

Kundt solutions of minimal massive 3D gravityNihat Sadik Deger^{*}*Department of Mathematics, Boğaziçi University, 34342 Bebek, Istanbul, Turkey*Özgür Sarioglu[†]*Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06800 Ankara, Turkey*

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We construct Kundt solutions of minimal massive gravity theory and show that, similar to topologically massive gravity (TMG), most of them are constant scalar invariant (CSI) spacetimes that correspond to deformations of round and warped (A)dS. We also find an explicit non-CSI Kundt solution at the merger point. Finally, we give their algebraic classification with respect to the traceless Ricci tensor (Segre classification) and show that their Segre types match with the types of their counterparts in TMG.

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I. INTRODUCTION

In principle, the celebrated AdS/CFT duality can be used in giving a “quantum description” of a gravity theory possessing an anti-de Sitter (AdS) vacuum, or at the very least, in ruling out the existence of such a “consistent” quantum theory. Yet, the construction of a physically viable theory that passes this powerful test has been a big challenge. Since two-dimensional conformal field theories (CFTs) are better understood, three-dimensional gravitational models are in a better shape for the application of this duality. One such 3D model is the “good old” cosmological “topologically massive gravity” (TMG) [1,2], where the “opposite sign” Einstein-Hilbert action with a cosmological constant is modified by the addition of the parity-odd gravitational Chern-Simons term. TMG has a single massive propagating graviton mode, and when the cosmological constant is negative it has an AdS vacuum. The renowned BTZ black hole [3] is also a solution to TMG, which makes the theory even more interesting. Unfortunately, the central charge of the dual CFT is negative when the bulk graviton has positive energy, and this destroys unitarity. This so-called “bulk vs boundary clash” problem persists also in the parity-preserving “new massive gravity” (NMG) model [4,5], which has an action described by the addition of a particular combination of curvature-squared terms to the usual Einstein-Hilbert term.

Remarkably, a “deformation” of TMG theory, recently proposed in [6] and termed as “minimal massive gravity” (MMG), solves the bulk vs boundary clash problem for a certain range of its parameters. This theory is “minimal” in the sense that it has only one massive spin-2 mode in the bulk, i.e., the same minimal local structure as TMG. These features considerably promote MMG and determining its

exact solutions is quite an interesting problem on its own. The exact solutions of MMG that have been found so far are as follows: Apart from static black hole solutions, which obviously exist for specific values of parameters, and an (A)dS₂ × S¹ vacuum, there is a warped (A)dS [7], a solution that describes gravitational waves [8], a two-parameter deformation of a BTZ black hole [9] and a non-BTZ AdS black hole [10]. It has been shown that all solutions of TMG that have Petrov types O, N, and D are also solutions of MMG after a proper redefinition of parameters [11]. In [11] it was also found that MMG inherits from TMG a specific type of Kundt solution.

In this article we construct and algebraically classify all Kundt solutions, i.e., spacetimes with an expansion-free null geodesic congruence, of MMG theory. Kundt spacetimes in three dimensions are quite special: They are known to be primary examples of spacetimes for which all scalar invariants constructed from the relevant curvature tensors and their covariant derivatives are constants. All such, so-called “constant scalar invariant” (CSI) spacetimes in three dimensions have been classified in [12]: Apart from Kundt solutions, there are only locally homogeneous geometries with this property. In our analysis to determine these special solutions, we will adapt the strategy followed by [13] to find the analogous solutions of TMG. We find that as TMG, the CSI Kundt solutions of MMG turn out to be deformations of round and warped (A)dS. However, not all Kundt solutions are CSI and indeed we also find an exact non-CSI Kundt solution for a specific fine-tuning of two of the four MMG parameters. Then, we algebraically classify the solutions we find with respect to their traceless Ricci tensor, i.e., present their algebraic Segre types, making use of the results of [14]. It turns out that their Segre types match with those of their relevant counterparts in TMG. In passing, it is worth noting that the Kundt solutions of NMG were studied in [15], where a non-CSI Kundt solution to TMG was also found at the chiral point [16].

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The paper is organized as follows: In Sec. II we briefly review MMG theory. Section III is devoted to the detailed derivation of the Kundt solutions obtained and forms the bulk of our paper. The algebraic Segre classification of the Kundt solutions found are given in Sec. IV. We conclude and discuss possible future problems in Sec. V. In two separate appendixes, we give certain technical details that we left out of the main text.

II. THE MMG MODEL

MMG theory itself is closely related to TMG [1,2], whose source-free field equation reads

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_0 g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \quad (1)$$

where the symmetric, traceless, parity-odd, and covariantly conserved Cotton tensor $C_{\mu\nu}$ is defined in terms of the Schouten tensor $S_{\sigma\nu}$ as

$$C^\mu{}_\nu \equiv \epsilon^{\mu\rho\sigma}\nabla_\rho S_{\sigma\nu}, \quad S_{\sigma\nu} \equiv R_{\sigma\nu} - \frac{1}{4}Rg_{\sigma\nu}, \quad (2)$$

and the Levi-Civita pseudotensor is defined as $\epsilon_{\mu\rho\sigma} = \sqrt{-g}\epsilon_{\mu\rho\sigma}$ in terms of the weight +1 tensor density $\epsilon_{\mu\rho\sigma}$, where we use the convention $\epsilon_{012} = +1$. Here Λ_0 is the cosmological constant, and μ is a mass parameter (with dimensions 1/length).

On the other hand, the field equation of source-free MMG theory reads

$$\bar{\sigma}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \bar{\Lambda}_0 g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} + \frac{\gamma}{\mu^2}J_{\mu\nu} = 0, \quad (3)$$

where the symmetric, curvature-squared tensor $J_{\mu\nu}$ is

$$\begin{aligned} J^{\mu\nu} &\equiv -\frac{1}{2}\epsilon^{\mu\rho\sigma}\epsilon^{\nu\tau\eta}S_{\rho\tau}S_{\sigma\eta}, \\ &= S^{\mu\rho}S^\nu{}_\rho - S^{\mu\nu}S - \frac{1}{2}g^{\mu\nu}(S^{\rho\sigma}S_{\rho\sigma} - S^2), \\ &= R^{\mu\rho}R^\nu{}_\rho - \frac{3}{4}R^{\mu\nu}R - \frac{1}{2}g^{\mu\nu}\left(R^{\rho\sigma}R_{\rho\sigma} - \frac{5}{8}R^2\right), \end{aligned} \quad (4)$$

with $S \equiv g^{\mu\nu}S_{\mu\nu}$. Even though the J -tensor is not covariantly conserved on its own, i.e., it does not satisfy a Bianchi identity so that the MMG field equation (3) cannot be derived from an action that only involves the metric and its curvature tensors, it is nevertheless conserved *on shell* as a consequence of (3) itself (see [6,7] for details). Moreover, it is possible to construct conserved charges for solutions of the theory [17]. In (3), the parameters $\bar{\sigma}$ and γ are nonzero dimensionless constants, whereas $\bar{\Lambda}_0$ is the cosmological constant with dimensions 1/length². Note that the trace of (3) implies that

$$\bar{\sigma}R - 6\bar{\Lambda}_0 + \frac{\gamma}{\mu^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\right) = 0. \quad (5)$$

The parameters showing up in the MMG field equation can be expressed in terms of those of TMG [6] as¹

$$\begin{aligned} \bar{\sigma} &= 1 + \alpha + \frac{\alpha^2\Lambda_0}{2\mu^2(1+\alpha)^2}, \\ \gamma &= -\frac{\alpha}{(1+\alpha)^2}, \\ \bar{\Lambda}_0 &= \Lambda_0\left(1 + \alpha - \frac{\alpha^3\Lambda_0}{4\mu^2(1+\alpha)^2}\right), \end{aligned} \quad (6)$$

where α is a dimensionless parameter such that one gets TMG in the $\alpha \rightarrow 0$ limit. For bulk and boundary unitarity it is necessary to have [18]

$$-1 < \alpha < 0, \quad \Lambda_0 < \frac{4\mu^2(1+\alpha)^3}{\alpha^3}. \quad (7)$$

Note that these conditions imply $\Lambda_0 < 0$ and $\Lambda_0\alpha > 0$. [See also Eqs. (5.13) and (5.15) of [6] and the discussion that leads to them.]

