PHYSICAL REVIEW D **92,** 105021 (2015)

Suppressing vacuum fluctuations with vortex excitations

J. F. de Medeiros Neto,[1,*] Rudnei O. Ramos,[2,†] Carlos Rafael M. Santos,[1,‡] Rodrigo F. Ozela,[1,§] Gabriel C. Magalhães,[1,∥] and Van Sérgio Alves[1,¶]

[1]*Faculdade de Física, Universidade Federal do Pará, Belém, 66075-110 Pará, Brazil*
[2]*Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, 20550-013 Rio de Janeiro, Brazil*

(Received 15 September 2014; revised manuscript received 22 July 2015; published 10 November 2015)

The Casimir force for a planar gauge model is studied considering perfect conducting and perfect magnetically permeable boundaries. By using an effective model describing planar vortex excitations, we determine the effect these can have on the Casimir force between parallel lines. Two different mappings between models are considered for the system under study, where generic boundary conditions can be more easily applied and the Casimir force can be derived in a more straightforward way. It is shown that vortex excitations can be an efficient suppressor of vacuum fluctuations. In particular, for the model studied here, a planar Chern-Simons type of model that allows for the presence of vortex matter, the Casimir force is found to be independent of the choice of boundary conditions, at least for the more common types, like Neumann, perfect conducting and magnetically permeable boundary conditions. We give an interpretation for these results and some possible applications for them are also discussed.

DOI: 10.1103/PhysRevD.92.105021 PACS numbers: 03.70.+k, 11.10.Ef, 11.15.Yc

I. INTRODUCTION

There has been considerable interest in studying the validity of Newton's gravitational law at submillimeter scales and well below that (for a recent review see, e.g., Ref. [1]). There is a possibility that with these experiments deviations from the standard power-law behavior could be found, thus, possibly probing phenomena like modified gravity scenarios predicted by string theory, or by physics beyond the standard model of particle physics. For example, compactified extra spatial dimensions in string theory could lead to a modification of the quadratic power-law behavior, depending on the number of extra dimensions. Also, physics beyond the standard model of particle physics can produce Yukawa-type corrections for the gravitational force (for a comprehensive review, see also, e.g., Ref. [2] and references therein).

Laboratory experiments measuring gravity related forces at extremely small scales pose some extraordinary challenges. One of these challenges for probing forces at such very small scales is to distinguish gravitationlike interactions from other effects that can come from quantum phenomena, most notably the Casimir force [3], which can potentially dominate gravity effects by several orders of magnitude at distances of the order of the micrometer and below that. In fact, the fast recent developments on laboratory experiments measuring the Casimir force [4,5] have also helped to put some strong constraints on the level of possible corrections to gravity [6]. On the other hand, it is also highly desirable to devise ways of either isolating the Casimir effect, or to suppress it up to the level of precision that can be found in those experiments. Recently, graphene [7] has been proposed for such a purpose due to its extraordinary absorption properties, which could effectively function as a shield for quantum vacuum fluctuations. It is also important to look for other types of materials that can be as versatile in terms of being easily produced and also with tunable properties under laboratory conditions. One such possibility could be, for example, the use of superconducting films.

It is known that superconducting films can have magnetic vortex excitations. Most of the properties of these systems can be described in terms of planar gauge systems. We recall that planar gauge field theories, in particular Chern-Simons (CS) type of models, have long been recognized as important for understanding several physical phenomena that can be well approximated as planar ones, like high-temperature superconductivity and the fractional quantum Hall effect, just to cite a few examples (see, e.g., Ref. [8] and references therein). The Casimir force in the presence of condensed vortices in a plane was studied previously in Ref. [9] from the point of view of the particle-vortex duality, where an effective description of vortex excitations was made in terms of a Maxwell-Proca-Chern-Simons (MPCS) model.

The study of the Casimir force in the presence of vortex excitations carried out in Ref. [9] was based on a particular mapping existing between the MPCS model and a model of two noninteracting massive scalar fields. Since the Casimir force is well known for the latter case, the corresponding

*jfmn@ufpa.br
†rudnei@uerj.br
‡carlos.santos@icen.ufpa.br
§ozelarf@gmail.com
∥gconduru@ufpa.br
¶vansergi@ufpa.br

result for the former could be easily determined. This mapping, however, severely restricted the form of the boundary conditions (BC) considered there. In particular, the connection between the different model Hamiltonians was only possible in the case of Neumann BC for the scalar field and, in this sense, the form of the mathematical transformations has implied in the consideration of a specific type of BC for the scalar and vector fields. Also, the connection was only possible for the simplest geometry treated there to compute the Casimir force, i.e., the force between parallel lines, and could not be generalized to other geometries.

It is a very desirable solution to explore appropriate mappings between models that can be used in the determination of the Casimir effect, where the above-mentioned restrictions can be avoided. In this paper, we consider two known relationships associated with the MPCS model: (a) the connection of the MPCS model with a sum of a self-dual and an antiself-dual Proca-Chern-Simons (PCS) model [10], and (b) the connection of the MPCS model with a sum of two Maxwell-Chern-Simons (MCS) models [10,11]. The advantage of both associations, as compared to that related to scalar degrees of freedom [9,12], is that a direct relation between the original and final fields can be made very clear. This in turn facilitates the connection between the BC and also the calculation of the Casimir force.

As we see in this paper, the difficulties met with the original mapping used in Ref. [9] are removed. In the case (a) listed above, we make use of the intrinsic properties of self duality and antiself duality of the PCS models. This allows us to define mathematically the BC in terms of the Green functions; then, performing the calculations again in the case (b) listed above, helps to confirm our results. As we show, the use of the relation (a) facilitates our calculations because we can make use of the symmetry of the resulting PCS models and the final form of the energy-momentum tensor. Another important benefit provided by the relation (a) is that the number of differential equations that we need to solve and the number of required Green's functions are smaller, when compared to the case (b), as we are going to see.

Our objectives in this work are twofold. First, by using more general mappings than the one used in Ref. [9], we can compute the Casimir force in the cases of more realistic and physically relevant BC and geometries. Secondly, with the use of a different BC, we can determine any possible effect that might have on the Casimir force. We derive results for two BC of interest, i.e., for perfect conducting and for perfect magnetically permeable boundaries. We also consider another type of (Neumann) BC, previously considered in Ref. [9], and confirm the result found there. We still use for convenience and simplicity the simplest geometry of parallel lines, but our results can be extended to other more complex geometries, which we leave for a future work. It is explicitly shown that, for the model studied here, the Casimir force is found to be independent of the choice of BC used.

Given the many approximations and considerations assumed in our calculation (which are discussed below), the use of the results that we have obtained in this work to the high precision gravity and Casimir experiments that were mentioned above may sound too optimistic and, thus, should not be taken literally in that context. However, the present results point to effects that can be of relevance in the future planning of these experiments. Nevertheless, the present work is of theoretical interest, where some novel aspects related to topological (vortex) excitations are considered, along also with issues regarding the use of different BC in the computation of the Casimir effect.

The remainder of this work is organized as follows. In Sec. II, we summarize the connection of the MPCS model as an effective vortex-particle dual to the Chern-Simons-Higgs (CSH) model. We also summarize the mathematical relations that connect the MPCS model in terms of a self-dual and an antiself-dual PCS model and also in terms of a sum of two MCS models. In Sec. III, we analyze the relation between the original vector field of the MPCS model and the new fields associated with the two PCS models and give the relevant equations needed to evaluate the Casimir force. This evaluation is done considering the cases of perfect conductor and also perfect magnetically permeable lines at the boundaries. In Sec. IV, we check and confirm our results to be independent of the mapping used, by considering this time the connection between the MPCS and two MCS models, rederiving our results again for both cases of perfect magnetically permeable and perfect conductor boundaries. In Sec. V, based on the symmetries and constraints of the models studied, we explain the reason for the independence of the Casimir force on the BC considered in the calculations. In Sec. VI we analyze and discuss the Casimir force obtained in the context of a vortex condensate. Finally, in Sec. VII, we give our concluding remarks and discuss other possible applications and implications of the results derived in this work.

II. THE MPCS MODEL AS AN EFFECTIVE DUAL VORTEX DESCRIPTION AND ITS MAPPING ONTO TWO PCS MODELS

It has been shown in Ref. [13] (for earlier derivations, see for example Ref. [14]) that vortex excitations in a CSH model can be expressed effectively in terms of a dual equivalent theory (for applications of similar duality ideas in planar systems of interest in condensed matter that also make use of the particle-vortex duality in Chern-Simons type of models, see Ref. [15] and references therein). This effective model for vortices, in turn, can be expressed in the form of a MPCS model, when both the scalar Higgs field and the vortex field are in their symmetry broken vacuum states, $\rho_0 \neq 0$ and $\psi_0 \neq 0$, respectively. The Lagrangian density of the MPCS model can be expressed as [9]

$$\mathcal{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \frac{m^2}{2}A^\alpha A_\alpha + \frac{\mu}{4}\epsilon^{\alpha\beta\lambda}A_\alpha\partial_\beta A_\lambda, \tag{2.1}$$

where

$$m \equiv 4\pi\rho_0\psi_0, \tag{2.2}$$

$$\mu \equiv 2e^2\rho_0^2/\Theta, \tag{2.3}$$

and Θ is the Chern-Simons parameter of the original CSH model, from which Eq. (2.1) is derived.

In addition to the connection of the above model with a dual vortex equivalent one, the MPCS model given by Eq. (2.1) can be mapped in a sum of a self-dual and an antiself-dual PCS model [10] or, also, in terms of a sum of two MCS models [10,11]. As we discuss later on, these associations will simplify considerably the calculation of the Casimir force. For completeness, let us briefly review below these two considerations concerning the model given by Eq. (2.1).

