Influence of a plasma on the shadow of a spherically symmetric black hole

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We analytically calculate the influence of a plasma on the shadow of a black hole (or of another compact object). We restrict to spherically symmetric and static situations, where the shadow is circular. The plasma is assumed to be nonmagnetized and pressureless. We derive the general formulas for a spherically symmetric plasma density on an unspecified spherically symmetric and static spacetime. Our main result is an analytical formula for the angular size of the shadow. As a plasma is a dispersive medium, the radius of the shadow depends on the photon frequency. The effect of the plasma is significant only in the radio regime. The formalism applies not only to black holes but also, e.g., to wormholes. As examples for the underlying spacetime model, we consider the Schwarzschild spacetime and the Ellis wormhole. In particular, we treat the case that the plasma is in radial free fall from infinity onto a Schwarzschild black hole. We find that for an observer far away from a Schwarzschild black hole, the plasma has a decreasing effect on the size of the shadow. The perspectives of actually observing the influence of a plasma on the shadows of supermassive black holes are discussed.

I. INTRODUCTION

In the last 20 years, increasing evidence has been found for the existence of a supermassive black hole at the center of our Galaxy. It is expected that a distant observer should "see" this black hole as a dark disk in the sky which is known as the "shadow." It is sometimes said that the shadow is an image of the event horizon, and the size of the shadow is estimated by the angle under which the observer would see the horizon according to Euclidean geometry. Actually, the boundary of the shadow corresponds to light rays that asymptotically approach the photon sphere (at $r = 3M$ in the Schwarzschild case) and not the horizon (at $r = 2M$ in the Schwarzschild case). Moreover, light rays do not follow straight lines in Euclidean geometry but they are bent. For these two reasons, the angular diameter of the shadow is actually bigger than the naive Euclidean estimate suggests. For the black hole at the center of our Galaxy, it amounts to about 53 μas whereas the Euclidean estimate gives only about 20 μas. At present, two projects are under way to observe this shadow which would give important information on the compact object at the center of our Galaxy. These projects, which are going to use (sub)millimeter, very long baseline interferometry (VLBI) observations with radio telescopes distributed over Earth, are the Event Horizon Telescope (http://eventhorizontelescope.org) and the BlackHoleCam (http://blackholecam.org).

On the theoretical side, the shadow is defined as the region of the observer's sky that is left dark if there are light sources distributed everywhere but not between the observer and the black hole. For a nonrotating black hole, the shadow is a circular disk. For a Schwarzschild black hole, the angular diameter of the shadow was calculated, as a function of the mass of the black hole and of the radius coordinate where the observer is situated, by Synge [1]. (Synge did not use the word shadow. He calculated what he called the "escape cones" of light. However, the complement of the opening angle of the escape cone at the observer gives exactly the diameter of the shadow.) For a rotating black hole the shadow is no longer circular. The first to correctly calculate the shape of the shadow of a Kerr black hole was Bardeen [2]. More generally, the shape and the size of the shadow can be calculated analytically for the whole class of Plebański-Demiański spacetimes [3,4]. All these calculations are based on the assumption that light propagates along lightlike geodesics, without being influenced directly by matter. If such an influence is to be taken into account, one usually resorts to numerical calculation. In particular, Falcke *et al.* [5] have numerically simulated the visual appearance of the black hole at the center of our Galaxy, assuming that it is a Kerr black hole, with scattering and the presence of emission regions between the observer and the black hole taken into account (at 0.6 and 1.3 mm wavelengths). Sophisticated ray tracing programs have been written for producing realistic images of a black hole surrounded by an accretion disk, e.g., for the movie *Interstellar*. The numerical techniques used for this movie are described in detail, along with a review of earlier work, by James *et al.* [6].

While the purely geometric calculation of the shadow can be done analytically, at least for spacetimes in which

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the equation for lightlike geodesics is completely integrable, virtually all of the work that takes the influence of matter into account was based on numerics so far. In this paper, we want to take the first steps towards an investigation of the shadow in matter based on analytical calculations. It is clear that this can be done only for highly idealized situations, but, nonetheless, we think that it is worthwhile to do so. In contrast to numerical simulations, which can depict the situation only for a particular choice of the parameters involved, analytical results demonstrate how exactly an effect depends on these parameters, and they bring out the general features more clearly. Also, analytical results are useful for testing numerical codes. In this paper, we begin with the simplest nontrivial case: We want to analytically calculate the influence of a nonmagnetized pressureless plasma on the size of the shadow of a nonrotating black hole.

A plasma is a dispersive medium; i.e., the light rays deviate from lightlike geodesics in a way that depends on the frequency. The Hamiltonian for the light rays can be derived from Maxwell's equations where the source of the electromagnetic field consists of two charged fluids, one modeling the ions and the other the electrons. For a plasma on a curved background, the transition from Maxwell's equations to ray optics has to be done by a two-scale method. A rigorous derivation of the Hamiltonian for light rays was provided by Breuer and Ehlers [7,8] who considered a magnetized pressureless plasma. For the much simpler case of a nonmagnetized pressureless plasma, a similar derivation can be found in Perlick [9]. In the latter case, the equation of light rays can be fully characterized by a scalar, i.e., direction-independent, index of refraction which is a function of the spacetime point and frequency. Then the resulting equation of motion for light rays belongs to a class that was studied in the textbook by Synge [10]. The resulting formula for the deflection angle of light in a plasma whose density is a function of the radius coordinate was calculated in the Schwarzschild spacetime (and more generally in the equatorial plane of the Kerr metric) by Perlick [9]. The same result was found and further discussed by Bisnovatyi-Kogan and Tsupko [11–13]. Morozova *et al.* [14] generalized the calculation to the Kerr metric off the equatorial plane, assuming that the black hole is slowly rotating. For recent suggestions of how the effects of a plasma on light rays could actually be observed, we refer to Er and Mao [15] and Rogers [16].