There are two special points in the parameter space of the MMG theory. The first one is called the ‘‘chiral point’’ [16] for which the central charges vanish and is given by [6,17]

$$\bar{\sigma} + \frac{\gamma}{2\mu^2\ell_{\text{ch}}^2} \pm \frac{1}{\mu\ell_{\text{ch}}} = 0, \quad \frac{1}{\mu^2\ell_{\text{ch}}^2} \equiv \bar{\sigma}^2 - \frac{\gamma\bar{\Lambda}_0}{\mu^2}. \quad (8)$$

The second one is called the ‘‘merger point’’ [7] where

$$\bar{\Lambda}_0 = \frac{\mu^2\bar{\sigma}^2}{\gamma}. \quad (9)$$

For this choice the quadratic equation for the effective cosmological constant of maximally symmetric vacua has a repeated root. (See [10] for further discussion.) Note that the use of (6) in the condition (9) implies that

$$\Lambda_0\alpha = -\mu^2(1+\alpha)^2 < 0, \quad (10)$$

which violates the unitarity conditions (7), since it demands $\Lambda_0\alpha > 0$.

III. KUNDT SOLUTIONS

The MMG field equation (3) is highly nonlinear and involves fourth-order derivatives, and we now want to look for Kundt solutions of the theory. To facilitate the comparison of possible Kundt solutions with the known Kundt

¹To be able to make smooth contact with the Kundt solutions of TMG [13] in the limit $\gamma \rightarrow 0$, we made the choice $\sigma = 1$ in these formulas.

solutions of TMG found in [13], we hence use the same strategy, notations, and conventions used there.

Thus, taking the orientation convention $\epsilon_{vur\rho} = 1$, we start with a general Kundt spacetime given by the metric

$$ds^2 = d\rho^2 + 2dudv + f(v, u, \rho)du^2 + 2W(v, u, \rho)dud\rho. \quad (11)$$

Note that the null vector required in the definition of a Kundt spacetime is $k^\mu = \delta^\mu_v$, which makes $k_\mu = \delta_{\mu u}$, and fulfills the following conditions:

- (a) expansion free: $\nabla_\mu k^\mu = 0$,
- (b) shear free: $(\nabla^\mu k^\nu)\nabla_{(\mu}k_{\nu)} - (\nabla_\mu k^\mu)^2 = 0$,
- (c) twist free: $(\partial^\mu k^\nu)\partial_{[\mu}k_{\nu]} = 0$.

The vv -component of the field equation (3) gives

$$\gamma\left(\frac{\partial^2 W}{\partial v^2}\right)^2 + 2\mu\frac{\partial^3 W}{\partial v^3} = 0. \quad (13)$$

The most general solution of (13) is

$$\begin{aligned} W(v, u, \rho) &= \frac{2\mu}{\gamma^2}(-v\gamma + (v\gamma - 2\mu W_2(u, \rho)) \ln(v\gamma - 2\mu W_2(u, \rho))) \\ &\quad + vW_1(u, \rho) + W_0(u, \rho). \end{aligned} \quad (14)$$

As shown in detail in Appendix A, unless one takes $v\gamma - 2\mu W_2(u, \rho) = 0$, one ends up in a dead end. Hence, as argued for the TMG case in [13] (and see also [19] for the analogous result on Kundt solutions in higher dimensions), we continue with a metric function W which is

linear in v rather than using the most general solution. Thus we take

$$W(v, u, \rho) = vW_1(u, \rho) + W_0(u, \rho). \quad (15)$$

Now the $v\rho$ -component of the field equation becomes²

$$\frac{\partial^3 f}{\partial v^3} = 0, \quad (16)$$

which is easily integrated as

$$f(v, u, \rho) = v^2 f_2(u, \rho) + v f_1(u, \rho) + f_0(u, \rho). \quad (17)$$

The form of the metric (11) [with metric functions W and f as in (15) and (17), respectively] is left invariant under the following coordinate transformations³:

$$\begin{aligned} v &= \frac{\tilde{v}}{\dot{u}(\tilde{u})} + F(\tilde{u}, \tilde{\rho}), & u &= u(\tilde{u}), \\ \dot{u}(\tilde{u}) &\equiv \frac{du}{d\tilde{u}}, & \rho &= \tilde{\rho} + G(\tilde{u}). \end{aligned} \quad (18)$$

These will be needed in the discussion that follows.

Using (15) and (17) in (5) and a bit of massaging, one now obtains

$$\begin{aligned} &\frac{\gamma}{4\mu^2} \left(f_2 - \frac{1}{4}W_1^2\right) \left(2\frac{\partial W_1}{\partial \rho} - f_2 - \frac{3}{4}W_1^2\right) \\ &\quad - \bar{\sigma} \left(\frac{\partial W_1}{\partial \rho} + f_2 - \frac{3}{4}W_1^2\right) + 3\bar{\Lambda}_0 \\ &= 0, \end{aligned} \quad (19)$$

a quadratic equation for f_2 unlike the case for TMG. One can easily solve it to write

$$\begin{aligned} f_2(u, \rho) &= -\frac{2\mu^2 \bar{\sigma}}{\gamma} + \frac{\partial W_1}{\partial \rho} - \frac{1}{4}W_1^2 \\ &\quad \pm \sqrt{\frac{12\mu^2}{\gamma} \left(\frac{\mu^2 \bar{\sigma}^2}{3\gamma} + \bar{\Lambda}_0\right) + \left(\frac{\partial W_1}{\partial \rho} - \frac{1}{2}W_1^2\right) \left(\frac{\partial W_1}{\partial \rho} - \frac{1}{2}W_1^2 - \frac{8\mu^2 \bar{\sigma}}{\gamma}\right)}, \end{aligned} \quad (20)$$

which makes the $\gamma \rightarrow 0$ limit, for comparison with the TMG case, intractable at first sight. It turns out that the trace equation can be written as a linear combination of the vu - and $\rho\rho$ -components of the field equation.⁴ Concentrating only on the $\rho\rho$ -component, one can write it as

$$-\frac{1}{\mu} \frac{\partial f_2}{\partial \rho} + \frac{1}{\mu} f_2 W_1 + \frac{\gamma}{4\mu^2} \left(f_2 - \frac{1}{4}W_1^2\right)^2 - \bar{\sigma} \left(f_2 - \frac{1}{4}W_1^2\right) + \bar{\Lambda}_0 = 0. \quad (21)$$

²See (A1) of Appendix A for its form in the generic case.

³Here we have directly reproduced Eq. (2.9) of [13]. Refer to Eqs. (2.9) to (2.11) of [13] for details.