A. The effective dual vortex description for the MPCS model

Chern-Simons gauge field theories can exhibit many features of relevance in different contexts. One of these features, which is of particular importance in our study, is the possibility of having topological vortex solutions when these models are coupled to symmetry broken scalar potentials [16]. For instance, we can consider the CSH model described by the Euclidean action

$$S_E[h_\mu, \eta, \eta^*] = \int d^3x \left[-i\frac{\Theta}{4}\epsilon_{\mu\nu\gamma}h_\mu H_{\nu\gamma} + |D_\mu\eta|^2 + V(|\eta|)\right], \tag{2.4}$$

where $H_{\mu\nu} = \partial_\mu h_\nu - \partial_\nu h_\mu$, $D_\mu \equiv \partial_\mu + ieh_\mu$ and η is a complex scalar field, with a non-null vacuum expectation value (VEV) obtained from a symmetry breaking polynomial potential $V(|\eta|)$. For instance, for a potential given by $V(|\eta|) = e^4(|\eta|^2 - \nu^2)^2|\eta|^2/\Theta^2$, the field equations for the model (2.4) have nontrivial vortex solutions given by [17]

$$\eta_{\text{vortex}} = \xi(r)\exp(in\chi), \qquad h_{\mu,\text{vortex}} = \frac{n}{e}h(r)\partial_\mu\chi, \tag{2.5}$$

where n is an integer that represents the vortex charge, while $\xi(r)$ and $h(r)$ are the (vortex profile) functions obtained from the solutions of the classical field differential equations, subjected to the BC $\lim_{r\to 0}\xi(r) = 0$, $\lim_{r\to\infty}\xi(r) = \nu$, $\lim_{r\to 0}h(r) = 0$ and $\lim_{r\to\infty}h(r) = 1$. The presence of vortex excitations means that the phase of the scalar field, $\phi = \rho\exp(i\chi)/\sqrt{2}$, is a multivalued function. The phase χ can then in general be expressed in terms of a regular (single valued) and a singular part as $\chi(x) = \chi_{\text{reg}}(x) + \chi_{\text{sing}}(x)$. The vortex excitations can be made explicit in the action by functionally integrating over the regular phase, while leaving explicitly the dependence of the singular phase in the action. This procedure can be done by the so-called dual transformations (see, e.g., Refs. [13,18] for a detailed account for this procedure). The final result can be expressed in terms of a dual action, written in terms of a complex scalar field ψ (representing quantized vortex excitations) and a new gauge field A_μ, which is related to the original fields by the relation $\rho^2(\partial_\mu\chi + eh_\mu) = (\sigma/(2\pi e))\epsilon_{\mu\nu\gamma}\partial_\mu A_\gamma$, where σ is an arbitrary parameter with mass dimension. The final dual action can be expressed in the form [13]

$$S_{\text{dual}} = \int d^3x \left[\frac{\sigma^2}{16\pi^2e^2\rho_0^2}F_{\mu\nu}^2 + i\frac{\sigma^2}{8\pi^2\Theta}\epsilon_{\mu\nu\gamma}A_\mu\partial_\nu A_\gamma + \left|\partial_\mu\psi + i\frac{2\sigma}{e}A_\mu\psi\right|^2 + V_{\text{vortex}}(|\psi|) + \mathcal{L}_G\right], \tag{2.6}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, $V(|\psi|)$ is the effective potential term for the vortex field, with a VEV ψ_0, and $\mathcal{L}_G$ is a gauge fixing term.

When the system is taken deep inside its vortex condensed phase, we can take the London-type approximation for the vortex field [19], where $|\psi| \to \psi_0/\sqrt{2}$. In this case, we can neglect the derivative of ψ that appears in Eq. (2.6). We can also choose $\sigma \equiv 2\pi e\rho_0$, so that Eq. (2.6) can then be finally rewritten in the form of the MPCS model with the (Minkowski) Lagrangian density given by Eq. (2.1).

B. Mapping the MPCS model onto two PCS models

To compute the Casimir force for the MPCS model, we could in principle start directly from Eq. (2.1) and use standard methods based on the vacuum expectation values for the space-space and time-time components of the energy-momentum tensor (like, e.g., those discussed in Ref. [20]). This procedure leads, however, to a hard to solve system of partial differential equations (PDE). It turns out that it is much simpler to express the original model, Eq. (2.1), in terms of an equivalent one that can be easily treated mathematically. In particular, we want to have a well-defined mapping between the fields in each model, such that we can unequivocally establish their behaviors at the physical boundaries of the system. Such mapping must imply in a direct correspondence between the BC considered for the MPCS and its equivalent model, resulting in a one-to-one mapping between the Casimir forces for the models involved. One such possibility is to follow the proposal of Refs. [10,11], where the MPCS of Eq. (2.1) is mapped into a doublet consisting of a self-dual and an antiself-dual PCS model in $2+1$ dimensions. One of the advantages of this procedure is that a direct relation between the original and final fields can be made very clear, which facilitates the connection between the BC. Besides, it also allows the use of different BC and, eventually, it can also be generalized to different geometries, as opposite to the case treated originally in Ref. [9].

Following in particular Ref. [10], we consider a doublet consisting of an antiself-dual and a self-dual PCS model, represented, respectively, by the Lagrangian densities,

$$\mathcal{L}_{-} = -\frac{1}{2}\epsilon_{\mu\nu\beta}g^{\mu}\partial^{\nu}g^{\beta} + \frac{m_{-}}{2}g_{\mu}g^{\mu}, \tag{2.7}$$

and

$$\mathcal{L}_{+} = \frac{1}{2}\epsilon_{\mu\nu\beta}f^{\mu}\partial^{\nu}f^{\beta} + \frac{m_{+}}{2}f_{\mu}f^{\mu}, \tag{2.8}$$

where f_μ and g_μ are two independent vector fields. By making use of a soldering field W_μ with no dynamics, it is a simple exercise to obtain, from the combination of $\mathcal{L}_+$ and $\mathcal{L}_-$, a final Lagrangian density that does not depend on W_μ. For example, we can define an intermediate Lagrangian density given by

$$\mathcal{L} = \mathcal{L}_{-}(g) + \mathcal{L}_{+}(f) - W_{\mu}[J^{\mu}_{-}(g) + J^{\mu}_{+}(f)] + \frac{1}{2}(m_{+} + m_{-})W_{\mu}W^{\mu}, \tag{2.9}$$

where $J^{\mu}_{\pm}$ are defined by

$$J^{\mu}_{+}(f) \equiv \sqrt{m_{+}}f^{\mu} + \epsilon^{\mu\alpha\beta}\partial_{\alpha}f_{\beta}, \tag{2.10}$$

$$J^{\mu}_{-}(g) \equiv \sqrt{m_{-}}g^{\mu} - \epsilon^{\mu\alpha\beta}\partial_{\alpha}g_{\beta}. \tag{2.11}$$

In the generating functional associated with (2.9), W^μ plays the role of an auxiliary field, which can be eliminated by a direct integration (another way of seeing the auxiliary role of W^μ is by the use of its equation of motion). The resulting final Lagrangian density can then be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{(m_{-} - m_{+})}{2}\epsilon_{\mu\nu\beta}A^{\mu}\partial^{\nu}A^{\beta} + \frac{1}{2}m_{+}m_{-}A_{\mu}A^{\mu}, \tag{2.12}$$

where A_μ is a new vector field, related to f_μ and g_μ by

$$A_{\mu} \equiv \frac{1}{\sqrt{m_{+} + m_{-}}}(f_{\mu} - g_{\mu}), \tag{2.13}$$

and m_+ and m_- are related to the original mass parameters μ and m of Eq. (2.1) by

$$m_{-} - m_{+} = \mu/2, \tag{2.14}$$

$$m_{+}m_{-} = m^{2}. \tag{2.15}$$

It is important to note that in Eq. (2.13) we consider that m_+ and m_- are both positive. This consideration implies that $m^2 > 0$. Thus, Eqs. (2.14)–(2.15) imply that

$$m_{\pm} = \mp\frac{\mu}{4} + \sqrt{\frac{\mu^{2}}{16} + m^{2}}. \tag{2.16}$$

The result of the detailed study of the relation between $\mathcal{L}$ and $\mathcal{L}_+ + \mathcal{L}_-$ shows a complete equivalence between them [10], i.e., $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$. Hence, it is straightforward to perceive that the Casimir force related to the original MPCS model can be written as the sum of the Casimir forces associated with $\mathcal{L}_+$ and $\mathcal{L}_-$. The relation between f_μ, g_μ and A_μ, given by Eq. (2.13), implies in a direct determination of the BC considered for f_μ and g_μ, in terms of those considered for A_μ. We can also conclude from Eq. (2.13) that, in principle, there is no restriction for the BC to be considered for A_μ (which will be associated with the BC for f_μ and g_μ), as long as they are mathematically and physically acceptable. We also note that determining the Casimir force related to a PCS model is rather simpler than determining the force for the MPCS model directly, as we discuss in the next section.

C. The MPCS model written in term of two MCS models

Alternatively, we can also use the equivalence between the MPCS model and a doublet of MCS models, given in Ref. [10]. These two MCS models will be written in terms of two gauge fields P_μ and Q_μ, respectively, which can be conveniently rescaled, when compared with their analogues considered in Ref. [10]. We can write the Lagrangian densities for the two MCS models as

$$\tilde{\mathcal{L}}_{-}(P) = -\frac{1}{4}P_{\mu\nu}P^{\mu\nu} + \frac{1}{2}m_{-}\epsilon_{\mu\nu\beta}P^{\mu}\partial^{\nu}P^{\beta}, \tag{2.17}$$

and

$$\tilde{\mathcal{L}}_{+}(Q) = -\frac{1}{4}Q_{\mu\nu}Q^{\mu\nu} - \frac{1}{2}m_{+}\epsilon_{\mu\nu\beta}Q^{\mu}\partial^{\nu}Q^{\beta}, \tag{2.18}$$

where $P^{\mu\nu} = \partial^\mu P^\nu - \partial^\nu P^\mu$, and $Q^{\mu\nu} = \partial^\mu Q^\nu - \partial^\nu Q^\mu$. The masses m_+ and m_- in Eqs. (2.17)–(2.18) are the same as the ones defined in Eq. (2.16).

The two gauge fields P_μ and Q_μ are connected to the original gauge field A_μ of the MPCS model by

$$A_{\mu} \equiv \frac{1}{\sqrt{m_{+} + m_{-}}}(\sqrt{m_{-}}P_{\mu} - \sqrt{m_{+}}Q_{\mu}). \tag{2.19}$$

The relation between the doublet of MCS models, Eqs. (2.17)–(2.18), with the MPCS model (2.1) is established in a similar fashion as in the case of the previous subsection. By using this time a tensor field $B_{\mu\nu}$ connecting the two Lagrangian densities (2.17)–(2.18), we have that

$$\mathcal{L} = \tilde{\mathcal{L}}_-(P) + \tilde{\mathcal{L}}_+(Q) - \frac{1}{2} B_{\mu\nu}[J_-^{\mu\nu}(P) + J_+^{\mu\nu}(Q)] - \frac{m_+ + m_-}{4m_+ m_-} B_{\mu\nu} B^{\mu\nu}, \quad (2.20)$$

where $J_\pm^\mu$ are defined by

$$J_{\mu\nu}^+(Q) \equiv -Q_{\mu\nu} - m_+ \epsilon_{\mu\nu\beta} Q^\beta, \quad (2.21)$$

$$J_{\mu\nu}^-(P) \equiv -P_{\mu\nu} + m_- \epsilon_{\mu\nu\beta} P^\beta. \quad (2.22)$$

Again, considering the relation between the fields given in Eq. (2.19), we can eliminate the auxiliary field $B_{\mu\nu}$, reproducing once again the original MPCS model.

It is important to realize that in both mappings described above, the number of degrees of freedom is preserved. It is noteworthy to realize that in a MCS model the mass term for the gauge field is of topological origin. Each MCS model has only one (transverse) polarization degree of freedom. However, in the MPCS model, the explicit mass term for the gauge field implies that there are now two polarization degrees of freedom for the gauge field. The number of degrees of freedom is preserved in the two mappings used. The duality between these different types of gauge models has also been discussed extensively in the literature before. For example, in Ref. [21] this issue is discussed in terms of an interpolating master action and how it explains the doubling of fields, yet preserves the number of degrees of freedom.

Finally, it is important to also note that while the association of the vortex excitations in the CSH model with the MPCS model given in Eq. (2.1) is only valid within the approximations considered in the previous subsection (e.g., for a special Higgs potential, no vortex interactions, and the use of a London-type limit for the Higgs and vortex fields), the relation between the MPCS and PCS models is exact. The same can be said with respect to the MCS models.

