

**Spin-axion coupling**Alexander B. Balakin<sup>\*</sup> and Vladimir A. Popov<sup>†</sup>*Department of General Relativity and Gravitation, Institute of Physics, Kazan Federal University,  
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We establish a new covariant phenomenological model, which describes an influence of pseudoscalar (axion) field on spins of test massive particles. The model includes general relativistic equations of particle motion and spin evolution in background pseudoscalar (axion), electromagnetic and gravitational fields. It describes both the direct spin-axion coupling of the gradient type and indirect spin-axion interaction mediated by electromagnetic fields. Special attention is paid to the direct spin-axion coupling caused by the gradient of the pseudoscalar (axion) field. We show that it describes a spin precession, when the pseudoscalar (axion) field is inhomogeneous and/or nonstationary. Applications of the model, which correspond to the three types of four-vectors attributed to the gradient of the pseudoscalar (axion) field (timelike, spacelike, and null), are considered in detail. These are the spin precessions induced by relic cosmological axions, axions distributed around spherically symmetric static objects, and axions in a gravitational wave field, respectively. We discuss features of the obtained exact solutions and some general properties of the axionically induced spin rotation.

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**I. INTRODUCTION**

The terms *angular momentum* and *spin* are considered in theories of gravity in two main aspects. First, we face with them, when we study the gravitational field formed by a single rotating extended body (e.g., black holes, neutron stars, etc. [1,2]), or a system of bodies rotating around an attracting center (e.g., stars in spiral galaxies [3,4]). Second, we deal with the polarization or spin, when we investigate the gravitationally induced dynamics of test particles, which possess vector (or pseudovector, respectively) degrees of freedom. This sector of investigations is usually connected with experiments in gravitational physics (see, e.g., [5,6] and references therein).

According to the standard terminology the rotating extended body or pointlike particle can be described by an antisymmetric tensor of total moment  $S_{ik}$ , which can be decomposed as  $S_{ik} = \delta_{ik}^{mn} L_m U_n - \epsilon_{ikmn} S^m U^n$ , using the unit velocity four-vector  $U^n$  of the body. The quantity  $L^m$  can be indicated as a four-vector of the orbital moment, and the (pseudo)vector  $S^m$  plays the role of a classical spin four-vector. The term  $\delta_{ik}^{mn}$  is the four-indices Kronecker tensor and  $\epsilon_{ikmn}$  is the Levi-Civita (pseudo)tensor. Evolutionary equations for the total moment  $S_{ik}$  and/or for its constituents  $L^m$  and  $S^m$  have been investigated in various contexts by many authors. The list of obtained results is very long, and we would like to attract attention to the following two only.

In 1959 Bargmann, Michel and Telegdi (BMT) obtained the covariant equations for classical spin particles with

anomalous magnetic moment [7]. This model describes a spin evolution in the framework of special relativity as a generalization of the nonrelativistic theory elaborated by Thomas [8], Frenkel [9] and Bloch [10]. Being completed by the spin-curvature coupling terms, introduced earlier by Mathisson [11] and Papapetrou [12], this model became a starting point for numerous investigations of spin particle dynamics in general relativity (see, e.g., [13–20] and references therein). When it was necessary to include electromagnetic interactions, these investigations used the Faraday-Maxwell electrodynamics. Scientific events of the two past decades attracted the attention to its generalization called *axion electrodynamics*. In light of the hypothesis about the axionic nature of the dark matter this theory seems to be more appropriate for describing the cosmic electrodynamics than the standard Faraday-Maxwell theory.

The story, how axions were associated with dark matter particles, is well known. The dark matter, a cosmic substance, which neither emits nor scatters the electromagnetic radiation, is assumed to accumulate about 23% of the Universe's energy. The mass density distribution of the dark matter is presented in the astrophysical catalogues as a result of observations and theoretical simulations (see, e.g., [21–25] for details, review and references). The origin of the dark matter is not yet established. One of the most attractive hypothesis links the dark matter with massive pseudo-Goldstone bosons. The particles of this type were postulated in 1977 by Peccei and Quinn [26] in order to solve the problem of strong *CP*-invariance, and were introduced into the high-energy physics as new light bosons by Weinberg [27] and Wilczek [28] in 1978. Later these pseudobosons were indicated as *axions* and

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now they are considered as the most appropriate candidate for dark matter particles [29–37].

The description of these axions in terms of the field theory is based on the introduction of a pseudoscalar field  $\phi$ , which is assumed to interact with electromagnetic and  $SU(2)$ -symmetric gauge fields. The theory of interaction between electromagnetic and pseudoscalar fields was elaborated by Ni in 1977 [38]. Due to the works of Sikivie (see, e.g., [39]) we recognize now this sector of science as *axion electrodynamics*; a number of effects such as the axionically induced optical activity [40,41], magnetoelectric cross effects [42,43], etc., were predicted in the framework of this theory.

Thus, on the one hand, the axions are associated in our mind with the cosmic dark matter, which regulates key gravitational processes in our Universe, for instance, provides specific (flat) distribution of star velocities in the spiral galaxies (see, e.g., [44]). On the other hand, the axions produce the photon polarization rotation, in other words, the axionic dark matter forms an electro-dynamically active chiral cosmic medium. Clearly, we could try to find fingerprints of the axionic dark matter in the electromagnetic signals associated with astrophysical and geophysical phenomena [45–49].

Returning to the problem of influence of the (axionic) dark matter on the orbital moment, polarization and spin, we can say the following. First, definitely, the dark matter influences the orbital moment of star gas in spiral galaxies, providing the observed flat profiles of the velocity distribution [44]. Second, the axionic dark matter is predicted to rotate the photon polarization [40]. Third, due to the first and second arguments, one can expect that the axionic dark matter should also influence the particle spin.

In fact, the theoretical discussions about a new branch in the axion physics, i.e., the theory of *spin-axion coupling*, were opened in 1980s of the past century; it was a discussion concerning the axionically induced spin dependent long range forces [50]. The long range spin dependent forces were in a focus of experimental investigations (see, e.g., [51]), and this circumstance gave the impetus for development of new experimental programs devoted to detection of axions, based on their interactions with nuclear spins, electric and magnetic moments, etc. (see, e.g., [52–60] and references therein). This branch of axion physics was indicated as spin-axion coupling by analogy with the axion-photon and the axion-gluon couplings. Nevertheless the term spin-axion coupling is rather wide and two theoretical approaches have to be distinguished. Experimental works in the terrestrial laboratories deal with bonded spin particles in the specific material media, and these *bonded* particles are, in average, at rest with respect to these media. For these purposes the nonrelativistic formalism of condensed matter physics is adequate and describes correctly the models of spin dynamics. When one deals with *free* spin particles, which move with high velocity

(e.g., in storage rings) or in a vicinity of sources of strong gravitational and axion fields (e.g., near axion stars), the covariant formalism of high energy physics is necessary for description of the model.

From the mathematical point of view, our goal is to develop the covariant version of the theory of spin-axion coupling, and we consider extended dynamic equations for relativistic spin particle. The corresponding extensions include axionic modifications of the Bargmann-Michel-Telegdy forcelike term, and gradient-type modifications of a rotatory term. From the physical point of view, adding the third entity, the axion, to the interacting pair of photon and charged spin particle, one can expect that three channels of interaction will be activated: first, the already known direct axion-photon coupling; second, the direct spin-axion coupling; third, the indirect spin-axion coupling mediated by the axion-photon interaction. In this paper we intend to discuss in detail the two last channels of the photon-axion-spin interactions.

The paper is organized as follows. In Sec. II we analyze some details of the well-known model of axion-photon coupling to motivate our new model of spin-axion coupling. In particular, we derive an equation of the photon polarization precession induced by a gradient of the pseudoscalar (axion) field. In Sec. III we generalize the results of Sec. II for massive particles and reconstruct phenomenologically equations of spin-particle dynamics in the axion environment. We divide the obtained terms into three types: first, the terms indicated as axionic generalizations of the Bargmann-Michel-Telegdy terms; second, the terms describing direct spin-axion coupling; third, the nonminimal terms. In Sec. IV we consider four applications of the model with the direct spin-axion coupling. In Sec. IV A we obtain an exact solution to the dynamic equations for the massive spin particle, which moves straightforwardly and interacts with the relic cosmic axions. In Sec. IV B we apply the formalism to the model of relativistic charged spin particle motion in a storage ring with magnetic field. In Sec. IV C we discuss the spin-axion coupling in the static spherically symmetric gravitational background. We studied in detail two cases, describing radial and circular particle motion, respectively. The exact solutions for the spin precession in the field of pp-wave gravitational field provided by axions are presented in Sec. IV D. In Sec. V we discuss the common details of the axionically induced spin precession, which were revealed in all three submodels.

## II. CLASSICAL ANALOGY: PHOTON POLARIZATION ROTATION INDUCED BY THE AXION FIELD

In 1977 Ni [38] obtained equations for the electromagnetic field coupled to a pseudoscalar field  $\phi$ . The initial form of the currentless electrodynamic equations was the following:

$$\nabla_l [F^{kl} + \phi F^{*kl}] = 0. \quad (1)$$

This equation is derived by variation with respect to the potential four-vector  $A_i$  of the electromagnetic part  $S_{(\text{EM})}$  of the total action functional, where

$$S_{(\text{EM})} = \int d^4x \sqrt{-g} \left[ \frac{1}{4} F_{mn} F^{mn} + \frac{1}{4} \phi F_{mn}^* F^{mn} \right]. \quad (2)$$

Here  $F_{mn}$  is the Maxwell tensor, and  $F^{*kl} = \frac{1}{2} \epsilon^{klmn} F_{mn}$  is its dual tensor. As usual,  $\epsilon^{klmn} \equiv \frac{E^{klmn}}{\sqrt{-g}}$  is the Levi-Civita tensor, based on the completely antisymmetric Levi-Civita symbol  $E^{klmn}$  with  $E^{0123} = 1$ . The Maxwell tensor  $F_{mn}$  is connected with the electromagnetic potential  $A_k$  in the regular way:

$$F_{mn} = \nabla_m A_n - \nabla_n A_m, \quad (3)$$

so, that the well-known relationship

$$\nabla_l F^{*kl} = 0 \quad (4)$$

converts into the identity. Due to (4) the basic equation (1) simplifies as follows:

$$\nabla_l F^{kl} = -F^{*kl} \nabla_l \phi. \quad (5)$$

In order to transform Eq. (5) to a one-photon form we use the standard procedure. First, we follow the approximation of geometrical optics, and represent the potential  $A_m$  as

$$A_m = a_m e^{i\Psi}. \quad (6)$$

Here  $\Psi$  is a rapidly varying phase, the so-called eikonal. Its gradient four-vector  $k_j \equiv \nabla_j \Psi$  has a sense of a wavelength four-vector of the photon. The amplitude  $a_m$  and its derivatives are assumed to be slowly varying functions, i.e.,  $\max |\nabla_j a_m| \ll \min |k_j a_m|$ . In these terms the Lorentz condition,

$$\nabla_m A^m = 0, \quad (7)$$

gives two relations. The first one,  $a_m k^m = 0$ , appears in the first order approximation and means that the polarization is orthogonal to the wavelength four-vector. In the next order we obtain  $\nabla_m a^m = 0$ . The leading order approximation in Eq. (5) in the context of (6) yields  $k_m k^m = 0$ , the ordinary eikonal equation, implying that the photons travel along null geodesics. The tangent vector to the photon world-line can be defined as  $k^m = \hbar \frac{dx^m}{d\tau}$ , where  $\tau$  is an affine parameter along the line, and  $\hbar$  is the Planck constant.