⁴The trace equation (19) equals the sum of the $\rho\rho$ - and twice the vu -component of the field equation.

Substituting (20) into (21), one obtains

$$\begin{aligned}
& -\frac{1}{\mu} \frac{\partial}{\partial \rho} \left[\chi \pm \sqrt{\left(\chi - \frac{4\mu^2 \bar{\sigma}}{\gamma}\right)^2 + \frac{12\mu^2}{\gamma} \left(\bar{\Lambda}_0 - \frac{\mu^2 \bar{\sigma}^2}{\gamma}\right)} \right] + \bar{\Lambda}_0 - \frac{\mu^2 \bar{\sigma}^2}{\gamma} \\
& + \frac{\gamma}{4\mu^2} \left[-\frac{4\mu^2 \bar{\sigma}}{\gamma} + \chi \pm \sqrt{\left(\chi - \frac{4\mu^2 \bar{\sigma}}{\gamma}\right)^2 + \frac{12\mu^2}{\gamma} \left(\bar{\Lambda}_0 - \frac{\mu^2 \bar{\sigma}^2}{\gamma}\right)} \right]^2 \\
& + \frac{W_1}{\mu} \left[-\frac{2\mu^2 \bar{\sigma}}{\gamma} + \frac{1}{2} \chi \pm \sqrt{\left(\chi - \frac{4\mu^2 \bar{\sigma}}{\gamma}\right)^2 + \frac{12\mu^2}{\gamma} \left(\bar{\Lambda}_0 - \frac{\mu^2 \bar{\sigma}^2}{\gamma}\right)} \right] = 0, \tag{22}
\end{aligned}$$

where

$$\chi(u, \rho) \equiv \frac{\partial W_1}{\partial \rho} - \frac{1}{2} W_1^2. \tag{23}$$

In principle, one can determine the function $W_1(u, \rho)$ from (22). If one has a solution which is only ρ -dependent, say $W_1(\rho)$, then for an arbitrary smooth enough function $c(u)$, $W_1(\rho + c(u))$ will also be a solution because of (18). So, out of the two “integration constants” present in the solution of (22), one can be discarded since the transformation $\rho \rightarrow \rho - c(u)$ can be employed to that effect⁵ for a generic ρ -dependent W_1 . Hence, in what follows we will take $W_1 = W_1(\rho)$ only.

Once $W_1(\rho)$ is determined from (22), the coordinate transformation $v \rightarrow v + F(u, \rho)$ can be employed to set $W_0 = 0$. Then the $u\rho$ -component of the field equation turns out to be of the form

$$A\left(W_1, \frac{dW_1}{d\rho}\right)v + B\left(W_1, \frac{dW_1}{d\rho}, \frac{\partial f_1}{\partial \rho}, \frac{\partial^2 f_1}{\partial \rho^2}\right) = 0, \tag{24}$$

where the coefficient functions A and B , with their indicated arguments, are rather long and complicated. Unlike the TMG case, where the coefficient function A identically vanishes, both A and B must equal zero for (24) to hold. For a generic $W_1(\rho)$, however, it turns out that the differential constraints (22) and $A = 0$ together are too strong and yield an overdetermined system with only the trivial solution.

Hence we trace back the steps we have taken from (19) to (22) and ask whether there are any “simple” choices one can make to find some “special” classes of solutions. Thus we are led to the following cases to consider:

$$(A) \chi(\rho) \equiv \frac{dW_1}{d\rho} - \frac{1}{2} W_1^2 = w_1 = \text{const}, \tag{25}$$

$$(B) W_1 = \omega = \text{const}, \tag{26}$$

$$(C) \bar{\Lambda}_0 = \frac{\mu^2 \bar{\sigma}^2}{\gamma} \quad (\text{merger point}). \tag{27}$$

We will show that the first two choices lead to CSI solutions, whereas there is a non-CSI solution for the last one. Let us also note that even though the case $f_2 - W_1^2/4 = \text{const}$ seems to be special at first sight, as we show in Appendix B, it turns out that this only leads to special instances of solutions obtained in other cases, which we now study in detail below.

A. $\chi = \text{const}$

As argued after (23), taking $W_1 = W_1(\rho)$, one easily integrates (25) to find⁶

$$W_1(\rho) = \sqrt{2w_1} \tan\left(\rho \sqrt{\frac{w_1}{2}}\right). \tag{28}$$

The solution of f_2 , which follows from (19) and replaces (20), is

$$\begin{aligned}
f_2(u, \rho) &= -\frac{2\mu^2 \bar{\sigma}}{\gamma} + \frac{w_1}{2} \left[1 + \sec^2\left(\rho \sqrt{\frac{w_1}{2}}\right) \right] \\
&\pm \sqrt{\left(w_1 - \frac{4\mu^2 \bar{\sigma}}{\gamma}\right)^2 + \frac{12\mu^2}{\gamma} \left(\bar{\Lambda}_0 - \frac{\mu^2 \bar{\sigma}^2}{\gamma}\right)}. \tag{29}
\end{aligned}$$

Note that the u -dependency of f_2 has effectively dropped altogether. Substituting (28) and (29) into the $\rho\rho$ -component of the field equation (21), one finds a long expression which is a linear function of $W_1(\rho)$. Setting the coefficient of $W_1(\rho)$ to zero, which can be shown to be equivalent to the condition

$$\gamma w_1^2 - 8\mu^2 \bar{\sigma} w_1 + 16\mu^2 \bar{\Lambda}_0 = 0, \tag{30}$$

⁵This immediately follows from the first equation in (2.11) of [13].

⁶Here we again use the transformation $\rho \rightarrow \rho - \text{const}$ to drop an integration constant.

one can fix the constant w_1 uniquely, depending on the sign in the solution of f_2 (29), as⁷

$$w_1 = \frac{4\mu^2\bar{\sigma}}{\gamma} \left(1 \mp \sqrt{1 - \frac{\gamma\bar{\Lambda}_0}{\mu^2\bar{\sigma}^2}} \right). \quad (31)$$

Substituting (31) back into (29), one finds a much simpler expression for f_2 :

$$f_2(u, \rho) = \frac{1}{4}(W_1(\rho))^2 + \frac{w_1}{2} = \frac{w_1}{2} \sec^2 \left(\rho \sqrt{\frac{w_1}{2}} \right). \quad (32)$$

Setting $f_1(u, \rho) = 0$ and $f_0(u, \rho) = 0$, one finds the “background metric,” which is itself a solution to MMG, as

$$ds^2 = d\rho^2 + 2dudv + \frac{w_1}{2} \sec^2 \left(\rho \sqrt{\frac{w_1}{2}} \right) v^2 du^2 + 2v\sqrt{2w_1} \tan \left(\rho \sqrt{\frac{w_1}{2}} \right) dud\rho, \quad (33)$$

with curvature scalar $R = 3w_1$ and $R^{\mu\nu}R_{\mu\nu} = 3w_1^2$.