III. THE CASIMIR FORCE FOR THE MPCS MODEL EXPRESSED IN TERMS OF A DOUBLET OF PCS MODELS

In this section, we use an analogous procedure as used, e.g., in Ref. [20] to calculate the Casimir forces associated with $\mathcal{L}_+$ and $\mathcal{L}_-$, given by Eqs. (2.8) and (2.7), respectively.

In the following, we have adopted the notation $X \equiv x^\mu = (t, x, y)$ and considered the metric tensor $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1)$. The physical boundaries are placed in $x = 0$ and $x = a$.

The Casimir force (per unit length) for the MPCS model is determined from the 11 component of the energy-momentum tensor,

$$f \equiv (\text{force/lenght})_{\text{MPCS}} = \langle T^{11}{}_{\text{MPCS}} \rangle|_{x=0 \text{ and } x=a}, \quad (3.1)$$

which can also be written, according to the results shown in the previous section, as

$$f = [\langle T_-^{11} \rangle + \langle T_+^{11} \rangle]|_{x=0 \text{ and } x=a}, \quad (3.2)$$

where T_-^{11} is the energy-momentum tensor component obtained from $\mathcal{L}_-$, given by Eq. (2.7), while T_+^{11} is the one obtained from $\mathcal{L}_+$, given by Eq. (2.8). As it is well known, the CS term does not contribute to the symmetric energy-momentum tensor, since it is given in terms of the derivative of the action with respect to the metric tensor and the CS term does not depend on this metric [20,22]. Thus, we obtain

$$T_-^{\mu\nu} = -\eta^{\mu\nu} \frac{m_-}{2} g_\alpha g^\alpha, \quad (3.3)$$

$$T_+^{\mu\nu} = -\eta^{\mu\nu} \frac{m_+}{2} f_\alpha f^\alpha. \quad (3.4)$$

Equation (3.2) can be written in terms of the Green functions for the gauge fields f^μ and g^μ, $G_+^{\mu\nu}(X, X') = i\langle \hat{T}[f^\mu(X) f^\nu(X')] \rangle$ and $G_-^{\mu\nu}(X, X') = i\langle \hat{T}[g^\mu(X) g^\nu(X')] \rangle$, respectively. For example, using Eq. (3.3), we can write

$$\langle T_-^{11}(X) \rangle = -i \frac{m_-}{2} \lim_{X' \to X} [G_-^{00}(X, X') - G_-^{11}(X, X') - G_-^{22}(X, X')], \quad (3.5)$$

and similarly for $\langle T_+^{11}(X) \rangle$.

The Green functions for f^μ and g^μ can be derived from the Euler-Lagrange equations for the fields as usual:

$$m_- g_\mu(X) - \epsilon_{\mu\beta\nu} \partial^\beta g^\nu(X) + J_{(-)\mu}(X) = 0, \quad (3.6)$$

$$m_+ f_\mu(X) + \epsilon_{\mu\beta\nu} \partial^\beta f^\nu(X) + J_{(+)\mu}(X) = 0, \quad (3.7)$$

where $J_{(-)\mu}$ and $J_{(+)\mu}$ are the source terms. The formal solutions to Eqs. (3.6)–(3.7) are

$$g^\mu(X) = \int G_-^{\mu\alpha}(X, X') J_{(-)\alpha}(X') dX', \quad (3.8)$$

$$f^\mu(X) = \int G_+^{\mu\alpha}(X, X') J_{(+)\alpha}(X') dX', \quad (3.9)$$

and

$$m_- G_-^{\mu\alpha} - \epsilon^\mu{}_{\beta\nu} \partial^\beta G_-^{\nu\alpha} + \delta(X - X') \eta^{\mu\alpha} = 0, \quad (3.10)$$

$$m_+ G_+^{\mu\alpha} + \epsilon^\mu{}_{\beta\nu} \partial^\beta G_+^{\nu\alpha} + \delta(X - X') \eta^{\mu\alpha} = 0. \quad (3.11)$$

Note that, unlike the calculations followed in Refs. [20,23] (where the Green functions for the field's duals were used), we work directly in terms of the Green functions for the fields themselves (f_μ and g_μ). This would also be the case if we had decided to work with the MPCS model directly. This fact can be seen as a consequence of the fact that the Proca term, $m^2 A_\mu A^\mu/2$, cannot be written in terms of the dual of A_μ. But if we had decided to work with the MPCS model directly (without "transforming" it to a doublet of PCS models beforehand as we are proceeding here), the resulting system of second-order differential equations would be more difficult to solve, when compared to the one that we have in the present case [20,23]. The transformations taken here simplify the calculations significantly, since the system of equations with which we have to deal with is relatively easier to solve, given by Eqs. (3.10)–(3.11).

Using the Fourier transforms in time and in the transverse coordinate y for $G_{\pm}^{\mu\nu}(X,X')$,

$$G_{\pm}^{\mu\nu}(X,X') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{dk}{2\pi} e^{ik(y-y')} \mathcal{G}_{\pm}^{\mu\nu}(k,\omega,x,x'), \tag{3.12}$$

we can write

$$\langle T_{\pm}^{11}\rangle = -i\frac{m_\pm}{2}\lim_{X'\to X}\int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{dk}{2\pi} e^{ik(y-y')} \times [\mathcal{G}_{\pm}^{00}(k,\omega,x,x') - \mathcal{G}_{\pm}^{11}(k,\omega,x,x') - \mathcal{G}_{\pm}^{22}(k,\omega,x,x')], \tag{3.13}$$

and the Casimir force (per unit length) can be expressed as

$$\begin{aligned} f &= \langle T^{11}{}_{\mathrm{MPCS}}\rangle|_{x=0 \text{ and } x=a} \\ &= [\langle T_{-}^{11}(X)\rangle + \langle T_{+}^{11}(X)\rangle]|_{x=0 \text{ and } x=a} \\ &= -i\lim_{X'\to X}\left\{\int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{dk}{2\pi} e^{ik(y-y')} \right. \\ &\quad \times \left[\frac{m_-}{2}(\mathcal{G}_{-}^{00} - \mathcal{G}_{-}^{11} - \mathcal{G}_{-}^{22}) \right. \\ &\quad \left.\left. + \frac{m_+}{2}(\mathcal{G}_{+}^{00} - \mathcal{G}_{+}^{11} - \mathcal{G}_{+}^{22})\right]\right\}\Bigg|_{x=0 \text{ and } x=a}. \end{aligned} \tag{3.14}$$

The components $\mathcal{G}_{\pm}^{00}$, $\mathcal{G}_{\pm}^{11}$ and $\mathcal{G}_{\pm}^{22}$ are obtained from the solutions of the following systems of PDE (where x stands for x^1):

$$\begin{cases} -ik\mathcal{G}_{-}^{01} + m_-\mathcal{G}_{-}^{11} + i\omega\mathcal{G}_{-}^{21} = \delta(x-x'), \\ m_-\mathcal{G}_{-}^{01} - ik\mathcal{G}_{-}^{11} + \partial_x\mathcal{G}_{-}^{21} = 0, \\ \partial_x\mathcal{G}_{-}^{01} - i\omega\mathcal{G}_{-}^{11} + m_-\mathcal{G}_{-}^{21} = 0, \end{cases} \tag{3.15}$$

$$\begin{cases} -m_-\mathcal{G}_{-}^{00} + ik\mathcal{G}_{-}^{10} - \partial_x\mathcal{G}_{-}^{20} = \delta(x-x'), \\ -ik\mathcal{G}_{-}^{00} + m_-\mathcal{G}_{-}^{10} + i\omega\mathcal{G}_{-}^{20} = 0, \\ \partial_x\mathcal{G}_{-}^{00} - i\omega\mathcal{G}_{-}^{10} + m_-\mathcal{G}_{-}^{20} = 0, \end{cases} \tag{3.16}$$

$$\begin{cases} \partial_x\mathcal{G}_{-}^{22} - ik\mathcal{G}_{-}^{12} + m_-\mathcal{G}_{-}^{02} = 0, \\ i\omega\mathcal{G}_{-}^{22} + m_-\mathcal{G}_{-}^{12} - ik\mathcal{G}_{-}^{02} = 0, \\ m_-\mathcal{G}_{-}^{22} - i\omega\mathcal{G}_{-}^{12} + \partial_x\mathcal{G}_{-}^{02} = \delta(x-x'), \end{cases} \tag{3.17}$$

$$\begin{cases} ik\mathcal{G}_{+}^{01} + m_+\mathcal{G}_{+}^{11} - i\omega\mathcal{G}_{+}^{21} = \delta(x-x'), \\ m_+\mathcal{G}_{+}^{01} + ik\mathcal{G}_{+}^{11} - \partial_x\mathcal{G}_{+}^{21} = 0, \\ -\partial_x\mathcal{G}_{+}^{01} + i\omega\mathcal{G}_{+}^{11} + m_+\mathcal{G}_{+}^{21} = 0, \end{cases} \tag{3.18}$$

$$\begin{cases} -m_+\mathcal{G}_{+}^{00} - ik\mathcal{G}_{+}^{10} + \partial_x\mathcal{G}_{+}^{20} = \delta(x-x'), \\ ik\mathcal{G}_{+}^{00} + m_+\mathcal{G}_{+}^{10} - i\omega\mathcal{G}_{+}^{20} = 0, \\ -\partial_x\mathcal{G}_{+}^{00} + i\omega\mathcal{G}_{+}^{10} + m_+\mathcal{G}_{+}^{20} = 0, \end{cases} \tag{3.19}$$

$$\begin{cases} -\partial_x\mathcal{G}_{+}^{22} + ik\mathcal{G}_{+}^{12} + m_+\mathcal{G}_{+}^{02} = 0, \\ -i\omega\mathcal{G}_{+}^{22} + m_+\mathcal{G}_{+}^{12} + ik\mathcal{G}_{+}^{02} = 0, \\ m_+\mathcal{G}_{+}^{22} + i\omega\mathcal{G}_{+}^{12} - \partial_x\mathcal{G}_{+}^{02} = \delta(x-x'). \end{cases} \tag{3.20}$$

The above equations are explicitly solved in the following for the two specific BC that we consider: for a perfect conductor (PC) and for magnetically permeable (MP) boundaries, respectively.

A. The Casimir force for PC boundaries

We now describe the mapping between the original BC that can be imposed on the original vector field A_μ of the MPCS model with the ones imposed on the fields f_μ and g_μ. The Casimir effect follows from Eq. (3.2). We first consider PC at the boundaries, which can be represented mathematically by $F_1 = 0$, where

$$F_\mu \equiv \epsilon_{\mu\nu\gamma}\partial^\nu A^\gamma \tag{3.21}$$

is the dual of A_μ. This is a BC that could not be treated for instance in Ref. [9], due to the specific form of the mathematical transformations used in that work, based on scalar degrees of freedom.