In this paper, we calculate the angular diameter of the shadow for a nonrotating black hole that is surrounded by a nonmagnetized pressureless plasma. Although we are mainly interested in the Schwarzschild case, in large parts of the paper, we work in an unspecified spherically symmetric and static spacetime, so the results can be applied also, e.g., to charged black holes or to wormholes.

In our work, the plasma manifests itself as a refractive and dispersive medium, which leads to a change of the light rays in comparison with vacuum. As a result, the geometrical size of the shadow is changed. We do not take into account the processes of absorption and scattering of the photons by plasma electrons. Also, we neglect the gravitational field of the plasma particles.

The paper is organized as follows. In Sec. II, we work out the equation of motion for light rays in a plasma on a spherically symmetric and static spacetime. In Sec. III, we determine the circular light orbits which are of crucial relevance for the formation of the shadow. In Sec. IV, we calculate the angular diameter of the shadow. We specify the results in Sec. V to the case of a low-density plasma and in Sec. VI to the case that the plasma has the particular density profile that corresponds to spherically symmetric accretion of a dust onto a Schwarzschild black hole. In the same section, we also discuss the perspectives of actually observing the influence of a plasma on the shadow of a supermassive black hole.

We use units such that $G = c = 1$, so the Schwarzschild radius is $R_S = 2M$. Our choice of signature is $\{-, +, +, +\}$. We use Einstein's summation convention for latin indices which take the values $i, k, \ldots = t, r, \vartheta, \varphi$.

II. EQUATIONS OF MOTION FOR LIGHT RAYS IN A NONMAGNETIZED PLASMA

We consider a spherically symmetric and static metric

$$
\begin{aligned}
g_{ik}dx^i dx^k = &-A(r)dt^2 + B(r)dr^2 \\
&+ D(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2),
\end{aligned} \tag{1}
$$

where $A(r)$, $B(r)$, and $D(r)$ are positive. We are mainly interested in the case of a Schwarzschild black hole, but for the time being, there is no need for specifying the metric any further.

We assume that the spacetime is filled with a non-magnetized cold plasma whose electron plasma frequency ω_p is a function of the radius coordinate only,

$$
\omega_p(r)^2 = \frac{4\pi e^2}{m}N(r). \tag{2}
$$

Here, e is the charge of the electron, m is the electron mass, and $N(r)$ is the number density of the electrons in the plasma. The refraction index n of this plasma depends on the radius coordinate r and on the frequency ω of the photon as it is measured by a static observer,

$$
n(r, \omega)^2 = 1 - \frac{\omega_p(r)^2}{\omega^2}. \tag{3}
$$

Because of the spherical symmetry, we may restrict to the equatorial plane $\vartheta = \pi/2$, $p_\vartheta = 0$. Then the Hamiltonian for light rays in the plasma reads

$$H = \frac{1}{2}(g^{ik}p_i p_k + \omega_p(r)^2)$$

$$= \frac{1}{2}\left(-\frac{p_t^2}{A(r)} + \frac{p_r^2}{B(r)} + \frac{p_\varphi^2}{D(r)} + \omega_p(r)^2\right). \quad (4)$$

For a derivation of the Hamiltonian (4) from Maxwell's equations with a two-fluid source, we refer to the literature given in the Introduction.

The light rays are the solutions to Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial x^i}, \qquad \dot{x}^i = \frac{\partial H}{\partial p_i}, \quad (5)$$

which, in this case, read

$$\dot{p}_t = -\frac{\partial H}{\partial t} = 0, \quad (6)$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0, \quad (7)$$

$$\dot{p}_r = -\frac{\partial H}{\partial r}$$
$$= \frac{1}{2}\left(-\frac{p_t^2 A'(r)}{A(r)^2} + \frac{p_r^2 B'(r)}{B(r)^2} + \frac{p_\varphi^2 D'(r)}{D(r)^2} - \frac{d}{dr}\omega_p(r)^2\right), \quad (8)$$

$$\dot{t} = \frac{\partial H}{\partial p_t} = -\frac{p_t}{A(r)}, \quad (9)$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{D(r)}, \quad (10)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{B(r)}, \quad (11)$$

with $H = 0$, i.e.,

$$0 = -\frac{p_t^2}{A(r)} + \frac{p_r^2}{B(r)} + \frac{p_\varphi^2}{D(r)} + \omega_p(r)^2. \quad (12)$$

Here a dot means differentiation with respect to an affine parameter λ, and a prime means differentiation with respect to r.

From (6) and (7) it follows that p_t and p_φ are constants of motion. We write $\omega_0 := -p_t$. If ω_0 has been fixed and if $A(r) \to 1$ for $r \to \infty$, which is the case if the spacetime is asymptotically flat, the frequency ω measured by a static observer becomes a function of r by the gravitational redshift formula,

$$\omega(r) = \frac{\omega_0}{\sqrt{A(r)}}. \quad (13)$$

By (12), a light ray with constant of motion ω_0 is restricted to the region where

$$\frac{\omega_0^2}{A(r)} > \omega_p(r)^2. \quad (14)$$

The restriction (14) physically means that the photon frequency at a given point $\omega(r)$ must be bigger than the plasma frequency $\omega_p(r)$ at the same point. This is always true for light propagation in a plasma.

