the case for the DBI action and hence one has to adopt another approach. Here we suggest such an alternative. We define the Fourier transform of $\mathcal{H}$ as

$$\sqrt{\mathcal{H}(x,t)} = \int \frac{dk}{\sqrt{2\pi}} \frac{[e^{-ikx}\tilde{\mathcal{F}}(k,t) + e^{ikx}\tilde{\mathcal{F}}^*(k,t)]}{2} \tag{A2}$$

and substitute it into the energy, so that

$$\begin{aligned} E &= \frac{1}{2\pi}\int dx \int dk \frac{[e^{-ikx}\tilde{\mathcal{F}}(k,t) + e^{ikx}\tilde{\mathcal{F}}^*(k,t)]}{2} \int dp \frac{[e^{-ipx}\tilde{\mathcal{F}}(p,t) + e^{ipx}\tilde{\mathcal{F}}^*(p,t)]}{2} \\ &= \int dk \frac{[2\tilde{\mathcal{F}}(k,t)\tilde{\mathcal{F}}^*(k,t) + \tilde{\mathcal{F}}(k,t)\tilde{\mathcal{F}}(-k,t) + \tilde{\mathcal{F}}^*(k,t)\tilde{\mathcal{F}}^*(-k,t)]}{4}. \end{aligned} \tag{A3}$$

For the theory of a free massless scalar in two spacetime dimensions $\mathcal{H}(x,t) = \frac{1}{2}[(\partial_x\phi)^2 + (\partial_0\phi)^2]$. In this case it is easy to see that $\tilde{\mathcal{F}}(k,t) = \sqrt{k}a(k)$ where the field $\phi(x,t)$ has a Fourier transform $\phi(x,t) = \int dk \frac{1}{\sqrt{2\pi k}}[a(k)e^{-ikx} + a^\dagger(k)e^{+ikx}]$. In the case of a massive free scalar field we get $\tilde{\mathcal{F}}(k,t) = \sqrt{\sqrt{k^2+m^2}}a(k)$. In these cases the only contributions to Eq. (A3) will be from the $\tilde{\mathcal{F}}(k,t)\tilde{\mathcal{F}}^*(k,t)$ term. For the general case one has to first determine $\tilde{\mathcal{F}}(k,t)$ and then E is given by Eq. (A3).

Following this approach we now have to find the Fourier transform of $\sqrt{\mathcal{H}(s)}$. We cannot find an exact analytic expression for either for the massless case or for the massive one. From the analysis of the energy as an integral over s one finds that the main contribution to the energy comes from the region of small s. Thus we can get an approximation of the dependence of the energy on γ using the the leading order in s expression of $\sqrt{\mathcal{H}(s)} \sim \frac{1}{l^2\sqrt{m}}\frac{t}{s^{3/4}} = \frac{1}{l^2\sqrt{m}}\frac{t}{(t^2-x^2)^{3/4}}$. Its Fourier transform reads

$$TF[\sqrt{\mathcal{H}(s)}] \sim \frac{1}{l^2\sqrt{m}} \frac{\sqrt[4]{2}\sqrt[4]{|k|}K_{-\frac{1}{4}}\left(\frac{|k|}{\sqrt{-\frac{1}{t^2}}}\right)}{(-\frac{1}{t^2})^{5/8}(t^2)^{3/4}\Gamma(\frac{3}{4})}. \tag{A4}$$

Expanding this expression in $\frac{1}{k}$ we get

$$\frac{\sqrt{\pi}e^{-\frac{|k|}{\sqrt{-\frac{1}{t^2}}}}\left(1 - \frac{\sqrt[3]{-\frac{1}{t^2}}}{32|k|}\right)}{\sqrt[4]{2}(-\frac{1}{t^2})^{3/8}(t^2)^{3/4}\Gamma(\frac{3}{4})\sqrt[4]{|k|}}. \tag{A5}$$

Substituting this expression in the energy and taking the integration region to be $\gamma m > k > m$ we finally get that

$$E \sim \sqrt{\gamma m}. \tag{A6}$$

The reason that this result does not match the result found in Sec. II is that we took a crude approximation of $\sqrt{\mathcal{H}(s)}$. Obviously this approximation can be systematically improved by improving the approximation of $\sqrt{\mathcal{H}(s)}$.