In our case, the BC $F_1 = 0$ will imply [due to Eq. (2.13)] in $\epsilon_{1\nu\gamma}\partial^\nu f^\gamma = \epsilon_{1\nu\gamma}\partial^\nu g^\gamma$, which can be written in terms of the dual fields $\tilde{f}_\mu$ and $\tilde{g}_\mu$, associated with f_μ and g_μ, respectively,

$$\tilde{f}_\mu \equiv \epsilon_{\mu\nu\gamma}\partial^\nu f^\gamma, \tag{3.22}$$

$$\tilde{g}_\mu \equiv \epsilon_{\mu\nu\gamma}\partial^\nu g^\gamma. \tag{3.23}$$

In terms of these fields, the BC $F_1 = 0$ implies in $\tilde{f}_1 = \tilde{g}_1$. But since the PCS models are self dual and antiself dual, $\tilde{f}_\mu$ and $\tilde{g}_\mu$ are proportional to f_μ and g_μ, respectively (this proportionality can be obtained if we use the Euler-Lagrange equations for f_μ and g_μ). Thus, we can write that the BC $\tilde{f}_1 = \tilde{g}_1$ implies in $f_1 = -g_1$ at the boundaries. Using then Eqs. (3.8)–(3.9) we obtain

$$\left[\int G_-^{1\alpha}(X,X')J_{(-)\alpha}(X')dX'\right]\Big|_{x=0\,\text{and}\,x=a} = -\left[\int G_+^{1\alpha}(X,X')J_{(+)\alpha}(X')dX'\right]\Big|_{x=0\,\text{and}\,x=a}. \tag{3.24}$$

Since the sources $J_{(-)\alpha}(X')$ and $J_{(+)\alpha}(X')$ are arbitrary, Eq. (3.24) implies that

$$G_-^{1\alpha}(X,X')|_{x=0\,\text{and}\,x=a} = G_+^{1\alpha}(X,X')|_{x=0\,\text{and}\,x=a} = 0. \tag{3.25}$$

Note that when taking the BC, we are interested only in the limit $X \to X'$ of $G_\pm^{\alpha\beta}(X,X')$, such that we can take for instance $\exp[-i\omega(t-t')] = \exp[ik(y-y')] = 1$, e.g., in Eq. (3.12). Then, Eq. (3.25), when expressed in terms of its Fourier transform, like in Eq. (3.12), gives that we can write the BC equivalently as

$$\mathcal{G}_\pm^{1\alpha}(k,\omega,x,x')|_{x=0\,\text{and}\,x=a} = 0. \tag{3.26}$$

Hence, we note that in the present case, due to the BC, only $\mathcal{G}_\pm^{00}$ and $\mathcal{G}_\pm^{22}$ will contribute to the Casimir force f, Eq. (3.14). To find the required functions, we use the standard method of continuity and also consider a notation similar to the one used in Ref. [20] for convenience. Thus, we define

$$\kappa_\pm^2 = \omega^2 - k^2 - m_\pm^2, \tag{3.27}$$

$$ss_\pm = \sin(\kappa_\pm x_<)\sin[\kappa_\pm(x_> - a)], \tag{3.28}$$

$$cc_\pm = \cos(\kappa_\pm x_<)\cos[\kappa_\pm(x_> - a)], \tag{3.29}$$

$$sc_\pm = \begin{cases} \sin(\kappa_\pm x)\cos[\kappa_\pm(x'-a)], & \text{if } x < x', \\ \cos(\kappa_\pm x')\sin[\kappa_\pm(x-a)], & \text{if } x > x', \end{cases} \tag{3.30}$$

$$cs_\pm = \begin{cases} \cos(\kappa_\pm x)\sin[\kappa_\pm(x'-a)], & \text{if } x < x', \\ \sin(\kappa_\pm x')\cos[\kappa_\pm(x-a)], & \text{if } x > x', \end{cases} \tag{3.31}$$

where $x_>$ ($x_<$) is the greater (smaller) value in the set $\{x, x'\}$.

To determine $\mathcal{G}_\pm^{22}$, it is useful to write it in terms of $\mathcal{G}_\pm^{1\alpha}$, over which the BC is imposed directly. Using Eqs. (3.17) and (3.20), we obtain

$$\mathcal{G}_\pm^{22}(k,\omega,x,x') = \frac{i}{k^2-\omega^2}\left[k\partial_x\mathcal{G}_\pm^{12}(k,\omega,x,x') - \mathcal{G}_\pm^{12}(k,\omega,x,x')\omega m_\pm + \frac{k^2}{m_\pm^2}\delta(x-x')\right]. \tag{3.32}$$

We can drop the spatial Dirac delta function in Eq. (3.32), since it gives no contribution to $\mathcal{G}_\pm^{22}$ (we are considering $x \neq x'$). Note that dropping the spatial Dirac delta function corresponds physically to a renormalization, where an infinite contribution proportional to $\delta(0)$, when evaluating the Green function at the same point, is removed from the Casimir force. While this procedure is perfectly fine for the present type of (rigid) BC and the Casimir force is independent of this renormalization process, the reader should be aware that this simple renormalization procedure may not work for other types of BC. For instance, it is known that for other types of geometries (like circular BC, or including the case of smooth backgrounds), when computing the Casimir energy special care must be taken with this renormalization procedure, as shown in detail in Refs. [24,25]. Physically, the restriction to the use of this BC approach to Casimir problems is related to the physical role of the BC: A real material at the boundaries cannot constrain all modes of a field, as may be assumed in the BC approach. In reality, the material that produces the BC should be modeled by suitable interactions, and the divergences must be removed by counterterms for these interactions; the renormalization group then ensures that the predictive power of the theory is not lost through the subtraction.

Next, we have to find a PDE for $\mathcal{G}_\pm^{12}$ subjected to the BC $\mathcal{G}_\pm^{12} = 0$ and to use this result in Eq. (3.32). With this aim, we use again Eqs. (3.17) and (3.20), obtaining

$$(\partial_x^2 + \kappa_+^2)\mathcal{G}_\pm^{12}(k,\omega,x,x') = i\left(\frac{k}{m_\pm}\partial_x \mp \omega\right)\delta(x-x'). \tag{3.33}$$

We use the discontinuity method to solve Eq. (3.33), obtaining

$$\mathcal{G}_\pm^{12}(k,\omega,x,x') = -\frac{i}{\sin(a\kappa_\pm)}\left(\frac{k}{m_\pm}sc_\pm \pm \frac{\omega}{\kappa_\pm}ss_\pm\right). \tag{3.34}$$

By substituting Eq. (3.34) in Eq. (3.32), it follows that

$$\mathcal{G}^{22}_{\pm}(k,\omega,x,x') = \frac{k^2\kappa^2_{\pm}cc_{\pm} + \omega^2 m^2_{\pm}ss_{\pm} \pm k\omega\kappa_{\pm}m_{\pm}(cs_{\pm}+sc_{\pm})}{(k^2-\omega^2)m_{\pm}\kappa_{\pm}\sin(a\kappa_{\pm})}. \tag{3.35}$$

Next, we follow an analogous procedure to find $\mathcal{G}^{00}_{\pm}$. First, we use Eqs. (3.16) and (3.19) to write these functions in terms of $\mathcal{G}^{10}_{\pm}$:

$$\mathcal{G}^{00}_{\pm}(k,\omega,x,x') = \frac{i}{k^2-\omega^2}\left[\pm\frac{k}{m_+}(\omega^2-k^2) + \omega\partial_x \pm \frac{k}{m_+}\partial^2_x\right]\mathcal{G}^{10}_{+}(k,\omega,x,x'). \tag{3.36}$$

Using Eqs. (3.16) and (3.19), we obtain

$$(\kappa^2_{\pm}+\partial^2_x)\mathcal{G}^{10}_{\pm}(k,\omega,x,x') = i\left(\frac{\omega}{m_{\pm}}\partial_x \mp k\right)\delta(x-x'). \tag{3.37}$$

From Eq. (3.37), we find

$$\mathcal{G}^{10}_{\pm}(k,\omega,x,x') = \frac{\mp i}{\sin(\kappa_{\pm}a)}\left(\frac{k}{\kappa_{\pm}}ss_{\pm} \mp \frac{\omega}{m_{\pm}}sc_{\pm}\right). \tag{3.38}$$

Substituting Eq. (3.38) in Eq. (3.36), we find

$$\mathcal{G}^{00}_{\pm}(k,\omega,x,x') = \frac{\omega^2\kappa^2_{\pm}cc_{\pm} + k^2 m^2_{\pm}ss_{\pm} \pm m_{\pm}\kappa_{\pm}k\omega(cs_{\pm}+sc_{\pm})}{(k^2-\omega^2)m_{\pm}\kappa_{\pm}\sin(a\kappa_{\pm})}. \tag{3.39}$$

Inserting the expressions for $\mathcal{G}^{00}_{\pm}$ and $\mathcal{G}^{22}_{\pm}$, together with $\mathcal{G}^{11}_{\pm}=0$, in Eq. (3.14), we can write the Casimir force for the PC BC case as

$$f_{\mathrm{PC}} = (\langle T^{11}_{-}\rangle + \langle T^{11}_{+}\rangle)|_{x=0\,\mathrm{and}\,x=a} = \frac{i}{2}\int\frac{d\omega}{2\pi}\int\frac{dk}{2\pi}[\kappa_+\cot(a\kappa_+) + \kappa_-\cot(a\kappa_-)]. \tag{3.40}$$

The integrals appearing in Eq. (3.40) can be evaluated in an analogous fashion as in Ref. [20]. First, we make a complex rotation $\omega \to i\zeta$, where ζ is real (this is possible since there are no poles in the first and in the third quadrants). The effect of this rotation is to turn $\kappa_{\pm} \equiv (\omega^2 - k^2 - m^2_{\pm})^{1/2}$ into a purely complex variable. Then we can redefine it as $\kappa_{\pm} = i\lambda_{\pm}$, where $\lambda_{\pm} = \sqrt{\zeta^2+k^2+m^2_{\pm}}$ is a real variable. Then, using the relation

$$\cot(\kappa_{\pm}a) = -i\left[1+\frac{2}{\exp(2\lambda_{\pm}a)-1}\right], \tag{3.41}$$

we can rewrite Eq. (3.40) as an integral defined entirely in the real (ζ,k) plane, where

$$\langle T^{11}_{\pm}\rangle|_{x=0\,\mathrm{and}\,x=a} = -\int\frac{d\zeta}{2\pi}\int\frac{dk}{2\pi}\frac{\lambda_{\pm}}{[\exp(2\lambda_{\pm}a)-1]}. \tag{3.42}$$

We can also write Eq. (3.42) in terms of polar coordinates (r,ϕ), defined by

$$\zeta = r\cos\phi, \tag{3.43}$$

$$k = r\sin\phi. \tag{3.44}$$

Substituting Eqs. (3.43)–(3.44) in Eq. (3.42) and performing the integration over ϕ, we obtain

$$\langle T^{11}_{\pm}\rangle|_{x=0\,\mathrm{and}\,x=a} = -\int_0^{\infty}\frac{dr}{2\pi}\frac{r\sqrt{m^2_{\pm}+r^2}}{[\exp(2a\sqrt{m^2_{\pm}+r^2})-1]} = -\int_{m_{\pm}}^{\infty}\frac{d\lambda}{2\pi}\frac{\lambda^2}{[\exp(2\lambda a)-1]}, \tag{3.45}$$

where to obtain the last expression on the right-hand side in Eq. (3.45), we have made a change of integration variables, using $\lambda^2 = r^2+m^2_{\pm}$. From this equation, we can write the Casimir force for the case of PC boundaries as (when making the change of variables $z = 2\lambda a$)

$$f_{\rm PC} = -\frac{1}{16\pi a^3}\left[\int_{2am_-}^{\infty} dz \frac{z^2}{e^z - 1} + \int_{2am_+}^{\infty} dz \frac{z^2}{e^z - 1}\right]. \tag{3.46}$$

B. The Casimir force for perfect MP boundaries

Following an analogous derivation as outlined in the previous subsection, we now derive the Casimir force for the case of perfect MP lines. The same mapping relating the MPCS with a doublet made of a self-dual and an antiself-dual PCS model is, of course, still applicable, as is the system of PDE, Eqs. (3.15)–(3.20), derived previously. Perfect MP lines at the boundaries are represented by the BC $F_0 = 0$. This BC, in turn, can be represented in terms of $\mathcal{G}^{\mu\nu}_{\pm}$, analogously to what we have done in the previous subsection to obtain the BC given in Eq. (3.26). Thus, we find that we can write the present BC as

$$\mathcal{G}^{0\alpha}_{\pm}|_{x=0\,\text{and}\,x=a} = 0. \tag{3.47}$$

Using Eqs. (3.14)–(3.47), we can see that $\mathcal{G}^{00}_{\pm}$ do not contribute to the Casimir force at the boundaries. Hence, we only need to obtain $\mathcal{G}^{11}_{\pm}$ and $\mathcal{G}^{22}_{\pm}$.

