Twisted conformal algebra related to κ-Minkowski space

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Twisted deformations of the conformal symmetry in the Hopf algebraic framework are constructed. The first one is obtained by a Jordanian twist built up from dilatation and momenta generators. The second is the lightlike κ-deformation of the Poincaré algebra extended to the conformal algebra, obtained by a twist corresponding to the extended Jordanian r-matrix. The κ-Minkowski spacetime is covariant quantum space under both of these deformations. The extension of the conformal algebra by the noncommutative coordinates is presented in two cases. The differential realizations for κ-Minkowski coordinates, as well as their left-right dual counterparts, are also included.

I. INTRODUCTION

The conformal symmetry is considered as the fundamental symmetry of spacetime. Even though it cannot describe massive particles and fields, many high-energy physics theories admit the conformal symmetry. It also includes two fundamental geometries—Poincaré and de Sitter—as subcases. The conformal algebra $\mathfrak{c}$ consists of the Lorentz generators $M_{\mu\nu}$, translations P_μ, dilatations D (which generate scaling transformations) and generators of the special conformal transformations K_μ. The metric tensor on the d-dimensional spacetime we denote as $g_{\mu\nu}$ (it does not need to be in the diagonal form; it only has to be symmetric and nondegenerate). The commutation relations of the conformal algebra $\mathfrak{c}$ for $d > 2$ [for example, for $d = 4$, we deal with $\mathfrak{c} = \mathfrak{so}(2,4)$], including the standard Poincaré ones, are the following:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}),$$
$$[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu),$$
$$[D, K_\mu] = -iK_\mu, \qquad [D, P_\mu] = iP_\mu,$$
$$[K_\mu, P_\nu] = 2i(g_{\mu\nu}D - M_{\mu\nu}),$$
$$[K_\mu, M_{\nu\rho}] = i(g_{\mu\nu}K_\rho - g_{\mu\rho}K_\nu),$$
$$[K_\mu, K_\nu] = 0, \qquad [M_{\mu\nu}, D] = 0, \qquad [P_\mu, P_\nu] = 0. \quad (1)$$

Together with the rise of the interest in the deformations of relativistic symmetries of spacetime [1,2], the quantum deformations of the conformal algebra were investigated already in the 1990s [3–10]. After the introduction of the κ-deformed Poincaré algebra (with $M_{\mu\nu}$ and P_μ as its generators) with the dimensionful deformation parameter κ,

the same classical r-matrix as in Ref. [1] ($r = \frac{i}{\kappa}M_{0\nu} \wedge P^\nu$ with the special choice of the basis for which the metric tensor is $g_{00} = 0$) was used in the quantum deformation of Poincaré–Weyl algebra [4]. Also, the deformations of the full $D = 4$ conformal symmetries were introduced [5,6], corresponding to the standard (i.e., timelike) version of the κ-deformation. The so-called null-plane (light-cone) deformation of Poincaré algebra [7] has been extended as well to the deformation of the Poincaré–Weyl group [8] and to the conformal group [9] as well. All of the above-mentioned deformations of conformal symmetry were corresponding to the κ-Minkowski spacetime noncommutativity: $[\hat{x}^\mu, \hat{x}^\nu] = i(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu)$ in either the timelike $a^\mu a_\mu < 0$, spacelike $a^\mu a_\mu > 0$, or lightlike $a^\mu a_\mu = 0$ case. The deformation parameter κ enters via $a^\mu = \frac{1}{\kappa}\tau^\mu$, with $\tau^2 \in \{-1, 0, 1\}$ for the metric tensor with the Lorentzian (mostly positive) signature.

Since the Poincaré–Weyl and conformal algebras contain dilatation D as an additional generator, one can also consider another classical r-matrix, $r = \frac{i}{\kappa}D \wedge P_0$, as an internal one for these algebras and the corresponding quantum deformations. For example, in Ref. [10], the so-called Jordanian deformations of the conformal algebra (together with anti-de Sitter and de Sitter ones) were considered. One should also mention that the same (timelike) Jordanian r-matrix was also used in the twisted deformation of the Poincaré–Weyl algebra [11] as well as in the twisting of the inhomogeneous general linear algebra [12] together with some applications found in field and gauge theories [13,14]. A large class of Abelian twists related to twisted statistics in κ-Minkowski spacetime was considered in Ref. [15].

For the conformal algebra, the twist type of the deformation was first considered in the Moyal–Weyl case [16] related with the Abelian classical r-matrix, $r = i\theta^{\mu\nu}P_\mu \wedge P_\nu$, corresponding to the constant noncommutativity of the

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spacetime coordinates $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$. The θ-deformation of the conformal algebra found some applications in noncommutative field theories, see, e.g., Ref. [17], and has been extended also to the deformation of the superconformal algebra [18] as well.

Recently, the Jordanian deformations have gained in popularity in the applications in AdS/CFT correspondence as some of the deformations of the Yang–Baxter sigma models were shown to preserve the integrability [19,20]. The Jordanian deformations related with the classical r-matrices (satisfying the classical or modified Yang–Baxter equation) were applied to the anti-de Sitter (AdS) part of the correspondence principle [21]. The κ-Minkowski spacetime was also considered in this context in Ref. [22].