To derive the orbit equation, we use (10) and (11) to find

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \frac{D(r)p_r}{B(r)p_\varphi}. \quad (15)$$

Upon substituting for p_r from (12), this results in

$$\frac{dr}{d\varphi} = \pm\frac{\sqrt{D(r)}}{\sqrt{B(r)}}\sqrt{\frac{\omega_0^2}{p_\varphi^2}h(r)^2 - 1}, \quad (16)$$

where we have defined the function

$$h(r)^2 = \frac{D(r)}{A(r)}\left(1 - A(r)\frac{\omega_p(r)^2}{\omega_0^2}\right). \quad (17)$$

In general, the orbit has to be decomposed into sections where r is increasing as a function of φ and sections where it is decreasing, and the sign in (16) has to be chosen appropriately. For a light ray that comes in from infinity, reaches a minimum at a radius R, and goes out to infinity again, integration over the orbit gives the formula for the bending angle δ,

$$\pi + \delta = 2\int_R^\infty \frac{\sqrt{B(r)}}{\sqrt{D(r)}}\left(\frac{\omega_0^2}{p_\varphi^2}h(r)^2 - 1\right)^{-1/2}dr. \quad (18)$$

As R corresponds to the turning point of the trajectory, the condition $dr/d\varphi|_R = 0$ has to hold. This equation relates R to the constant of motion p_φ/ω_0,

$$h(R)^2 = \frac{p_\varphi^2}{\omega_0^2}. \quad (19)$$

Then the deflection angle can be rewritten as a function of only R and ω_0 (for a given plasma distribution) as

$$\pi + \delta = 2\int_R^\infty \frac{\sqrt{B(r)}}{\sqrt{D(r)}}\left(\frac{h(r)^2}{h(R)^2} - 1\right)^{-1/2}dr. \quad (20)$$

A. Example 1: Schwarzschild spacetime

For the Schwarzschild spacetime,

$$A(r) = B(r)^{-1} = 1 - \frac{2M}{r}, \qquad D(r) = r^2, \quad (21)$$

the function $h(r)$ specifies to

$$h(r)^2 = r^2 \left(\frac{r}{r - 2M} - \frac{\omega_p(r)^2}{\omega_0^2} \right). \tag{22}$$

Then the bending angle reads

$$\pi + \delta = 2 \int_R^\infty \left(\frac{r^2 \left(\frac{r}{r-2M} - \frac{\omega_p(r)^2}{\omega_0^2} \right)}{R^2 \left(\frac{R}{R-2M} - \frac{\omega_p(R)^2}{\omega_0^2} \right)} - 1 \right)^{-1/2} \frac{dr}{\sqrt{r}\sqrt{r - 2M}}. \tag{23}$$

This formula for the bending angle in a plasma on Schwarzschild spacetime was derived in [9]. In [13] it was rederived using Synge's approach and rewritten in terms of an elliptic integral for a homogeneous plasma; there also the strong deflection limit ($\delta \gg 1$) was investigated.

B. Example 2: Ellis wormhole

As a second example, we choose the Ellis wormhole [17] which is a traversable wormhole of the Morris-Thorne class [18]. It is true that the existence of such wormholes is questionable because they need exotic matter [18], and, at least for a certain kind of perturbation, it has been shown that the Ellis wormhole is unstable [19–21]. On the other hand, wormholes have met with great interest because they make time travel possible, and the Ellis wormhole is an instructive example for illustrating the applicability of our results. In this case, the metric coefficients are

$$A(r) = B(r) = 1, \qquad D(r) = r^2 + a^2, \tag{24}$$

where the coordinate r ranges from $-\infty$ to ∞, and a is a constant that determines the radius of the "neck" of the wormhole. Note that this is an example of a spherically symmetric and static spacetime where we cannot make a transformation of the radius coordinate, $r \to \tilde{r}$, such that $\tilde{D}(\tilde{r}) = \tilde{r}^2$. The reason is that the function $D(r)$ has vanishing derivative at $r = 0$ (i.e., at the neck), so the desired transformation fails to be a good coordinate transformation on any radius interval that contains the point $r = 0$. For the Ellis wormhole, the function $h(r)$ reads

$$h(r)^2 = (r^2 + a^2) \left(1 - \frac{\omega_p(r)^2}{\omega_0^2} \right), \tag{25}$$

and the bending angle is given by

$$\pi + \delta = 2 \int_R^\infty \left(\frac{(r^2 + a^2)(\omega_0^2 - \omega_p(r)^2)}{(R^2 + a^2)(\omega_0^2 - \omega_p(R)^2)} - 1 \right)^{-1/2}$$
$$\times \frac{dr}{\sqrt{r^2 + a^2}}. \tag{26}$$

III. CIRCULAR LIGHT ORBITS

We now derive the condition for circular light orbits which will be crucial for determining the shadow. Along a circular light orbit, we must have $\dot{r} = 0$ and $\ddot{r} = 0$. The first condition by (11) implies $p_r = 0$; from (12) we get the equation

$$0 = -\frac{\omega_0^2}{A(r)} + \frac{p_\varphi^2}{D(r)} + \omega_p(r)^2. \tag{27}$$

On the other hand, (11) implies

$$\dot{p}_r = \frac{d}{d\lambda}(B(r)\dot{r}) = \ddot{r}B(r) + \dot{r}^2 B'(r). \tag{28}$$

From this equation, we read that $\dot{r} = 0$ together with $\ddot{r} = 0$ leads to $\dot{p}_r = 0$, and from (8) we get the second equation for circular light orbits,

$$0 = -\frac{\omega_0^2 A'(r)}{A(r)^2} + \frac{p_\varphi^2 D'(r)}{D(r)^2} - \frac{d}{dr}\omega_p(r)^2. \tag{29}$$

We solve each of these two equations (27) and (29) for p_φ^2,

$$p_\varphi^2 = D(r) \left(\frac{\omega_0^2}{A(r)} - \omega_p(r)^2 \right), \tag{30}$$

$$p_\varphi^2 = \frac{D(r)^2}{D'(r)} \left(\frac{\omega_0^2 A'(r)}{A(r)^2} + \frac{d}{dr}\omega_p(r)^2 \right). \tag{31}$$

Subtracting these two equations from each other yields, after some elementary rearrangements, the equation for the radius of a circular light orbit in the form

$$0 = \frac{d}{dr}h(r)^2 \tag{32}$$

with the function $h(r)^2$ from (17). Any solution $r = r_{\text{ph}}$ of (32) determines the radius of a *photon sphere*. If a light ray starts tangentially to such a sphere, it will stay on a circular path with radius r_{ph} forever. If the spacetime is asymptotically flat, and if $\omega_p(r) \to 0$ for $r \to \infty$, the outermost photon sphere is always unstable with respect to radial perturbations. This means that the circular photon orbits in this photon sphere can serve as limit curves for light rays that approach them asymptotically. The radius r_{ph} of the outermost photon sphere is the critical value of the minimal radius R mentioned above. If a light ray comes in from infinity and reaches a minimum radius R bigger than r_{ph}, it will go out to infinity again. The case $R = r_{\text{ph}}$ corresponds to a light ray that spirals asymptotically towards a circular photon orbit in the sphere of radius r_{ph}. All other rays cross the photon sphere, and we exclude the case that they can come back. The latter case can occur only if there is a second photon sphere.