Using an analogous procedure as the one used in the previous subsection, and noting that the BC is imposed on $\mathcal{G}^{0\alpha}_{\pm}$, we first find a relation between $\mathcal{G}^{22}_{\pm}$ and $\mathcal{G}^{02}_{\pm}$. Analogously, we need to find a relation between $\mathcal{G}^{11}_{\pm}$ and $\mathcal{G}^{01}_{\pm}$. For example, for $\mathcal{G}^{22}_{\pm}$, we can write (and again dropping a space Dirac delta function for the same reason explained in the previous subsection)

$$\mathcal{G}^{22}_{\pm}(k,\omega,x,x') = \frac{(k\omega \mp m\partial_x)}{\omega^2 - m^2_{\pm}}\mathcal{G}^{02}_{\pm}(k,\omega,x,x'), \tag{3.48}$$

and

$$(\partial_x^2 + \kappa_{\pm}^2)\mathcal{G}^{02}_{\pm}(k,\omega,x,x') = -\left(\frac{k\omega}{m_{\pm}} \pm \partial_x\right)\delta(x-x'), \tag{3.49}$$

which has the solution

$$\mathcal{G}^{02}_{\pm}(k,\omega,x,x') = -\frac{(\omega k ss_{\pm} \mp m_{\pm}\kappa_{\pm} sc_{\pm})}{m_{\pm}\kappa_{\pm}\sin(a\kappa_{\pm})}. \tag{3.50}$$

Hence,

$$\mathcal{G}^{22}_{\pm}(k,\omega,x,x') = \frac{m^2_{\pm}\kappa^2_{\pm} cc_{\pm} + k^2\omega^2 ss_{\pm} \mp m_{\pm}\kappa_{\pm}k\omega(cs_{\pm} + sc_{\pm})}{(m^2_{\pm} - \omega^2)m_{\pm}\kappa_{\pm}\sin(a\kappa_{\pm})}. \tag{3.51}$$

The procedure to find $\mathcal{G}^{11}_{\pm}$ is completely analogous, leading to the result

$$\mathcal{G}^{11}_{\pm}(k,\omega,x,x') = \frac{\omega^2_{\pm}\kappa^2_{\pm} cc_{\pm} + k^2 m^2_{\pm} ss_{\pm} \mp m_{\pm}\kappa_{\pm}k\omega(cs_{\pm} + sc_{\pm})}{(m^2_{\pm} - \omega^2)m_{\pm}\kappa_{\pm}\sin(a\kappa_{\pm})}. \tag{3.52}$$

Using the above expressions for $\mathcal{G}^{11}_{\pm}$ and $\mathcal{G}^{22}_{\pm}$, together with $\mathcal{G}^{00}_{\pm} = 0$, in Eq. (3.14), it can be easily verified that this results again in the same Casimir force as derived in the previous subsection, Eq. (3.40), leading also to Eq. (3.46), i.e., $f_{MP} = f_{PC}$. In the next two sections we try to understand this rather surprising result.

IV. CASIMIR FORCE FROM THE MAPPING BETWEEN THE MPCS MODEL AND A DOUBLET OF MCS MODELS

In the previous section we have obtained that the Casimir force for the MPCS model is independent of the two types of BC considered, i.e., for PC and MP lines at the boundaries. In this section we verify whether this result is not a consequence of the particular mapping that we have used, involving the relation of the MPCS model with a self-dual and an antiself-dual PCS model, described in Sec. II B. For this, we use the second relationship discussed in Sec. II C, relating the MPCS model with a doublet of MCS models, expressed by Eqs. (2.17)–(2.18).

A. Casimir force for perfect MP boundaries

We here specialize to the case of the perfect MP BC $F_0 = 0$. This analysis is made easier by the fact that the Casimir force for a MCS model under the BC $F_0 = 0$ was already studied in Ref. [23]. The results found in that reference can be easily extended to the Lagrangian densities given by Eqs. (2.17)–(2.18), as we show below.

The Casimir force for the MPCS model can be obtained from the sum of the 11 component of the total energy-momentum tensor determined from the Lagrangian densities (2.17)–(2.18), i.e.,

$$f = [\langle T^{11}_{(P)}\rangle + \langle T^{11}_{(Q)}\rangle]|_{x=0\,\text{and}\,x=a}, \tag{4.1}$$

where $T^{11}_{(P)}$ and $T^{11}_{(Q)}$ are the 11 component of the total energy-momentum tensor associated with $\tilde{\mathcal{L}}_{-}(P)$ and $\tilde{\mathcal{L}}_{+}(Q)$, Eqs. (2.17)–(2.18), respectively.

Let us first consider $T^{11}_{(P)}$. Our considerations can be easily extended to $T^{11}_{(Q)}$. Using analogous procedures to the ones used in the previous section, we can write

$$\langle T^{\mu\nu}_{(P)}\rangle|_{x=0\,\text{and}\,x=a}=\left(\langle\tilde{P}^{\mu}\tilde{P}^{\nu}\rangle-\frac{1}{2}\eta^{\mu\nu}\langle\tilde{P}_{\alpha}\tilde{P}^{\alpha}\rangle\right)\Bigg|_{x=0\,\text{and}\,x=a}, \tag{4.2}$$

where $\tilde{P}^{\mu}=\epsilon^{\mu\alpha\beta}\partial_{\alpha}P_{\beta}$. Analogously, we define $\tilde{Q}^{\mu}=\epsilon^{\mu\alpha\beta}\partial_{\alpha}Q_{\beta}$. The VEV $\langle\tilde{P}^{\mu}\tilde{P}^{\nu}\rangle$ can be obtained from $\langle\tilde{P}^{\mu}(X)\tilde{P}^{\nu}(X')\rangle$ as

$$\langle\tilde{P}^{\mu}\tilde{P}^{\nu}\rangle=\lim_{X\to X'}\langle\tilde{P}^{\mu}(X)\tilde{P}^{\nu}(X')\rangle, \tag{4.3}$$

and $\langle\tilde{P}^{\mu}(X)\tilde{P}^{\nu}(X')\rangle$ can be related to the Green function $G^{\mu\rho}_{(P)}$ for $\tilde{P}^{\mu}$, as we show below.

We know that $G^{\mu\rho}_{(P)}$ can be obtained from the Euler-Lagrange equation associated with $\tilde{\mathcal{L}}_{-}(P)$, written in terms of $\tilde{P}^{\mu}$, plus a source term:

$$\tilde{\mathcal{L}}_{-}(P)=-\frac{1}{2}\tilde{P}_{\mu}\tilde{P}^{\mu}+\frac{1}{2}m_{-}\tilde{P}^{\mu}P_{\mu}+J^{\mu}P_{\mu}. \tag{4.4}$$

Considering the equation of motion

$$-\epsilon^{\mu\alpha\beta}\partial_{\alpha}\tilde{P}_{\beta}+m_{-}\tilde{P}^{\mu}+J^{\mu}=0, \tag{4.5}$$

with formal solution

$$\tilde{P}^{\mu}=\int G^{\mu\rho}_{(P)}(X,X')J_{\rho}(X')dX', \tag{4.6}$$

we obtain the differential equation satisfied by $G^{\mu\rho}_{(P)}(X,X')$:

$$(\epsilon_{\nu\alpha\beta}\partial^{\alpha}-m_{-}\eta_{\nu\beta})G^{\beta\rho}_{(P)}(X,X')=\delta^{\rho}_{\nu}\delta(X-X'). \tag{4.7}$$

We then solve Eq. (4.7) to find the functions $G^{\beta\rho}_{(P)}(X,X')$ that will be necessary to compute $\langle T^{\mu\nu}_{(P)}\rangle$ in Eq. (4.2). First, we need a relation between $G^{\beta\rho}_{(P)}(X,X')$ and $\langle\tilde{P}^{\beta}(X)\tilde{P}^{\rho}(X')\rangle$. For this purpose, we consider the propagator for P^{μ},

$$\Delta^{\beta\rho}(X,X')=i\langle P^{\beta}(X)P^{\rho}(X')\rangle, \tag{4.8}$$

where $\langle P^{\beta}(X)P^{\rho}(X')\rangle$ is the Green function for P^{μ}, which can be obtained directly from the equation of motion generated by Eq. (2.17), when including a source term $J^{\mu}P_{\mu}$, as above. Hence, we can write [20]

$$G^{\beta\rho}_{(P)}(X,X')=\epsilon^{\beta\alpha\nu}\partial_{\alpha}\Delta_{\nu}{}^{\rho}(X,X')=i\langle\tilde{P}^{\beta}(X)P^{\rho}(X')\rangle. \tag{4.9}$$

From Eq. (4.9), we obtain

$$\langle\tilde{P}^{\beta}(X)\tilde{P}^{\rho}(X')\rangle=-i\epsilon^{\rho\alpha\gamma}\partial'_{\alpha}G^{\beta}_{(P)\gamma}(X,X'). \tag{4.10}$$

Using Eq. (4.2), we can write

$$\langle T^{11}_{(P)}\rangle|_{x=0\,\text{and}\,x=a}=\frac{1}{2}(\langle P^{0}P^{0}\rangle+\langle P^{1}P^{1}\rangle-\langle P^{2}P^{2}\rangle)|_{x=0\,\text{and}\,x=a}, \tag{4.11}$$

where

$$\langle P^{\mu}P^{\nu}\rangle|_{x=0\,\text{and}\,x=a}=-i\lim_{X\to X'}\epsilon^{\nu}{}_{\lambda\rho}\partial'^{\lambda}G^{\mu\rho}_{(P)}(X,X')|_{x=0\,\text{and}\,x=a}, \tag{4.12}$$

and $G^{\mu\rho}_{(P)}(X,X')$ satisfies

$$(\epsilon_{\nu\alpha\beta}\partial^{\alpha}-m_{-}\eta_{\nu\beta})G^{\beta\rho}_{(P)}(X,X')=\delta^{\rho}_{\nu}\delta(X-X'). \tag{4.13}$$