Our aim in this paper is to present the quantum deformations of the conformal algebra which are described by the classical r-matrices satisfying the classical Yang–Baxter equation. For such cases, the Drinfeld twists (satisfying the cocycle condition) provide explicitly the star product in the algebra of spacetime coordinates. We are interested in the Jordanian κ-deformation of the conformal symmetry and the lightlike κ-deformation of the Poincaré algebra extended to the conformal algebra which will correspond to Jordanian and extended Jordanian r-matrices, respectively. The deformed conformal algebra is considered as Hopf algebra with twisted coproducts and antipodes. Conformal invariance is compatible with the κ-Minkowski spacetime which constitutes the covariant quantum space under both of these deformations.

The paper is organized as follows. We start, in Sec. II, with recalling the basics of the twist deformations and the conditions for noncommutative spacetime covariance with respect to the twisted symmetries. In Sec. III, we consider the Jordanian twist [11,23] in the covariant form [24], providing, for the metric with the Lorentzian signature, three kinds (time-, light-, and spacelike) of deformations of the conformal algebra depending on the type of the vector a^μ. The twisted coalgebra sector is presented together with the corresponding κ-Minkowski spacetime realization consistent with the Hopf-algebraic actions. Later, in Sec. IV, we investigate another twist [25–27] related by transposition to the so-called extended Jordanian twist [28,29]. The twist is built from only Poincaré generators, and therefore it provides the extension of the lightlike κ-deformation of the Poincaré algebra to the conformal algebra. For the Poincaré subalgebra, the $a^2 = 0$ deformation reduces to the null-plane deformation of Refs. [7,29]. The realization for κ-Minkowski coordinates is also presented. In both cases, in Sec. III and IV, we include the cross-commutation relations between the conformal algebra generators and the noncommutative coordinates. Also, the so-called left-right dual κ-Minkowski realizations are constructed from the transposed twists in Secs. III and IV, respectively. The last section concludes the paper with some remarks.

II. TWIST DEFORMATIONS OF THE CONFORMAL ALGEBRA

The twist deformation framework of spacetime symmetries requires us to deal with the Hopf algebras instead of Lie algebras corresponding to the given symmetry. To introduce this notion, we need to extend the Lie algebra [we are interested in the conformal algebra $\mathfrak{c}$ described by (1)] into the universal enveloping algebra $U(\mathfrak{c})$ which can be equipped with the Hopf algebra structures on its generators $L = \{M_{\mu\nu}, P_\mu, D, K_\mu\}$ in the following standard way:

$$\text{coproduct:} \quad \Delta(L) = L \otimes 1 + 1 \otimes L \qquad (2)$$

$$\text{counit:} \quad \epsilon(L) = 0 \qquad (3)$$

$$\text{antipode:} \quad S(L) = -L. \qquad (4)$$

The above maps are then extended to the whole $U(\mathfrak{c})$. Such undeformed Hopf algebra can be seen as the conformal symmetry of the usual Minkowski spacetime in the algebraic form given by an Abelian algebra of coordinate functions $x^\mu \in \mathcal{A}$, which is itself a subalgebra of undeformed Heisenberg algebra H:

$$[x^\mu, x^\nu] = 0 \qquad (5)$$

$$[x^\mu, P_\nu] = i\delta^\mu_\nu \qquad (6)$$

$$[P_\mu, P_\nu] = 0. \qquad (7)$$

The conformal algebra has the following representation:

$$\begin{aligned} M^{\mu\nu} &= -x^\mu P^\nu + x^\nu P^\mu \\ D &= x \cdot P \\ K^\mu &= 2x^\mu (x \cdot P) - x^2 P^\mu = x^\mu D + x_\alpha M^{\alpha\mu}, \end{aligned} \qquad (8)$$

where $P^\mu = g^{\mu\nu} P_\nu$. In general the compatibility of the spacetime with its symmetry in this "algebralized" setting is via the action $\mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$ of the Hopf algebra $\mathcal{H}$ on the spacetime (module) algebra $\mathcal{A}$ such that

$$L \triangleright (f \cdot g) = \mu[\Delta(L)(\triangleright \otimes \triangleright)(f \otimes g)]. \qquad (9)$$

The multiplication in the module algebra $\mu \colon \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is compatible[1] with the coproduct in the Hopf algebra $\Delta \colon \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ and $L(1) = \epsilon(L) \cdot 1$, $1(f) = f$ for $L \in \mathcal{H}$ and $f \in \mathcal{A}$.

One can easily check that the above condition (9) is satisfied for the undeformed spacetime described by Abelian algebra (5) and the conformal Hopf algebra (1)

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[1]It is also common to rewrite the condition (9) as $L \triangleright (f \cdot g) = (L_{(1)} \triangleright f) \cdot (L_{(2)} \triangleright g)$ where Sweedler notation for the coproduct is used $\Delta(L) = L_{(1)} \otimes L_{(2)}$.

with (2)–(4) as its symmetry and the condition reduces to the usual Leibniz rule,

$$L \triangleright (x^\mu \cdot x^\nu) = (L \triangleright x^\mu)x^\nu + x^\mu(L \triangleright x^\nu),$$
$$L = \{M_{\mu\nu}, P_\mu, D, K_\mu\}, \qquad (10)$$

for any of the generators of the conformal algebra due to (2).