Turning $f_1(u, \rho)$ and $f_0(u, \rho)$ on, the $u\rho$ -component of the field equation reads⁸

$$\frac{\partial^2 f_1}{\partial \rho^2} + \left[\mu\bar{\sigma} - \frac{w_1\gamma}{4\mu} - \sqrt{\frac{w_1}{2}} \tan \left(\rho \sqrt{\frac{w_1}{2}} \right) \right] \frac{\partial f_1}{\partial \rho} = 0. \quad (34)$$

This can be thought of as a first-order linear partial differential equation for $\partial f_1/\partial \rho$, which can easily be solved as

$$\frac{\partial f_1}{\partial \rho} = f_{11}(u) e^{a\rho} \sec \left(\rho \sqrt{\frac{w_1}{2}} \right), \quad a \equiv \frac{w_1\gamma}{4\mu} - \mu\bar{\sigma}.$$

Thus the most general solution of (34) reads

$$f_1(u, \rho) = F(\rho) f_{11}(u) + f_{12}(u), \quad (35)$$

$$F(\rho) \equiv \int^\rho d\tilde{\rho} e^{a\tilde{\rho}} \sec \left(\tilde{\rho} \sqrt{\frac{w_1}{2}} \right).$$

A coordinate transformation of the form $v \rightarrow v/(du/d\tilde{u})$, $u \rightarrow u(\tilde{u})$, can be employed to set $f_{12}(u) = 0$.⁹ Finally the uu -component of the field equation reads

$$\begin{aligned} & \frac{\partial^3 f_0}{\partial \rho^3} + \left(\frac{3}{2} W_1(\rho) + \mu\bar{\sigma} - \frac{w_1\gamma}{4\mu} \right) \left[\frac{\partial^2 f_0}{\partial \rho^2} + W_1(\rho) \frac{\partial f_0}{\partial \rho} + \frac{1}{2} (W_1(\rho))^2 f_0(u, \rho) \right] \\ & + 2w_1 \frac{\partial f_0}{\partial \rho} + \left(\frac{3}{2} w_1 W_1(\rho) + \mu(4\bar{\Lambda}_0 - w_1\bar{\sigma}) \right) f_0(u, \rho) \\ & = \frac{dF}{d\rho} \left[\frac{(f_{11}(u))^2}{2} \left(\frac{\gamma}{\mu} \frac{dF}{d\rho} - F(\rho) \right) - \frac{df_{11}}{du} \right], \end{aligned} \quad (36)$$

which is a linear partial differential equation for the remaining metric function $f_0(u, \rho)$. Given $f_{11}(u)$, one can in principle solve it to find $f_0(u, \rho)$. Hence the generic Kundt solution for this case is

$$ds^2 = d\rho^2 + 2dudv + \left[\frac{w_1}{2} \sec^2 \left(\rho \sqrt{\frac{w_1}{2}} \right) v^2 + f_1(u, \rho)v + f_0(u, \rho) \right] du^2 + 2v\sqrt{2w_1} \tan \left(\rho \sqrt{\frac{w_1}{2}} \right) dud\rho, \quad (37)$$

with a real w_1 determined as in (31), $f_1(u, \rho)$ as given by (35) [with $f_{11}(u)$ arbitrary and $f_{12}(u) = 0$] and $f_0(u, \rho)$ satisfying (36).

Inspired by the discussion given in subsection 4.3.2 of [13], if one defines a new coordinate

$$\hat{v} = u - \frac{4\cos^2(\rho\sqrt{w_1/2})}{w_1 v},$$

then the solution (33) takes the form

$$ds^2 = d\rho^2 + \frac{8\cos^2(\rho\sqrt{w_1/2})}{w_1(u - \hat{v})^2} dud\hat{v}, \quad (38)$$

which is the metric for the round (A)dS. Thus (37) can be thought of as the deformation of the round (A)dS metric, cast in the generalized Kerr-Schild form with an (A)dS background.

⁷Note that w_1 is twice the value of the allowed cosmological constant for maximally symmetric vacua of MMG, found in Eq. (2.3) of [7]. (31) simplifies considerably at the chiral point (8) and the merger point (9).

⁸For clarity of argument, w_1 found in (31) has not been used in the remainder of this subsection.

⁹See (2.11) of [13] for details.

B. $W_1 = \text{const}$

When one sets $W_1 = \omega = \text{const}$, (19) can be arranged to read

$$f_2^2 + f_2 \left(\frac{\omega^2}{2} + \frac{4\mu^2 \bar{\sigma}}{\gamma} \right) - 3 \left(\frac{\omega^4}{16} + \frac{\mu^2}{\gamma} (\bar{\sigma} \omega^2 + 4\bar{\Lambda}_0) \right) = 0, \quad (39)$$

which clearly implies that f_2 must also be a constant. Using this in the $\rho\rho$ -component of the field equation (21), one finds that this can be written as

$$f_2^2 - f_2 \left(\frac{\omega^2}{2} + \frac{4\mu}{\gamma} (\mu \bar{\sigma} - \omega) \right) + \left(\frac{\omega^4}{16} + \frac{\mu^2}{\gamma} (\bar{\sigma} \omega^2 + 4\bar{\Lambda}_0) \right) = 0. \quad (40)$$

Since (39) and (40) must be satisfied simultaneously, one can simply solve for $\bar{\Lambda}_0$ using either one of these and use this in the other to solve for f_2 . Doing so, one finds that there are two possibilities:

$$f_2 = 0 \quad \text{or} \quad f_2 = \frac{\omega^2}{4} - \frac{3\mu\omega}{\gamma} + \frac{2\mu^2 \bar{\sigma}}{\gamma}. \quad (41)$$

If one sets $f_2 = 0$ in either (39) or (40), one finds that these are satisfied only if

$$\omega^2 = -\frac{8\mu^2 \bar{\sigma}}{\gamma} \left(1 \mp \sqrt{1 - \frac{\gamma \bar{\Lambda}_0}{\mu^2 \bar{\sigma}^2}} \right), \quad (42)$$

which simplifies both at the chiral point (8) and the merger point (9). At this stage if one were to further set $f_1(u, \rho) = 0$ and $f_0(u, \rho) = 0$, one would simply obtain the spacetime of constant curvature

$$ds^2 = d\rho^2 + 2dudv + 2\omega v dud\rho, \quad (43)$$

with curvature scalar $R = -3\omega^2/2$ and $R^{\mu\nu} R_{\mu\nu} = 3\omega^4/4$. This, of course, is a solution to MMG with any real ω satisfying (42).

However, one can dress up this “background metric” further in the following way: The $u\rho$ -component of the field equation yields¹⁰

$$\frac{\partial^2 f_1}{\partial \rho^2} + \left[\mu \bar{\sigma} + \frac{\gamma \omega^2}{8\mu} - \frac{\omega}{2} \right] \frac{\partial f_1}{\partial \rho} = 0. \quad (44)$$

Using a coordinate transformation [similar to the one employed after (35) in Sec. III A] to set an arbitrary function of u to zero, we take the solution of (44) as

¹⁰For clarity, neither of the roots in (42) has been used explicitly in the remainder of this subsection.

$$f_1(u, \rho) = F(\rho) f_{11}(u), \quad \text{where } F(\rho) = e^{-(\mu \bar{\sigma} + \frac{\gamma \omega^2}{8\mu} - \frac{\omega}{2})\rho}, \quad (45)$$

with $f_{11}(u)$ an arbitrary function.¹¹ Then the uu -component of the field equation is

$$\begin{aligned} & \frac{\partial^3 f_0}{\partial \rho^3} + \left(\frac{3\omega}{2} + \mu \bar{\sigma} + \frac{\gamma \omega^2}{8\mu} \right) \left[\frac{\partial^2 f_0}{\partial \rho^2} + \omega \frac{\partial f_0}{\partial \rho} \right] - \omega^2 \frac{\partial f_0}{\partial \rho} \\ &= \frac{dF}{d\rho} \left[\frac{(f_{11}(u))^2}{2} \left(\frac{\gamma dF}{\mu d\rho} - F(\rho) \right) - \frac{df_{11}}{du} \right], \end{aligned} \quad (46)$$

a linear partial differential equation for the metric function $f_0(u, \rho)$. As in Sec. III A, one can in principle solve it to find $f_0(u, \rho)$ given $f_{11}(u)$. Putting it all together, the Kundt solution found is

$$\begin{aligned} ds^2 &= d\rho^2 + 2dudv + 2\omega v dud\rho \\ &+ (vf_1(u, \rho) + f_0(u, \rho)) du^2, \end{aligned} \quad (47)$$

where ω is any one of the real roots of (42), $f_1(u, \rho)$ is given by (45) with $f_{11}(u)$ arbitrary, and $f_0(u, \rho)$ satisfies (46).

