Metric of two balancing Kerr particles in physical parametrization

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The present paper aims at elaborating a completely physical representation for the general 4-parameter family of the extended double-Kerr spacetimes describing two spinning sources in gravitational equilibrium. This involved problem is solved in a concise analytical form by using the individual Komar masses and angular momenta as arbitrary parameters, and the simplest equatorially symmetric specialization of the general expressions obtained by us yields the physical representation for the well-known Dietz-Hoenselaers superextreme case of two balancing identical Kerr constituents. The existence of the physically meaningful "black-hole–superextreme-object" equilibrium configurations permitted by the general solution may be considered as a clear indication that the spin-spin repulsion force might actually be by far stronger than expected earlier, when only the balance between two superextreme Kerr sources was thought possible. We also present the explicit analytical formulas relating the equilibrium states in the double-Kerr and double-Reissner-Nordström configurations.

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I. INTRODUCTION

The well-known double-Kerr solution [1] was discovered three and a half decades ago by Kramer and Neugebauer as a nontrivial application to Einstein's equations of the modern solution generating techniques in the form of Bäcklund transformations [2]. It gave the researchers an attractive possibility to study binary systems of interacting Kerr black holes [3], and in particular to answer an important question of whether the gravitational attraction of the rotating black holes can be counterbalanced by their spin-spin interaction. The equilibrium conditions were first worked out by Kihara and Tomimatsu [4,5], and later Tomimatsu [6] obtained the expressions for the individual Komar [7] masses and angular momenta of the constituents in a generic double-Kerr configuration. Restricted to the subextreme case, the algebraic system of the balance equations was solved analytically by Hoenselaers [8], who also conjectured, after analyzing numerically the formulas of Komar masses, that equilibrium between two Kerr black holes endowed with positive masses is impossible. At this point, it should be noted that the parametrization employed in [1] does not describe configurations involving superextreme Kerr constituents, and that is why, for being able to consider a system of two identical superextreme Kerr sources, Dietz and Hoenselaers [9] used a special complex trick to pass from the subextreme to the superextreme case. Remarkably, they were able to demonstrate analytically that such a pair of superspinning Kerr constituents with positive Komar masses can be in stationary equilibrium.

Two decades after the publication of the paper [1], a unified description of the binary equilibrium configurations composed of arbitrary combinations of the subextreme and superextreme Kerr constituents became possible due to the so-called extended double-Kerr (EDK) solution [10] constructed with the aid of Sibgatullin's integral method [11,12]. The set of parameters used in the paper [10] turned out to be very advantageous not only for solving analytically the equilibrium conditions in the general case, which led in particular to the discovery of the physically meaningful "subextreme-superextreme" equilibrium configurations, but also for giving a rigorous proof [13] to Hoenselaers' conjecture on the nonexistence of balance between two black-hole Kerr constituents with positive Komar masses. Moreover, in recent years there has been a renewed interest in the double-Kerr solution, mostly related to the issues of the black-hole configurations with struts and the geometrical inequalities for black holes. With regard to the former issue, the research has been principally directed to the study of the physical properties of two interacting Kerr black holes [14–16], while the latter issue gave birth to a series of papers by Neugebauer and Hennig [17–19] in which the aforementioned nonexistence proof [13] was reexamined on the basis of the area–angular-momentum inequality [20] (the validity of this inequality in the multiple-black-hole case has been proven by Chruściel *et al.* [21]). It should be emphasized that the Neugebauer-Hennig analysis, which employs our solution of the equilibrium problem [10], is in full agreement with the earlier nonexistence proof [13]:

the balance of two Kerr black holes with positive Komar masses is impossible, while a subextreme constituent with negative mass is unphysical (it develops a massless ring singularity outside the symmetry axis).

Curiously, although the general equilibrium problem for the EDK solution was solved more than a decade ago [10,13,22], the physical parametrization of the 4-parameter family of equilibrium configurations in terms of the Komar quantities has not yet been obtained to date. This can be explained by numerous technical difficulties that one has to overcome for being able to express all the "canonical" parameters of the EDK solution and various associated constant quantities in terms of the physical parameters. Recently, nonetheless, we have succeeded in finding the desired reparametrization for a 3-parameter equilibrium configuration [23] that describes a Schwarzschild black hole levitating in the field of a superextreme Kerr source, and have studied physical effects in that binary system. To reach a more ambitious goal, in the present paper we are going to reparametrize the entire 4-parameter family of the EDK equilibrium configurations in terms of the Komar physical quantities. This will be done with the aid of two sets of the inversion formulas involving parameters of the solution and individual physical characteristics of the constituents.