For the deformation, we will use the (Drinfeld) twist technique, which will provide the deformation of the universal enveloping algebra of the conformal algebra $U(\mathfrak{c})$ as Hopf algebra $\mathcal{H} = (U(\mathfrak{c}), \Delta, \epsilon, S)$. The twist $\mathcal{F}$ is, in general, an invertible element of $\mathcal{H} \otimes \mathcal{H}$ satisfying cocycle and normalization conditions:

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}, \qquad (11)$$

$$(id \otimes \epsilon)\mathcal{F} = (\epsilon \otimes id)\mathcal{F} = 1 \otimes 1. \qquad (12)$$

One gets the new Hopf algebra structure $\mathcal{H}^{\mathcal{F}} = (U(\mathfrak{c}), \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}})$ via modifying the coproduct and antipode maps in the following way:

$$\Delta^{\mathcal{F}}(L) = \mathcal{F}\Delta(L)\mathcal{F}^{-1}, \qquad L \in \mathcal{H}$$
$$\epsilon(L) = 0, \qquad S^{\mathcal{F}}(L) = \mathrm{f}^\alpha S(\mathrm{f}_\alpha)S(L)S(\bar{\mathrm{f}}^\beta)\bar{\mathrm{f}}_\beta. \qquad (13)$$

Here, we use the short notation for the twist as $\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha$, $\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$. Both of the twisted deformations considered in this paper will be compatible with the κ-Minkowski spacetime with the defining commutation relations as

$$[\hat{x}^\mu, \hat{x}^\nu] = i(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu). \qquad (14)$$

This algebra will constitute the module algebra over the deformed conformal Hopf algebra; i.e., it is its covariant quantum space.

The cocycle condition (11) for the twist guarantees the coassociativity of the deformed coproduct $\Delta^{\mathcal{F}}$ and also associativity of the corresponding twisted star product in the twisted module algebra $\mathcal{A}^{\mathcal{F}}(\mathcal{A}, \mu_\star)$,

$$f \star g = \mu_\star(f \otimes g) = \mu \circ \mathcal{F}^{-1}(\triangleright \otimes \triangleright)(f \otimes g)$$
$$= (\bar{\mathrm{f}}^\alpha \triangleright f) \cdot (\bar{\mathrm{f}}_\alpha \triangleright g), \qquad (15)$$

for $f, g \in \mathcal{A}$. Additionally, to a given twist, we can associate the so-called realization of noncommuting coordinate functions as follows:

$$\hat{x}^\mu = \mu[\mathcal{F}^{-1}(\triangleright \otimes 1)(x^\mu \otimes 1)] = (\bar{\mathrm{f}}^\alpha \triangleright x^\mu) \cdot \bar{\mathrm{f}}_\alpha, \qquad x^\mu \in \mathcal{A}. \qquad (16)$$

For the twisted case, the compatibility between the deformed coproduct $\Delta^{\mathcal{F}}$ and the star product in the module algebra is analogous to (9)

$$L \triangleright (\mu_\star(f \otimes g)) = \mu_\star(\Delta^{\mathcal{F}}(L)(\triangleright \otimes \triangleright)(f \otimes g)). \qquad (17)$$

In the literature, this condition is known under twisted covariance, and, for example, it was investigated in more detail in the context of the conformal algebra undergoing the Moyal–Weyl deformation with theta-deformed spacetime [16]. The covariance under twisted symmetry was first proved in Ref. [2] for the Moyal–Weyl deformation of the Poincaré symmetry and theta spacetime. In the Hopf algebraic framework, when the noncommutative spacetimes are Hopf modules and their deformed symmetry is the Hopf algebra, the condition of covariance is automatically satisfied via the requirements (9) and (17).

III. JORDANIAN DEFORMATION OF THE CONFORMAL ALGEBRA

We can deform the conformal Hopf algebra $U(\mathfrak{c})$ (1), (2), (3), (4) with the Jordanian twist [11,23,24,27],

$$\mathcal{F}_J = \exp[i\ln(1 - a \cdot P) \otimes D], \qquad (18)$$

where $a \cdot P = a^\mu P_\mu$. The corresponding classical r-matrix is $r = ia^\mu D \wedge P_\mu$. For the metric with the Lorentzian signature, we can distinguish here three cases when vector a^μ can be either timelike, lightlike or spacelike; nevertheless, the formulas presented below are valid for arbitrary, symmetric, and nondegenerate metric.

For simplicity, we introduce the shortcut notation $Z = 1 - a \cdot P$.

The algebra relations (1) and counits (3) stay undeformed. The deformed coproducts are

$$\Delta^{\mathcal{F}_J}(P_\mu) = P_\mu \otimes 1 + Z \otimes P_\mu \qquad (19)$$

$$\Delta^{\mathcal{F}_J}(M_{\mu\nu}) = \Delta(M_{\mu\nu}) - (a_\mu P_\nu - a_\nu P_\mu)Z^{-1} \otimes D \qquad (20)$$

$$\Delta^{\mathcal{F}_J}(D) = D \otimes 1 + Z^{-1} \otimes D \qquad (21)$$

$$\Delta^{\mathcal{F}_J}(K_\mu) = K_\mu \otimes 1 + Z^{-1} \otimes K_\mu$$
$$+ 2[a^\alpha M_{\alpha\mu} + a_\mu D]Z^{-1} \otimes D$$
$$- [2a_\mu(a \cdot P) + a^2 P_\mu]Z^{-2} \otimes iD(iD + 1), \qquad (22)$$

and the deformed antipodes are

$$S^{\mathcal{F}_J}(P_\mu) = -Z^{-1}P_\mu \qquad (23)$$

$$S^{\mathcal{F}_J}(M_{\mu\nu}) = -M_{\mu\nu} - (a_\mu P_\nu - a_\nu P_\mu)D \qquad (24)$$

$$S^{\mathcal{F}_J}(D) = -ZD \qquad (25)$$

$$S^{\mathcal{F}_J}(K_\mu) = -Z\{K_\mu - 2[a^\alpha M_{\alpha\mu} + a_\mu D]D$$
$$+ [2a_\mu(a \cdot P) + a^2 P_\mu]iD(iD + 1)\}. \qquad (26)$$