The differential consequence of the eikonal equation  $k_m k^m = 0$  is  $k^m \nabla_n k_m = 0$ . Since the quantity  $k^m$  is the gradient four-vector, the relation  $\nabla_m k_n = \nabla_n k_m$  is valid, so we obtain additionally that

$$k^m \nabla_m k^n = 0. \quad (8)$$

For an illustration of the main idea we assume that the wavelength vector  $k^m$  is divergence-free, i.e.,  $\nabla_m k^m = 0$  (or equivalently,  $\nabla^m \nabla_m \Psi = 0$ ). This can be realized in many models, e.g. in the Minkowski space-time, in the space-time with plane-wave symmetry. Then the first order approximation in Eq. (5) gives the following equation for the evolution of the amplitude  $a^j$ :

$$k^m \nabla_m a^j = \frac{1}{2} \epsilon^{jlpq} \nabla_l \phi a_p k_q. \quad (9)$$

The convolution of this equation with  $a_j$  yields

$$\frac{1}{2} k^m \nabla_m (a^j a_j) = 0, \quad (10)$$

i.e.,  $a^j a_j \equiv -a^2$  is constant along the photon world-line with the tangent vector  $k^l$ . This fact allows us to introduce the unit spacelike polarization four-vector  $\xi^j$  as follows:

$$a^j = a \xi^j, \quad \xi^j \xi_j = -1, \quad (11)$$

and to exclude the amplitude  $a$  from Eq. (9):

$$k^m \nabla_m \xi^j = \frac{1}{2} \epsilon^{jlpq} \nabla_l \phi \xi_p k_q. \quad (12)$$

Using the covariant differential operator

$$\frac{D}{D\tau} \equiv \frac{dx^m}{d\tau} \nabla_m, \quad (13)$$

one can rewrite Eqs. (8) and (12) as a pair of basic equations of photon evolution: first, the dynamic equation

$$\frac{Dk^j}{D\tau} = 0, \quad (14)$$

and second, the equation of polarization rotation

$$\frac{D\xi^j}{D\tau} = \frac{1}{2} \epsilon^{jlpq} \nabla_l \phi \xi_p \frac{dx^q}{d\tau}. \quad (15)$$

It is clear that in terms of one-photon description three scalar quantities,  $k^m k_m$ ,  $k_m \xi^m$ , and  $\xi^m \xi_m$ , are the integrals of motion for the set of equations (14), (15), i.e., they remain constant along the photon world-line,  $k^m k_m = \text{const} = 0$ ,  $k_m \xi^m = \text{const} = 0$ ,  $\xi^m \xi_m = -1$ .

There is an obvious analogy between equations of motion for the massless photons and massive fermions. Based on this analogy, we can replace the polarization four-vector  $\xi^j$  by the spin four-(pseudo)vector  $S^j$ , and introduce the timelike particle momentum four-vector  $p^k = mU^k = m \frac{dx^k}{ds}$  instead of the null wave four-vector  $k^j$  (here and below we consider the system of units, in which  $c = 1$ ). As

a natural extension of Eqs. (14) and (15) we obtain the following evolutionary equations for the fermion particle:

$$\frac{Dp^j}{D\tau} = 0, \quad \frac{DS^j}{D\tau} = \frac{\alpha}{2m} \epsilon^{jl}{}_{pq} \nabla_l \phi S^p p^q, \quad (16)$$

where  $\alpha$  is some parameter introduced phenomenologically. The presented analogy could explain the appearance of the rotational term in the right-hand side of Eq. (16). However, some fermions possess electric charges and thus interact with electromagnetic fields, and a phenomenological generalization of Eqs. (14) and (15) requires more sophisticated efforts. We will take into account these interactions in the next section.

### III. SPIN-PARTICLE DYNAMICS IN AN AXION ENVIRONMENT

#### A. Basic equations

Let us consider the evolution of a relativistic point particle with an electric charge and a spin four-vector. Dynamic equations for the particle momentum  $p^i$  and for the spin four-vector  $S^i$  can be written as

$$\frac{Dp^i}{D\tau} = \mathcal{F}^i, \quad \frac{DS^i}{D\tau} = \mathcal{G}^i, \quad (17)$$

i.e., the rates of change of the given quantities are predetermined by the corresponding forcelike terms,  $\mathcal{F}^i$  and  $\mathcal{G}^i$ , respectively. Let us present, first, three general properties of the four-vectors  $\mathcal{F}^i$  and  $\mathcal{G}^i$ .

(i) The mass of the particle,  $m$ , defined from the normalization law  $p^i p_i = m^2 c^2$ , is assumed to be conserved quantity, providing the four-vector  $\mathcal{F}^i$  to be orthogonal to the momentum:

$$p_i \mathcal{F}^i = p_i \frac{Dp^i}{D\tau} = \frac{1}{2} \frac{D}{D\tau} (p_i p^i) = 0. \quad (18)$$

(ii) Similarly, we assume that the scalar square of the spacelike spin four-vector is constant, i.e.,  $S^i S_i = \text{const} = -S^2$ . Then using the second equation from (17) we obtain that

$$S_i \mathcal{G}^i = S_i \frac{DS^i}{D\tau} = \frac{1}{2} \frac{D}{D\tau} (S_i S^i) = -\frac{1}{2} \frac{D}{D\tau} S^2 = 0, \quad (19)$$

or in other words, that the forcelike term  $\mathcal{G}^i$  is orthogonal to the spin four-vector.

(iii) Finally, we assume that the spin four-vector is orthogonal to the momentum four-vector,  $S^i p_i = \text{const} = 0$ ; then one obtains that

$$\frac{D}{D\tau} (p^i S_i) = 0 \Rightarrow \mathcal{F}^i S_i + \mathcal{G}_i p^i = 0. \quad (20)$$

In the minimal theory the quantities  $\mathcal{F}^i$  and  $\mathcal{G}^i$  depend on the particle momentum  $p^k$ , spin  $S^l$ , Maxwell tensor  $F_{mn}$  and its dual  $F_{mn}^*$ , as well as on the pseudoscalar (axion) field  $\phi$  and its gradient four-vector  $\nabla_k \phi$ . In the non-minimally extended theory the quantities  $\mathcal{F}^i$  and  $\mathcal{G}^i$  can include the Riemann tensor  $R_{klm}^i$ , Ricci tensor  $R_{ik}$  and Ricci scalar  $R$ ; also the covariant derivatives of the Maxwell tensor,  $\nabla_s F_{mn}$ , and of the Riemann tensor  $\nabla_s R_{ikmn}$  can appear in the decomposition of  $\mathcal{F}^i$  and  $\mathcal{G}^i$ . Three equations (18)–(20) are satisfied identically, when

$$\mathcal{F}^i = \omega^{ik} p_k, \quad \mathcal{G}^i = \omega^{ik} S_k, \quad (21)$$

where  $\omega^{ik}$  is an arbitrary antisymmetric tensor,  $\omega^{ik} = -\omega^{ki}$ . As a first step, we remind the reader what this tensor is for the most known example, the Bargmann-Michel-Telegdi model [7].

#### B. Bargmann-Michel-Telegdi model

Let us consider the Bargmann-Michel-Telegdi model, for which the relevant forcelike term  $\mathcal{F}^i$  is the Lorentz force

$$\mathcal{F}^i = \frac{e}{m} F^i{}_k p^k, \quad (22)$$

and  $\mathcal{G}^i$  is of the form

$$\mathcal{G}^i = \frac{e}{2m} \left[ g F^i{}_k S^k + \frac{(g-2)}{m^2} p^i F_{kl} S^k p^l \right]. \quad (23)$$

Here  $g$  denotes the so-called  $g$ -factor; the case  $g \neq 2$  means that the particle possesses an anomalous magnetic moment. Clearly, these terms can be written in the form (21), when

$$\omega^{ik} = \omega_{(0)}^{ik} = \frac{e}{2m} \left[ g F^{ik} + \frac{(g-2)}{m^2} \delta_{mn}^{ik} p_j F^{jm} p^n \right], \quad (24)$$

where  $\delta_{mn}^{ik} = \delta_m^i \delta_n^k - \delta_n^i \delta_m^k$  is the four-indices Kronecker tensor. The important detail of this model is that the force  $\mathcal{F}^i$  does not contain the spin four-vector, and the term  $\mathcal{G}^i$  is linear in  $S^k$ . There is a simple motivation of such model construction. In the quasiclassical approach one uses the decomposition of microscopic equations with respect to small quantity  $\hbar$ , the Planck constant. Although the spin of particle enters the microscopic equations in the product  $\hbar \cdot S$ , the Planck constant is not involved into the classical dynamic equations. Therefore one has to exclude the quantity  $\hbar \cdot S$  from (24) to provide that both left-hand and right-hand sides of Eq. (17) are of the same order in  $\hbar$ , and the multiplier  $\hbar$  can be eliminated.

### C. Electrically charged spin particle in electromagnetic and axion fields

#### 1. Reconstruction of the force-type sources

Keeping in mind the general relationships (21) we reconstruct the tensor  $\omega^{ik}$  using the following ansatz:

- (A) The tensor  $\omega^{ik}$  does not contain the four-vector  $S^i$ .
- (B) The tensor  $\omega^{ik}$  is up to the first order in the Maxwell tensor  $F_{ik}$ .
- (C) The tensor  $\omega^{ik}$  is linear in the pseudoscalar (axion) field  $\phi$ , or in its gradient four-vector  $\nabla_i\phi$ .
- (D) The tensor  $\omega^{ik}$  is linear in the Riemann tensor.

The first two requirements are the same as in the Bargmann-Michel-Telegdi model. The third point is a new detail, which appears as a natural extension of this model for the case of particles interacting with the pseudoscalar field. The last point implies that the model can be minimal, when  $R^i{}_{klm}$  does not enter the tensor  $\omega^{ik}$ , and nonminimal, when there are terms containing the Riemann tensor and its linear convolutions. Below we consider all the appropriate constructions, which satisfy the requirements (A)–(D) and can be added to the tensor  $\omega_{(0)}^{ik}$  (24).

Minimal (curvature free) terms linear in the axion field  $\phi$  itself can be represented as follows:

$$\omega_{(1)}^{ik} = \frac{e\lambda}{2m} \phi \left[ g_A F^{*ik} + \frac{(g_A - 2)}{m^2} \delta_{mn}^i{}^k p_j F^{*jm} p^n \right]. \quad (25)$$

In fact,  $\omega_{(1)}^{ik}$  can be obtained from  $\omega_{(0)}^{ik}$  with replacements  $F_{mn} \rightarrow F_{mn}^*$  and  $g \rightarrow g_A$ , where the coupling constant  $g_A$  is an axionic analog of  $g$ -factor. Dimensionless parameter  $\lambda$  is equal to one, if the nature admits this coupling term, and  $\lambda = 0$ , if it does not admit.