Considering the Fourier transform of $G^{\mu\rho}_{(P)}$ (with respect to t and y),

$$G^{\mu\rho}_{(P)}(X,X')=\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{dk}{2\pi}e^{ik(y-y')}\mathcal{G}^{\mu\rho}_{(P)}(k,\omega,x,x'), \tag{4.14}$$

we can write, using Eqs. (4.11)–(4.12), that

$$\langle T^{11}_{(P)}\rangle|_{x=0\,\text{and}\,x=a}=\lim_{X\to X'}\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{dk}{2\pi}e^{ik(y-y')}t^{11}_{(P)}\Big|_{x=0\,\text{and}\,x=a}, \tag{4.15}$$

where

$$t^{11}_{(P)}=\frac{i}{2}\frac{\partial}{\partial x'}(\mathcal{G}^{02}_{(P)}-\mathcal{G}^{20}_{(P)})-\frac{k}{2}(\mathcal{G}^{01}_{(P)}+\mathcal{G}^{10}_{(P)})+\frac{\omega}{2}(\mathcal{G}^{12}_{(P)}+\mathcal{G}^{21}_{(P)}). \tag{4.16}$$

The required functions $\mathcal{G}^{\mu\nu}_{(P)}$ can be obtained from Eqs. (4.13)–(4.14), analogously to what we have done in the previous sections. We can write

$$\begin{cases}-ik\mathcal{G}^{01}_{(P)}+m_{-}\mathcal{G}^{11}_{(P)}+i\omega\mathcal{G}^{21}_{(P)}=\delta(x-x'),\\ m_{-}\mathcal{G}^{01}_{(P)}-ik\mathcal{G}^{11}_{(P)}+\partial_{x}\mathcal{G}^{21}_{(P)}=0,\\ \partial_{x}\mathcal{G}^{01}_{(P)}-i\omega\mathcal{G}^{11}_{(P)}+m_{-}\mathcal{G}^{21}_{(P)}=0,\end{cases} \tag{4.17}$$

$$\begin{cases} -m_-\mathcal{G}^{00}_{(P)} + ik\mathcal{G}^{10}_{(P)} - \partial_x\mathcal{G}^{20}_{(P)} = \delta(x-x'), \\ -ik\mathcal{G}^{00}_{(P)} + m_-\mathcal{G}^{10}_{(P)} + i\omega\mathcal{G}^{20}_{(P)} = 0, \\ \partial_x\mathcal{G}^{00}_{(P)} - i\omega\mathcal{G}^{10}_{(P)} + m_-\mathcal{G}^{20}_{(P)} = 0, \end{cases} \tag{4.18}$$

$$\begin{cases} \partial_x\mathcal{G}^{22}_{(P)} - ik\mathcal{G}^{12}_{(P)} + m_-\mathcal{G}^{02}_{(P)} = 0, \\ i\omega\mathcal{G}^{22}_{(P)} + m_-\mathcal{G}^{12}_{(P)} - ik\mathcal{G}^{02}_{(P)} = 0, \\ m_-\mathcal{G}^{22}_{(P)} - i\omega\mathcal{G}^{12}_{(P)} + \partial_x\mathcal{G}^{02}_{(P)} = \delta(x-x'). \end{cases} \tag{4.19}$$

We note that the above equations are the same ones as those treated in Ref. [23] and, also, the forms of $T^{11}_{(P)}$ and $t^{11}_{(P)}$ are analogous to the ones derived in that reference. In the present case, where we are considering the BC $F_0 = 0$, using Eq. (2.19), we obtain that $\sqrt{m_-}\tilde{P}_0(X) = \sqrt{m_+}\tilde{Q}_0(X)$ at the boundaries. Hence, using an analogous procedure as used to obtain Eq. (3.26) and considering Eq. (4.6), we can write the BC in the present case as

$$\mathcal{G}^{0\rho}_{(P)}(k,\omega,x,x')|_{x=0\,\text{and}\,x=a} = \mathcal{G}^{0\rho}_{(Q)}(k,\omega,x,x')|_{x=0\,\text{and}\,x=a} = 0. \tag{4.20}$$

Hence, we conclude from Eqs. (4.15)–(4.16) and (4.20) that we only need to find $\frac{\partial}{\partial x'}\mathcal{G}^{20}_{(P)}$, $\frac{\partial}{\partial x'}\mathcal{G}^{02}_{(P)}$, $\mathcal{G}^{10}_{(P)}$, $\mathcal{G}^{12}_{(P)}$ and $\mathcal{G}^{21}_{(P)}$ to compute $\langle T^{11}_{(P)}\rangle$ at $x=0$ and $x=a$. As already commented on in the introduction, we note that the number of functions that we need to find, in the case of the mapping treated in this section, is greater than the number of required functions in the case considered in the previous section (where we considered the mapping between the MPCS model and the two PCS models).

The solutions to Eq. (4.13), considering the BC given in Eq. (4.20), are given by [23]

$$\mathcal{G}^{21}_{(P)}(k,\omega,x,x') = \frac{-i\omega}{(\omega^2 - m_-^2)\sin(a\kappa_-)}\left(\frac{k^2}{\kappa_-}ss_- + \frac{k\omega}{m_-}sc_- + \frac{km_-}{\omega}cs_- + \kappa_- cc_-\right), \tag{4.21}$$

$$\mathcal{G}^{12}_{(P)}(k,\omega,x,x') = \frac{i\omega}{(\omega^2 - m_-^2)\sin(a\kappa_-)}\left(\frac{k^2}{\kappa_-}ss_- + \frac{k\omega}{m_-}cs_- + \frac{km_-}{\omega}sc_- + \kappa_- cc_-\right), \tag{4.22}$$

$$\mathcal{G}^{20}_{(P)}(k,\omega,x,x') = \frac{-1}{\sin(a\kappa_-)}\left(\frac{\omega k}{m_-\kappa_-}ss_- + cs_-\right), \tag{4.23}$$

$$\mathcal{G}^{10}_{(P)}(k,\omega,x,x') = \frac{i}{\sin(a\kappa_-)}\left(\frac{\omega}{m_-}cs_- + \frac{k}{\kappa_-}ss_-\right). \tag{4.24}$$

Substituting Eqs. (4.21)–(4.24) in Eqs. (4.15)–(4.16), we obtain

$$\langle T^{11}_{(P)}\rangle|_{x=0\,\text{and}\,x=a} = \frac{i}{2}\int\frac{d\omega}{2\pi}\int\frac{dk}{2\pi}\kappa_-\cot(a\kappa_-). \tag{4.25}$$

The derivation of $\langle T^{11}_{(Q)}\rangle$ is completely analogous and the result found is

$$\langle T^{11}_{(Q)}\rangle|_{x=0\,\text{and}\,x=a} = \frac{i}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\kappa_+\cot(a\kappa_+). \tag{4.26}$$

Thus, from Eqs. (4.1) and (4.25), we obtain again Eq. (3.40). This confirms our previous result and at the same time it shows that the result obtained for the Casimir force is independent of the mapping used for the case of a MP BC.

B. Casimir force for PC boundaries

We can also use the mapping between the MPCS model and $\tilde{\mathcal{L}}_-(P) + \tilde{\mathcal{L}}_+(Q)$ to also confirm our result for the Casimir force in the case of a PC BC, $F_1 = 0$. The MCS model under this BC was considered in Ref. [20] and the results found there can be easily extended to the case treated here, in the same way as we did in the previous subsection.

In this case Eqs. (4.15)–(4.16) still remain valid, as do the PDE satisfied by $\mathcal{G}^{\mu\nu}_{(P)}$ and $\mathcal{G}^{\mu\nu}_{(Q)}$. We then have that

$$\langle T^{11}_{(P)}\rangle|_{x=0\,\text{and}\,x=a} = \lim_{X\to X'}\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{dk}{2\pi}e^{ik(y-y')}t^{11}_{(P)}\Big|_{x=0\,\text{and}\,x=a}, \tag{4.27}$$

where

$$t^{11}_{(P)} = \frac{1}{2i}\frac{\partial}{\partial x'}(\mathcal{G}^{02}_{(P)} - \mathcal{G}^{20}_{(P)}) + \frac{k}{2}(\mathcal{G}^{01}_{(P)} + \mathcal{G}^{10}_{(P)}) + \frac{\omega}{2}(\mathcal{G}^{12}_{(P)} + \mathcal{G}^{21}_{(P)}). \quad (4.28)$$

As in the previous cases, we can conclude that the BC $F_1 = 0$ implies in

$$\mathcal{G}^{1\nu}_{(P)}(k,\omega,x,x')|_{x=0\,\text{and}\,x=a} = \mathcal{G}^{1\nu}_{(Q)}(k,\omega,x,x')|_{x=0\,\text{and}\,x=a} = 0. \quad (4.29)$$

To obtain the Casimir force f at the boundaries, we need $\frac{\partial}{\partial x'}\mathcal{G}^{02}_{(P)}$, $\frac{\partial}{\partial x'}\mathcal{G}^{20}_{(P)}$, $\mathcal{G}^{01}_{(P)}$, $\mathcal{G}^{21}_{(P)}$ and the corresponding $\mathcal{G}^{\mu\nu}_{(Q)}$. The required functions are now found to be given by

$$\mathcal{G}^{02}_{(P)}(k,\omega,x,x') = \frac{1}{\sin(a\kappa_-)}\left(\frac{km_-\omega}{\kappa_-\rho_-^2}ss_- + \frac{k^2}{\rho_-^2}sc_- + \frac{\omega^2}{\rho_-^2}cs_- + \frac{\omega k\kappa_-}{m_-\rho_-^2}cc_-\right), \quad (4.30)$$

$$\mathcal{G}^{20}_{(P)}(k,\omega,x,x') = \frac{1}{\sin(a\kappa_-)}\left(\frac{km_-\omega}{\kappa_-\rho_-^2}ss_- + \frac{k^2}{\rho_-^2}cs_- + \frac{\omega^2}{\rho_-^2}sc_- + \frac{\omega k\kappa_-}{m_-\rho_-^2}cc_-\right), \quad (4.31)$$

$$\mathcal{G}^{01}_{(P)}(k,\omega,x,x') = \frac{i}{\sin(a\kappa_-)}\left(\frac{k}{\kappa_-}ss_- + \frac{\omega}{m_-}cs_-\right), \quad (4.32)$$

$$\mathcal{G}^{21}_{(P)}(k,\omega,x,x') = \frac{i}{\sin(a\kappa_-)}\left(\frac{\omega}{\kappa_-}ss_- + \frac{k}{m_-}cs_-\right). \quad (4.33)$$

Using Eq. (4.28) and Eqs. (4.30)–(4.33), we obtain

$$\langle T^{11}_{(P)}\rangle|_{x=0\,\text{and}\,x=a} = \frac{i}{2}\int\frac{d\omega}{2\pi}\int\frac{dk}{2\pi}\kappa_-\cot(a\kappa_-). \quad (4.34)$$

The procedure to find $\langle T^{11}_{(Q)}\rangle$ is again completely analogous and we do not need to repeat it again here. The final result that we find is once again the same one given in Eq. (4.26). Thus, we are again lead to the very same previous result for the Casimir force, given by Eq. (3.46).