The corresponding covariant quantum spacetime is the κ-Minkowski one (14) with the realization for coordinates given via (16) as

$$\hat{x}^{\mu}_{J} = \mu[\mathcal{F}^{-1}_{J}(\rhd \otimes 1)(x^{\mu} \otimes 1)]$$
$$= x^{\mu} - a^{\mu}D = x^{\mu} - a^{\mu}(x \cdot P). \qquad (27)$$

It is called right covariant realization [30,31], and commutators with generators of the conformal algebra are

$$[P^{\mu}, \hat{x}^{\nu}_{J}] = -i(g^{\mu\nu} - a^{\nu}P^{\mu})$$
$$[D, \hat{x}^{\mu}_{J}] = -ix^{\mu}$$
$$[M^{\mu\nu}, \hat{x}^{\lambda}_{J}] = i(x^{\mu}g^{\nu\lambda} - x^{\nu}g^{\mu\lambda})$$
$$[K^{\mu}, \hat{x}^{\nu}_{J}] = i(2x^2 g^{\mu\nu} - x^{\mu}x^{\nu} - a^{\nu}K^{\mu}), \qquad (28)$$

where

$$x^{\mu} = \hat{x}^{\mu}_{J} + a^{\mu}D. \qquad (29)$$

Note that the commutators are closed in the conformal algebra and noncommutative coordinates $\hat{x}^{\mu}_{J}$.

The spacetime algebra (14) obtained via (15) is invariant under the twisted conformal transformations which can be seen from action of the conformal symmetry generators on the algebra of functions of κ-Minkowski coordinates, i.e., via the compatibility condition (17). One can check that indeed the twisted case of the Leibniz rule:

$$L \rhd [\mu \circ \mathcal{F}^{-1}_{J}(\rhd \otimes \rhd)(x^{\mu} \otimes x^{\nu} - x^{\nu} \otimes x^{\mu})] =$$
$$L \rhd [i(a^{\mu}x^{\nu} - a^{\nu}x^{\mu})] \qquad (30)$$

is satisfied for any of the generators $L = \{M_{\mu\nu}; P_{\mu}; D; K_{\mu}\}$.

Transposed twist $\tilde{\mathcal{F}}_{J} = \tau_0 \mathcal{F}_J \tau_0$ is obtained from $\mathcal{F}_J$ by interchanging the left and right sides of the tensor product [i.e., $\tau_0(a \otimes b) = b \otimes a$], and it is also a Drinfeld twist satisfying the cocycle (11) and normalization condition

(12). A set of left-right dual generators of κ-Minkowski space can be obtained from the transposed twist:

$$\hat{y}^{\mu}_{J} = \mu[\tilde{\mathcal{F}}_{J}^{-1}(\rhd \otimes 1)(x^{\mu} \otimes 1)] = x^{\mu}(1 - a \cdot P). \qquad (31)$$

Generators $\hat{y}^{\mu}_{J}$ satisfy κ-Minkowski algebra with $a_{\mu} \to -a_{\mu}$,

$$[\hat{y}^{\mu}_{J}, \hat{y}^{\nu}_{J}] = -i(a^{\mu}\hat{y}^{\nu}_{J} - a^{\nu}\hat{y}^{\mu}_{J}), \qquad (32)$$

and they commute with generators $\hat{x}^{\mu}_{J}$:

$$[\hat{x}^{\mu}_{J}, \hat{y}^{\nu}_{J}] = 0. \qquad (33)$$

IV. LIGHTLIKE κ-DEFORMATION OF POINCARÉ ALGEBRA EXTENDED TO THE CONFORMAL ALGEBRA

For the purpose of this section, we consider the metric tensor with Lorentzian signature and use mostly positive sign convention, i.e., $g_{\mu\nu} = \text{diag}(-, +, +, \cdots, +)$. We deform the conformal Hopf algebra $U(\mathfrak{c})$ (1), (2), (3), (4) with the twist $\mathcal{F}_{LL}$ leading to lightlike κ-deformation of the Poincaré algebra [14,25,27,29] (which is related to the extended Jordanian twist [28,32])[2]:

$$\mathcal{F}_{LL} = \exp\left[-ia_{\alpha}P_{\beta}\frac{\ln(1 + a \cdot P)}{a \cdot P} \otimes M^{\alpha\beta}\right]. \qquad (34)$$

The above twist satisfies the cocycle condition (11) [27] with the lightlike vector a^{μ} [25,27], and the classical r-matrix is $r = a^{\mu}M_{\mu\nu} \wedge P^{\nu}$. We also introduce the following notation:

$$\tilde{Z} = \frac{1}{1 + a \cdot P}, \qquad m_{\mu} = a^{\alpha}M_{\alpha\mu}. \qquad (35)$$