Minimal terms linear in the gradient four-vector of the axion field can be written as follows:

$$\begin{aligned} \omega_{(2)}^{ik} = & \frac{e\mu}{2m} p^l \nabla_l \phi \left[ g_G F^{*ik} + \frac{(g_G - 2)}{m^2} \delta_{mn}^i{}^k p_j F^{*jm} p^n \right. \\ & \left. + \omega_{23} \epsilon^{ikjn} p_n p^s F_{js} \right] \\ & + \frac{\nabla_l \phi}{m} [\omega_{24} \epsilon^{ikmn} p_n F_m{}^l + \omega_{25} \epsilon^{ikln} p_n]. \end{aligned} \quad (26)$$

Here, in addition to the axionic analog of the  $g$ -factor  $g_A$ , we introduced its gradient-type analog  $g_G$ . The constant  $\mu$  in (26) plays the same role as the constant  $\lambda$  in (25). Other constants have no direct analogs, and we indicated them as  $\omega_{23}$ ,  $\omega_{24}$ , etc., where the first index 2 is an indicator that the decomposition relates to the term  $\omega_{(2)}^{ik}$ . The last term in (26) is a unique element of the presented irreducible decomposition, which does not contain the Maxwell tensor  $F^{mn}$ .

Note that this term with  $\omega_{25}$  in front corresponds to the right-hand side of Eq. (16) with the multiplier  $\alpha$ .

In order to classify *nonminimal* terms in the theory of spin-axion coupling we can use the following procedure. First, we replace the Maxwell tensor  $F^{ik}$  in (24), (25), and (26) with the tensor of nonminimal polarization-magnetization  $M^{ik} = \mathcal{R}^{ikmn} F_{mn}$ , where the so-called nonminimal susceptibility tensor  $\mathcal{R}^{ikmn}$  is introduced according to the rule

$$\begin{aligned} \mathcal{R}^{ikmn} \equiv & \frac{1}{2} q_1 R (g^{im} g^{kn} - g^{in} g^{km}) + q_3 R^{ikmn} \\ & + \frac{1}{2} q_2 (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}). \end{aligned} \quad (27)$$

Here  $R^{mn}$  is the Ricci tensor,  $R$  is the Ricci scalar,  $q_1, q_2, q_3$  are the nonminimal coupling constants (see, e.g., [61–64] for details). Similarly, we replace the dual Maxwell tensor  $F^{*ik}$  with the tensor  $\mathcal{M}^{ik}$ :

$$\mathcal{M}^{ik} = [\alpha_1 {}^* \mathcal{R}^{ikmn} + \alpha_2 \mathcal{R}^{*ikmn}] F_{mn}, \quad (28)$$

where  ${}^* \mathcal{R}^{ikmn}$  and  $\mathcal{R}^{*ikmn}$  are the left-dual and right-dual tensors of nonminimal susceptibility, respectively. The corresponding coupling parameters  $\omega_{ab}$  should be replaced with  $\tilde{\omega}_{ab}$ . Concerning the last term in (26), which does not contain the Maxwell tensor, our strategy is to add the following nonminimal analogs:

$$\begin{aligned} \omega_{(\text{NM})}^{ik} = & \nabla_l \phi \{ \tilde{\omega}_{25} [\tilde{\alpha}_1 {}^* \mathcal{R}^{iklj} + \tilde{\alpha}_2 \mathcal{R}^{*iklj}] p_j \\ & + \tilde{\omega}_{26} p^l \epsilon_{mn}^{ik} \mathcal{R}^{mq} p^n p_q \}. \end{aligned} \quad (29)$$

Here  $\mathcal{R}^{mq} = \mathcal{R}^{mnpq} g_{ns}$ . As a result of the described procedure, we deal with a large number of phenomenologically introduced coupling parameters. As a first step, below we restrict ourselves by the minimal model, i.e., by the case when  $\tilde{\omega}_{ab} = 0$ ; the nonminimal model contains a lot of specific details and will be discussed in a special paper.

#### 2. Dynamic equations

Following the representation of the total tensor  $\omega^{ik} = \omega_{(0)}^{ik} + \omega_{(1)}^{ik} + \omega_{(2)}^{ik}$ , described above, we obtain the minimal dynamic equation in the form

$$\frac{Dp^i}{D\tau} = \frac{e}{m} \{ F^{ik} p_k + F^{*ik} p_k [\lambda \phi + \mu (p^l \nabla_l \phi)] \}. \quad (30)$$

The first term in the right-hand side of this equation is the usual Lorentz force. The second term can be interpreted as an axionic analog of Lorentz force, in which the Maxwell tensor  $F^{ik}$  is replaced by its dual tensor  $F^{*ik}$ , and the electric

charge  $e$  is replaced by an effective pseudocharge  $e\phi$ . If  $\lambda = 0$  identically, the model is characterized by vanishing pseudocharge. The coupling constant  $\mu$  introduces a completely new term, which contains the gradient four-vector of an effective pseudocharge  $e\phi$ . We faced with similar situation in the  $SU(2)$  and  $SU(3)$ -symmetric gauge models, in which the isospin and color charges, respectively, were considered as functions, and their derivatives also entered the master equations [65,66].

The evolutionary equation for the spin with the redefinitions, mentioned above, takes the form

$$\begin{aligned} \frac{DS^i}{D\tau} = & \frac{e}{2m} \left[ gF^{ik} S_k + \frac{(g-2)}{m^2} p^i F_{kl} S^k p^l \right] \\ & + \frac{e\lambda}{2m} \phi \left[ g_A F^{*ik} S_k + \frac{(g_A-2)}{m^2} p^i F_{*kl} S^k p^l \right] \\ & + \frac{e\mu}{2m} (p^l \nabla_l \phi) \left[ g_G F^{*ik} S_k + \frac{(g_G-2)}{m^2} p^i F_{*kl} S^k p^l \right] \\ & + \frac{e\mu}{2m} (p^l \nabla_l \phi) \omega_{23} \epsilon^{ikmn} S_k p_n p^j F_{mj} \\ & + \frac{\omega_{24}}{m} \nabla^l \phi \epsilon^{ikmn} S_k p_n F_{ml} + \frac{\omega_{25}}{m} \nabla_l \phi \epsilon^{ikln} S_k p_n. \end{aligned} \quad (31)$$

The first term in the right-hand side of this equation is the usual term attributed to the Bargmann-Michel-Telegdi model with anomalous magnetic moment; other terms can be indicated as its axionic analogs. Only one term with  $\omega_{25}$  in front does not contain the Maxwell tensor thus describing the direct spin-axion coupling.

### 3. Our further strategy

We have established the phenomenological model, in which three channels of spin-axion coupling can be distinguished. The first channel is direct: it works even if there are no electromagnetic fields ( $F_{mn} = 0$ ), and the tidal (nonminimal) interactions are absent ( $\mathcal{R}^{ikmn} = 0$ ). The second channel is indirect, the corresponding spin-axion coupling is mediated by the Maxwell field ( $F_{mn} \neq 0$ ,  $\mathcal{R}^{ikmn} = 0$ ). The third channel is also indirect, and it can be opened when the model is nonminimal ( $\mathcal{R}^{ikmn} \neq 0$ ). Below we put  $F_{mn} = 0$  in (31) and consider effects of a direct spin-axion interaction only. The next papers will be devoted to a systematic study of effects mediated by electromagnetic fields of various structures. In the future we also intend to add nonminimal couplings to the direct and indirect models of spin-axion coupling.

## IV. APPLICATIONS OF THE MODEL WITH DIRECT SPIN-AXION COUPLING

We focus now on the direct spin-axion interactions, i.e., we assume that  $F_{mn} = 0$  and  $\mathcal{R}^{ikmn} = 0$ , and the gradient four-vector  $\nabla_i \phi$  is nonvanishing due to the coupling of

axions to the gravity field. In this case Eqs. (30) and (31) obtain the simple form

$$\frac{Dp^i}{D\tau} = 0, \quad \frac{DS^i}{D\tau} = \frac{\omega_{25}}{m} \nabla_l \phi \epsilon^{ikln} S_k p_n. \quad (32)$$

Below we study three examples. The first one relates to the cosmological context, and the gradient four-vector  $\nabla_i \phi$  is timelike (i.e.,  $\nabla_i \phi \nabla^i \phi > 0$ ). The second example relates to the case with the spacelike gradient four-vector (i.e.,  $\nabla_i \phi \nabla^i \phi < 0$ ), which can be realized, e.g., in a spherically symmetric static space-time. The third example corresponds to the case  $\nabla_i \phi \nabla^i \phi = 0$ , which can be realized in space-times with plane-wave symmetry (gravitational waves). In all three cases we consider the spin particle as a test one. It moves in a given space-time through the pseudoscalar field, which obeys the equation

$$\nabla^m \nabla_m \phi + \mathcal{V}'(\phi^2) \phi = 0. \quad (33)$$

Here  $\mathcal{V}(\phi^2)$  is the potential of the pseudoscalar (axion) field. This equation is derived from the axionic part of the total Lagrangian

$$S_{(\text{axion})} = \int d^4x \sqrt{-g} \frac{\Psi_0^2}{2} [\mathcal{V}(\phi^2) - \nabla_k \phi \nabla^k \phi], \quad (34)$$

where the constant  $\Psi_0$  is reciprocal to the coupling constant of the axion-photon interaction  $\rho_{A\gamma\gamma}$ , i.e.,  $\frac{1}{\Psi_0} = \rho_{A\gamma\gamma}$  (see, e.g., [64]). Since the direct spin-axion effect appears if and only if the gradient four-vector  $\nabla_k \phi$  is nonvanishing, i.e.,  $\nabla_k \phi \neq 0$ , we assume that just the gravitational field produces the inhomogeneity or nonstationarity of the axionic field  $\phi$ .

### A. Spin coupling to relic dark matter axions

Dark matter hypothetically contains relic axions born in the early Universe, and in the cosmological context the pseudoscalar (axion) field  $\phi$  can be considered as a function of cosmological time only,  $\phi(t)$ . The time variable  $t$  corresponds to the following choice of the background space-time metric:

$$ds^2 = dt^2 - a^2(t) \left[ \left( dx^1 \right)^2 + \left( dx^2 \right)^2 + \left( dx^3 \right)^2 \right]. \quad (35)$$

In this model the gradient four-vector  $\nabla_i \phi$  is of the form  $\nabla_i \phi = U_i \dot{\phi}$ , where  $U_i \equiv \delta_i^0$  is the global velocity four-vector. In the context of cosmological application we consider the axion field potential  $\mathcal{V}(\phi^2)$  to be of the form  $\mathcal{V}(\phi^2) = m_{(a)}^2 \phi^2$ , where  $m_{(a)}$  is the axion mass. Then the equation of the axion field evolution (33) is

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m_{(a)}^2\phi = 0, \quad (36)$$

where the dot denotes a derivative with respect to time.