Note that the coordinate transformation $\hat{v} = ve^{\omega\rho}$ turns (43) into

$$ds^2 = d\rho^2 + 2e^{-\omega\rho} dud\hat{v}, \quad (48)$$

which is simply the round AdS [13]. Thus (47) can be considered as a deformation of the round AdS, cast again in a generalized Kerr-Schild form with an AdS background.

Now one can also use the other choice of f_2 in (41), i.e., that

$$f_2 = \frac{\omega^2}{4} - \frac{3\mu\omega}{\gamma} + \frac{2\mu^2 \bar{\sigma}}{\gamma}. \quad (49)$$

Before proceeding any further, note that this can also be written as¹²

$$\begin{aligned} f_2 &= \frac{\omega^2}{4} + \xi, \quad \text{where } \xi \equiv -\frac{3\mu\omega}{\gamma} + \frac{2\mu^2 \bar{\sigma}}{\gamma} \quad \text{or} \\ \omega &= \frac{2\mu \bar{\sigma}}{3} - \frac{\gamma \xi}{3\mu}. \end{aligned} \quad (50)$$

In this case both (19) and (21) imply

¹¹A careful scrutiny of Eqs. (2.9) to (2.11) of [13] shows that the function $f_{11}(u)$ cannot be “gauged away” by the coordinate transformations (18).

¹²This alternative form will be of use for the discussion in Appendix B.

$$\frac{\omega^3}{4\mu} - \frac{3\omega^2}{4\gamma} + \frac{2\mu\bar{\sigma}\omega}{\gamma} + \left(\bar{\Lambda}_0 - \frac{\mu^2\bar{\sigma}^2}{\gamma}\right) = 0, \quad (51)$$

or for easier access to the $\gamma \rightarrow 0$ limit

$$\begin{aligned} \frac{\gamma^3}{108\mu^4}\xi^3 + \frac{\gamma}{6\mu^2}\left(\frac{1}{2} - \frac{\gamma\bar{\sigma}}{3}\right)\xi^2 + \frac{\bar{\sigma}}{3}\left(1 + \frac{\gamma\bar{\sigma}}{3}\right)\xi \\ - \left(\bar{\Lambda}_0 + \frac{2\mu^2\bar{\sigma}^3}{27}\right) = 0, \end{aligned} \quad (52)$$

a cubic equation in ω (or ξ), the explicit roots of which are better left not displayed. One can now “play the game” of setting $f_1(u, \rho) = 0$ and $f_0(u, \rho) = 0$, and obtain the background metric

$$\begin{aligned} ds^2 = d\rho^2 + 2dudv + 2\omega v dud\rho \\ + \left(\frac{\omega^2}{4} - \frac{3\mu\omega}{\gamma} + \frac{2\mu^2\bar{\sigma}}{\gamma}\right)v^2 du^2, \end{aligned} \quad (53)$$

which has a curvature scalar

$$R = -\omega^2 + 2\xi \quad \text{and} \quad R^{\mu\nu}R_{\mu\nu} = \frac{1}{8}(2\omega^4 + (\omega^2 - 4\xi)^2),$$

with ω any one of the real root(s) of (51).

Turning f_1 and f_0 on is straightforward: The $u\rho$ -component of the field equation is

$$\frac{\partial^2 f_1}{\partial \rho^2} + \omega \frac{\partial f_1}{\partial \rho} = 0, \quad (54)$$

which is readily integrated, with an arbitrary function $f_{12}(u) = 0$ as before, as

$$f_1(u, \rho) = e^{-\omega\rho} f_{11}(u). \quad (55)$$

However, the coordinate transformation

$$v \rightarrow v - \frac{1}{2f_2} e^{-\omega\rho} f_{11}(u)$$

lets one take $f_{11}(u) = 0$, so that one can set $f_1(u, \rho) = 0$ for good. Finally, the uu -component of the field equation becomes

$$\begin{aligned} \frac{\partial^3 f_0}{\partial \rho^3} + \left(2\mu\bar{\sigma} + \frac{\gamma\omega^2}{4\mu}\right) \frac{\partial^2 f_0}{\partial \rho^2} \\ - \left(\frac{\omega^2}{2} - \frac{3\mu\omega}{\gamma} + \frac{2\mu^2\bar{\sigma}}{\gamma} + \gamma\bar{\Lambda}_0 - \mu^2\bar{\sigma}^2\right) \frac{\partial f_0}{\partial \rho} = 0, \end{aligned} \quad (56)$$

whose most general solution is easy to obtain:

$$\begin{aligned} f_0(u, \rho) = f_{01}(u) + f_{02}(u)e^{-(\Omega+3\omega+\gamma f_2/\mu)\rho/2} \\ + f_{03}(u)e^{(\Omega-3\omega-\gamma f_2/\mu)\rho/2}, \end{aligned}$$

with

$$\Omega \equiv \sqrt{10\omega^2 + \frac{6\gamma f_2 \omega}{\mu} + \frac{\gamma^2 f_2^2}{\mu^2} + 4(f_2 + \gamma\bar{\Lambda}_0 - \mu^2\bar{\sigma}^2)}, \quad (57)$$

for arbitrary functions $f_{01}(u)$, $f_{02}(u)$, and $f_{03}(u)$ with f_2 given in (49). Thus, with $f_0(u, \rho)$ determined as in (57), the generic Kundt solution for this case is

$$\begin{aligned} ds^2 = d\rho^2 + 2dudv + 2\omega v dud\rho \\ + \left[\left(\frac{\omega^2}{4} - \frac{3\mu\omega}{\gamma} + \frac{2\mu^2\bar{\sigma}}{\gamma}\right)v^2 + f_0(u, \rho)\right] du^2. \end{aligned} \quad (58)$$

A specific solution of this type with $f_0 = 1$ has already been found in [11].