Coproducts are

$$\Delta^{\mathcal{F}_{LL}}(P_{\mu}) = \Delta(P_{\mu}) + \left[P_{\mu}a^{\alpha} - a_{\mu}\left(P^{\alpha} + \frac{1}{2}a^{\alpha}P^2\right)\tilde{Z}\right] \otimes P_{\alpha} \qquad (36)$$

$$\Delta^{\mathcal{F}_{LL}}(M_{\mu\nu}) = \Delta(M_{\mu\nu}) + (\delta^{\alpha}_{\mu}a_{\nu} - \delta^{\alpha}_{\nu}a_{\mu})\left(P^{\beta} + \frac{1}{2}a^{\beta}P^2\right)\tilde{Z} \otimes M_{\alpha\beta} \qquad (37)$$

$$\Delta^{\mathcal{F}_{LL}}(m_{\alpha}) = \Delta(m_{\alpha}) + a_{\mu}\left(P^{\alpha} + \frac{1}{2}a^{\alpha}P^2\right)\tilde{Z} \otimes m_{\alpha} \qquad (38)$$

$$\Delta^{\mathcal{F}_{LL}}(D) = \Delta(D) - P_{\alpha}\tilde{Z} \otimes m^{\alpha} \qquad (39)$$

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[2]The explicit relation between the twist (34) and the standard extended Jordanian twist corresponding to the lightlike case (up to the transposition) is presented in detail in Sec. VIII B in Ref. [27].

$$\Delta^{\mathcal{F}_{LL}}(K_\mu) = K_\mu \otimes 1 + \left[\delta_\mu^\alpha + P_\mu a^\alpha - a_\mu \left(P^\alpha + \frac{1}{2} a^\alpha P^2 \right) \tilde{Z} \right] \otimes K_\alpha$$
$$+ \left\{ [2(a_\mu(iD + \tilde{Z}) - im_\mu)P^\alpha + iM_\mu{}^\alpha] - 2Dg_{\mu\alpha} \right\} \otimes m_\alpha$$
$$+ [iP_\mu g^{\alpha\beta} - 2i(\delta_\mu^\alpha - a_\mu P^\alpha \tilde{Z})P^\beta \tilde{Z}] \otimes m_\alpha m_\beta. \tag{40}$$

Antipodes are

$$S^{\mathcal{F}_{LL}}(P_\mu) = -\left[P_\mu + a_\mu \left(P^\alpha + \frac{1}{2} a^\alpha P^2 \right) P_\alpha \right] \tilde{Z} \tag{41}$$

$$S^{\mathcal{F}_{LL}}(M_{\mu\nu}) = -M_{\mu\nu} - (\delta_\mu^\alpha a_\nu - \delta_\nu^\alpha a_\mu) \left(P_\beta + \frac{1}{2} a_\beta P^2 \right) M_{\alpha\beta} \tag{42}$$

$$S^{\mathcal{F}_{LL}}(m_\mu) = -m_\mu - a_\mu \left(P^\alpha + \frac{1}{2} a^\alpha P^2 \right) m_\alpha \tag{43}$$

$$S^{\mathcal{F}_{LL}}(D) = -\tilde{Z}^{-1}D \tag{44}$$

$$S^{\mathcal{F}_{LL}}(K_\mu) = -\left[\delta_\mu^\gamma + P^\gamma a^\mu - a^\gamma \left(P_\mu + \frac{1}{2} a_\mu P^2 \right) \tilde{Z} \right]$$
$$\times \left\{ K_\gamma + [2(a_\gamma(iD + \tilde{Z}) - im_\gamma)P^\alpha + iM_\gamma{}^\alpha]S(m_\alpha) - 2DS(m_\gamma) \right.$$
$$\left. + iP_\gamma S(m^2) - 2i(\delta_\gamma^\alpha - a_\gamma P^\alpha \tilde{Z})P^\beta \tilde{Z}S(m_\alpha m_\beta) \right\}. \tag{45}$$

In this case, the realization (16) is given as

$$\hat{x}_{LL}^\mu = \mu[\mathcal{F}_{LL}^{-1}(\triangleright \otimes 1)(x^\mu \otimes 1)] = x^\mu + a_\alpha M^{\alpha\mu} = x^\mu(1 + a \cdot P) - (a \cdot x)P^\mu. \tag{46}$$

It corresponds to the natural realization of κ-Minkowski space [30,31]. Commutators with generators of the conformal algebra are

$$[P^\mu, \hat{x}_{LL}^\nu] = -i[g^{\mu\nu}(1 + a \cdot P) - a^\mu P^\nu]$$
$$[D, \hat{x}_{LL}^\mu] = -ix^\mu$$
$$[M^{\mu\nu}, \hat{x}_{LL}^\lambda] = i(\hat{x}_{LL}^\mu g^{\nu\lambda} - \hat{x}_{LL}^\nu g^{\mu\lambda} - a^\mu M^{\nu\lambda} + a^\nu M^{\mu\lambda})$$
$$[K^\mu, \hat{x}_{LL}^\nu] = i(2x^2 g^{\mu\nu} - x^\mu x^\nu + a^\mu K^\nu - g^{\mu\nu}(a \cdot K)), \tag{47}$$

where

$$x^\mu = \hat{x}_{LL}^\mu - a_\alpha M^{\alpha\mu}. \tag{48}$$

Note that, like in the Jordanian case, the above commutators (47) are also closed in the conformal algebra and noncommutative coordinates $\hat{x}_{LL}^\mu$.