The rest of the paper is organized as follows. In the next section we briefly review the EDK equilibrium configurations and present the first set of inversion formulas. The Komar individual characteristics of the constituents and their relation to the canonical parameters of the EDK solution are discussed in Sec. III. The general 4-parameter family of equilibrium binary systems determined by the EDK solution is reparametrized in physical parameters in Sec. IV, and the reparametrized quantities σ_u and σ_d play a crucial role in this process; here, in particular, we obtain a physical representation for the Dietz-Hoenselaers (DH) solution [9] describing two identical corotating superextreme Kerr sources in equilibrium. In Sec. V we give a simple new proof of the absence of balance between two Kerr black holes, and also derive explicit analytical formulas relating the equilibrium states in the EDK and double-Reissner-Nordström [24] solutions. Section VI presents concluding remarks.

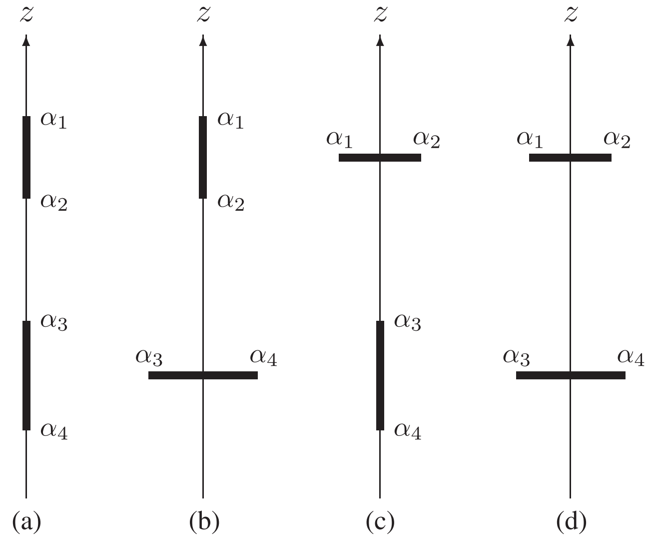

FIG. 1. Four different types of the equilibrium configurations of two Kerr sources.

II. SOLUTION OF THE EDK EQUILIBRIUM PROBLEM IN CANONICAL PARAMETERS AND THE FIRST SET OF INVERSION FORMULAS

The main advantage of the EDK solution over the nonextended one originally obtained by Kramer and Neugebauer for two black-hole constituents [1] consists in a remarkable possibility of its use for solving in a unified manner the equilibrium problem for any combination of two Kerr sources—black holes or superextreme objects. Such a possibility becomes feasible due to the presence in the EDK solution of the parameters α_i which can assume arbitrary real values or occur in complex conjugate pairs. A pair of two real α_i then naturally determines an underextreme Kerr constituent (a black hole if its mass is positive), while a complex conjugate pair defines a superextreme constituent (the four main types of binary configurations are shown in Fig. 1).

The equilibrium configurations in the EDK solution are defined by an Ernst complex potential $\mathcal{E}$ [25] of the following form [13]:

$$\begin{aligned}
&\mathcal{E} = \frac{\Lambda + \Gamma}{\Lambda - \Gamma}, \qquad \Lambda = \sum_{1\le i<j\le 4} \lambda_{ij} r_i r_j, \qquad \Gamma = \sum_{i=1}^{4} \gamma_i r_i, \\
&\lambda_{ij} = (-1)^{i+j}(\alpha_i - \alpha_j)(\alpha_{i'} - \alpha_{j'}) X_i X_j, \qquad (i', j' \neq i, j; i' < j') \\
&\gamma_i = (-1)^i (\alpha_{i'} - \alpha_{j'})(\alpha_{i'} - \alpha_{k'})(\alpha_{j'} - \alpha_{k'}) X_i, \qquad (i', j', k' \neq i; i' < j' < k') \\
&r_i = \sqrt{\rho^2 + (z - \alpha_i)^2}, \qquad (1)
\end{aligned}$$

where the parameters α_i, $i = 1,2,3,4$, as was already mentioned, occur as arbitrary real constants or complex conjugate pairs, and X_i are given by the formulas

$$X_1 = \varphi\frac{\epsilon_1\omega_1 - \varphi}{1 - \epsilon_1\omega_1\varphi}, \qquad X_2 = \varphi\frac{1 - \epsilon_1\omega_1\varphi}{\epsilon_1\omega_1 - \varphi}, \qquad X_3 = -\varphi\frac{1 + i\epsilon_4\omega_4\varphi}{i\epsilon_4\omega_4 - \varphi}, \qquad X_4 = \varphi\frac{i\epsilon_4\omega_4 - \varphi}{1 + i\epsilon_4\omega_4\varphi},$$
$$\omega_1 = \sqrt{\frac{(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4)}}, \qquad \omega_4 = \sqrt{\frac{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)}}, \tag{2}$$

the complex constant φ being subject to the constraint $|\varphi|^2 \equiv \varphi\bar{\varphi} = 1$ (a bar over a symbol means complex conjugation), while $\epsilon_1 = \pm 1$ and $\epsilon_4 = \pm 1$.