V. INTERPRETING THE INDEPENDENCE OF THE RESULTS FOR DIFFERENT BOUNDARY CONDITIONS

Casimir forces are, in general, sensible to the BC changes. However, in the previous calculations, we have shown that, for the MPCS model, it does not depend whether we have MP or PC BC. In this section, we are willing to find an argument that sustains this coincidence, as well as to find out some other equivalent BC. The fact that the Casimir force obtained with both the PC and MP boundaries is the same can be understood as a consequence of the fact that the components f_μ (or g_μ) are not independent from each other (since there are three components A_μ and just two degrees of freedom). To see this interdependence more clearly, we can use the relations obtained for the canonical momenta in the model,

$$\pi_\nu = \frac{\partial\mathcal{L}}{\partial\dot{A}^\nu}, \quad (5.1)$$

where $\mathcal{L}$ is given in Eq. (2.1). The MPCS model has two constraints:

$$\pi_0 \approx 0 \quad (5.2)$$

and

$$\partial_i\pi_i - \frac{\mu}{4}\epsilon_{ij}\partial_iA_j - m^2A_0 \approx 0, \quad (5.3)$$

where the "≈" symbol is used to emphasize that both constraints are secondary and $\pi_i = F_{0i} + (\mu/4)\epsilon_{ji}A_j$. The second constraint, Eq. (5.3), shows us that A_0 is not an independent variable (the same can be said about f_0 and also for g_0). Indeed, using Eqs. (5.2)–(5.3), we can write the generating functional Z only in terms of $\{A_i,\pi_i\}$ (and analogously for f_μ and g_μ).

Another important conclusion about the Casimir force, concerning the interdependence of f_μ and g_μ, in the case of the BC $F_0 = 0$, can be obtained as follows. Using the equations of motion for g_μ and f_μ, given by Eqs. (3.6)–(3.7), respectively, we can obtain [10]

$$m_+\epsilon^{\mu\nu\gamma}\partial_\nu f_\gamma = -\partial_\alpha f^{\mu\alpha}, \quad (5.4)$$

$$m_-\epsilon^{\mu\nu\gamma}\partial_\nu g_\gamma = \partial_\alpha g^{\mu\alpha}. \quad (5.5)$$

Thus, we find the following relations satisfied by the vector field f^μ:

$$m_+f^0 = f^{21}, \quad (5.6)$$

$$m_+ f^1 = f^{20}, \tag{5.7}$$

$$m_+ f^2 = f^{01}, \tag{5.8}$$

$$m_+^2 f^1 = \partial_\mu f^{1\mu}, \tag{5.9}$$

$$\partial_\mu f^\mu = 0, \tag{5.10}$$

where $f_{\alpha\beta} = \partial_\alpha f_\beta - \partial_\beta f_\alpha$. Similar relations also follow for the vector field g^μ when considering Eq. (5.5).

Considering the BC $f_0 = 0$, we obtain, from Eq. (5.6) that $\partial^1 f^2 = \partial^2 f^1$ or, using Eqs. (5.7)–(5.8), that

$$\partial^0(\partial^1 f^1 + \partial^2 f^2) - \partial^1\partial^1 f^0 - \partial^2\partial^2 f^0 = 0. \tag{5.11}$$

We make use of a transverse Fourier transform for f_0, similar to the one used in Eq. (3.12),

$$f_0(x,y,t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{dk}{2\pi} e^{iky} \mathcal{F}_0(k,\omega,x). \tag{5.12}$$

Since we are considering $f_0 = 0$ at the boundaries, we can write

$$f_0(x,y,t)|_{x=0 \text{ and } x=a} = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{dk}{2\pi} e^{iky} \mathcal{F}_0(k,\omega,x)\Big|_{x=0 \text{ and } x=a} = 0. \tag{5.13}$$

Since Eq. (5.13) must be valid for all y and t, we conclude that $\mathcal{F}_0(k,\omega,x) = 0$ at $x = 0$ and $x = a$. Thus, we can write

$$\partial_2 f_0(x,y,t)|_{x=0 \text{ and } x=a} = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{dk}{2\pi} e^{iky} ik\mathcal{F}_0(k,\omega,x)\Big|_{x=0 \text{ and } x=a} = 0. \tag{5.14}$$

The condition above has a simple geometric interpretation: $f_0(x) = 0$ for all points $(0,y)$ and (a,y). Therefore, at $x = 0$ and at $x = a$ the variation of $f_0(x,y,t)$ with respect to y $(\partial f/\partial y)$ is null. In a similar way we can conclude that (the following expressions are to be assumed to be implicitly valid always at the boundaries, unless specified otherwise)

$$\partial_2 g_0 = 0, \tag{5.15}$$

$$\partial_0\partial_0 f_0 = 0, \tag{5.16}$$

$$\partial_2\partial_2 f_0 = 0. \tag{5.17}$$

From Eqs. (2.13) and (5.14)–(5.15), we can conclude that the imposition of the BC $F_0 = 0$ is equivalent to the BC $\partial_2 A_0 = 0$. Analogously, we can obtain that $\partial_0 A_0 = 0$.

Also, from Eqs. (5.11) and (5.17), we can write

$$\partial^0(\partial^1 f^1 + \partial^2 f^2) - \partial^1\partial^1 f^0 = 0. \tag{5.18}$$

Using Eq. (5.10), we can rewrite Eq. (5.18) as $\partial^0\partial_0 f^0 - \partial^1\partial^1 f^0 = 0$. Thus, using Eq. (5.16), we can conclude that

$$\partial^1\partial^1 f^0 = 0. \tag{5.19}$$

Making analogous considerations as the ones that lead to Eqs. (5.14) and (5.16)–(5.17), we can conclude from Eq. (5.19) that

$$\partial_0\partial^1\partial^1 f^0 = 0. \tag{5.20}$$

Using now Eqs. (5.6) and (5.9), we can write $m_+^2 f^1 = \partial_0 f^{10} - m_+\partial_2 f_0$. But since $\partial_2 f_0 = 0$, we obtain that

$$m_+^2 f^1 = \partial_0 f^{10} \Rightarrow m_+^2 \partial^1 f^1 = \partial_0\partial^1\partial^1 f^0 - \partial_0\partial^0\partial^1 f^1. \tag{5.21}$$

Using Eq. (5.20), we conclude, from Eq. (5.21), that

$$m_+^2 \partial_1 f^1 = -\partial_0\partial^0\partial_1 f^1. \tag{5.22}$$

We can now also use a transverse Fourier transform for $f^{1\prime} = \partial_1 f^1$ to write

$$f^{1\prime}(x,y,t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{dk}{2\pi} e^{iky} \mathcal{F}^{1\prime}(k,\omega,x). \tag{5.23}$$

Using Eqs. (5.22)–(5.23), we conclude that

$$m_+^2 \mathcal{F}^{1\prime} = \omega^2 \mathcal{F}^{1\prime}. \tag{5.24}$$

Since Eq. (5.24) must be valid for all ω, we conclude that $\mathcal{F}^{1\prime} = 0$. Thus, from Eq. (5.23), we obtain that $f^{1\prime} = \partial_1 f^1 = 0$. We can also draw analogous conclusions as applied for the field g_μ. Thus, we can conclude that the imposition of the BC $F_0 = 0$ (which is here seen in terms of the equivalent strong BC imposed on the fields g and f, i.e., $\tilde{f}_0 = \tilde{g}_0 = 0$ and $f_0 = g_0 = 0$) is equivalent to the BC $\partial_1 F^1 = 0$. Therefore, the same Casimir force should be obtained in the cases of these two BC.

We can collect all the results found up to now to study the behavior of F_1. First, using the BC $\partial_2 A_0 = 0$ in the definition (3.21), we obtain

$$F_1 = -\partial^0 A^2. \tag{5.25}$$

Also, from Eq. (3.21), we deduce that $\partial_\mu F^\mu = 0$. But since $\partial_1 F^1 = 0$ (at the boundaries), we obtain

$$\partial_0 F^0 + \partial_2 F^2 = 0. \tag{5.26}$$

We are considering the BC $F_0 = 0$. Then, analogously to what we have done in Eqs. (5.13)–(5.14), we can conclude that (recalling that the relations below are meant to be valid at the boundaries) $\partial_0 F^0 = 0$. Then, from Eq. (5.26), we can write that $\partial_2 F^2 = 0$. Using again the reasoning that lead us from Eq. (5.13) to Eq. (5.14), we obtain that $F^2 = \epsilon^{2\nu\gamma}\partial_\nu A_\gamma = 0$. Using the relation (2.13), we conclude that $\epsilon^{2\nu\gamma}\partial_\nu g_\gamma = 0$ (analogously to f_γ). We can use then the self duality of g_γ [represented by Eq. (3.6), with $J_{(-)\mu} = 0$] to obtain $g_2 = 0$ (analogously to f_2).

Hence we conclude that $A_2 = 0$ is also a BC for our model. Analogously to what we have done above [Eqs. (5.13)–(5.14)], we conclude then that $\partial^0 A_2 = 0$ and, hence, using Eq. (5.25), we obtain an equivalent BC: $F_1 = 0$.

Summarizing, we can conclude that the BC $F_1 = 0$, $F_2 = 0$, $\partial_1 F^1 = 0$ and $\partial_2 F^2 = 0$ are all equivalent to the BC $F_0 = 0$. Therefore, the same Casimir force is expected to be obtained for all these cases. Here, we have made explicit calculations for the BC $F_0 = 0$ and $F_1 = 0$, confirming that the results obtained are the same in both cases. We note that the particular case for the Neumann BC $\partial_1 F^1 = 0$ was studied in Ref. [9], where it was shown to also lead to the same result for the Casimir force, Eq. (3.46).[1]

VI. SUPPRESSION OF THE CASIMIR FORCE IN THE PRESENCE OF VORTEX PARTICLELIKE EXCITATIONS

As shown in the previous sections, the Casimir force for the cases of PC ($F_1 = 0$), MP ($F_0 = 0$) and also Neumann ($\partial_1 F^1 = 0$) BC all leads to the same result,

$$f = -\frac{1}{16\pi a^3}\left[\int_{2am_-}^{\infty} dz \frac{z^2}{e^z - 1} + \int_{2am_+}^{\infty} dz \frac{z^2}{e^z - 1}\right]. \tag{6.1}$$

Note that Eq. (6.1) is of the form of a second Debye function [26],

$$\int_b^\infty dz \frac{z^n}{e^z - 1} = \sum_{k=1}^{\infty} e^{-kb}\left(\frac{b^n}{k} + n\frac{b^{n-1}}{k^2} + n(n-1)\frac{b^{n-2}}{k^3} + \cdots + \frac{n!}{k^{n+1}}\right), \tag{6.2}$$

indicating that the Casimir force for both cases decays exponentially with $am_\pm$.

Specific limits for $am_\pm$, like for small or large values, can be easily derived using directly the expression (3.46) or from (6.2). These results can also be readily expressed in terms of the Proca and Chern-Simons masses, m and μ, respectively, using Eq. (2.16), or also from Eqs. (2.2)–(2.3), relating these masses to the original parameters of the effective particle-vortex dual Lagrangian density model.