Let us also comment on the fact that, even though the above twist (34) is written in a covariant form (valid for the a^μ as a time-, light-, and spacelike vector), it satisfies the cocycle condition (11) only for the lightlike case, $a^2 = 0$ [25,27]. Therefore, only in this case, it corresponds to an associative star product (15) of κ-Minkowski coordinates (14). The two remaining cases (time- and spacelike) lead to deformations of κ-Snyder type with a nonassociative star product.

Again, one can easily check that the noncommutative spacetime (14) is invariant under this twisted conformal symmetry via the analogous condition as (30) in Sec. III:

$$L \triangleright [\mu \circ \mathcal{F}_{LL}^{-1}(\triangleright \ \otimes \triangleright)(x^\mu \otimes x^\nu - x^\nu \otimes x^\mu)]$$
$$= L \triangleright [i(a^\mu x^\nu - a^\nu x^\mu)]. \tag{49}$$

Transposed twist $\tilde{\mathcal{F}}_{LL} = \tau_0 \mathcal{F}_{LL} \tau_0$ is obtained from $\mathcal{F}_{LL}$ by interchanging the left and right sides of the tensor product, and it is also a Drinfeld twist satisfying the cocycle (11) and normalization condition (12). A set of left-right dual generators of κ-Minkowski space can be obtained from transposed twist:

$$\hat{y}_{LL}^\mu = \mu[\tilde{\mathcal{F}}_{LL}^{-1}(\triangleright \ \otimes 1)(x^\mu \otimes 1)]$$
$$= x^\mu + (a \cdot x)P^\mu - a^\mu \left(D + \frac{a \cdot x}{2} P^2 \right) \tilde{Z}. \tag{50}$$

Generators $\hat{y}_{LL}^{\mu}$ satisfy κ-Minkowski algebra with $a_{\mu} \rightarrow -a_{\mu}$,

$$[\hat{y}_{LL}^{\mu}, \hat{y}_{LL}^{\nu}] = -i(a^{\mu}\hat{y}_{LL}^{\nu} - a^{\nu}\hat{y}_{LL}^{\mu}), \qquad (51)$$

and they commute with generators $\hat{x}_{LL}^{\mu}$:

$$[\hat{x}_{LL}^{\mu}, \hat{y}_{LL}^{\nu}] = 0. \qquad (52)$$

Realizations $\hat{y}_{J}^{\mu}$ and $\hat{y}_{LL}^{\mu}$ cannot be expressed in terms of x^{μ} and generators of conformal algebra (whereas realizations $\hat{x}_{J}^{\mu}$ and $\hat{x}_{LL}^{\mu}$ are expressed in terms of these generators).

Note that $\hat{x}_{J}^{\mu}$ [Eq. (27)] and $\hat{x}_{LL}^{\mu}$ [Eq. (46)] are different realizations of κ-Minkowski space $\hat{x}_{J}^{\mu} \neq \hat{x}_{LL}^{\mu}$, related by similarity transformation. There is also another point of view, so that, for $a^2 = 0$, $\hat{x}_{J}^{\mu}$ and $\hat{x}_{LL}^{\mu}$ can be identified, but generators x^{μ} and generators of conformal algebra have different realizations in two cases (Secs. III and IV), related by similarity transformation. In this case, let us denote as $(x_{J}^{\mu}, P_{J}^{\mu})$ and $(x_{LL}^{\mu}, P_{LL}^{\mu})$ two pairs of commutative coordinates and momenta, each satisfying undeformed Heisenberg algebra (5)–(7), which are related by similarity transformation:

$$P_{J}^{\mu} = \left(P_{LL}^{\mu} + \frac{a^{\mu}}{2}P_{LL}^2\right)\tilde{Z}_{LL} \qquad (53)$$

$$P_{LL}^{\mu} = \left(P_{J}^{\mu} - \frac{a^{\mu}}{2}P_{J}^2\right)Z_{J}^{-1} \qquad (54)$$

$$x_{J}^{\mu} = [x_{LL}^{\mu} + a^{\mu}(x_{LL} \cdot P_{LL})]\tilde{Z}_{LL}^{-1}$$
$$- (a \cdot x_{LL})\left(P_{LL}^{\mu} + \frac{a^{\mu}}{2}P_{LL}^2\right) \qquad (55)$$

$$x_{LL}^{\mu} = [x_{J}^{\mu} - a^{\mu}(x_{J} \cdot P_{J})]Z_{J} + (a \cdot x_{J})\left(P_{J}^{\mu} - \frac{a^{\mu}}{2}P_{J}^2\right), \qquad (56)$$

where

$$Z_{J} \equiv 1 - a \cdot P_{J} = \frac{1}{1 + a \cdot P_{LL}} \equiv \tilde{Z}_{LL}. \qquad (57)$$

Hence, $\hat{x}^{\mu} = x_{J}^{\mu} - a^{\mu}(x_{J} \cdot P_{J}) = x_{LL}^{\mu}(1 + a \cdot P_{LL}) - (a \cdot x_{LL})P_{LL}^{\mu}$ and two sets of conformal generators, $\{P_{J}^{\mu}, M_{J}^{\mu\nu}, D_{J}, K_{J}^{\mu}\}$ and $\{P_{LL}^{\mu}, M_{LL}^{\mu\nu}, D_{LL}, K_{LL}^{\mu}\}$, are related by similarity transformation.