APPENDIX B: SCALAR SOLUTIONS IN 0 + 1 DIMENSIONS

Here for completeness we write down solutions of the Heisenberg action in $0+1$ dimensions. The equations of motion in this case are

$$\ddot{\phi} + m^2\phi + l(\dot{\phi})^2 \frac{\ddot{\phi} - m^2\phi}{1 - l[(\dot{\phi})^2 - m^2\phi^2]} = 0. \tag{B1}$$

For the massless case the equation reduces to $\ddot{\phi} = 0$ and hence the solution takes the form $\phi = at + b$. For the massive case the solution takes the form

$$\begin{aligned} y(x) &= \frac{i\mathrm{sn}(im\sqrt{lc_1+1}x + im\sqrt{lc_1+1}c_2|\frac{lc_1}{lc_1+1})}{\sqrt{lm}}, \\ y(x) &= -\frac{i\mathrm{sn}(i(m\sqrt{lc_1+1}x + m\sqrt{lc_1+1}c_2)|\frac{lc_1}{lc_1+1})}{\sqrt{lm}}. \end{aligned} \tag{B2}$$

The solution takes the form in Figs. 3–5 for various values of $\frac{m}{l} = 0.1, 1, 5$.

For the one-dimensional case the Hamiltonian density (2.11) is the Hamiltonian and hence we can write a first-order differential equation which is its conservation in time instead of the equation of motion. The Hamiltonian for this case reads

$$Hl = \frac{1}{\sqrt{1 - l[(\dot{\phi})^2 - m^2\phi^2]}}[1 + lm^2\phi^2)] - 1. \tag{B3}$$

Thus the first-order differential equation is

$$(\dot{\phi})^2 = \frac{H(lH+2)}{(lH+1)^2} + m^2\left(1 - \frac{2}{(lH+1)^2}\right)\phi^2 - \frac{lm^4\phi^4}{(lH+1)^2}, \tag{B4}$$

or in an integral form

$$\int \frac{d\phi}{\sqrt{\frac{H(lH+2)}{(lH+1)^2} + m^2(1 - \frac{2}{(lH+1)^2})\phi^2 - \frac{lm^4\phi^4}{(lH+1)^2}}} = t + c. \tag{B5}$$

APPENDIX C: SCALAR SOLUTIONS IN $1+1$ DIMENSIONS: STATIC AND DEPENDING INDEPENDENTLY ON x^+ AND x^-

Before discussing a genuine two-dimensional case let us check the equation for a (soliton) static solution. For that case the equation takes the form

$$\partial_x^2\phi - m^2\phi - l^2(\partial_x\phi)^2 \frac{\partial_x^2\phi + m^2\phi}{1 + l^2[(\partial_x\phi)^2 + m^2\phi^2]} = 0. \tag{C1}$$

This equation admits an analytic solution similar to the one of the one-dimensional case, namely

$$y(x) = \frac{i\,\mathrm{sn}(im\sqrt{l^2c_1 - 1}x + im\sqrt{l^2c_1 - 1}c_2 | \frac{l^2c_1}{l^2c_1 - 1})}{lm},$$
$$y(x) = -\frac{i\,\mathrm{sn}(i(m\sqrt{l^2c_1 - 1}x + m\sqrt{l^2c_1 - 1}c_2) | \frac{l^2c_1}{l^2c_1 - 1})}{lm}. \tag{C2}$$

This soliton solution is similar to the solution of the one-dimensional case discussed above. It can be seen from the equations of motion that the map $l \to -l^2$ and $m^2 \to -m^2$ maps the one-dimensional equation to the solitonic two-dimensional one.