The potential $\mathcal{E}$ defined by (1) and (2) is an exact solution of the Ernst equation [25] obtained via Sibgatullin's method, and the entire metric associated with this potential has the form [10,26]

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2,$$
$$f = \frac{\Lambda\bar{\Lambda} - \Gamma\bar{\Gamma}}{(\Lambda - \Gamma)(\bar{\Lambda} - \bar{\Gamma})}, \qquad e^{2\gamma} = \frac{\Lambda\bar{\Lambda} - \Gamma\bar{\Gamma}}{\lambda_0\bar{\lambda}_0 r_1 r_2 r_3 r_4}, \qquad \omega = 2\mathrm{Im}(\sigma_0) - \frac{2\mathrm{Im}[(G(\bar{\Lambda} - \bar{\Gamma})]}{\Lambda\bar{\Lambda} - \Gamma\bar{\Gamma}},$$
$$G = z\Gamma + \sum_{1\le i<j\le 4}(\alpha_i + \alpha_j)\lambda_{ij} r_i r_j - \sum_{i=1}^{4}(\alpha_{i'} + \alpha_{j'} + \alpha_{k'})\gamma_i r_i,$$
$$\lambda_0 = \sum_{1\le i<j\le 4}\lambda_{ij}, \qquad \gamma_0 = \sum_{i=1}^{4}\gamma_i, \qquad \sigma_0 = \frac{1}{\lambda_0}\left[\gamma_0 + \sum_{1\le i<j\le 4}(\alpha_i + \alpha_j)\lambda_{ij}\right]. \tag{3}$$

Mention that the Weyl-Papapetrou cylindrical coordinates ρ and z enter into the potential $\mathcal{E}$ from (1) and into the metric coefficients f, γ, ω from (3) only through the functions r_i.

Formulas (1)–(3) represent a canonical form of the solution describing equilibrium configurations in the EDK spacetime. In order to rewrite them in physical parameters, we find it helpful first to express the parameters α_i in terms of the quantities ω_1 and ω_4. For this purpose we introduce two additional constants, z_0 and s, defined as

$$z_0 \equiv \frac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \qquad s \equiv \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4), \tag{4}$$

the constant z_0 permitting one to make an appropriate choice of the origin of coordinates on the symmetry axis, and s being the relative coordinate distance between the centers of the two constituents.

The sets of α's describing each type of the binary system in Fig. 1 are the following (the notation is obvious):

$$A_{BB} = \{\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4\},$$
$$A_{BS} = \{\alpha_1 > \alpha_2 > \mathrm{Re}(\alpha_3) = \mathrm{Re}(\alpha_4), \mathrm{Im}(\alpha_3) < 0, \alpha_4 = \bar{\alpha}_3\},$$
$$A_{SB} = \{\mathrm{Re}(\alpha_1) = \mathrm{Re}(\alpha_2) > \alpha_3 > \alpha_4, \mathrm{Im}(\alpha_1) < 0, \alpha_2 = \bar{\alpha}_1\},$$
$$A_{SS} = \{\mathrm{Re}(\alpha_1) = \mathrm{Re}(\alpha_2) > \mathrm{Re}(\alpha_3) = \mathrm{Re}(\alpha_4),$$
$$\mathrm{Im}(\alpha_1) < 0, \mathrm{Im}(\alpha_3) < 0, \alpha_2 = \bar{\alpha}_1, \alpha_4 = \bar{\alpha}_3\}. \tag{5}$$

The proposed change of parametrization is going to transform the above sets into the new ones, namely,

$$A_{BB} \to \Omega_{(0,0)},$$
$$A_{BS} \to \Omega_{(0,-1)} \cup \Omega_{(0,+1)},$$
$$A_{SB} \to \Omega_{(-1,0)} \cup \Omega_{(+1,0)},$$
$$A_{SS} \to \Omega_{(-1,-1)} \cup \Omega_{(+1,-1)} \cup \Omega_{(-1,+1)}, \tag{6}$$