Note first that the metric (53) can be written in the simple form

$$ds^2 = 2dudv + (d\rho + \omega v du)^2 - qv^2 du^2, \quad (59)$$

where

$$q \equiv \omega^2 - f_2 = \frac{3\omega^2}{4} + \frac{3\mu\omega}{\gamma} - \frac{2\mu^2\bar{\sigma}}{\gamma}.$$

The coordinate transformations $\hat{u} = qu/2$ and $\hat{v} = 1/v + qu/2$ take (59) into the form

$$ds^2 = -\frac{4d\hat{u}d\hat{v}}{q(\hat{u} - \hat{v})^2} + \left(d\rho - \frac{2\omega d\hat{u}}{q(\hat{u} - \hat{v})}\right)^2. \quad (60)$$

Finally the coordinates $t = (\hat{u} + \hat{v})/2$, $x = (\hat{v} - \hat{u})/2$, and $z = q\rho/\omega - \ln(\hat{v} - \hat{u})$ can be employed to cast (60) into

$$ds^2 = \frac{1}{q} \left[\frac{-dt^2 + dx^2}{x^2} + \frac{\omega^2}{q} \left(dz + \frac{dt}{x} \right)^2 \right], \quad (61)$$

which renders this solution as the spacelike-warped (A)dS [7,13]. Thus (58) can be thought of as the deformation of spacelike-warped (A)dS, written in a generalized Kerr-Schild form with a spacelike-warped (A)dS background.

C. Merger point: $\bar{\Lambda}_0 = \mu^2\bar{\sigma}^2/\gamma$

For this special fine-tuning of the parameters Eq. (19) factorizes:

$$\frac{\gamma}{4\mu^2} \left(f_2 - \frac{1}{4}W_1^2 - \frac{2\mu^2\bar{\sigma}}{\gamma} \right) \left(f_2 - 2\frac{dW_1}{d\rho} + \frac{3}{4}W_1^2 + \frac{6\mu^2\bar{\sigma}}{\gamma} \right) = 0, \quad (62)$$

which lets f_2 be determined in terms of W_1 .

The substitution of the first choice

$$f_2 = \frac{1}{4} W_1^2 + \frac{2\mu^2 \bar{\sigma}}{\gamma} \quad (63)$$

into the $\rho\rho$ -component of the field equation (21) gives

$$W_1 \left(\frac{dW_1}{d\rho} - \frac{1}{2} W_1^2 - \frac{4\mu^2 \bar{\sigma}}{\gamma} \right) = 0. \quad (64)$$

If one takes $W_1 = 0$, so that $f_2 = 2\mu^2 \bar{\sigma}/\gamma$, then one ends up finding the solution described in the second half of Sec. III B with $\omega=0$. If, however [using our earlier notation of (25)], one takes $\chi = 4\mu^2 \bar{\sigma}/\gamma = \text{const}$, one is led to the general solution described in Sec. III A with $w_1 = 4\mu^2 \bar{\sigma}/\gamma$.

The substitution of the second choice, namely,

$$f_2 = 2 \frac{dW_1}{d\rho} - \frac{3}{4} W_1^2 - \frac{6\mu^2 \bar{\sigma}}{\gamma}, \quad (65)$$

into the $\rho\rho$ -component of the field equation (21) is a bit subtler. One finds

$$\begin{aligned} \frac{d}{d\rho} \left(\frac{dW_1}{d\rho} - \frac{1}{2} W_1^2 - \frac{4\mu^2 \bar{\sigma}}{\gamma} \right) - \frac{\gamma}{2\mu} \left(\frac{dW_1}{d\rho} - \frac{1}{2} W_1^2 - \frac{4\mu^2 \bar{\sigma}}{\gamma} \right)^2 \\ - \frac{3}{4} W_1 \left(\frac{dW_1}{d\rho} - \frac{1}{2} W_1^2 - \frac{4\mu^2 \bar{\sigma}}{\gamma} \right) = 0, \end{aligned} \quad (66)$$

a *nonlinear* second-order ordinary differential equation from which $W_1(\rho)$ can in principle be solved. It is easy to see that the particular choices, the cases $W_1 = 0$ and $\chi = 4\mu^2 \bar{\sigma}/\gamma$ that we have already covered, solve it. Now we present a ‘‘particular’’ solution of (66) which is a non-CSI spacetime as we will see. Provided that out of the three remaining ‘‘free’’ parameters, $\bar{\sigma}$, μ , and γ , of the MMG theory, the pair $\bar{\sigma}$ and γ are further fine-tuned as $2\bar{\sigma}\gamma + 1 = 0$, such that $\bar{\Lambda}_0 = \mu^2/(4\gamma^3)$ now, one finds that (66) is identically satisfied if

$$\frac{dW_1}{d\rho} - \frac{1}{2} W_1^2 + \frac{\mu}{2\gamma} W_1 + \frac{\mu^2}{\gamma^2} = 0. \quad (67)$$

Employing the transformation $\rho \rightarrow \rho - \text{const}$ to get rid of an integration constant as previously argued, the solution of this equation reads

$$W_1(\rho) = \frac{\mu}{\gamma} \left(\frac{2 + e^{3\mu\rho/(2\gamma)}}{1 - e^{3\mu\rho/(2\gamma)}} \right), \quad (68)$$

which determines f_2 through (65) as

$$f_2 = \left(\frac{1}{2} W_1 - \frac{\mu}{\gamma} \right)^2 = \frac{9\mu^2}{4\gamma^2} \frac{e^{3\mu\rho/\gamma}}{(1 - e^{3\mu\rho/(2\gamma)})^2}. \quad (69)$$

Thus one arrives at the background metric

$$\begin{aligned} ds^2 = d\rho^2 + 2dudv + \frac{9\mu^2}{4\gamma^2} \frac{e^{3\mu\rho/\gamma}}{(1 - e^{3\mu\rho/(2\gamma)})^2} v^2 du^2 \\ + 2 \frac{\mu}{\gamma} \left(\frac{2 + e^{3\mu\rho/(2\gamma)}}{1 - e^{3\mu\rho/(2\gamma)}} \right) v dud\rho, \end{aligned} \quad (70)$$

with f_1 and f_0 switched off. This metric has curvature scalar

$$\begin{aligned} R = -\frac{3\mu^2}{\gamma^2} \left(\frac{2 + e^{3\mu\rho/(2\gamma)}}{1 - e^{3\mu\rho/(2\gamma)}} \right) \quad \text{and} \\ R^{\mu\nu} R_{\mu\nu} = \frac{3\mu^4}{8\gamma^4} \frac{(32 + 32e^{3\mu\rho/(2\gamma)} + 17e^{3\mu\rho/\gamma})}{(1 - e^{3\mu\rho/(2\gamma)})^2}. \end{aligned}$$

Note that there is a curvature singularity at $\rho = 0$ and as $\rho \rightarrow \infty$ the metric (70) approaches to the CSI metric given in (53) with $\omega = -\mu/\gamma$ (and hence $\xi = 2\mu^2/\gamma^2$), which is a root of (51), at this special setting. Note that the asymptotic CSI metric has positive curvature scalar $R = 3\mu^2/\gamma^2 > 0$ then.