In the vacuum case, $\omega_p(r) = 0$, the condition for a circular light orbit in an unspecified spherically symmetric and static spacetime was first given by Atkinson [22]. It is easy to check that our condition (32) reduces, indeed, to Atkinson's if $\omega_p(r) = 0$.

A. Example 1: Schwarzschild spacetime

For the Schwarzschild spacetime, where $h(r)$ is given by (22), condition (32) for a photon sphere reads

$$0 = \frac{r(r - 3M)}{(r - 2M)^2} - \frac{\omega_p(r)^2}{\omega_0^2} - r\frac{\omega_p(r)\omega_p'(r)}{\omega_0^2}. \qquad (33)$$

For the special case that ω_p depends on r via a power law, this condition was already derived by Rogers [16]. If there is no plasma, $\omega_p(r) = 0$, (33) gives the well-known result $r_{\text{ph}} = 3M$. For the sake of curiosity, notice that (33) is *identically* satisfied if

$$h(r)^2 = r^2\left(\frac{r}{r - 2M} - \frac{\omega_p(r)^2}{\omega_0^2}\right) = C, \qquad (34)$$

$$\text{i.e.,} \qquad \omega_p(r)^2 = \omega_0^2\left(\frac{r}{r - 2M} - \frac{C}{r^2}\right) \qquad (35)$$

with a positive constant C. For this particular density profile of the plasma, there is a circular photon orbit with constant of motion ω_0 at *any* radius r for which

$$\frac{r^3}{r - 2M} > C. \qquad (36)$$

B. Example 2: Ellis wormhole

For the Ellis wormhole, $h(r)$ is given by (25), and condition (32) for a photon sphere reads

$$0 = r\left(1 - \frac{\omega_p(r)^2}{\omega_0^2}\right) - (r^2 + a^2)\frac{\omega_p(r)\omega_p'(r)}{\omega_0^2}. \qquad (37)$$

Without a plasma, $\omega_p(r) = 0$, there is a unique photon sphere at the neck of the wormhole, $r_{\text{ph}} = 0$. The same is true for a homogeneous plasma, $\omega_p(r) = \text{constant}$. However, in an inhomogeneous plasma there may be arbitrarily many photon spheres. For an observer at a large positive r coordinate, the outermost photon sphere is relevant for the formation of the shadow which will be discussed in the next section.

IV. RADIUS OF THE SHADOW

The shadow of a spherically symmetric and static black hole is defined in the following way; see Fig. 1. Consider light rays sent from an observer at radius coordinate r_O into the past. As we want to take the influence of a plasma into
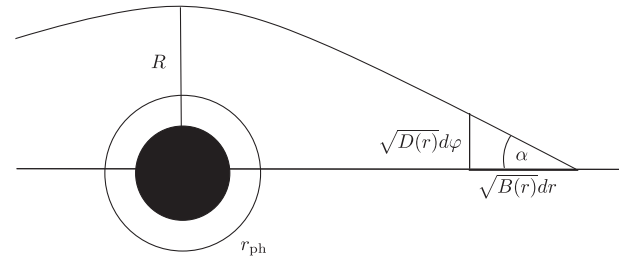

FIG. 1. For $R \to r_{\text{ph}}$, the angle α approaches the angular radius α_{sh} of the shadow.

account, by a light ray we mean a solution to the equations of motion discussed in Sec. II. These light rays can be divided into two classes: Light rays of the first class go to infinity after being deflected by the black hole. Light rays of the second class go towards the horizon of the black hole. If we assume that there are no light sources between the observer and the black hole, initial directions of the second class correspond to darkness on the observer's sky. This dark circular disk on the observer's sky is called the *shadow* of the black hole. The boundary of the shadow is determined by the initial directions of light rays that asymptotically spiral towards the outermost photon sphere. Here it is crucial that the light rays in the photon sphere are unstable with respect to radial perturbations because, otherwise, they could not serve as limit curves. For this reason, the construction of the shadow works for any spherically symmetric and static spacetime that admits an unstable photon sphere. This includes not only black holes but also, e.g., wormholes. We have already mentioned that in a spherically symmetric and static spacetime that is asymptotically flat, the outermost photon sphere is always unstable, provided that the plasma density tends to zero for $r \to \infty$. We will now calculate, for this situation, the angular radius α_{sh} of the shadow. The observer is assumed to be static somewhere between the outermost photon sphere and infinity.

We consider a light ray that is sent from the observer's position at r_O into the past under an angle α with respect to the radial direction. From Fig. 1, we read that α is given by

$$\cot\alpha = \frac{\sqrt{g_{rr}}}{\sqrt{g_{\varphi\varphi}}}\frac{dr}{d\varphi}\bigg|_{r=r_O} = \frac{\sqrt{B(r)}}{\sqrt{D(r)}}\frac{dr}{d\varphi}\bigg|_{r=r_O}. \qquad (38)$$

If the light ray goes out again after reaching a minimum radius R, the orbit equation (16) can be rewritten, with the help of (19), as

$$\frac{dr}{d\varphi} = \pm\frac{\sqrt{D(r)}}{\sqrt{B(r)}}\sqrt{\frac{h^2(r)}{h^2(R)} - 1}. \qquad (39)$$

For the angle α, we obtain

$$\cot^2\alpha = \frac{h^2(r_O)}{h^2(R)} - 1. \qquad (40)$$

Using

$$1 + \cot^2\alpha = \frac{1}{\sin^2\alpha}, \tag{41}$$

we get

$$\sin^2\alpha = \frac{h(R)^2}{h(r_O)^2}. \tag{42}$$

The boundary of the shadow α_{sh} is determined by light rays that spiral asymptotically towards a circular light orbit at radius r_{ph}. Therefore, the angular radius of the shadow is given by sending $R \to r_{ph}$ in (42),

$$\sin^2\alpha_{sh} = \frac{h(r_{ph})^2}{h(r_O)^2}, \tag{43}$$

where $h(r)$ is given by the formula (17).