where

$$\begin{aligned}
\Omega_{(0,0)} &= \{\omega_1 > 1, \omega_4 > 1\},\\
\Omega_{(0,-1)} &= \{\omega_1 > 1, \omega_4\bar{\omega}_4 = 1, \mathrm{Im}(\omega_4) < 0, \mathrm{Re}(\omega_4) > 0\},\\
\Omega_{(0,+1)} &= \{\omega_1 > 1, \omega_4\bar{\omega}_4 = 1, \mathrm{Im}(\omega_4) > 0, 1/\omega_1 > \mathrm{Re}(\omega_4) \geq 0\},\\
\Omega_{(-1,0)} &= \{\omega_4 > 1, \omega_1\bar{\omega}_1 = 1, \mathrm{Im}(\omega_1) < 0, \mathrm{Re}(\omega_1) > 0\},\\
\Omega_{(+1,0)} &= \{\omega_4 > 1, \omega_1\bar{\omega}_1 = 1, \mathrm{Im}(\omega_1) > 0, 1/\omega_4 > \mathrm{Re}(\omega_1) \geq 0\},\\
\Omega_{(-1,-1)} &= \{\omega_1\bar{\omega}_1 = \omega_4\bar{\omega}_4 = 1, \mathrm{Im}(\omega_1) < 0, \mathrm{Im}(\omega_4) < 0, \mathrm{Re}(\omega_1) > 0, \mathrm{Re}(\omega_4) > 0\},\\
\Omega_{(+1,-1)} &= \{\omega_1\bar{\omega}_1 = \omega_4\bar{\omega}_4 = 1, \mathrm{Im}(\omega_1) > 0, \mathrm{Im}(\omega_4) < 0, \mathrm{Re}(\omega_4) > \mathrm{Re}(\omega_1) \geq 0\},\\
\Omega_{(-1,+1)} &= \{\omega_1\bar{\omega}_1 = \omega_4\bar{\omega}_4 = 1, \mathrm{Im}(\omega_1) < 0, \mathrm{Im}(\omega_4) > 0, \mathrm{Re}(\omega_1) > \mathrm{Re}(\omega_4) \geq 0\},
\end{aligned} \tag{7}$$

and also

$$-\infty < z_0 < +\infty, \qquad s > 0 \tag{8}$$

for all Ω's. Note that the subindexes in Ω's have been designed in such a way that they provide one with the information about the presence of a black-hole constituent (0) and the sign of the imaginary part of ω_1 or ω_4.

The inverse parameter change, i.e. the one that maps Ω's into the original A's, can be described by means of the following bivalued relations ($\delta = \pm 1$):

$$\begin{aligned}
\alpha_1 &= z_0 + \frac{s}{2} + s\frac{\delta\omega_4(\omega_1^2 - 1)}{(\omega_1 + \delta\omega_4)(1 + \delta\omega_1\omega_4)},\\
\alpha_2 &= z_0 + \frac{s}{2} - s\frac{\delta\omega_4(\omega_1^2 - 1)}{(\omega_1 + \delta\omega_4)(1 + \delta\omega_1\omega_4)},\\
\alpha_3 &= z_0 - \frac{s}{2} + s\frac{\omega_1(\omega_4^2 - 1)}{(\omega_1 + \delta\omega_4)(1 + \delta\omega_1\omega_4)},\\
\alpha_4 &= z_0 - \frac{s}{2} - s\frac{\omega_1(\omega_4^2 - 1)}{(\omega_1 + \delta\omega_4)(1 + \delta\omega_1\omega_4)}.
\end{aligned} \tag{9}$$

It is of course understood that for each Ω one has to use only one of the two branches in the above formulas, and the criterion of choosing the appropriate branch is very simple: if one of the two subindexes of an Ω is equal to +1 (the imaginary part of any of the two Ω's is positive) then one has to use the branch $\delta = -1$, if not—then the branch $\delta = +1$.

We now turn to the consideration of the physical Komar quantities associated with the EDK solution.

III. KOMAR MASSES AND ANGULAR MOMENTA. THE SECOND SET OF INVERSION FORMULAS

Explicit analytical formulas for the physical masses and angular momenta of the balancing constituents in the EDK solution were obtained in the paper [13]. The Komar masses m_u and m_d (the subindexes "u" and "d" are abbreviations from "up" and "down," referring to the location of the upper and lower constituents on the symmetry axis) are given by the formulas

$$\begin{aligned}
m_u &= -s\frac{C(C_1 - C)}{CC_1 + SC_4 - 1 + \epsilon\delta CS},\\
m_d &= -s\frac{S(C_4 - S)}{CC_1 + SC_4 - 1 + \epsilon\delta CS},
\end{aligned} \tag{10}$$