We consider the space-time background to be fixed by the corresponding gravity field equations, and the scale factor  $a(t)$  is a known function of time [67]. For instance, in the de Sitter-type regime of cosmological expansion the scale factor is of the form  $a(t) = a(t_0) \exp[H_0(t - t_0)]$  with the constant Hubble function  $H(t) \equiv \frac{\dot{a}}{a} = H_0$ . When  $m_{(a)} > \frac{3}{2}H_0$  the solution to Eq. (36) is

$$\phi(t) = e^{-\frac{3}{2}H_0(t-t_0)} \left\{ \phi(t_0) \cos \Omega_{(a)}(t-t_0) + \frac{1}{\Omega_{(a)}} \left[ \dot{\phi}(t_0) + \frac{3}{2}H_0\phi(t_0) \right] \sin \Omega_{(a)}(t-t_0) \right\}, \quad (37)$$

where the effective axionic frequency is introduced as

$$\Omega_{(a)} \equiv \sqrt{m_{(a)}^2 - \frac{9}{4}H_0^2}. \quad (38)$$

When the electromagnetic field is absent, the equations of particle dynamics (32) for the metric (35) are reduced to

$$\frac{dp_j}{d\tau} = \frac{1}{2m} \delta_j^0 p^k p^l \dot{g}_{kl}. \quad (39)$$

The solution is known to be the following:

$$\begin{aligned} p_1(t) &= p_1(t_0), & p_2(t) &= p_2(t_0), & p_3(t) &= p_3(t_0), \\ p_0(t) &= \sqrt{m^2 + \frac{q^2}{a^2(t)}}, \end{aligned} \quad (40)$$

where  $q^2 \equiv p_1^2(t_0) + p_2^2(t_0) + p_3^2(t_0)$  is the constant quantity. Since the space-time is spatially isotropic, we assume that the particle had only one nonvanishing component at  $t = t_0$ , say,  $p_3(t_0) \neq 0$ , and consider (for the illustration) the following initial data:

$$p_1(t_0) = 0, \quad p_2(t_0) = 0, \quad S^3(t_0) = 0. \quad (41)$$

Clearly, for this case at an arbitrary time moment  $p_1(t) = p_2(t) = 0$  and  $p_3(t) = p_3(t_0)$ , so that the cosmological time  $t$  and the proper time  $\tau$  along the particle world-line are linked by the relationship

$$\tau = \int_{t_0}^t \frac{ma(t)dt}{\sqrt{m^2 a^2(t) + q^2}}. \quad (42)$$

The equation of spin dynamics (32) is now of the form

$$\frac{DS^i}{D\tau} = \frac{\omega_{25}}{m} e^{ik03} S_k p_3 \dot{\phi}. \quad (43)$$

Its right-hand side does not equal zero only for two values of indices:  $i = 1$  and  $i = 2$ . This means that using the property  $S^k p_k = S^0 p_0 + S^3 p_3 = 0$  we can write the equations for the components  $S^3$  and  $S^0$  as follows:

$$\frac{dS^3}{dt} + H_0 S^3 \left( 1 + \frac{p_3^2}{a^2 p_0^2} \right) = 0, \quad S^0 = -\frac{S^3 p_3}{p_0}. \quad (44)$$

For the initial value  $S^3(t_0) = 0$  these equations have the only trivial solution  $S^3(t) = 0$ ,  $S^0(t) = 0$ . In other words the transverse spin components only are influenced by the axion environment and the longitudinal component is not touched.

From the normalization condition  $S^k S_k = -S^2$  we obtain  $a^{-2}(t)(S_1^2 + S_2^2) = S^2$ , which is the hint to introduce two convenient variables,

$$S_+(t) = \frac{S_1}{a(t)}, \quad S_-(t) = \frac{S_2}{a(t)}, \quad (45)$$

so that

$$S_+^2 + S_-^2 = S^2 = \text{const}. \quad (46)$$

In these terms the equations for the transverse components take the form

$$\dot{S}_+ = -\Omega(t)S_-, \quad \dot{S}_- = \Omega(t)S_+, \quad (47)$$

where

$$\Omega(t) = \frac{\omega_{25} p_3(t_0) \dot{\phi}(t)}{a(t) p^0(t)}. \quad (48)$$

The solutions to (47) are of harmonic type

$$S_+ = S \cos \Psi(t), \quad S_- = S \sin \Psi(t), \quad (49)$$

where the phase of rotation is presented as a formal integral,

$$\Psi(t) = \int_{t_0}^t \Omega(t) dt + \Psi(t_0). \quad (50)$$

For the illustration of the obtained exact solution we assume, first, that the particle is ultrarelativistic [ $q^2 \gg m^2 a^2(t)$ ] and  $p_3(t_0)$  is positive; second, we choose the time moment  $t_0$  so that  $\dot{\phi}(t_0) = 0$ ; third, we put for simplicity  $\Psi(t_0) = \omega_{25} \phi(t_0)$ . Then, keeping in mind that nowadays  $m_{(a)} \gg H_0$ , we find the phase  $\Psi(t)$  in the explicit form [see (37)]

$$\Psi(t) = \sigma \cos m_{(a)}(t - t_0), \quad \sigma \equiv \omega_{25} \phi(t_0). \quad (51)$$

The solutions for the spin components can be now presented as follows:

$$\begin{aligned}
S_+ &= \mathcal{S} \cos\{\sigma \cos[m_{(a)}(t - t_0)]\} \\
&= J_0(\sigma) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(\sigma) \cos[2nm_{(a)}(t - t_0)],
\end{aligned} \tag{52}$$

$$\begin{aligned}
S_- &= \mathcal{S} \sin\{\sigma \cos m_{(a)}(t - t_0)\} \\
&= 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\sigma) \cos[(2n+1)m_{(a)}(t - t_0)],
\end{aligned} \tag{53}$$

where  $J_m(\sigma)$  are the Bessel functions of the first kind. We deal with sophisticated spin precession, for which the phase of precession  $\Psi(t)$  is the harmonic function oscillating with the axionic frequency  $\Omega_{(a)} = m_{(a)}$  and the amplitude  $\sigma = \omega_{25}\phi(t_0)$ .

### B. Spin precession of relativistic charged particle in storage rings

When a relativistic charged spin particle moves in the constant magnetic field  $B$ , the motion is known to be circular, and the quantity reciprocal to the Larmor frequency,  $\omega_L^{-1} = \frac{m}{eB}$ , predetermines the time scale of dynamic processes. In fact, for such motion the cosmological phenomena can be considered as extremely slow, and one can put  $H(t) \rightarrow 0$ . The scale factor  $a(t)$  can be replaced by constant  $a(t_0)$  and absorbed into the redefined coordinates [in fact, one can put  $a(t) \rightarrow 1$ ]. Since the size of the storage ring is much smaller than the typical size of the dark matter inhomogeneity, we can neglect a spatial dependence and consider the axion field as a function of time only. In our case the quantity  $\dot{\phi}$  can be expressed in terms of the energy density scalar  $W_{(a)}$  and pressure  $P_{(a)}$  attributed to the axionic dark matter as follows (see, e.g., [68]):

$$\dot{\phi} = \pm \frac{1}{\Psi_0} \sqrt{W_{(a)}(t) + P_{(a)}(t)}. \tag{54}$$

When the axionic dark matter is cold, i.e.,  $P_{(a)} = 0$ , and  $W_{(a)} = \rho_{(a)}$ , where  $\rho_{(a)}$  is the mass density of the dark matter, this formula is simplified, respectively, as

$$\dot{\phi} = \pm \frac{1}{\Psi_0} \sqrt{\rho_{(a)}(t)}. \tag{55}$$

Also, for the sake of simplicity, we can neglect the deformation of the initial magnetic field by the axionic field, and put equal to zero all the new coupling constants except  $\omega_{25}$ .

Let the magnetic field be directed along the  $x^3 \equiv z$  axis, i.e., only one component of the Maxwell tensor,  $F_{12} = \text{const}$ , is nonvanishing. In the cylindrical coordinates  $\{\rho, \varphi, z\}$  with the metric

$$ds^2 = dt^2 - (dz^2 + d\rho^2 + \rho^2 d\varphi^2) \tag{56}$$

the equation of particle motion,

$$\frac{Dp_j}{D\tau} = \frac{e}{m} F_{jk} P^k, \tag{57}$$

can be transformed as

$$\frac{dp_j}{dt} = -\frac{\rho}{p_0} \delta_j^\rho p^{\varphi^2} + \frac{e}{p_0} F_{jk} P^k, \tag{58}$$

and gives the evident solutions

$$p_z(t) = p_z(0) = 0, \quad p_\rho(t) = 0, \tag{59}$$

$$p_\varphi = -e\rho F_{\rho\varphi} = -e\rho^2 F_{12}, \tag{60}$$

$$\rho(t) = R = \text{const}, \quad \varphi(t) = \Omega_{(B)} t, \tag{61}$$

$$p_0 = \sqrt{m^2 + e^2 R^2 F_{12}^2}, \quad t = \tau \sqrt{1 + \frac{e^2 R^2 F_{12}^2}{m^2}}. \tag{62}$$

Here  $R$  is the radius of the circular orbit, and the quantity  $\Omega_{(B)}$  given by

$$\Omega_{(B)} = \frac{eF_{12}}{p_0} = \text{const} \tag{63}$$

is the relativistic angular frequency of rotation (the relativistic Larmor frequency).

The equations for the spin evolution

$$\frac{DS^i}{D\tau} = \frac{\omega_{25}\dot{\phi}}{m} \epsilon^{ik0n} S_k P_n + \frac{e}{m} F^i{}_k S^k \tag{64}$$

rewritten as

$$p_0 \frac{dS^i}{dt} + \frac{p_\varphi}{\rho} (g^{i\varphi} S^\rho - g^{i\rho} S^\varphi) = \frac{\omega_{25}\dot{\phi}}{\rho} E^{ik0\varphi} S_k P_\varphi + e F^i{}_k S^k \tag{65}$$

give

$$\dot{S}^0 = 0, \quad \dot{S}^\varphi = 0, \tag{66}$$

$$\dot{S}^z = \omega_{25}\dot{\phi}\Omega_{(B)} R S^\rho, \tag{67}$$

$$\dot{S}^\rho = -\omega_{25}\dot{\phi}\Omega_{(B)} R S^z. \tag{68}$$

Physically motivated solutions to these equations are

$$S^0(t) = 0, \quad S^\varphi(t) = 0, \tag{69}$$

$$S^z(t) = -\mathcal{S} \cos \Psi_{(H)}(t), \tag{70}$$

$$S^\rho(t) = \mathcal{S} \sin \Psi_{(H)}(t), \tag{71}$$



where the hybrid precession phase  $\Psi_{(H)}$  is given by

$$\Psi_{(H)}(t) = \Psi(t)\Omega_{(B)}R = \Psi(t) \frac{eF_{12}R}{\sqrt{m^2 + e^2R^2F_{12}^2}}, \quad (72)$$

and  $\Psi(t)$  is the axionic phase (we put here  $t_0 = 0$ )

$$\begin{aligned} \Psi(t) &= \omega_{25}[\phi(t) - \phi(0)] \\ &= \omega_{25} \left[ \phi(0)(\cos \Omega_{(a)}t - 1) + \frac{\dot{\phi}(0)}{\Omega_{(a)}} \sin \Omega_{(a)}t \right] \\ &\simeq \omega_{25}\dot{\phi}(0)t. \end{aligned} \quad (73)$$

Clearly, the spin four-vector is orthogonal to the particle momentum four-vector, i.e.,  $p_k S^k = 0$ . This solution can be illustrated as follows. If  $\omega_{25}\dot{\phi} = 0$ , the particle has the spin three-vector directed along the magnetic field, and this direction is conserved during the particle circular motion. If  $\omega_{25}\dot{\phi} \neq 0$  the spins start to precess in the plane  $\rho Oz$  according the law described by formulas (69)–(71); the frequency of the precession depends on the particle energy, or, equivalently, on the orbit radius  $R$ . From the experimental point of view, if the polarized beam of electrons is formed in a storage ring, and all the spins are initially directed perpendicularly to the ring plane, one can expect that the axionically induced spin rotation will start, and the distribution of the angles between the spins and the ring plane will be a predicted function of time and the particle energy. In the ultrarelativistic regime, when  $m \rightarrow 0$  effectively, one obtains from (72) that  $\Psi_{(H)}(t) \rightarrow \Psi(t)$  and the dependence on the parameter  $R$  disappears (some specific details of ultrarelativistic spin particle motion can be found also in [69]).