For many applications, we may assume that the observer is in a region where the plasma density is negligibly small. Then (17) implies

$$h(r_O)^2 = \frac{D(r_O)}{A(r_O)}, \tag{44}$$

and (43) reduces to

$$\sin^2\alpha_{sh} = \frac{D(r_{ph})A(r_O)}{A(r_{ph})D(r_O)}\left(1 - \frac{A(r_{ph})\omega_p(r_{ph})^2}{\omega_0^2}\right),$$
$$N(r_O) \ll N(r_{ph}). \tag{45}$$

This demonstrates that, under the assumptions stated, the plasma always has a *decreasing* effect on the size of the shadow.

We emphasize that the preceding calculation applies not only to black holes but also to other spherically symmetric and static spacetimes with an unstable photon sphere, e.g., to ultracompact stars and to wormholes. As long as there is no light coming from the direction of the central object to the observer (which means, in particular, that the central object must not have a bright surface), these objects would cast a shadow in the same way as a black hole.

Let us summarize the results of this section. To find α_{sh} for a given metric of the form (1), a given plasma concentration $N(r)$, a given photon frequency at infinity ω_0, and a given observer position r_O, we have to calculate r_{ph} using Eq. (32) and substitute the result into formula (43). Note that the photon frequency at the observer position is $\omega(r_O)$ according to (13).

A. Example 1: Schwarzschild spacetime

For the Schwarzschild case, (43) specifies to

$$\sin^2\alpha_{sh} = \frac{r_{ph}^2\left(\frac{r_{ph}}{r_{ph}-2M} - \frac{\omega_p(r_{ph})^2}{\omega_0^2}\right)}{r_O^2\left(\frac{r_O}{r_O-2M} - \frac{\omega_p(r_O)^2}{\omega_0^2}\right)}, \tag{46}$$

where r_{ph} has to be determined from (33). In Sec. VI below, we will evaluate this equation for a particular plasma density profile. For vacuum $\omega_p(r) = 0$, our consideration gives

$$h(r)^2 = \frac{r^2}{1 - 2M/r}, \qquad r_{ph} = 3M,$$

$$\sin^2\alpha_{sh} = \frac{27M^2(1 - 2M/r_O)}{r_O^2}. \tag{47}$$

This is Synge's [1] formula for the radius of the shadow of a Schwarzschild black hole which was mentioned already in the Introduction.

B. Example 2: Ellis wormhole

We now calculate the radius of the shadow for the Ellis wormhole. In this case, our assumption of no light coming towards the observer from the direction of the central object means, in particular, that there are no light sources in the region $r < 0$. For the Ellis wormhole, the function $h(r)$ is given by (25), so Eq. (43) for the angular radius of the shadow specifies to

$$\sin^2\alpha_{sh} = \frac{(r_{ph}^2 + a^2)}{(r_O^2 + a^2)} \frac{(\omega_0^2 - \omega_p(r_{ph})^2)}{(\omega_0^2 - \omega_p(r_O)^2)}. \tag{48}$$

For a homogeneous plasma $\omega_p(r) = $ constant, we have $r_{ph} = 0$ and, thus,

$$\sin^2\alpha_{sh} = \frac{a^2}{r_O^2 + a^2}, \tag{49}$$

so a homogeneous plasma has no influence on the size of the shadow.

V. THE SHADOW IN A LOW-DENSITY PLASMA

If the plasma frequency is much smaller than the photon frequency, the equations for the photon sphere and for the radius of the shadow can be linearized about the corresponding values for vacuum light rays. To work this out, we rewrite (17) as

$$h(r)^2 = Q(r)(1 - \epsilon\beta(r)), \tag{50}$$

where

$$Q(r) = \frac{D(r)}{A(r)}, \qquad \beta(r) = \frac{A(r)\omega_p(r)^2}{\omega_0^2} = \frac{\omega_p(r)^2}{\omega(r)^2}. \qquad (51)$$

Here we have introduced a bookkeeping parameter ε which will be set equal to unity after all equations have been linearized with respect to ε.

The equation for a photon sphere (32) now reads

$$0 = Q'(r)(1 - \varepsilon\beta(r)) - \varepsilon Q(r)\beta'(r). \qquad (52)$$

We write the solution to this equation in the form

$$r_{\mathrm{ph}} = r_{\mathrm{ph}}^0 + \varepsilon r_{\mathrm{ph}}^1 + \cdots, \qquad (53)$$

where r_{ph}^0 is a solution in the case without plasma, i.e.,

$$Q'(r_{\mathrm{ph}}^0) = 0. \qquad (54)$$

After inserting these expressions into (52) and comparing coefficients of ε we find

$$r_{\mathrm{ph}}^1 = \frac{Q(r_{\mathrm{ph}}^0)\beta'(r_{\mathrm{ph}}^0)}{Q''(r_{\mathrm{ph}}^0)}. \qquad (55)$$

Depending on the signs of $\beta'(r_{\mathrm{ph}})$ and $Q''(r_{\mathrm{ph}}^0)$, this expression can be positive or negative; i.e., the plasma can shift the photon sphere inwards or outwards.