while the Komar angular momenta j_u and j_d are defined by the expressions

$$\begin{aligned}
a_u &\equiv \frac{j_u}{m_u} = s\frac{\epsilon\delta C[(C - \epsilon\delta S)C_1 - 1 + \epsilon\delta CS]}{(C_1 + \epsilon\delta C_4)(CC_1 + SC_4 - 1 + \epsilon\delta CS)},\\
a_d &\equiv \frac{j_d}{m_d} = s\frac{S[(S - \epsilon\delta C)C_4 - 1 + \epsilon\delta CS]}{(C_1 + \epsilon\delta C_4)(CC_1 + SC_4 - 1 + \epsilon\delta CS)},
\end{aligned} \tag{11}$$

where the new constants C, S, C_1, C_4 and ϵ are introduced via the relations

$$\begin{aligned}
&\varphi \equiv C + iS, \qquad C_1 \equiv \frac{1}{2}\epsilon_1\left(\omega_1 + \frac{1}{\omega_1}\right),\\
&C_4 \equiv \frac{1}{2}\epsilon_4\left(\omega_4 + \frac{1}{\omega_4}\right), \qquad \epsilon \equiv \epsilon_1\epsilon_4.
\end{aligned} \tag{12}$$

The above Komar quantities (10) and (11) constitute a set of four parameters with a clear physical meaning. Then a question arises: Can these quantities be used for parametrizing the equilibrium solution? Remarkably, the answer is yes, and the best practical way to do this is by means of the following inversion formulas:

$$
\begin{aligned}
C_1 &= C - \epsilon\delta \frac{m_u}{M+s} S,\\
C_4 &= S - \epsilon\delta \frac{m_d}{M+s} C,\\
C &= \kappa \frac{M+s+\epsilon\delta a_u}{\sqrt{(M+s+\epsilon\delta a_u)^2 + (M+s+\epsilon\delta a_d)^2}},\\
S &= \kappa \frac{\epsilon\delta(M+s+\epsilon\delta a_d)}{\sqrt{(M+s+\epsilon\delta a_u)^2 + (M+s+\epsilon\delta a_d)^2}}, \qquad (13)
\end{aligned}
$$

where $\kappa = \pm 1$, while s satisfies the quadratic equation

$$
\begin{aligned}
&s^2 + [2M + \epsilon\delta(a_u + a_d)]s + M^2 + \epsilon\delta J = 0,\\
&M \equiv m_u + m_d, \qquad J \equiv j_u + j_d. \qquad (14)
\end{aligned}
$$

Note that Eq. (14), after rewriting it in the form

$$\epsilon\delta(M+s)^2 + s(a_u + a_d) + J = 0, \qquad (15)$$

can be immediately recognized as the equilibrium law for two arbitrary Kerr constituents originally derived in our paper [22].

Mention that the κ sign, to be congruent with all our previous conventions, has to be chosen in such a way that $C > 0$. It is also clear that, since $s > 0$, the admissible values of the masses and angular momenta are those that correspond to at least one positive s in Eq. (14).

Therefore, the set $(z_0, m_u, m_d, j_u, j_d)$ can be used for parametrizing the equilibrium class of the EDK solution. Apparently, the constant z_0 can be always fixed at some specific value, for instance if one wants to bring the origin of coordinates into the center of mass or into some other point related to a concrete binary configuration that might look attractive from the physical standpoint.

IV. PHYSICAL PARAMETRIZATION OF α_i AND X_i. THE METRIC FUNCTIONS

In order to rewrite the complex potential (1) and corresponding metric (3) in the physical parameters, it is necessary to find the reparametrized form of the quantities α_i and X_i. As it follows from (9), the constants α_i can be written in the form

$$
\begin{aligned}
\alpha_1 &= z_0 + \frac{s}{2} + \sigma_u, \qquad \alpha_2 = z_0 + \frac{s}{2} - \sigma_u,\\
\alpha_3 &= z_0 - \frac{s}{2} + \sigma_d, \qquad \alpha_4 = z_0 - \frac{s}{2} - \sigma_d, \qquad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_u &= s\frac{\delta\omega_4(\omega_1^2 - 1)}{(\omega_1 + \delta\omega_4)(1 + \delta\omega_1\omega_4)},\\
\sigma_d &= s\frac{\omega_1(\omega_4^2 - 1)}{(\omega_1 + \delta\omega_4)(1 + \delta\omega_1\omega_4)}. \qquad (17)
\end{aligned}
$$