Turning f_1 and f_0 on is straightforward. One finds that f_1 must satisfy

$$\frac{\partial^2 f_1}{\partial \rho^2} - \frac{\mu}{\gamma} \frac{\partial f_1}{\partial \rho} = 0, \quad (71)$$

which is easily integrated, with an arbitrary function $f_{12}(u) = 0$ as before, as

$$f_1(u, \rho) = e^{\mu\rho/\gamma} f_{11}(u). \quad (72)$$

On the other hand, the coordinate transformation

$$v \rightarrow v - \frac{1}{2f_2} e^{\mu\rho/\gamma} f_{11}(u)$$

can be employed to set $f_{11}(u) = 0$, so that one has $f_1(u, \rho) = 0$ for good. The equation that f_0 must satisfy is

$$\begin{aligned} \frac{\partial^3 f_0}{\partial \rho^3} + \frac{3\mu}{4\gamma} \left(\frac{4 + e^{3\mu\rho/(2\gamma)}}{1 - e^{3\mu\rho/(2\gamma)}} \right) \frac{\partial^2 f_0}{\partial \rho^2} \\ + \frac{\mu^2}{4\gamma^2} \frac{(8 + 47e^{3\mu\rho/(2\gamma)} - 10e^{3\mu\rho/\gamma})}{(1 - e^{3\mu\rho/(2\gamma)})^2} \frac{\partial f_0}{\partial \rho} \\ + \frac{9\mu^3}{8\gamma^3} \frac{e^{3\mu\rho/(2\gamma)}(14 + e^{3\mu\rho/(2\gamma)})}{(1 - e^{3\mu\rho/(2\gamma)})^3} f_0 = 0. \end{aligned} \quad (73)$$

Thus the general Kundt solution for this case is

$$\begin{aligned}
ds^2 &= d\rho^2 + 2dudv \\
&+ \left[\frac{9\mu^2}{4\gamma^2} \frac{e^{3\mu\rho/\gamma}}{(1 - e^{3\mu\rho/(2\gamma)})^2} v^2 + f_0(u, \rho) \right] du^2 \\
&+ 2\frac{\mu}{\gamma} \left(\frac{2 + e^{3\mu\rho/(2\gamma)}}{1 - e^{3\mu\rho/(2\gamma)}} \right) v dud\rho, \tag{74}
\end{aligned}$$

where it is understood that f_0 satisfies the linear partial differential equation (73).

IV. ALGEBRAIC CLASSIFICATION

In three dimensions the algebraic classification of curvature can be achieved by using either the traceless Ricci tensor

$$\hat{R}^\mu{}_\nu \equiv R^\mu{}_\nu - \frac{1}{3}R\delta^\mu{}_\nu$$

(Segre classification) or the Cotton tensor (2) (Petrov classification). Even though these two classifications coincide for TMG, this is not so for MMG. Since the Cotton tensor involves the derivative of the Ricci tensor, and thus can be considered “less fundamental,” we concentrate on the traceless Ricci tensor for the algebraic classification of the Kundt solutions of MMG. In this case, one must thus find the eigenvalues of $\hat{R}^\mu{}_\nu$ and their multiplicities, if any. Since the Jordan normal form encodes all of this information, we calculate this for each of the solutions obtained thus far. However it will be convenient to first recall the basics of the Segre notation we use: The symbols 1, 2, and 3 indicate the sizes of Jordan blocks. Parentheses are used for grouping Jordan blocks that belong to the same eigenvalue. A comma is used for splitting the Jordan blocks corresponding to spacelike eigenvectors from those corresponding to the timelike ones, where the former are always written before the comma. (Refer to Table 1 of [14] for details.)

A generic Kundt spacetime (11), where W and f are given as in (15) and (17), respectively, is of Segre type [12], where the eigenvalues of $\hat{R}^\mu{}_\nu$ are β and -2β with algebraic multiplicities 2 and 1, respectively, and

$$\beta \equiv \frac{1}{6} \left(2f_2 - \frac{\partial W_1}{\partial \rho} \right). \tag{75}$$

For the solutions found in Sec. III A, it turns out that $\beta = 0$. The solution (33) is of Segre type [(12)], whereas (37) is of Segre type [3]. Since $W_1 = \text{const}$ and f_2 must be as in (41) for the solutions found in Sec. III B, there are two possibilities for each choice of f_2 : For the trivial case $f_2 = 0$, it follows that $\beta = 0$ as well. Then the solution (43) is of Segre type [(11,1)], whereas (47) is of Segre type [3]. For the nontrivial f_2 given as in (49), it is obvious that $\beta = f_2/3 \neq 0$ and it turns out that the solution (53) is of Segre type [1(1,1)], whereas the solution (58) is of Segre

type [12]. Note that solutions presented in Appendix B, namely, (B3) and its dressed-up version (B4), are both closely related to their counterparts (53) and (58) of Sec. III B, respectively, and are of the same Segre type as their cousins. Finally, for the non-CSI solution presented in Sec. III C, namely, (70) and its sister with f_0 turned on (74), the specific value of the eigenvalue is

$$\beta = -\frac{3\mu^2}{4\gamma^2} \left(\frac{e^{3\mu\rho/(2\gamma)}}{1 - e^{3\mu\rho/(2\gamma)}} \right),$$

and both of these solutions are again of Segre type [12].

Comparing the Segre types of these solutions to the Segre types of the Kundt solutions of TMG, one sees that the presence of the parameter γ does not alter the general picture. The Kundt solutions of MMG also fall into two broad classes of Segre type [12] and Segre type [3], with special cases of Segre types [(11,1)], [1(1,1)], and [(12)] also occurring.

V. DISCUSSION

In this paper we investigated the Kundt solutions of MMG theory. All of the explicit solutions presented in Secs. III A and III B turn out to be CSI spacetimes, which are deformations of round and warped (A)dS. We also showed the existence of Kundt solutions at the merger point (9) in Sec. III C and found an explicit non-CSI solution. Even though the fine-tuned parameters of the latter violate the unitarity conditions (7), there are no restrictions on the parameters of the former and they remain indifferent to the unitarity requirements. The chiral point (8) does not seem to be special for Kundt solutions except for simplifying relevant expressions. Since the algebraic classification is a useful tool for identifying seemingly different spacetimes, we also gave the Segre classification of these solutions and found that their Segre types coincide with their cousins in TMG.

There are various possible directions to extend our work. First of all, one may try to relax the expansion-free condition on the null vector k^μ in (12) and look for Robinson-Trautman solutions. One may also try to find all locally homogeneous solutions as was done for TMG [20,21]. This is an interesting problem, since then all CSI solutions of MMG can be completed. Another open question is whether the Goldberg-Sachs theorem is also valid for MMG. This was recently proven for TMG in [22]. Studying such solutions in models that are related to MMG would also be interesting [23,24]. We hope to investigate these and related issues in the near future.

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APPENDIX A: THE MOST GENERAL SOLUTION OF (13)

As argued in the beginning of Sec. III, here we show that the Kundt spacetimes exist only if one takes $Y(v, u, \rho) \equiv v\gamma - 2\mu W_2(u, \rho) = 0$.

Recall that the most general solution of (13) is given by (14). Using (13), the $v\rho$ -component of the field equation (3) reads

$$\left[8\mu^2\bar{\sigma} + \gamma \left(\frac{\partial W}{\partial v} \right)^2 - 2\gamma \frac{\partial^2 f}{\partial v^2} \right] \frac{\partial^2 W}{\partial v^2} - 4\mu \frac{\partial^3 f}{\partial v^3} = 0, \quad (\text{A1})$$

which after the substitution of (14) can be cast into

$$8\gamma\mu^2\bar{\sigma} + (\gamma W_1(u, \rho) + 2\mu \ln(v\gamma - 2\mu W_2(u, \rho)))^2 - 2\gamma \frac{\partial}{\partial v} \left((v\gamma - 2\mu W_2(u, \rho)) \frac{\partial^2 f}{\partial v^2} \right) = 0. \quad (\text{A2})$$