V. CONCLUDING REMARKS

We have presented the two different κ-deformations of the conformal symmetry within the Drinfeld twist framework. Both twists provide the κ-Minkowski star product; therefore, the κ-Minkowski spacetime stays covariant under

the twisted conformal symmetries. Thanks to the twist, we are also able to obtain the differential realization for the noncommutative coordinates. The extension of the conformal algebra by the noncommutative coordinates is also presented, and it includes the deformed phase space (deformed Heisenberg algebra) as subalgebra. For an alternative point of view, where the phase space stays undeformed but the realizations of the conformal algebra generators are modified, see, e.g., Ref. [33]. Additionally, we have constructed, from transposed twists, another set of realizations satisfying the κ-Minkowski relations (with $a^{\mu} \rightarrow -a^{\mu}$). Both of the deformations presented in this paper (Jordanian and extended Jordanian) provide the so-called triangular deformation as the corresponding classical r-matrices satisfy the classical Yang–Baxter equation. Interestingly, the Jordanian and extended Jordanian deformations can be generated by other (than already mentioned) classical r-matrices. One can notice that the form of the conformal algebra (1) does not change if we exchange the generators (see also a similar comment in Ref. [6]) in the following way:

$$P_{\mu} \rightarrow K_{\mu}, \qquad K_{\mu} \rightarrow P_{\mu}, \qquad D \rightarrow -D, \qquad \kappa \rightarrow \frac{1}{\tilde{\kappa}}. \qquad (58)$$

This allows us, to distinguish yet another classical r-matrix for the conformal algebra (besides, for example, the one investigated in Sec. III for the Jordanian case $r = ia^{\mu}D \wedge P_{\mu}$), i.e., $r = -i\tilde{a}^{\mu}D \wedge K_{\mu}$ with a new deformation parameter $\tilde{\kappa}$ and $\tilde{a}^{\mu} = \tilde{\kappa}^2 a^{\mu}$. Such an r-matrix is satisfying the classical Yang–Baxter equation, and the classical limit is obtained for $\tilde{\kappa} \rightarrow 0$ (which corresponds to $\kappa \rightarrow \infty$). The new Jordanian twist (18) with (58) for any $\tilde{a}^{\mu}$ will satisfy the cocycle condition (11) as well. Formal expressions for the twisted deformation of coproducts and antipodes in $(U(\mathfrak{c}), \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}})$ (as twisted conformal Hopf algebra) generated by this r-matrix will stay the same up to (58).

One way of interpreting the exchange in the deformation parameter $\kappa \rightarrow \frac{1}{\tilde{\kappa}}$ (related with $a_{\mu} \rightarrow \tilde{a}_{\mu}$ as above) could be the following. Instead of considering the minimal length, as it happens when introducing the noncommutative coordinates $\hat{x}^{\mu}$, we should consider the minimal momentum and introduce the noncommutative momenta $\hat{p}^{\mu}$. This way the $\tilde{\kappa}$-deformation would appear in the momentum space $[\hat{p}^{\mu}, \hat{p}^{\nu}] = i(\tilde{a}^{\mu}\hat{p}^{\nu} - \tilde{a}^{\nu}\hat{p}^{\mu})$ instead of (14). Other physical consequences of such an exchange are still an open issue.

Nevertheless, the deformations of the conformal symmetry introduced in this paper can be of interest in many physical applications. For example, the Jordanian deformations are also appearing in the context of AdS/CFT correspondence [21,22]; therefore, the corresponding deformations of the conformal field theory part in the

twisted framework could be of interest as well. Another point to consider would be, for example, the extension of the deformations introduced in this paper to the supersymmetric case, as it was already considered for the Moyal–Weyl deformation of the conformal superalgebra [18]. Additionally extending the presented framework into the Hopf algebroid language [24] would allow us to introduce yet another example for the twisted deformation of Hopf algebroids as well. Also, the deformations of the conformal symmetry presented here could be considered as a starting point in the study on deformed (noncommutative) cosmology. Recently a short review on models of the inflating Universe based on conformal symmetry was presented [34]. The straightforward way to make them noncommutative would be to introduce the star product (15) related with the twists in the conformally invariant actions corresponding to different models. This way, one could investigate, for example, if introducing the deformation parameter (as a quantum gravity scale) would have any influence on the scale invariance of the power spectrum of the scalar perturbations.

Our results, however, provide only the starting point for such investigations.

[1] J. Lukierski, A. Nowicki, H. Ruegg, and V. N. Tolstoy, Phys. Lett. B **264**, 331 (1991).

[2] M. Chaichian, P. Kulish, K. Nishijima, and A. Tureanu, Phys. Lett. B **604**, 98 (2004); M. Chaichian, P. Prešnajder, and A. Tureanu, Phys. Rev. Lett. **94**, 151602 (2005).

[3] J. Lukierski, A. Nowicki, and H. Ruegg, Phys. Lett. B **279**, 299 (1992).