The desired "physical" form of σ_u and σ_d is then obtainable with the aid of formulas (12)–(14), yielding after tedious but straightforward algebraic manipulations the following final expressions:

$$
\begin{aligned}
\sigma_u &= \sqrt{m_u^2 - a_u^2 + m_d a_u \frac{a_u(M + m_u + 2s) - 2m_u[a_d + \varepsilon(M+s)]}{(M+s)^2}},\\
\sigma_d &= \sqrt{m_d^2 - a_d^2 + m_u a_d \frac{a_d(M + m_d + 2s) - 2m_d[a_u + \varepsilon(M+s)]}{(M+s)^2}}, \qquad (18)
\end{aligned}
$$

where $\varepsilon \equiv \epsilon\delta$. The above σ_u and σ_d differ considerably from $\sigma = \sqrt{m^2 - a^2}$ of a single Kerr source [3] due to interaction of the constituents. It is worth mentioning that in the case of the real-valued α's, say α_1 and α_2, the corresponding $\sigma_u^2 > 0$; however, if $\alpha_2 = \bar{\alpha}_1$ then $\sigma_u^2 < 0$ and one must use the convention $\sigma_u = -i\sqrt{-\sigma_u^2}$ if one wants to pass to a positive definite radicand in (18). In Fig. 2 we have shown two reparametrized equilibrium configurations for which the origin of coordinates is chosen at the center of the lower constituent ($z_0 = s/2$).

In a similar manner, by using (12)–(14), it is possible to rewrite formulas (2) in terms of the Komar quantities; below we give the resulting reparametrized form of X_i:

$$
\begin{aligned}
X_1 &= \frac{(M+s+\varepsilon a_d)(M+s-i\varepsilon m_u) + i\varepsilon(M+s)\sigma_u}{(M+s+\varepsilon a_d)(M+s+i\varepsilon m_u) - i\varepsilon(M+s)\sigma_u},\\
X_2 &= \frac{(M+s+\varepsilon a_d)(M+s-i\varepsilon m_u) - i\varepsilon(M+s)\sigma_u}{(M+s+\varepsilon a_d)(M+s+i\varepsilon m_u) + i\varepsilon(M+s)\sigma_u},\\
X_3 &= -\frac{(M+s+\varepsilon a_u)(M+s+i\varepsilon m_d) + i\varepsilon(M+s)\sigma_d}{(M+s+\varepsilon a_u)(M+s-i\varepsilon m_d) - i\varepsilon(M+s)\sigma_d},\\
X_4 &= -\frac{(M+s+\varepsilon a_u)(M+s+i\varepsilon m_d) - i\varepsilon(M+s)\sigma_d}{(M+s+\varepsilon a_u)(M+s-i\varepsilon m_d) + i\varepsilon(M+s)\sigma_d}. \qquad (19)
\end{aligned}
$$

An alternative way of writing X_i which may be advantageous for some calculations is this:

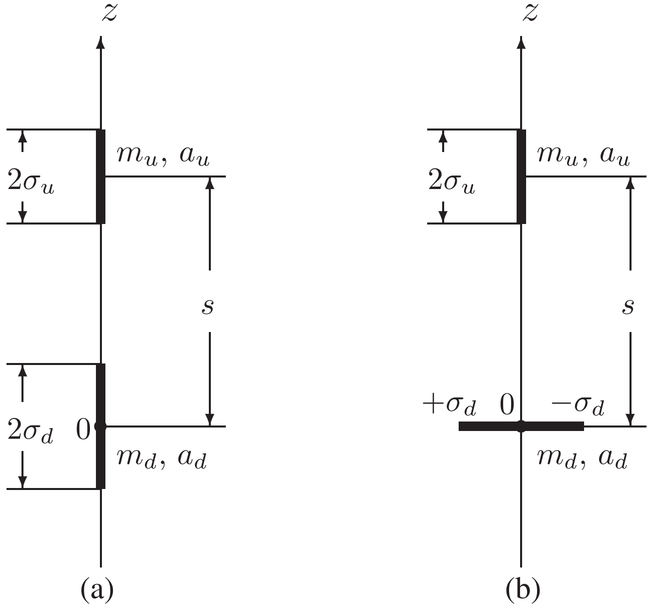

FIG. 2. Physical reparametrization of the equilibrium configurations (a) and (b) from Fig. 1.

$$\begin{aligned}
X_1 &= \frac{1}{m_u\Delta}[(M+s)(m_u+i\varepsilon\sigma_u)-\varepsilon(sa_u-m_ua_d)],\\
X_2 &= \frac{1}{m_u\Delta}[(M+s)(m_u-i\varepsilon\sigma_u)-\varepsilon(sa_u-m_ua_d)],\\
X_3 &= \frac{i}{m_d\Delta}[(M+s)(\varepsilon m_d+i\sigma_d)-sa_d+m_da_u)],\\
X_4 &= \frac{i}{m_d\Delta}[(M+s)(\varepsilon m_d-i\sigma_d)-sa_d+m_da_u)],\\
\Delta &\equiv -(M+s+\varepsilon a_u)+i[\varepsilon(M+s)+a_d]. \qquad (20)
\end{aligned}$$