Assuming that $Y(v, u, \rho) \equiv v\gamma - 2\mu W_2(u, \rho) \neq 0$, this partial differential equation for the metric function f can be solved to determine its v dependency as

$$\begin{aligned} f(v, u, \rho) = v^2 & \left\{ \frac{\mu^2}{2\gamma^2} [17 + 4\gamma\bar{\sigma} - (10 - 2 \ln Y) \ln Y] - \frac{\mu}{2\gamma} W_1(u, \rho)(5 - 2 \ln Y) + \frac{1}{4} W_1^2(u, \rho) \right\} \\ & + v \left\{ f_1(u, \rho) - \frac{f_2(u, \rho)}{\gamma} (1 - \ln Y) - \frac{2\mu^3}{\gamma^3} W_2(u, \rho)[17 + 8\gamma\bar{\sigma} - (10 + 4\gamma\bar{\sigma} - 2 \ln Y) \ln Y] \right. \\ & + \left. \frac{2\mu^2}{\gamma^2} W_1(u, \rho) W_2(u, \rho)(7 - 4 \ln Y) - \frac{\mu}{\gamma} W_1^2(u, \rho) W_2(u, \rho)(2 - \ln Y) \right\} \\ & + f_0(u, \rho) - \frac{2\mu}{\gamma^2} f_2(u, \rho) W_2(u, \rho) \ln Y - \frac{2\mu^4}{\gamma^4} W_2^2(u, \rho)[5 + 2 \ln Y(5 + 4\gamma\bar{\sigma} - \ln Y)] \\ & - \frac{2\mu^3}{\gamma^3} W_1(u, \rho) W_2^2(u, \rho)(1 - 6 \ln Y) - \frac{2\mu^2}{\gamma^2} W_1^2(u, \rho) W_2^2(u, \rho) \ln Y. \end{aligned} \quad (\text{A3})$$

Using (14) and (A3) in (5), one obtains a very long expression, better not displayed here at all, whose general form is

$$\sum_{m=0}^2 \sum_{n=0}^3 a_{mn}(u, \rho) v^m (\ln(v\gamma - 2\mu W_2(u, \rho)))^n = 0.$$

Obviously each of the coefficients $a_{mn}(u, \rho)$ must vanish for this equality to hold, but the vanishing of the ‘‘highest’’ one, $a_{23}(u, \rho) = 96\gamma^2\mu^4 = 0$, implies that either $\gamma = 0$ or $\mu = 0$, both of which take us out of MMG theory.

APPENDIX B: THE CASE $f_2 - W_1^2/4 = \text{const}$ IN (19)

In this appendix, we study what happens when one chooses $f_2 - W_1^2/4 = \text{const}$ in equation (19) and afterwards. Dropping the u -dependency of f_2 and setting $f_2(\rho) - W_1(\rho)^2/4 = \xi = \text{const}$, (19) simplifies to

$$\xi(\gamma\xi + 4\mu^2\bar{\sigma}) - 12\mu^2\bar{\Lambda}_0 - 2\chi(\rho)(\gamma\xi - 2\mu^2\bar{\sigma}) = 0, \quad (\text{B1})$$

where we have used the definition of $\chi(\rho)$ as given in (25), and taken $W_1 = W_1(\rho)$ as argued after (23). The generic solution of (B1) falls precisely into the $\chi = \text{const}$ case already discussed in detail in Sec. III A.¹³

¹³The field equations then impose the identification of the constant w_1 of Sec. III A as $w_1 = 2\xi$.

Note also that if one were to set $\xi = 0$ to start with, and solve for the relevant version of (B1) keeping $W_1 = W_1(\rho)$ generic, then one immediately finds that (21) can only be satisfied if $\bar{\Lambda}_0 = 0$. This at once leads to the ‘‘trivial’’ solution: Both W_1 and f_2 must vanish and one simply obtains the flat Minkowski metric

$$ds^2 = d\rho^2 + 2dudv, \quad (\text{B2})$$

when one keeps both f_1 and f_0 switched off. When one chooses to turn f_1 and f_0 on, the analysis flows exactly as in the second paragraph of Sec. III B with the substitution $\omega = 0$.

Still keeping $\xi = 0$, if one instead takes $W_1 = \tilde{\omega} = \text{const}$, one finds from (19) that¹⁴ $\tilde{\omega}^2 = -6\bar{\Lambda}_0/\bar{\sigma}$, which makes $f_2 = -3\bar{\Lambda}_0/(2\bar{\sigma})$. Substituting these in (21), one ends up with either $\bar{\Lambda}_0 = 0$ or $\bar{\Lambda}_0 = -2\mu^2\bar{\sigma}^3/27$. The former case goes ahead as mentioned in the previous paragraph. The latter gives $W_1 = 2\mu\bar{\sigma}/3$ and $f_2 = \mu^2\bar{\sigma}^2/9$.¹⁵ If one chooses to set $f_1(u, \rho) = 0$ and $f_0(u, \rho) = 0$ here, then one simply obtains the spacetime of constant negative curvature

¹⁴Of course, the constraint $\bar{\Lambda}_0/\bar{\sigma} < 0$ (or $\bar{\Lambda}_0\bar{\sigma} < 0$) is implicitly assumed in what follows.

¹⁵Note that these choices correspond to taking $\xi = 0$ in (50) and (52).

$$ds^2 = d\rho^2 + 2dudv + \frac{\mu^2\bar{\sigma}^2}{9}v^2du^2 + \frac{4\mu\bar{\sigma}}{3}vdud\rho, \quad (\text{B3})$$

with $R = -4\mu^2\bar{\sigma}^2/9$ and $R^{\mu\nu}R_{\mu\nu} = 2\mu^4\bar{\sigma}^4/27$, as a solution to MMG. The story with the turning on of the other functions f_1 and f_0 follows similar steps as before, but we skip the details for clarity and simply present the final result: The general Kundt solution for this case is¹⁶

$$ds^2 = d\rho^2 + 2dudv + \left(\frac{\mu^2\bar{\sigma}^2}{9}v^2 + f_0(u, \rho)\right)du^2 + \frac{4\mu\bar{\sigma}}{3}vdud\rho, \quad (\text{B4})$$

¹⁶This reduces to the metric given in Eq. (3.4) of [13] in the limit $\gamma \rightarrow 0$, $\bar{\sigma} \rightarrow 1$, and $\Lambda \rightarrow -2\mu^2/27$.

where a coordinate transformation as in (18) has been employed on v to set $f_1(u, \rho) = 0$ for good and $f_0(u, \rho)$ reads

$$f_0(u, \rho) = f_{01}(u) + f_{02}(u)e^{-\mu\bar{\sigma}(18+\gamma\bar{\sigma}+\Xi)\rho/18} + f_{03}(u)e^{-\mu\bar{\sigma}(18+\gamma\bar{\sigma}-\Xi)\rho/18},$$

with $\Xi \equiv \sqrt{\gamma^2\bar{\sigma}^2 + 12\gamma\bar{\sigma} + 72}$, (B5)

for arbitrary functions $f_{01}(u)$, $f_{02}(u)$, and $f_{03}(u)$.

To cover all bases, if one were to examine the assumption $W_1(\rho) = \bar{\omega} = \text{const}$ with $\xi \neq 0$, then one soon finds that this reduces to the case covered in the first half of Sec. III B, with the identification $\bar{\omega} \rightarrow \omega$ and $f_2 = 0$, etc.

Finally, if one were to set the coefficient of $\chi(\rho)$ in (B1) to zero, i.e., set $\xi = 2\mu^2\bar{\sigma}/\gamma$, then the remainder of (B1) demands $\bar{\Lambda}_0 = \mu^2\bar{\sigma}^2/\gamma$, which is examined in detail in Sec. III C.

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