[4] P. Kosinski and P. Maslanka, in *From Field Theory to Quantum Groups*, edited by B. Jancewicz and J. Sobczyk (World Scientific, Singapore, 1996), p. 41.

[5] J. Lukierski and M. Klimek, Acta Phys. Pol. B **26**, 7 (1995).

[6] J. Lukierski, P. Minnaert, and M. Mozrzymas, Phys. Lett. B **371**, 215 (1996).

[7] A. Ballesteros, F. J. Herranz, M. A. del Olmo, and M. Santander, Phys. Lett. B **351**, 137 (1995); A. Ballesteros, F. J. Herranz, and C. M. Perena, Phys. Lett. B **391**, 71 (1997).

[8] P. Kosinski, J. Lukierski, and P. Maslanka, *Symmetries in Gravity and Field Theory, Salamanca, Spain, 2003*, edited by V. Aldaya and J. M. Cervero (Ediciones Universidad de Salamanca, Salamanca, 2004).

[9] J. Lukierski, V. Lyakhovsky, and M. Mozrzymas, Phys. Lett. B **538**, 375 (2002).

[10] F. J. Herranz, Phys. Lett. B **543**, 89 (2002); A. Ballesteros, N. R. Bruno, and F. J. Herranz, Phys. Lett. B **574**, 276 (2003); *XI International Conference on Symmetry Methods in Physics, Prague, 2004*, Symmetry Methods in Physics, edited by C. Burdik, O. Navratil, and S. Posta (Joint Institute for Nuclear Research, Dubna, Russia, 2004), p. 1.

[11] A. Borowiec and A. Pachoł, Phys. Rev. D **79**, 045012 (2009).

[12] A. Borowiec and A. Pachoł, SIGMA **6**, 086 (2010).

[13] M. Dimitrijević, L. Jonke, and A. Pachoł, SIGMA **10**, 063 (2014).

[14] T. Jurić, S. Meljanac, D. Pikutić, and R. Štrajn, J. High Energy Phys. 07 (2015) 055.

[15] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac, and D. Meljanac, Phys. Rev. D **77**, 105010 (2008).

[16] P. Matlock, Phys. Rev. D **71**, 126007 (2005).

[17] F. Lizzi, S. Vaidya, and P. Vitale, Phys. Rev. D **73**, 125020 (2006); F. Lizzi and P. Vitale, Mod. Phys. Lett. A **23**, 3307 (2008).

[18] M. Irisawa, Y. Kobayashi, and S. Sasaki, Prog. Theor. Phys. **118**, 83 (2007).

[19] C. Klimcik, J. Math. Phys. (N.Y.) **50**, 043508 (2009).

[20] F. Delduc, M. Magro, and B. Vicedo, J. High Energy Phys. 11 (2013) 192.

[21] I. Kawaguchi, T. Matsumoto, and K. Yoshida, J. High Energy Phys. 04 (2014) 153; T. Matsumoto and K. Yoshida, J. High Energy Phys. 06 (2014) 135; I. Kawaguchi, T. Matsumoto, and K. Yoshida, J. High Energy Phys. 06 (2014) 146; T. Matsumoto and K. Yoshida, J. High Energy Phys. 03 (2015) 137.

[22] S. J. van Tongeren, arXiv:1506.01023.

[23] O. Ogievetsky, *Proceedings of the Winter School "Geometry and Physics," Palermo: Circolo Matematico di Palermo, Charles University, Faculty of Mathematics and Physics; Union of Czech Mathematicians and Physicists, Zdíkov, 1993*, edited by J. Bureš and V. Souček, Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 37 (Circolo Matematico di Palermo, Palermo, 1994), p. 185; http://dml.cz/dmlcz/701555.

[24] T. Jurić, S. Meljanac, and R. Štrajn, Int. J. Mod. Phys. A **29**, 1450022 (2014); T. Jurić, D. Kovačević, and S. Meljanac, SIGMA **10**, 106 (2014).

[25] T. Jurić, S. Meljanac, and R. Štrajn, Int. J. Mod. Phys. A **29**, 1450121 (2014); S. Meljanac and D. Pikutić, Proceedings of the XXXIII M. Born Symposium, Wroclaw, Poland, 2014 (unpublished).

[26] T. Jurić, S. Meljanac, and A. Samsarov, J. Phys. Conf. Ser. **634**, 012005 (2015).

[27] T. Jurić, S. Meljanac, and A. Pikutić, arXiv:1506.04955.

[28] P. P. Kulish, V. D. Lyakhovsky, and A. I. Mudrov, J. Math. Phys. (N.Y.) **40**, 4569 (1999).

[29] A. Borowiec and A. Pachoł, Eur. Phys. J. C **74**, 2812 (2014).

[30] D. Kovačević and S. Meljanac, J. Phys. A **45**, 135208 (2012).

[31] S. Krešić-Jurić, S. Meljanac, and M. Stojić, Eur. Phys. J. C **51**, 229 (2007).

[32] V. N. Tolstoy, VII International Workshop Lie Theory and its Applications in Physics, Varna, Bulgaria, 2007 (unpublished).

[33] S. Ghosh and P. Pal, Phys. Rev. D **75**, 105021 (2007).

[34] M. Libanov, V. Rubakov, and G. Rubtsov, Report No. INR-TH/2015-022.