We now insert the expansion (53) into the equation for the shadow (43). If we neglect all terms of quadratic and higher order in ε, and set ε equal to unity again, we find

$$\sin^2\alpha_{\mathrm{sh}} = \frac{Q(r_{\mathrm{ph}}^0)}{Q(r_{\mathrm{O}})}(1 - \beta(r_{\mathrm{ph}}^0) + \beta(r_{\mathrm{O}})), \qquad (56)$$
$$\omega_p(r) \ll \omega(r).$$

Note that r_{ph}^1 has dropped out from the equation for α_{sh} to within this order. According to (56), the plasma has a decreasing effect on the shadow as long as $\beta(r_{\mathrm{O}}) < \beta(r_{\mathrm{ph}}^0)$.

Let us summarize the results of this section. To find the radius of the photon sphere r_{ph} for a given metric (1), a given photon frequency at infinity ω_0, and a given plasma frequency that satisfies the condition $\omega_p(r) \ll \omega(r)$, one has to proceed in the following way: Write down the functions $Q(r)$ and $\beta(r)$ [see (51)]; calculate r_{ph}^0 from (54) and r_{ph}^1 from (55); then calculate r_{ph} according to (53) with $\varepsilon = 1$. To find the angular radius α_{sh} of the shadow for a given observer position r_{O} in this case, we have to insert r_{ph}^0 into (56).

A. Example 1: Schwarzschild spacetime

As an example of the approximation formalism developed in this section, we consider the Schwarzschild spacetime for the case that the plasma electron density is given by a power law,

$$\frac{\omega_p(r)^2}{\omega_0^2} = \beta_0 \frac{M^k}{r^k}, \qquad (57)$$

where β_0 and k are positive dimensionless constants. Then, $r_{\mathrm{ph}}^0 = 3M$ and

$$r_{\mathrm{ph}}^1 = \frac{\beta_0 M}{3^{k+1}}\left(1 - \frac{k}{2}\right), \qquad (58)$$

i.e., depending on the density profile, the radius of the photon sphere may become smaller ($k > 2$) or bigger ($k < 2$) than $3M$. In the case that $k = 2$, we obtain $r_{\mathrm{ph}}^1 = 0$. Note that in this case, the equation for the photon sphere can be solved exactly and gives the same result, $r_{\mathrm{ph}} = 3M$. The first-order equation for the radius of the shadow (56) yields

$$\sin^2\alpha_{\mathrm{ph}} = \frac{27M^2}{r_{\mathrm{O}}^2}\left(1 - \frac{2M}{r_{\mathrm{O}}}\right)$$
$$\times \left(1 - \frac{\beta_0}{3^{k+1}} + \left(1 - \frac{2M}{r_{\mathrm{O}}}\right)\frac{\beta_0 M^k}{r_{\mathrm{O}}^k}\right). \qquad (59)$$

If the observer is far away from the black hole, $r_{\mathrm{O}} \gg M$, this can be simplified to

$$\sin^2\alpha_{\mathrm{ph}} = \frac{27M^2}{r_{\mathrm{O}}^2}\left(1 - \frac{\beta_0}{3^{k+1}}\right). \qquad (60)$$

B. Example 2: Ellis wormhole

For the Ellis wormhole,

$$Q(r) = r^2 + a^2, \qquad \beta(r) = \frac{\omega_p(r)^2}{\omega_0^2}. \qquad (61)$$

With $r_{\mathrm{ph}}^0 = 0$, the first-order approximation yields

$$r_{\mathrm{ph}}^1 = \frac{a^2\omega_p(0)\omega_p'(0)}{\omega_0^2},$$
$$\sin^2\alpha_{\mathrm{sh}} = \frac{a^2}{(r_0^2 + a^2)}\left(1 - \frac{\omega_p(0)^2}{\omega_0^2} + \frac{\omega_p(r_{\mathrm{O}})^2}{\omega_0^2}\right). \qquad (62)$$

VI. SPHERICALLY SYMMETRIC ACCRETION OF A PLASMA ONTO A SCHWARZSCHILD BLACK HOLE

In this section, we consider in greater detail the special case that the underlying spacetime is the Schwarzschild spacetime and that the plasma electron density corresponds to spherically symmetric accretion of dust. As usual, we use the word "dust" as a synonym for a pressureless perfect fluid. Vanishing pressure implies that the flow lines are geodesics, i.e., that the fluid particles are freely falling.

Note that for our plasma model, the Hamiltonian and, thus, the equations of motion for light rays do not depend on the velocity of the plasma. This is, of course, different for other types of media; see Synge's book [10]. In the case of an infalling plasma, we have to determine the plasma density which is the only quantity that enters into the equation for the plasma frequency.

We start out from the continuity equation

$$\partial_i(\sqrt{-g}\rho u^i) = 0. \tag{63}$$

Here, ρ is the rest-mass density of the plasma, u^i is the 4-velocity, and g is the determinant of the metric. For the sake of simplicity, we consider a neutral hydrogen plasma, and we assume that the electrons have the same 4-velocity as the protons. As the electron mass is negligibly small in comparison to the proton mass, ρ can be written as $\rho = m_p N$ where m_p is the proton rest mass and N is the number density of the protons (which, as the plasma is assumed to be neutral, coincides with the number density of the electrons).

For spherically symmetric and stationary accretion in the Schwarzschild spacetime, the continuity equation reduces to

$$\frac{d}{dr}(r^2\rho(r)u^r(r)) = 0. \tag{64}$$

Integration gives

$$4\pi r^2\rho(r)u^r(r) = -\dot{M}_A = \text{const}, \tag{65}$$

where $\dot{M}_A$ is a stationary mass flux.