### C. Spin dynamics in the field of an axion star

In this application we consider a spherically symmetric static axionically active object, which is characterized by the metric

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (74)$$

The axion field is assumed to depend on the radial coordinate  $r$  only. The gradient four-vector  $\nabla_i\phi = \delta'_i\phi'(r)$  (the prime denotes the derivative with respect to  $r$ ) is now spacelike, i.e.,  $\nabla_i\phi\nabla^i\phi = -\frac{1}{A}\phi'^2 < 0$ . When we consider the distribution of the pseudoscalar (axion) field in this static model, and the gravity field is assumed to be strong, it seems to be reasonable to use the extended potential

$$\mathcal{V}(\phi^2) = m_{(a)}^2\phi^2 + \frac{1}{2}\nu_{(a)}(\phi^2 - \phi_*^2)^2. \quad (75)$$

The function  $\phi(r)$  satisfies now the equation

$$\begin{aligned} \phi'' + \phi' \left[ \frac{1}{2} \left( \frac{B'}{B} - \frac{A'}{A} \right) + \frac{2}{r} \right] \\ = A(r)\phi[m_{(a)}^2 + \nu_{(a)}(\phi^2 - \phi_*^2)], \end{aligned} \quad (76)$$

which is (due to the absence of electromagnetic field) formally the same as for the gravitating scalar field  $\Phi(r)$  (see, e.g., [70–72]). There is no need to present explicit solutions of (76) for our purposes. Examples of solutions with asymptotically flat space-time and scalar fields vanishing at  $r \rightarrow \infty$  can be found in [70–72].

In the static spherically symmetric case the plane of the particle motion can be chosen as the equatorial one  $\theta = \frac{\pi}{2}$ , and the first equation in (32) is known to give four integrals of motion [73]:

$$p_0 = K = \text{const}, \quad p_\theta = 0, \quad p_\varphi = -J = \text{const}, \quad (77)$$

$$p^r = \frac{1}{\sqrt{A}} \sqrt{\frac{K^2}{B} - \frac{J^2}{r^2} - E^2}, \quad E = \text{const}. \quad (78)$$

For the asymptotically flat space-time with  $A(\infty) = 1$  and  $B(\infty) = 1$  the normalization condition  $p_i p^i = m^2$  yields

$$K = p_0(\infty) = \sqrt{m^2 + p_r^2(\infty)}, \quad (79)$$

i.e., the constant  $K$  usually describes the particle energy at infinity. The constant  $J$  relates to the conserved particle angular momentum. The constant  $E$  regulates the parameter along the particle world-line, e.g., when  $d\tau = ds$ , we see that  $E = m$ .

In order to analyze properly the equations of spin evolution (32) we distinguish two specific types of particle motion: first, the radial motion; second, the motion along a circular orbit.

#### 1. Radial particle motion

For this type of motion  $p_\theta = 0$ ,  $p_\varphi = 0$  and the equations of spin evolution

$$\frac{DS^i}{D\tau} = \frac{\omega_{25}}{m} \phi'(r) \epsilon^{ikr0} S_k p_0 \quad (80)$$

can be split into two independent subsets. The first subset contains the components  $S^0$  and  $S^r$  only, and does not include the coupling term proportional to  $\omega_{25}$ . Keeping in mind that the condition  $S^k p_k = 0$  leads to  $S^0 p_0 + S^r p_r = 0$ , we can write this subset of equations as follows:

$$\frac{dS^0}{d\tau} + S^0 \left\{ \frac{B'(r)}{2mABp^r} \left[ 2 \frac{K^2}{B(r)} - m^2 \right] \right\} = 0, \quad (81)$$

$$\frac{dS^r}{d\tau} + S^r \left\{ \frac{p^r}{2m} \left( \frac{A'}{A} + \frac{B'}{B} \right) \right\} = 0. \quad (82)$$

As in the first application we assume that  $S^r(0) = 0$ , providing that  $S^0(0) = 0$  from the orthogonality condition. Then the solutions to the equations (81) and (82) are trivial  $S^0(\tau) = 0$  and  $S^r(\tau) = 0$ .

The second subset of (80),

$$\frac{dS^\theta}{d\tau} + S^\theta \left( \frac{p^r}{mr} \right) = -\frac{\omega_{25}\phi'}{m\sqrt{AB}r^2} p_0 S_\phi, \quad (83)$$

$$\frac{dS^\phi}{d\tau} + S^\phi \left( \frac{p^r}{mr} \right) = \frac{\omega_{25}\phi'}{m\sqrt{AB}r^2} p_0 S_\theta, \quad (84)$$

can be transformed into

$$\frac{dS_+}{dr} = -\Omega(r)S_-, \quad \frac{dS_-}{dr} = \Omega(r)S_+, \quad (85)$$

using the relation between  $\tau$  and  $r$  [see (78)],

$$\tau = \int_\infty^r dr \frac{m\sqrt{AB(r)}}{\sqrt{K^2 - E^2B(r)}}, \quad (86)$$

and the following definitions:

$$S_+ \equiv rS^\theta, \quad S_- \equiv rS^\phi, \quad (87)$$

$$\Omega(r) \equiv \frac{K\omega_{25}\phi'(r)}{\sqrt{K^2 - E^2B(r)}}. \quad (88)$$

Clearly, the solutions to (85) are

$$S_+(r) = \mathcal{S} \cos \Psi(r), \quad S_- = \mathcal{S} \sin \Psi(r), \quad (89)$$

$$\Psi(r) = \Psi(\infty) + \int_\infty^r dr \Omega(r). \quad (90)$$

Thus, when the particle moves in the radial direction, one deals with a spin turn in the plane  $(\theta, \phi)$ ; the quantity  $\Omega(r)$  plays the role of the rate of turn with respect to radial variable  $r$ , and  $\Psi(r)$  describes the cumulative angle of turn.

## 2. Circular particle motion

The circular motion is characterized by  $r = R = \text{const}$ , so that  $p^r = 0$  and thus

$$\frac{K^2}{B(R)} - \frac{J^2}{R^2} = E^2. \quad (91)$$

We consider a stable orbit and thus the radial component of the gravitational force should vanish on the orbit, providing  $\frac{Dp^r}{D\tau} = 0$ , or

$$\frac{B'(R)}{B^2(R)} = \frac{2J^2}{K^2R^3}. \quad (92)$$

Also, as previously, we have that  $p_0 = K$ ,  $p_\theta = 0$ ,  $p_\phi = -J$ , but now the particle cannot reach infinity, and we have to redefine the constant  $K$ . For instance, we obtain from (91) and (92) that

$$J^2 = \frac{E^2R^3B'(R)}{2B - RB'(R)}, \quad K = E\sqrt{\frac{2B^2(R)}{2B - RB'(R)}}. \quad (93)$$

The first equation of (93) gives implicitly the radius of the circular orbit as a function of orbital moment,  $R(J)$ . The second equation defines the energy of the particle on the given orbit as a function of obtained radius,  $K(R)$ . We assume that the inequality  $2B(R) > RB'(R)$  is satisfied [for instance, for the Schwarzschild metric  $B(r) = 1 - \frac{2GM}{r}$  this inequality means that  $R > \frac{3}{2}r_g = 3GM$ ].

The orthogonality condition  $S^k p_k = 0$  and the integrals of motion (77) provide that the components  $S^0$  and  $S^\phi$  are proportional one to another,  $S^0 = S^\phi \left( \frac{J}{K} \right)$ . For the circular orbit one can replace the differentiation with respect to proper time  $\tau$  with the azimuthal angle  $\phi$  due to the relationship  $\frac{d\phi}{ds} = \frac{p^\phi}{m} = \frac{J}{mR^2}$  (recall, that in the case  $s = \tau$  we obtain  $E = m$ ). With this replacement three independent equations for spin evolution take the following form:

$$\begin{aligned} \frac{dS^r}{d\phi} &= S^\phi \cdot H^r(R), & \frac{dS^\theta}{d\phi} &= S^\phi \cdot H^\theta(R), \\ \frac{dS^\phi}{d\phi} + \frac{S^r}{R} &= -S^\theta \cdot H^\phi(R). \end{aligned} \quad (94)$$

Here, for short, we introduced three auxiliary functions of the radius  $R$ :

$$\begin{aligned} H^r(R) &= \frac{RE^2B(R)}{K^2A(R)}, & H^\theta(R) &= H^\theta(R) \frac{K^2}{E^2B(R)}, \\ H^\phi(R) &= \frac{\omega_{25}R^2E^2\sqrt{B(R)}\phi'(R)}{KJ\sqrt{A(R)}}. \end{aligned} \quad (95)$$

The evident differential consequence of (94) and (95) is the following equation of the second order for  $S^\phi$ :

$$\frac{d^2S^\phi}{d\phi^2} + I^2(R)S^\phi = 0, \quad (96)$$

where

$$I^2(R) = \frac{1}{R}H^r + H^\theta H^\phi = \frac{E^2B}{K^2A} + \frac{\omega_{25}^2R^4E^2\phi'^2}{J^2A}. \quad (97)$$

For the positive metric functions  $B(R) > 0$  and  $A(R) > 0$  the solution to this equation is a harmonic function of the azimuthal angle

$$S^\varphi(\varphi) = C_1 \cos I\varphi + C_2 \sin I\varphi, \quad (98)$$

providing the following solutions in terms of  $\tau$ :

$$\begin{aligned} S^\varphi(\tau) &= C_1 \cos \Omega\tau + C_2 \sin \Omega\tau, \\ S^0(\tau) &= \frac{J}{K} [C_1 \cos \Omega\tau + C_2 \sin \Omega\tau]. \end{aligned} \quad (99)$$

The quantity  $\Omega = \frac{LJ}{mR^2}$  plays the role of frequency of the spin turn. Other components of the spin four-vector are, respectively,

$$\begin{aligned} S^r(\tau) &= \frac{H^r(R)}{I(R)} [C_1 \sin \Omega\tau - C_2 \cos \Omega\tau] + C_3, \\ S^\theta(\tau) &= \frac{H^\theta(R)}{I(R)} [C_1 \sin \Omega\tau - C_2 \cos \Omega\tau] + C_4. \end{aligned} \quad (100)$$