By expressing $m_\pm$ in terms of the original parameters of the particle-vortex dual Lagrangian density model, i.e., in terms of the vacuum expectation values for the Higgs field, ρ_0, for the vortex field, ψ_0, and the CS parameter Θ, we have that

$$m_\pm = \frac{e^2\rho_0^2}{2\Theta}\left[\sqrt{1 + \left(8\pi\frac{\Theta\psi_0}{e^2\rho_0}\right)^2} \mp 1\right]. \tag{6.3}$$

As it was shown in Ref. [13], vortices are energetically favored to condense for values of the CS parameter below a critical value $\Theta_c \approx (e^2/\pi)\ln 6 \simeq 0.57e^2$. For $\Theta < \Theta_c$ the vortex condensate can be written as $\psi_0^2 \approx (e^2\rho_0^2/\Theta)\sqrt{6 - \exp(\pi\Theta/e^2)}$. The condensed vortex phase can be interpreted as being equivalent to the Shubnikov phase for type-II superconductors in the presence of a magnetic field [27], with a Ginzburg-Landau parameter $\kappa \equiv e\rho_0/\Theta > 1/\sqrt{2}$. In the analysis that follows, we remain within parameter values satisfying these conditions. In Fig. 1 we show the result for the Casimir force

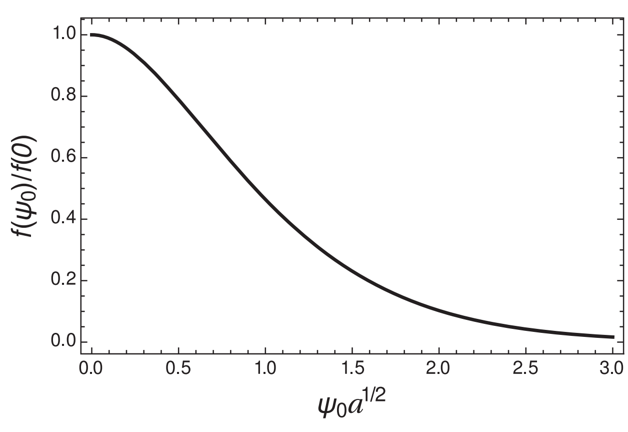

FIG. 1. The (normalized) Casimir force as a function of the vortex condensate ψ_0. The following representative values of parameters were used: $\Theta/e^2 = 0.1$ and $\rho_0 a^{1/2} = 1$.

[1]It should be noticed that in Ref. [9] there is a misprint in the expression for the masses m_1 and m_2 considered there by a factor two. With this correction, those two masses considered in that reference just correspond to $m_\pm$ considered here. This in turn corresponds to a correction in Sec. IV of that reference, where the CS parameter considered there should be replaced by 2Θ instead.

Eq. (6.1) as a function of the vortex condensate ψ_0, normalized by the Casimir force in the absence of a vortex condensate, $f(\psi_0 = 0)$. The result shows that the Casimir force can become strongly suppressed in the presence of vortex matter as compared to the absence of it. This suppression of the Casimir force can be interpreted as a result of the repelling force between vortices, analogously to what happens in the phenomenology of type-II superconductors, when in the Shubnikov phase [27], which opposes the attractive Casimir force.

VII. CONCLUSIONS

In this work, we have analyzed the Casimir force for the MPCS model. As explained in Sec. II, this model can be interpreted as an effective (dual) model describing vortex excitations for a CSH model. We have obtained the Casimir force for the cases of perfect conductor and perfect magnetically permeable BC. This has been possible by mapping the MPCS model into a doublet consisting of a self-dual and an antiself-dual PCS model. We found that the Casimir force remains the same when computed using the two forms of BC considered in this paper. The result found here for the Casimir force also agrees with the case of considering the Neumann BC, which was derived previously in Ref. [9]. The reason for these results being the same has been explained to be a consequence of the symmetry and constraints satisfied by these models involving a CS term. These results have also been confirmed by using the mapping of the MPCS model in a doublet of MCS models. The derivation using these two independent mappings also helps to show that the result obtained for the Casimir force (for the type of BC considered here) is not some particular consequence of the mapping used. Thus, our results also highlight a symmetry found when we consider various types of BC in the computation of the Casimir effect.

Even though it can be argued that the model we have studied here, which can be associated with the vacuum state of a system of vortex excitations in a plane, is mostly of theoretical interest and might be far from describing real physical systems of interest, our results are indicative of a behavior that can manifest in these systems. As such, our results might be of relevance for the next generation of experiments involving the Casimir effect [28], or those involving, for example, vortex-based superconducting detectors [29,30]. Usually, such systems involve nanometer scales, in which the Casimir force turns out to be relevant, and possibly also alter the microscopic parameters of the detectors [31]. Our results can also be of relevance when devising materials based on superconducting films to work as possible suppressors of the Casimir force, such as in those laboratory experiments that require performing extremely careful force measurements near surfaces. This might be the case of the searches for possible deviations of the Newtonian gravity.

The study performed here for the MPCS model also has its own merits, independent of its connection to a vortex model. The MPCS model constitutes massive gauge particles, with mass terms that have both topological and nontopological origins. Also, the Maxwell-Proca and the MCS models can be seen as particular cases of the MPCS model. So, we expect that a better comprehension of the roles of the mass terms, either of topological or nontopological origin, in the derivation of the Casimir force might eventually provide arguments in favor of one or the other, when using these models with the objective of understanding some of the properties of real planar systems with massive excitations. This also includes, of course, deriving the Casimir force under different BC, as we have studied in this work.

ACKNOWLEDGMENTS

R. O. R. is partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil) and by Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ). V. S. A. is partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES-Brazil). R. F. O. and C. R. M. S are partially supported by CNPq-Brazil. J. F. M. N. thanks J. A. Helayël Neto, S. Perez, M. C. de Lima and D. T. Alves for very fruitful commentaries. We are also grateful to the anonymous referee for the many helpful comments that allowed us to improve the manuscript from its first version. We also thank the anonymous referee for pointing out to us the Refs. [24,25] and for the comments on the renormalization issues involved when computing the Casimir effect.

[1] C. M. Will, The confrontation between general relativity and experiment, Living Rev. Relativity **17**, 4 (2014).

[2] E. G. Adelberger, B. R. Heckel, and A. E. Nelson, Tests of the gravitational inverse square law, Annu. Rev. Nucl. Part. Sci. **53**, 77 (2003).

[3] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).

[4] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009).

[5] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, The Casimir force between real materials: experiment and theory, Rev. Mod. Phys. **81**, 1827 (2009).

[6] V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Advance and prospects in constraining the Yukawa-type corrections to Newtonian gravity from the Casimir effect, Phys. Rev. D **81**, 055003 (2010).

[7] S. Ribeiro and S. Scheel, Shielding vacuum fluctuations with graphene, Phys. Rev. A **88**, 042519 (2013); **89**, 039904 (E) (2014).

[8] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. **83**, 1057 (2011); Y. H. Chen, F. Wilczek, E. Witten, and B. I. Halperin, On anyon superconductivity, Int. J. Mod. Phys. B **03**, 1001 (1989).

[9] J. F. de Medeiros Neto, R. O. Ramos, and C. R. M. Santos, Casimir force due to condensed vortices in a plane, Phys. Rev. D **86**, 125034 (2012).

[10] R. Banerjee and S. Kumar, Self-dual models and mass generation in planar field theory, Phys. Rev. D **63**, 125008 (2001).

[11] R. Banerjee and C. Wotzasek, Bosonization and duality symmetry in the soldering formalism, Nucl. Phys. **B527**, 402 (1998); R. Banerjee and S. Kumar, Self duality and soldering in odd dimensions, Phys. Rev. D **60**, 085005 (1999).

[12] J. F. Medeiros Neto, R. F. Ozela, R. O. Correa, and R. O. Ramos, Casimir force for a Maxwell-Chern-Simons system via model transformation, Braz. J. Phys. **44**, 798 (2014).

[13] R. O. Ramos and J. F. Medeiros Neto, Transition point for vortex condensation in the Chern-Simons Higgs model, Phys. Lett. B **666**, 496 (2008).

[14] X. G. Wen and A. Zee, Quantum Disorder, Duality and Fractional Statistics in $(2+1)$ Dimensions, Phys. Rev. Lett. **62**, 1937 (1989).

[15] C. P. Burgess and B. P. Dolan, Particle-vortex duality and the modular group: applications to the quantum Hall effect and other two-dimensional systems, Phys. Rev. B **63**, 155309 (2001).

[16] S. K. Paul and A. Khare, Charged vortices in an abelian Higgs model with Chern-Simons term, Phys. Lett. B **174**, 420 (1986); **182**, 414(E) (1986).

[17] R. Jackiw and E. J. Weinberg, Self Dual Chern-Simons Vortices, Phys. Rev. Lett. **64**, 2234 (1990).

[18] Y. Kim and K.-M. Lee, Vortex dynamics in self dual Chern-Simons Higgs systems, Phys. Rev. D **49**, 2041 (1994).

[19] H. Kleinert, *Gauge Fields in Condensed Matter, Vol. I: Superflow and Vortex Lines* (World Scientific, Singapore, 1989); *Gauge Fields in Condensed Matter, Vol. II: Stresses and Defects, Differential geometry, Crystal Defects* (World Scientific, Singapore, 1989).

[20] K. A. Milton and Y. J. Ng, Maxwell-Chern-Simons Casimir effect, Phys. Rev. D **42**, 2875 (1990).

[21] D. Dalmazi, Generalized duality between local vector theories in $D = 2 + 1$, J. High Energy Phys. 08 (2006) 040.

[22] G. V. Dunne, Aspects of Chern-Simons theory, arXiv:hep-th/9902115.

[23] D. T. Alves, E. R. Granhen, J. F. Medeiros Neto, and S. Perez, Repulsive Maxwell-Chern-Simons Casimir effect, Phys. Lett. A **374**, 2113 (2010).

[24] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, and H. Weigel, Calculating vacuum energies in renormalizable quantum field theories: a new approach to the Casimir problem, Nucl. Phys. **B645**, 49 (2002).

[25] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, and H. Weigel, Casimir energies in light of quantum field theory, Phys. Lett. B **572**, 196 (2003).

[26] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (Dover Publications, Inc.,, New York, 1972).

[27] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1996).

[28] A. Allocca, G. Bimonte, D. Born, E. Calloni, G. Esposito, U. Huebner, E. Il'ichev, L. Rosa, and F. Tafuri, Results of measuring the influence of Casimir energy on superconducting phase transitions, J. Supercond. **25**, 2557 (2012).

[29] A. M. Kadin, M. Leung, A. D. Smith, and J. M. Murduck, Photofluxonic detection: A new mechanism for infrared detection in superconducting thin films, Appl. Phys. Lett. **57**, 2847 (1990).

[30] A. D. Semenov *et al.*, An energy-resolving superconducting nanowire photon counter, Supercond. Rev. **20**, 919 (2007).

[31] D. Brandt, G. W. Fraser, D. J. Raine, and C. Binns, Superconducting Detectors and the Casimir Effect, J. Low Temp. Phys. **151**, 25 (2008).