We assume that the particles are dropped in radial free fall from rest at infinity, i.e., that our plasma is so "cold" that the pressure can be neglected. Then the integral curves of u^μ are radial geodesics, so the radial component of the 4-velocity is [23]

$$u^r(r) = \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}. \tag{66}$$

Thereupon, Eq. (65) gives us the rest-mass density,

$$\rho(r) = \frac{\dot{M}_A}{4\pi\sqrt{2M}}\frac{1}{r^{3/2}}. \tag{67}$$

Equations (66) and (67) also follow from Michel's [24] pioneering paper on fully relativistic spherically symmetric accretion where we have to specify Michel's equations (9) and (10) to the case of a dust.

The rest-mass density $\rho(r)$ gives us the ratio of frequencies

$$\frac{\omega_p(r)^2}{\omega_0^2} = \frac{4\pi e^2 N(r)}{m_e\omega_0^2} = \frac{4\pi e^2\rho(r)}{m_e m_p\omega_0^2} = \beta_0\frac{M^{3/2}}{r^{3/2}}, \tag{68}$$

where

$$\beta_0 = \frac{e^2\dot{M}_A}{m_e m_p\omega_0^2\sqrt{2M}M^{3/2}}. \tag{69}$$

For this function $\omega_p(r)$, the angular radius (46) of the shadow is plotted as a function of β_0 for different observer positions in Fig. 2.

Assuming that the approximation of a low-density plasma is justified, we can use formula (59) and find

$$\sin^2\alpha_{\text{sh}} = \frac{27M^2}{r_O^2}\left(1 - \frac{2M}{r_O}\right)$$
$$\times \left(1 - \frac{\beta_0}{3^{5/2}} + \left(1 - \frac{2M}{r_O}\right)\frac{\beta_0 M^{3/2}}{r_O^{3/2}}\right). \tag{70}$$

In this case, the function $-\beta(r_{\text{ph}}^0) + \beta(r_O)$ which enters into (56) is equal to zero when $r_O \simeq 3.768M$, and it takes its maximum when $r_O = 3.333M$. When $r_O > 3.768M$, the function is negative. Therefore, if the approximation of a low-density plasma is justified, the shadow becomes bigger if $r_O < 3.768M$ and it becomes smaller if $r_O > 3.768M$.

As an aside, we mention that the 4-velocity component u^r directly gives the velocity of the infalling particles as measured by a static observer. To demonstrate this, following [23] (see, also, [25]), we first observe that (66) together with the normalization condition $g_{ij}u^i u^j = -1$ implies that
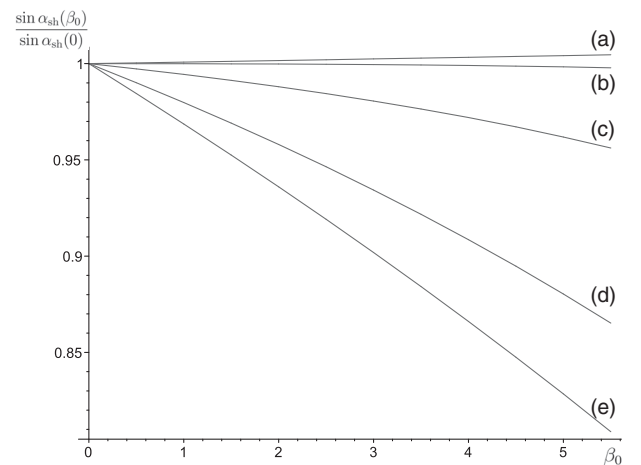

FIG. 2. Dependence of the radius of the shadow on β_0 for different observer positions r_O: $r_O/M = 3.333$ (a), 3.768 (b), 5 (c), 10 (d), 50 (e). The dependence is calculated using the exact formulas. It is shown that for a distant observer, the radius of the shadow becomes smaller.

$$u^0 = \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}, \tag{71}$$

hence,

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = -\sqrt{\frac{2M}{r}}\left(1 - \frac{2M}{r}\right). \tag{72}$$

For a static observer at radius coordinate r, a coordinate time interval dt corresponds to a proper time interval

$$dt' = \left(1 - \frac{2M}{r}\right)^{1/2} dt. \tag{73}$$

On the other hand, a radial coordinate separation dr corresponds to a proper radial distance equal to

$$dr' = \left(1 - \frac{2M}{r}\right)^{-1/2} dr. \tag{74}$$

As a consequence, the velocity of the radially infalling particle, as measured by a static observer at r, is given by

$$\frac{dr'}{dt'} = \left(1 - \frac{2M}{r}\right)^{-1}\frac{dr}{dt} = -\sqrt{\frac{2M}{r}} = u^r. \tag{75}$$

We will now use the results of this section for estimating the numerical values of the plasma effects on the shadow of the supermassive black holes at the center of our Galaxy and of M87.

For these applications, we can always use the formulas with $r_O \gg R_S$, so the angular radius of the shadow (60) can be written as

$$\sin^2\alpha_{\rm sh} = \frac{27R_S^2}{4r_O^2}(1 - Z), \qquad Z = \frac{\beta_0}{3^{5/2}}. \tag{76}$$

β_0 was defined in (69). For estimating the plasma correction numerically, we reexpress all quantities in Gaussian cgs units; i.e., we restore the factors of c and G. Then β_0 reads

$$\beta_0 = \frac{e^2\dot{M}_A c^3}{m_e m_p \omega_0^2 \sqrt{2GM}(GM)^{3/2}}. \tag{77}$$

The mass accretion rate $\dot{M}_A$ can be estimated via the observed luminosity of the Galactic center. We write

$$L \simeq \eta\dot{M}_A c^2, \tag{78}$$

where η is a nondimensional coefficient characterizing the accretion efficiency. We obtain for the correction due to the presence of the plasma:

$$Z = \frac{2^{3/2}}{3^{5/2}}\frac{e^2}{m_e m_p c}\frac{L}{\eta c^2}\frac{1}{R_S^2\omega_0^2}. \tag{79}$$

Note that the circular frequency ω_0 is connected with the ordinary frequency ν_0 by the usual relation, $\omega_0 = 2\pi\nu_0$, and that in the expression for Z, we can assume, as a valid approximation, that ω_0 is related to the wavelength λ_0 by the vacuum relation $\omega_0 = 2\pi c/\lambda_0$.