The constants of integration  $C_1, C_2, C_3, C_4$  are connected by the normalization condition

$$BS^{02} - AS^{r2} - r^2S^{\theta2} - r^2S^{\varphi2} = -S^2 = \text{const}, \quad (101)$$

which at  $r = R$  yields two relationships:

$$\begin{aligned} S^2 &= \frac{BE^2R^2}{K^2} (C_1^2 + C_2^2) + R^2C_4^2 \left[ 1 + \frac{\omega_{25}^2 R^4 K^2 \phi'^2}{J^2 B} \right], \\ C_3 &= -C_4 \frac{\omega_{25} R^3 \phi'(R) K}{J \sqrt{AB(R)}}. \end{aligned} \quad (102)$$

On the other hand, three independent constants of integration  $C_1, C_2, C_4$ , are connected with initial values  $S^\varphi(0), S^\theta(0), S^r(0)$  as follows:

$$\begin{aligned} C_1 &= S^\varphi(0), \quad C_2 = -\frac{S^r(0)}{RI(R)} - S^\theta(0) \frac{K^2 H^\theta}{E^2 IB(R)}, \\ C_4 &= -S^r(0) \frac{H^\theta}{RI^2} + S^\theta(0) \frac{H^r}{RI^2}. \end{aligned} \quad (103)$$

The problem is solved completely. The formulas (99) and (100) with constants given by (102) and (103) and auxiliary quantities (95) and (97) describe the axionically induced turn of the spin four-vector of the particle moving along the circular orbit around the static spherically symmetric gravitating object. When  $\omega_{25}\phi' = 0$ , we see that  $I = \frac{E}{K} \sqrt{\frac{B}{A}}$ , and the corresponding frequency  $\Omega = \frac{LJ}{mR^2}$  takes the form  $\Omega = \Omega_{(\text{geodesic})} = \frac{J}{KR^2} \sqrt{\frac{B}{A}}$ , describing the geodesic precession. This fact allows us to indicate the term  $\Omega_{(\text{axion})} = \frac{\omega_{25}\phi'(R)}{\sqrt{A(R)}}$  as the axionic frequency. With this terminology we can say that the total frequency  $\Omega$  satisfies the equality

$$\Omega = \sqrt{\Omega_{(\text{geodesic})}^2 + \Omega_{(\text{axion})}^2}, \quad (104)$$

and can be called a hybrid frequency of the geodesic-axionic precession. Finally, it should be mentioned that one can introduce local frequency  $\omega$  instead of  $\Omega$ , using the equality  $\omega dt = \Omega d\tau$ . Clearly, we obtain that  $\omega = \Omega B(R) \frac{m}{K}$ .

#### D. Spin precession induced by plane-symmetric axion-gravitational waves

The third application of the model relates to the case when the gradient four-vector  $\nabla_i\phi$  is the null one, i.e.,  $\nabla_i\phi\nabla^i\phi = 0$ . It can be realized, e.g., in the model with plane-wave symmetry [64,74,75]. The corresponding space-time metric is of the form

$$ds^2 = 2dudv - L^2[e^{2\beta}(dx^2)^2 + e^{-2\beta}(dx^3)^2], \quad (105)$$

where  $u = \frac{t-x^1}{\sqrt{2}}$  and  $v = \frac{t+x^1}{\sqrt{2}}$  are the retarded and advanced times, respectively, and two metric functions  $L(u)$  and  $\beta(u)$  depend on the retarded time  $u$  only. On the plane-wave front  $u = 0$  the initial data are fixed in the form

$$L(0) = 1, \quad L'(0) = 0, \quad \beta(0) = 0. \quad (106)$$

We assume that the background pseudoscalar (axion) field also depends on retarded time only,  $\phi = \phi(u)$ , providing the condition  $\nabla_i\phi\nabla^i\phi = 0$  automatically. Exact solutions of this type (in particular, the solution linear in the retarded time) can be found in [64].

The equations of particle dynamics in the metric (105) are known to yield (see, e.g., [66])

$$\begin{aligned} p^u &= p_v = C_v, \quad p_2 = C_2, \quad p_3 = C_3, \\ p^v &= p_u = \frac{m^2 + L^{-2}(e^{-2\beta}C_2^2 + e^{2\beta}C_3^2)}{2C_v}. \end{aligned} \quad (107)$$

Here  $C_v, C_2, C_3$  are constants of integration.

Our aim is to solve the equations of the spin evolution, which can be reduced now to the following three independent equations:

$$\frac{d}{du} S^u = 0 \Rightarrow S^u = S_v = \text{const} \quad (108)$$

$$\begin{aligned} \frac{e^{-\beta}}{L} \frac{d}{du} (L e^\beta S^2) + \frac{C_2}{2C_v} S_v (L^{-2} e^{-2\beta})' \\ = \frac{\omega_{25}\phi'}{L^2} \left( S_v \frac{C_3}{C_v} - S_3 \right), \end{aligned} \quad (109)$$

$$\begin{aligned} \frac{e^\beta}{L} \frac{d}{du} (L e^{-\beta} S^3) + \frac{C_3}{2C_v} S_v (L^{-2} e^{2\beta})' \\ = -\frac{\omega_{25}\phi'}{L^2} \left( S_v \frac{C_2}{C_v} - S_2 \right). \end{aligned} \quad (110)$$

Here we used the relationship  $u = \tau \frac{C_v}{m}$  between the parameter  $\tau$  and the retarded time  $u$ , which is the consequence of the equation  $m \frac{du}{ds} = p^u = C_v$  (we restrict ourselves by the case, when  $u = 0$  corresponds to  $\tau = 0$ ). The prime denotes here the derivative with respect to retarded time. We present only three equations from four, since the component  $S^v = S_u$  can be found from one of the two integrals,

$$2S_u S_v = (L e^\beta S^2)^2 + (L e^{-\beta} S^3)^2 - S^2, \quad (111)$$

$$S_u C_v + S_v p_u + S^2 C_2 + S^3 C_3 = 0. \quad (112)$$

Clearly, when  $S_v = 0$ , one can extract  $S_u$  from the second relationship only ( $C_v \neq 0$  for massive particles).

In order to solve the key equations (109) and (110), it is convenient to use the following auxiliary functions:

$$\begin{aligned} S_-(u) &= L e^\beta S^2, & S_+(u) &= L e^{-\beta} S^3, \\ \Omega(u) &= \omega_{25} \phi'(u), \\ f(u) &= S_v \frac{C_2}{C_v} \left( \frac{e^{-\beta}}{L} \right), & g(u) &= S_v \frac{C_3}{C_v} \left( \frac{e^\beta}{L} \right). \end{aligned} \quad (113)$$

In these terms the equations (109) and (110) take the form

$$\begin{aligned} \frac{d}{du} S_- &= \Omega(u) S_+ + \Omega g(u) - f'(u), \\ \frac{d}{du} S_+ &= -\Omega(u) S_- - \Omega f(u) - g'(u), \end{aligned} \quad (114)$$

and their solutions happen to be very simple:

$$\begin{aligned} S_+ &= \mathcal{A} \cos \Psi(u) - g(u), \\ S_- &= \mathcal{A} \sin \Psi(u) - f(u), \\ \Psi(u) &= \Psi(0) + \omega_{25} [\phi(u) - \phi(0)]. \end{aligned} \quad (115)$$

Here  $\mathcal{A}$  is an integration constant. When the integral  $S_v$  is nonvanishing, the last unknown function  $S_u(u)$  reads

$$S_u = \frac{1}{2S_v} [f^2 + g^2 - 2\mathcal{A}(g \cos \Psi + f \sin \Psi) + \mathcal{A}^2 - S^2], \quad (116)$$

and the constant  $\mathcal{A}$  can be found from the orthogonality condition (112) yielding

$$\mathcal{A} = \sqrt{S^2 - \frac{m^2 S_v^2}{C_v^2}}. \quad (117)$$

Below we illustrate the obtained exact solutions by the examples of longitudinal and transversal particle motion with respect to the plane front of the gravitational wave.

## 1. Longitudinal motion

Let the spinning particle start to move along the  $x^1$  axis, i.e.,  $p_2(0) = C_2 = 0$ ,  $p_3(0) = C_3 = 0$ . According to (107) at  $u > 0$  the particle keeps the direction of motion,  $p_2(u) = p_3(u) = 0$ . From (108) and (111) it follows that the components  $S_u$  and  $S_v$  do not feel the influence of axions, and we assume that  $S_u(0) = S_v(0) = 0$ , i.e., at  $u = 0$  there were only two nonvanishing spin four-vector components,  $S^2(0) \neq 0$  and  $S^3(0) \neq 0$ . For such initial data we obtain immediately that  $f(u) = 0$ ,  $g(u) = 0$ ,  $\mathcal{A} = S$  and thus

$$\begin{aligned} S_u(u) &= 0, & S_v(u) &= 0, \\ S^2(u) &= S \left( \frac{e^{-\beta}}{L} \right) \sin \Psi(u), \\ S^3(u) &= S \left( \frac{e^\beta}{L} \right) \cos \Psi(u). \end{aligned} \quad (118)$$

Again we deal with axionically induced spin rotation with the frequency  $\Omega(u) = \omega_{25} \phi'(u)$ .

## 2. Example of a transversal motion

Let the particle start to move at  $u = 0$  in the direction  $x^2$  (in the front plane of the gravitational wave), and have initially only one nonvanishing component of the spin four-vector  $S^3(0) \neq 0$ . Mathematically this is possible if

$$S_v = 0, \quad \Psi(0) = 0 \Rightarrow f(u) = g(u) = 0, \quad \mathcal{A} = S. \quad (119)$$

Then the exact solution obtained above gives

$$\begin{aligned} S_v(u) &= 0, & S_u(u) &= -\frac{C_2}{C_v} S \left( \frac{e^{-\beta}}{L} \right) \sin \Psi(u), \\ S^2(u) &= S \left( \frac{e^{-\beta}}{L} \right) \sin \Psi(u), \\ S^3(u) &= S \left( \frac{e^\beta}{L} \right) \cos \Psi(u). \end{aligned} \quad (120)$$

The first and second formulas in (120) yield

$$S^1(u) = \frac{1}{\sqrt{2}} (S_u - S_v) = -\frac{C_2}{\sqrt{2} C_v} S \left( \frac{e^{-\beta}}{L} \right) \sin \Psi(u). \quad (121)$$

One can see that due to the spin-axion coupling the longitudinal and the second transversal components of the spin appear  $S^1(u > 0) \neq 0$ ,  $S^2(u > 0) \neq 0$ . The axionically induced spin rotation is characterized by the frequency  $\Omega(u) = \omega_{25} \phi'(u)$ .

## V. DISCUSSION

We have established the model of the pseudoscalar (axion) field action on the spinning particle. In its minimal (curvature independent) version this model includes the dynamic equation (30) and the equation of the spin evolution (31). The terms, which include the Maxwell tensor  $F_{mn}$  and its dual  $F_{ik}^*$  are constructed phenomenologically by analogy with (and as generalization of) the well-known Bargmann-Michel-Telegdi (BMT) model. As in the BMT model, the dynamic equation (30) does not contain the spin four-vector, and the equation (31) is linear in  $S^k$ . Is it possible to extend (31) by introducing the spin four-vector quadratically, e.g., as it was made by Bander and Yee in [17]? For sure, the scheme, based on the representations (17) of the master equations and on the decomposition of the basic tensor  $\omega^{ik}$  introduced in (21), gives us such a tool. For instance, when  $\omega^{ik}$  is linear in the spin, the corresponding dynamic equation is also linear, and the equation of the spin evolution is quadratic in the spin four-vector.