Now we are able to write down the reparametrized complex potential (1) and (2); its new simple representation is the following:

$$\begin{aligned}
\mathcal{E} &= E_-/E_+,\\
E_\mp &= [s^2-(\sigma_u+\sigma_d)^2](X_1r_1-X_2r_2\mp 2\sigma_u)\\
&\quad\times(X_3r_3-X_4r_4\mp 2\sigma_d)\\
&\quad-4\sigma_u\sigma_d[X_2r_2-X_3r_3\mp(s-\sigma_u-\sigma_d)]\\
&\quad\times[X_1r_1-X_4r_4\mp(s+\sigma_u+\sigma_d)],\\
r_i &= \sqrt{\rho^2+(z-\alpha_i)^2},\\
\alpha_1 &= \frac{s}{2}+\sigma_u, \qquad \alpha_2=\frac{s}{2}-\sigma_u,\\
\alpha_3 &= -\frac{s}{2}+\sigma_d, \qquad \alpha_4=-\frac{s}{2}-\sigma_d, \qquad (21)
\end{aligned}$$

where σ_u, σ_d and X_i are determined by (18) and (19) or (20), and where we have set $z_0=0$ in the expressions of α's.

The above potential $\mathcal{E}$ can be also written in the form

$$\begin{aligned}
\mathcal{E} &= \frac{\Lambda+\Gamma}{\Lambda-\Gamma},\\
\Lambda &= [s^2-(\sigma_u+\sigma_d)^2](X_1r_1-X_2r_2)(X_3r_3-X_4r_4)\\
&\quad-4\sigma_u\sigma_d(X_2r_2-X_3r_3)(X_1r_1-X_4r_4),\\
\Gamma &= 2\sigma_d\{[(s+\sigma_u)^2-\sigma_d^2]X_2r_2-[(s-\sigma_u)^2-\sigma_d^2]X_1r_1\}\\
&\quad+2\sigma_u\{[(s-\sigma_d)^2-\sigma_u^2]X_4r_4\\
&\quad-[(s+\sigma_d)^2-\sigma_u^2]X_3r_3\}, \qquad (22)
\end{aligned}$$

and below we will use the functions Λ and Γ for presenting the reparametrized coefficients f, γ and ω in the metric (3):

$$\begin{aligned}
f &= \frac{\Lambda\bar\Lambda-\Gamma\bar\Gamma}{(\Lambda-\Gamma)(\bar\Lambda-\bar\Gamma)}, \qquad e^{2\gamma}=\frac{\Lambda\bar\Lambda-\Gamma\bar\Gamma}{K_0r_1r_2r_3r_4}, \qquad \omega=\omega_0-\frac{2\mathrm{Im}[(G(\bar\Lambda-\bar\Gamma)]}{\Lambda\bar\Lambda-\Gamma\bar\Gamma},\\
G &= z\Gamma+4s\sigma_u\sigma_d[(X_3r_3+\alpha_3)(X_4r_4+\alpha_4)-(X_1r_1+\alpha_1)(X_2r_2+\alpha_2)]\\
&\quad+(\sigma_u+\sigma_d)[s^2-(\sigma_u-\sigma_d)^2][(X_1r_1+\alpha_1)(X_3r_3+\alpha_3)-(X_2r_2+\alpha_2)(X_4r_4+\alpha_4)]\\
&\quad+(\sigma_u-\sigma_d)[s^2-(\sigma_u+\sigma_d)^2][(X_2r_2+\alpha_2)(X_3r_3+\alpha_3)-(X_1r_1+\alpha_1)(X_4r_4+\alpha_4)],\\
K_0 &= \frac{64}{m_1^2m_2^2}s^2|\sigma_u|^2|\sigma_d|^2(M+s)^2, \qquad \omega_0=-2\varepsilon(M+s). \qquad (23)
\end{aligned}$$