The value of η depends on the specific model of accretion. For estimations, we consider $\eta_1 = 10^{-4}$ (see [26,27]) and $\eta_2 = 0.1$ (see [28]).

Let us consider the Galactic center source Sgr A*. For estimations, we put $M = 4.3 \times 10^6 M_\odot$, $r_O = 8.3$ kpc (see [29,30]). Then Synge's formula (47) gives for the angular radius of the shadow in vacuum a value of $\alpha_{\rm sh} \simeq 27$ μas, i.e., a diameter of about 53 μas which is expected to be resolvable with VLBI soon [31,32]. By contrast, a naive Euclidean estimate of the horizon size $\alpha \approx \tan\alpha = R_S/r_O$ yields $\alpha \approx 10$ μas; i.e., a diameter of about 20 μas, as was already mentioned in the Introduction. For a rotating (Kerr) black hole, the shadow is flattened on one side, but its vertical diameter is still given by Synge's formula if $r_O \gg R_S$; see [4].

For estimating the correction Z, we use $L = 10^6 L_\odot$ (see, for example, [33]). For $\lambda_0 = 1$ mm, we obtain

$$Z = 0.8 \times 10^{-5} \quad \text{for } \eta_1, \qquad Z = 0.8 \times 10^{-8} \quad \text{for } \eta_2. \tag{80}$$

Note that observations of the shadow are planned in the (sub)millimeter regime because at wavelengths of more than about 1.3 mm, it is expected that the shadow is washed out by scattering [5]. We see that in this regime, the effect of a plasma is rather small for the chosen values of η. However, for bigger radio wavelengths, the effect can be significant. For $\lambda_0 = 10$ cm, we obtain

$$Z = 0.8 \times 10^{-1} \quad \text{for } \eta_1, \qquad Z = 0.8 \times 10^{-4} \quad \text{for } \eta_2. \tag{81}$$

We now turn to the Galactic center source M87. For estimations, we put $M = 3 \times 10^9 M_\odot$, $r_O = 18$ Mpc, $L = 7 \times 10^{40}$ ergs s^{-1} (see, for example, [34]). Then Synge's formula leads to an angular radius of the shadow in vacuum of $\alpha_{\rm sh} \simeq 9$ μas.

For $\lambda_0 = 1$ mm, we obtain

$$Z = 0.3 \times 10^{-9} \quad \text{for } \eta_1, \qquad Z = 0.3 \times 10^{-12} \quad \text{for } \eta_2. \tag{82}$$

For $\lambda_0 = 100$ cm we obtain

$$Z = 0.3 \times 10^{-3} \quad \text{for } \eta_1, \qquad Z = 0.3 \times 10^{-6} \quad \text{for } \eta_2. \tag{83}$$

VII. CONCLUSIONS

In this paper, the first steps towards an investigation of the shadow in the presence of matter based on analytical calculations have been taken. We have analytically calculated the influence of a nonmagnetized pressureless plasma on the angular size of the shadow. It was our goal to derive all relevant formulas for an unspecified spherically symmetric and static spacetime before specializing to the case of a Schwarzschild black hole. The gravitational field was not supposed to be weak. We have worked in the framework of geometrical optics; in this approximation, the presence of the plasma leads to a change of the geometrical size of the shadow via a change of the light rays in this medium.

The equation of motion for light rays in a plasma with a spherically symmetric density distribution on a spherically symmetric and static spacetime (1) has been derived; see (16). In particular, the formula for the photon deflection angle has been shown for unbound orbits; see (20). We have also found the radii of the circular light orbits (radii of the photon spheres); see (32). The central result of our paper is an analytical formula for the angular size of the shadow; see (43).[1]

Special attention has been given to the realistic case when the plasma frequency is much smaller than the photon frequency; see Sec. V. We have shown that, in this case, the

plasma has a decreasing effect on the size of the shadow provided that the plasma density is higher at the photon sphere than at the observer position; see (56). In particular, a simple formula for the size of the shadow has been presented for the case of a power-law density distribution and an observer at a large distance; see (60).

In the presence of a plasma, the size of the shadow depends on the wavelength at which the observation is made, in contrast to the vacuum case where it is the same for all wavelengths. For the underlying spacetime, we have treated two examples, the Schwarzschild black hole and the Ellis wormhole. In particular, the case of spherically symmetric accretion of plasma onto a Schwarzschild black hole has been considered in detail; see Sec. VI. We have found that for an observer far away from the Schwarzschild black hole, the plasma makes the shadow smaller. As examples, we have considered Sgr A* and M87. For the specific accretion model used here, we have found that the effect of the presence of a plasma on the size of the shadow can be significant only for wavelengths of at least a few centimeters. At such wavelengths, the observation of the shadow is made difficult because of scattering.

Apart from considering more complicated plasma models, an obvious next step would be to generalize the analysis to the case of axially symmetric and stationary situations to include rotating black holes. Then the shadow is no longer circular, and the plasma has an effect not only on its size but also on its shape. We are planning to work out the details in future work.

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[1]Soon after a preprint of the present paper was made public on the arXiv [35], a paper by Atamurotov and Ahmedov [36] appeared, again on the arXiv, where the shadow was calculated for the case of a *homogeneous* plasma. Their results are in obvious disagreement with ours if the latter are specialized to the case of a homogeneous plasma. The reason is in the fact that they erroneously treated the index of refraction as a constant. Actually, the index of refraction involves the plasma density, which is constant in a homogeneous plasma, and the photon frequency, which in a gravitational field depends on the space point. Therefore, the index of refraction is *not* a constant for a homogeneous plasma. This has the consequence that in the Schwarzschild spacetime, the photon sphere in a homogeneous plasma is different from the vacuum case. In the meantime, a strongly revised version of their paper appeared in PRD [37] where they are now considering a plasma density proportional to r^{-1}.

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