We next consider the truly two-dimensional case, thought of as an approximation for the four-dimensional system of colliding shock waves in the limit of zero width for the shock wave, and in a limit of azimuthal symmetry in the plane of the shock. It is convenient in two dimensions to use light-cone coordinates $x^\pm = t \pm x$, with

$$\partial_+ = \partial_{x^+} = \frac{1}{2}(\partial_t + \partial_x), \qquad \partial_- = \partial_{x^-} = \frac{1}{2}(\partial_t - \partial_x). \tag{C3}$$

In these light-cone coordinates

$$\partial_\mu\phi\partial^\mu\phi = (\dot{\phi})^2 - (\phi')^2 = 4\partial_+\phi\partial_-\phi,$$
$$\partial_\mu\partial^\mu\phi = \ddot{\phi} - \phi'' = 4\partial_+\partial_-\phi. \tag{C4}$$

We now define the following coordinates:

$$s = t^2 - x^2 = x^+x^-, \qquad q = \frac{x^-}{x^+}. \tag{C5}$$

For these coordinates we find that

$$\partial_\mu\phi\partial^\mu\phi = +4\left[s(\partial_s\phi)^2 - \frac{q^2}{s}(\partial_q\phi)^2\right],$$
$$\partial_\mu\partial^\mu\phi = +4\left[\partial_s(s\partial_s\phi) - \frac{q}{s}[\partial_q(q\partial_q\phi)\right], \tag{C6}$$

and also

$$\begin{aligned}(2(\partial_\mu\phi)(\partial_\nu\phi)(\partial^\mu\partial^\nu\phi) =)\partial_\mu\phi\partial^\mu(\partial_\nu\phi\partial^\nu\phi)\\ = 16\Bigg[s(\partial_s\phi)^3 + 2s^2(\partial_s\phi)^2\partial_s^2\phi + \frac{q^2}{s}(\partial_q\phi)^2(\partial_s\phi)\\ - 2q^2(\partial_s\phi)(\partial_q\phi)(\partial_q\partial_s\phi) + 2\frac{q^3}{s^2}(\partial_q\phi)^3\\ + 2\frac{q^4}{s^2}(\partial_q\phi)^2(\partial_q^2\phi)\Bigg].\end{aligned} \tag{C7}$$

Substituting Eqs. (C6) and (C7) into the equation of motion (2.2) we get that for the variables s and q the equation of motion takes the form

$$\begin{aligned}4\left[\partial_s(s\partial_s\phi) - \frac{q}{s}\partial_q\phi - \frac{q^2}{s}\partial_q^2\phi\right] + m^2\phi\\ = 4l^4m^2\phi\frac{[s(\partial_s\phi)^2 - \frac{q^2}{s}(\partial_q\phi)^2]}{1 + l^4[m^2\phi^2 - 4[s(\partial_s\phi)^2 - \frac{q^2}{s}(\partial_q\phi)^2]]} - 8l^4\frac{[s(\partial_s\phi)^3 + 2s^2(\partial_s\phi)^2\partial_s^2\phi + \frac{q^2}{s}(\partial_q\phi)^2(\partial_s\phi)}{1 + l^4[m^2\phi^2 - 4[s(\partial_s\phi)^2 - \frac{q^2}{s}(\partial_q\phi)^2]]}\\ + \frac{-2q^2(\partial_s\phi)(\partial_q\phi)(\partial_q\partial_s\phi) + 2\frac{q^3}{s^2}(\partial_q\phi)^3 + 2\frac{q^4}{s^2}(\partial_q\phi)^2(\partial_q^2\phi)}{1 + l^4[m^2\phi^2 - 4[s(\partial_s\phi)^2 - \frac{q^2}{s}(\partial_q\phi)^2]]}.\end{aligned} \tag{C8}$$

For the special case of $\phi(s,q)=\phi(s)$ the equation of motion reduces to

$$4\partial_s(s\partial_s\phi)+m^2\phi=4l^4s(\partial_s\phi)^2\frac{m^2\phi+2[(\partial_s\phi)+2s\partial_s^2\phi]}{1+l^4[m^2\phi^2-4[s(\partial_s\phi)^2]]}, \tag{C9}$$

which is the same as Eq. (2.6), so it reduces to the equation of motion of Heisenberg.

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