However, even in the simplest BMT-like form (30) the dynamic equation includes two new force terms, which points to the axionic extension of the theory. The first novelty is the term in which the tensor  $eF_{mn}$  is replaced with the tensor  $e\phi F_{mn}^*$ , where the pseudoscalar multiplier  $\phi$  compensates the pseudotensorial nature of the dual Maxwell tensor  $F_{mn}^*$ . The second novelty is the term with the gradient-type multiplier  $p^i \nabla_i \phi$  in front of  $F_{mn}^*$ . If the axionic environment indeed produces forces of these kinds, they could be tested in high-energy experiments with polarized beams, e.g. in the LHC. We hope to present the corresponding work and discuss exact solutions to the whole system of equations of axion electrodynamics, particle dynamics and spin evolution in the next paper.

As for this paper, we consider the contribution from the only new term, which is free of the Maxwell tensor and linear in the gradient four-vector  $\nabla_i \phi$ . This term describes the direct action of the axion field on the particle spin. This term,  $\frac{\omega_{25}}{mc} \nabla_i \phi \epsilon^{ikln} S_k p_n$ , was introduced phenomenologically, and the dimensionless coupling constant  $\omega_{25}$  should be recognized. One of the ways to find  $\omega_{25}$  is to make the reduction from the axionically extended Dirac theory; we will return to this problem in the future. The second way is to use the analogy with the axion-photon coupling, which has been considered in Sec. II. To follow the hypothesis of universality and to compare the evolutionary equations for the photon polarization (15) and for the spin rotation (16), one can assume that  $\omega_{25} = \alpha = 1$ . Nevertheless, one should repeat that this coupling constant has to be found experimentally.

One can mention that the term  $\frac{\omega_{25}}{m} \nabla_i \phi \epsilon^{ikln} S_k p_n$  describing the direct spin-axion coupling can be represented in the BMT-like form. Indeed, let us take the main term  $\frac{e}{m} F^{ik} S_k$  appearing in the BMT model for the case  $g = 2$ , and

consider the decomposition of the Maxwell tensor in the reference frame associated with the particle moving with the velocity  $\frac{p^i}{m}$ . In addition, let us assume that in this frame the electric field  $E^k$  is absent, then we obtain that  $F^{ik} = -\frac{1}{m} \epsilon^{ikln} B_l p_n$ , where  $B^k$  is the four-vector of the corresponding magnetic excitation. Comparing the terms  $\frac{\omega_{25}}{m} \nabla_i \phi \epsilon^{ikln} S_k p_n$  and  $-\frac{e}{m^2} \epsilon^{ikln} S_k B_l p_n$ , we can see that they formally coincide, when  $\omega_{25} \nabla_i \phi = -\frac{e}{m} B_i$ . In other words, this analogy hints that the gradient of the pseudoscalar (axion) field can produce the spin rotation similar to the well-known effect induced by the magnetic field.

Of course, the mentioned analogy is incomplete, since  $B^k$  is the spacelike four-(pseudo)vector, while the gradient four-(pseudo)vector  $\nabla_k \phi$  can be timelike ( $\nabla_i \phi \nabla^i \phi > 0$ ), spacelike ( $\nabla_i \phi \nabla^i \phi < 0$ ) or null ( $\nabla_i \phi \nabla^i \phi = 0$ ). However, in all three cases, as it was demonstrated using the obtained exact solutions to the master equations, we deal with the same phenomenon, the axionically induced spin precession.

The typical example for the timelike gradient four-vector  $\nabla_k \phi$  is given by the spatially homogeneous cosmological model according to which the pseudoscalar field corresponds to the relic dark matter axions. In this case the gradient four-vector reduces to  $\delta_i^0 \dot{\phi}$  and the spin rotates in the plane orthogonal to the direction of the particle motion. The corresponding time-dependent frequency  $\Omega(t) = \omega_{25} \dot{\phi}(t) \frac{V}{c}$  ( $V$  is the modulus of the velocity three-vector) is a direct analog of the Larmor frequency.

The static spherically symmetric model of gravitational and axion fields gives the typical example for the spacelike gradient,  $\nabla_i \phi \nabla^i \phi < 0$ . When the particle moves in the radial direction, we deal again with the spin rotation in the transverse plane. Now it is more reasonable to speak about the spin turn with respect to radial variable  $r$  rather than with respect to time [see (86)–(89)]. The rate of spin turn depends on the distance to the center; in particular, when  $p_r(\infty) = 0$  and thus  $K = m = E$ , we obtain  $\Omega(r) = \omega_{25} \frac{\phi'(r)}{\sqrt{1-B(r)}}$ . Far from the center the metric function

$B(r)$  has the standard behavior,  $B(r) \rightarrow (1 - \frac{2GM}{r})$ , so the asymptotic behavior of the frequency  $\Omega(r)$ ,  $\Omega(r \rightarrow \infty) \rightarrow \frac{\omega_{25}}{\sqrt{2GM}} \sqrt{r} \phi'(r)$  is predetermined by the function  $\sqrt{r} \phi'(r)$ .

One can expect that the zone of strong gravitation gives the maximal contribution into the total turn of the spin, however, this question should be analyzed in its own right.

When the particle moves along a circular orbit around the axionically active object, one can split the total effect in the spin rotation into geodesic precession and axionic precession. The frequency of rotation is given by hybrid formula (104), and the axionic frequency  $\Omega_{(\text{axion})} = \frac{\omega_{25} \phi'(R)}{\sqrt{A(R)}}$  is constant on the orbit, but depends on the radius of the orbit  $R$ . The behavior of the function  $\Omega_{(\text{axion})}(R)$  can be

studied only when the distribution of axion field  $\phi(r)$  is found. We hope to return to this question in the next paper in the context of discussion of qualitative and numerical study of the total system of master equations.

The case  $\nabla_i \phi \nabla^i \phi = 0$  is typical for the model with a plane-wave symmetry, for which the axion field depends on the retarded time only,  $\phi(u)$ . Again the exact solutions to the master equations demonstrate the spin rotation with the frequency  $\Omega(u) = \omega_{25} \phi'(u)$ . Two cases have to be distinguished in this model. First, when the particle moves orthogonally to the front of the gravitational wave (the so-called longitudinal motion), we deal with simple spin rotation in the front plane ( $S^2 \neq 0$  and  $S^3 \neq 0$ ). When the projection of the particle momentum on the front plane is nonvanishing ( $p_2 \neq 0$ , transversal motion), the spin rotation becomes more sophisticated (the additional, third component of the spin four-vector appears).

The first obvious conclusion for all three applications is that the pseudoscalar (axion) field makes the space-time chiral, so that the left-hand and right-hand rotations of the particle spin four-vector become nonequivalent. The spin precession can be indicated as the first general property of the model.

The second general property is that the gravitational field, providing the nonvanishing gradient of the axion field [ $\dot{\phi} \neq 0$ ,  $\phi'(r) \neq 0$ ,  $\phi'(u) \neq 0$ ], activates the spin-axion coupling. In this sense, when the gravity field is strong, it displays the phenomenon of spin rotation more effectively.

Since the phase of the spin turn is described by the integral formulas of the type  $\Psi = \int d\xi \Omega(\xi)$  ( $\xi = t$ ,  $\xi = r$  or  $\xi = u$ ), the effect of spin rotation is cumulative in the space-time domains, in which the quantities  $\dot{\phi}(t)$ ,  $\phi'(r)$  or  $\phi'(u)$ , respectively, hold the sign. In this sense the phase accumulation can be treated as the third general property of the model.

Finally, we would like to say a couple of words about estimation of the described effect. We prefer to do it on the example of relic dark matter axions, which seem to be

distributed everywhere, using the model of relativistic charged spin particle motion in a storage ring with magnetic field (see Sec. IV B). In the cosmological context the quantity  $\dot{\phi}(t_0)$  can be estimated as  $\dot{\phi}(t_0) = \frac{1}{\Psi_0} \sqrt{\rho_{(a)}(t_0)}$  using the mass density of dark matter axions  $\rho_{(a)}(t_0)$ . Thus, for the relic cold dark matter axions with the mass density of the order  $\rho_{(DM)} \approx 0.033 M_{(\text{Sun})} \text{pc}^{-3}$ , for the ultrarelativistic particle with  $V \rightarrow c$ , for the coupling constant  $\frac{1}{\Psi_0} = \rho_{A\gamma\gamma} \approx 10^{-9} \text{ GeV}^{-1}$ , we obtain that an optimistic estimation for the spin rotation frequency (in Hz) is

$$\begin{aligned} \Omega_{(H)} &= \dot{\Psi}_{(H)} \rightarrow \dot{\Psi} \\ &\approx 10^{-6} \left( \frac{\omega_{25}}{1} \right) \left( \frac{\rho_{A\gamma\gamma}}{10^{-9} \text{ GeV}^{-1}} \right) \left( \frac{\sqrt{\rho_{(DM)}}}{\sqrt{1.25 \text{ GeV cm}^{-3}}} \right). \end{aligned}$$

In order to estimate the possible total axionically induced phase variation,  $\Delta\Psi$ , for other examples, we have to know the time period, during which the sign of the frequency is nonchanged and is, say, positive. When we deal with homogeneous cosmological model, according to (52) and (53) this time period is about of  $T = \frac{\pi}{m_{(a)}}$ , and thus is negligibly small. The application to the static spherically symmetric gravitation field is much more promising, since now the derivative  $\phi'(r)$  is a monotonic function, and the accumulation of the phase of the spin turn can continue for a long time for both radial and circular motion of the test particle.

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- [1] *The Kerr Spacetime: Rotating Black Holes in General Relativity*, edited by D. L. Wiltshire, M. Visser, and S. M. Scott (Cambridge University Press, New York, 2009).
  - [2] J. L. Friedman and N. Stergioulas, *Rotating Relativistic Stars* (Cambridge University Press, New York, 2013).
  - [3] E. Athanassoula, The spiral structure of galaxies, *Phys. Rep.* **114**, 319 (1984).
  - [4] A. J. Benson, Galaxy formation theory, *Phys. Rep.* **495**, 33 (2010).
  - [5] C. M. Will, The confrontation between general relativity and experiment, *Living Rev. Relativity* **17**, 4 (2014).
  - [6] I. Ciufolini and E. C. Pavlis, A confirmation of the general relativistic prediction of the Lense-Thirring effect, *Nature (London)* **431**, 958 (2004).
  - [7] V. Bargmann, L. Michel, and V. L. Telegdi, Precession of the Polarization of Particles Moving in a Homogeneous Electromagnetic Field, *Phys. Rev. Lett.* **2**, 435 (1959).
  - [8] L. H. Thomas, The motion of a spinning electron, *Nature (London)* **117**, 514 (1926).
  - [9] J. Frenkel, Die elektrodynamik des rotierenden elektrons, *Z. Phys.* **37**, 243 (1926).
  - [10] F. Bloch, Nuclear induction, *Phys. Rev.* **70**, 460 (1946).

- [11] M. Mathisson, Neue mechanik materieller systeme, *Acta Phys. Pol.* **6**, 163 (1937).
- [12] A. Papapetrou, Spinning test particles in general relativity. I., *Proc. R. Soc. A* **209**, 248 (1951).
- [13] W. Dixon, Dynamics of extended bodies in general relativity. I. Momentum and angular momentum, *Proc. R. Soc. A* **314**, 499 (1970).
- [14] R. M. Wald, Gravitational spin interaction, *Phys. Rev. D* **6**, 406 (1972).
- [15] I. Khriplovich and A. Pomeransky, Equations of motion of spinning relativistic particle in external fields, *Surveys in High Energy Physics* **14**, 145 (1999); Gravitational interaction of spinning bodies, center-of-mass coordinate and radiation of compact binary systems, *Phys. Lett. A* **216**, 7 (1996).
- [16] R. Rietdijk and J. van Holten, Spinning particles in Schwarzschild space-time, *Classical Quantum Gravity* **10**, 575 (1993); J. W. van Holten, Relativistic dynamics of spin in strong external fields, [arXiv:hep-th/9303124](https://arxiv.org/abs/hep-th/9303124).
- [17] M. Bander and K. Yee, Equations of motion for spinning particles in external electromagnetic and gravitational fields, *Phys. Rev. D* **48**, 2797 (1993).
- [18] A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, Spinning particles in general relativity, *Phys. Lett.* **89B**, 199 (1980).
- [19] G. Cognola, L. Vanzo, S. Zebrini, and R. Soldati, On the Lagrangian formulation of a charged spinning particle in an external electromagnetic field, *Phys. Lett.* **104B**, 67 (1981).
- [20] W.-T. Ni, Searches for the role of spin and polarization in gravity, *Rep. Prog. Phys.* **73**, 056901 (2010).
- [21] G. Bertone, *Particle Dark Matter: Observations, Models and Searches* (Cambridge University Press, Cambridge, 2010).
- [22] J. Silk, The invisible universe: Dark matter and dark energy, *Lect. Notes Phys.* **720**, 101 (2007).
- [23] V. Trimble, Existence and nature of dark matter in the universe, *Annu. Rev. Astron. Astrophys.* **25**, 425 (1987).
- [24] M. Khlopov, *Fundamentals of Cosmic Particle Physics* (CISP-Springer, Cambridge, 2012).
- [25] L. Sadeghian, F. Ferrer, and C. M. Will, Dark matter distributions around massive black holes: A general relativistic analysis, *Phys. Rev. D* **88**, 063522 (2013).
- [26] R. D. Peccei and H. R. Quinn, *CP Conservation in the Presence of Pseudoparticles*, *Phys. Rev. Lett.* **38**, 1440 (1977).
- [27] S. Weinberg, A New Light Boson?, *Phys. Rev. Lett.* **40**, 223 (1978).
- [28] F. Wilczek, Problem of Strong  $P$  and  $T$  Invariance in the Presence of Instantons, *Phys. Rev. Lett.* **40**, 279 (1978).
- [29] M. S. Turner, Windows on the axion, *Phys. Rep.* **197**, 67 (1990).
- [30] G. G. Raffelt, Astrophysical methods to constrain axions and other novel particle phenomena, *Phys. Rep.* **198**, 1 (1990).
- [31] E. P. S. Shellard and R. A. Battye, Origin of dark matter axions, *Phys. Rep.* **307**, 227 (1998).
- [32] P. Sikivie, Axion cosmology, *Lect. Notes Phys.* **741**, 19 (2008).
- [33] R. Battesti, B. Beltran, H. Davoudiasl, M. Kuster, P. Pugnati, R. Rabadan, A. Ringwald, N. Spooner, and K. Zioutas, Axions, *Lect. Notes Phys.* **741**, 199 (2008).
- [34] E. di Valentino, E. Giusarma, M. Lattanzi, A. Melchiorri, and O. Mena, Axion cold dark matter: Status after Planck and BICEP2, *Phys. Rev. D* **90**, 043534 (2014).
- [35] L. Visinelli and P. Gondolo, Axion Cold Dark Matter in View of BICEP2 Results, *Phys. Rev. Lett.* **113**, 011802 (2014).
- [36] T. Noumi, K. Saikawa, R. Sato, and M. Yamaguchi, Effective gravitational interactions of dark matter axions, *Phys. Rev. D* **89**, 065012 (2014).
- [37] N. Banik and P. Sikivie, Axions and the galactic angular momentum distribution, *Phys. Rev. D* **88**, 123517 (2013).
- [38] W.-T. Ni, Equivalence Principles and Electromagnetism, *Phys. Rev. Lett.* **38**, 301 (1977).
- [39] P. Sikivie, Experimental Tests of the “Invisible” Axion, *Phys. Rev. Lett.* **51**, 1415 (1983).
- [40] W.-T. Ni, From equivalence principles to cosmology: Cosmic polarization rotation, CMB observation, neutrino number asymmetry, Lorentz invariance and  $CPT$ , *Prog. Theor. Phys. Suppl.* **172**, 49 (2008).
- [41] W.-T. Ni, Cosmic polarization rotation, cosmological models, and the detectability of primordial gravitational waves, *Int. J. Mod. Phys. A* **24**, 3493 (2009).
- [42] Y. N. Obukhov and F. W. Hehl, Measuring a piecewise constant axion field in classical electrodynamics, *Phys. Lett. A* **341**, 357 (2005).
- [43] A. M. Essin, J. E. Moore, and D. Vanderbilt, Magnetolectric Polarizability and Axion Electrodynamics in Crystalline Insulators, *Phys. Rev. Lett.* **102**, 146805 (2009).
- [44] J. F. Navarro, C. S. Frenk, and S. D. M. White, The structure of cold dark matter halos, *Astrophys. J.* **462**, 563 (1996).
- [45] A. B. Balakin and L. V. Grunskaya, Axion electrodynamics and dark matter fingerprints in the terrestrial magnetic and electric fields, *Rep. Math. Phys.* **71**, 45 (2013).
- [46] A. B. Balakin and W.-T. Ni, Anomalous character of the axion-photon coupling in a magnetic field distorted by a pp-wave gravitational background, *Classical Quantum Gravity* **31**, 105002 (2014).
- [47] A. B. Balakin, R. K. Muharlyamov, and A. E. Zayats, Electromagnetic waves in an axion-active relativistic plasma non-minimally coupled to gravity, *Eur. Phys. J. C* **73**, 2647 (2013).
- [48] A. B. Balakin, V. V. Bochkarev, and N. O. Tarasova, Gradient models of the axion-photon coupling, *Eur. Phys. J. C* **72**, 1895 (2012).
- [49] A. B. Balakin and N. O. Tarasova, Extended axion electrodynamics: Optical activity induced by nonstationary dark matter, *Gravitation Cosmol.* **18**, 54 (2012).
- [50] J. E. Moody and F. Wilczek, New macroscopic forces?, *Phys. Rev. D* **30**, 130 (1984).
- [51] N. Ramsey, The tensor force between two protons at long range, *Physica A (Amsterdam)* **96**, 285 (1979).
- [52] R. Barbieri, M. Cerdonio, G. Fiorentini, and S. Vitale, Axion to magnon conversion. A scheme for the detection of galactic axions, *Phys. Lett. B* **226**, 357 (1989).
- [53] F. Giuliani and T. A. Girard, Model-independent limits from spin-dependent WIMP dark matter experiments, *Phys. Rev. D* **71**, 123503 (2005).
- [54] A. G. Glenday, C. E. Cramer, D. F. Phillips, and R. L. Walsworth, Limits on Anomalous Spin-Spin Couplings between Neutrons, *Phys. Rev. Lett.* **101**, 261801 (2008).
- [55] G. Vasilakis, J. M. Brown, T. W. Kornack, and M. V. Romalis, Limits on New Long Range Nuclear Spin-Dependent Forces

- Set with a K<sup>3</sup>He<sup>3</sup> Comagnetometer, *Phys. Rev. Lett.* **103**, 261801 (2009).
- [56] D. Chelouche and E. I. Guendelman, Cosmic analogues of the Stern-Gerlach experiment and the detection of light bosons, *Astrophys. J.* **699**, L5 (2009).
- [57] P.-H. Chu, A. Dennis, C. B. Fu, H. Gao, R. Khatiwada, G. Laskaris, K. Li, E. Smith, W. M. Snow, H. Yan, and W. Zheng, Laboratory search for spin-dependent short-range force from axionlike particles using optically polarized He<sup>3</sup> gas, *Phys. Rev. D* **87**, 011105(R) (2013).
- [58] P. W. Graham and S. Rajendran, New observables for direct detection of axion dark matter, *Phys. Rev. D* **88**, 035023 (2013).
- [59] Y. V. Stadnik and V. V. Flambaum, Axion-induced effects in atoms, molecules and nuclei: Parity nonconservation, anapole moments, electric dipole moments, and spin-gravity and spin-axion momentum couplings, *Phys. Rev. D* **89**, 043522 (2014).
- [60] D. Budker, P. W. Graham, M. Ledbetter, S. Rajendran, and A. Sushkov, Cosmic Axion Spin Precession Experiment (CASPER), *Phys. Rev. X* **4**, 021030 (2014).
- [61] I. T. Drummond and S. J. Hathrell, QED vacuum polarization in a background gravitational field and its effect on the velocity of photons, *Phys. Rev. D* **22**, 343 (1980).
- [62] F. W. Hehl and Yu. N. Obukhov, How does the electromagnetic field couple to gravity, in particular to metric, non-metricity, torsion, and curvature?, *Lect. Notes Phys.* **562**, 479 (2001).
- [63] A. B. Balakin and J. P. S. Lemos, Nonminimal coupling for the gravitational and electromagnetic fields: A general system of equations, *Classical Quantum Gravity* **22**, 1867 (2005).
- [64] A. B. Balakin and W.-T. Ni, Nonminimal coupling of photons and axions, *Classical Quantum Gravity* **27**, 055003 (2010).
- [65] S. K. Wong, Field and particle equations for the classical Yang-Mills field and particles with isotopic spin, *Nuovo Cimento* **65**, 689 (1970).
- [66] A. B. Balakin, V. R. Kurbanova, and W. Zimdahl, Parametric phenomena of the particle dynamics in a periodic gravitational wave field, *J. Math. Phys. (N.Y.)* **44**, 5120 (2003).
- [67] A. A. Starobinsky, Future and origin of our universe: Modern view, *Gravitation Cosmol.* **6**, 157 (2000).
- [68] S. Nojiri and S. D. Odintsov, Unified cosmic history in modified gravity: From  $F(R)$  theory to Lorentz noninvariant models, *Phys. Rep.* **505**, 59 (2011).
- [69] A. A. Deriglazov and W. G. Ramrez, Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations in ultra-relativistic regime and gravimagnetic moment, *arXiv*: 1509.05357.
- [70] D. J. Kaup, Klein-Gordon geon, *Phys. Rev.* **172**, 1331 (1968).
- [71] R. Ruffini and F. Bonazzola, System of self-gravitating particles in general relativity and the concept of an equation of state, *Phys. Rev.* **187**, 1767 (1969).
- [72] J.-W. Lee and I.-G. Koh, Galactic halos as boson stars, *Phys. Rev. D* **53**, 2236 (1996).
- [73] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
- [74] C. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [75] J. B. Griffiths and J. Podolsky, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, New York, 2009).