Therefore, we have obtained a physical representation for the general family of equilibrium configurations in the EDK solution. Its interesting particular case which we would like to mention in conclusion of this section is the DH configuration for two balancing identical corotating superextreme Kerr particles [9] possessing an additional symmetry with respect to the equatorial plane [27,28]. For this specific two-body system $m_u=m_d=m$, $a_u=a_d=a$, $\sigma_u=\sigma_d=\sigma$, and it is convenient to solve Eq. (14) for a, yielding ($\delta=+1$)

$$a = -\frac{\epsilon(s+2m)^2}{2(s+m)}, \tag{24}$$

which means that m and s are chosen as arbitrary parameters of the solution. Then we readily obtain for X_i the expressions

$$\begin{aligned}
X_1 &= \frac{s+(2-i\epsilon)m+\epsilon\mu}{s+(2+i\epsilon)m-\epsilon\mu},\\
X_2 &= \frac{i\epsilon[s+(2+i\epsilon)m-\epsilon\mu]}{s+(2-i\epsilon)m+\epsilon\mu},\\
X_3 &= \frac{i\epsilon[s+(2-i\epsilon)m+\epsilon\mu]}{s+(2+i\epsilon)m-\epsilon\mu},\\
X_4 &= -\frac{s+(2+i\epsilon)m-\epsilon\mu}{s+(2-i\epsilon)m+\epsilon\mu},
\end{aligned} \tag{25}$$

while σ becomes a pure imaginary quantity (since $m > 0$, $s > 0$) whose explicit form is the following:

$$\sigma = -\frac{is\mu}{2(s+m)}, \qquad \mu \equiv \sqrt{s^2+6ms+7m^2}. \tag{26}$$

For α_i and r_i in the equatorially symmetric case one has

$$\begin{aligned}
&\alpha_1 = -\alpha_4 = \frac{s}{2}+\sigma, &&\alpha_2 = -\alpha_3 = \frac{s}{2}-\sigma,\\
&r_1 = \sqrt{\rho^2+(z-\alpha_1)^2}, &&r_2 = \sqrt{\rho^2+(z-\alpha_2)^2},\\
&r_3 = \sqrt{\rho^2+(z+\alpha_2)^2}, &&r_4 = \sqrt{\rho^2+(z+\alpha_1)^2},
\end{aligned} \tag{27}$$

and the potential $\mathcal{E}$ of the DH equilibrium configuration, after the substitutions into formulas (22) and subsequent simplifications, finally takes the form

$$\begin{aligned}
\mathcal{E} &= \frac{\Lambda+\Gamma}{\Lambda-\Gamma},\\
\Lambda &= (s^2-4\sigma^2)(\mu_- r_2-\mu_+ r_1)(\mu_+ r_3-\mu_- r_4)\\
&\quad -4\sigma^2(\mu_- r_2 - i\epsilon\mu_+ r_3)(i\epsilon\mu_+ r_1+\mu_- r_4),\\
\Gamma &= 2ms\sigma[(1-i\epsilon)(s-2\sigma)(\mu_- r_4+i\epsilon\mu_+ r_1)\\
&\quad -(1+i\epsilon)(s+2\sigma)(\mu_- r_2 - i\epsilon\mu_+ r_3)],
\end{aligned} \tag{28}$$

whereas the corresponding metric functions f, γ and ω can be written as

$$\begin{aligned}
f &= \frac{\Lambda\bar\Lambda-\Gamma\bar\Gamma}{(\Lambda-\Gamma)(\bar\Lambda-\bar\Gamma)}, \qquad e^{2\gamma} = \frac{\Lambda\bar\Lambda-\Gamma\bar\Gamma}{K_0 r_1 r_2 r_3 r_4},\\
\omega &= \omega_0 - \frac{2\mathrm{Im}[(G(\bar\Lambda-\bar\Gamma)]}{\Lambda\bar\Lambda-\Gamma\bar\Gamma},\\
G &= z\Gamma + s\sigma\{2s(\mu_-^2 r_2 r_4 - \mu_+^2 r_1 r_3)\\
&\quad - 8i\epsilon m^2\sigma(r_1 r_2 + r_3 r_4) + (1-i\epsilon)m(s^2-4\sigma^2)\\
&\quad \times[\mu_+(r_3+i\epsilon r_1) - \mu_-(r_4+i\epsilon r_2)]\},\\
K_0 &= 256 s^2\sigma^4(s+2m)^2, \qquad \omega_0 = -2\epsilon(s+2m),\\
\mu_\pm &\equiv s+3m\pm\epsilon\mu.
\end{aligned} \tag{29}$$

To consider a particular DH configuration, one only needs to choose the values of m and s, and find from (24) the corresponding value of a at which the balance occurs. Formulas (26)–(29) will then describe the spacetime for that parameter choice.

V. DISCUSSION

Although the general formulas worked out in the previous section are applicable to all four types of the two-Kerr configurations from Fig. 1, the equilibrium states with $m_u > 0$, $m_d > 0$ are only possible for the systems (b), (c) and (d) containing at least one superextreme component. Various particular equilibrium configurations between a black hole and superextreme constituents, or between two unequal superextreme constituents were considered in the paper [10], and recently we have shown [23] that balance can be achieved even between a Schwarzschild black hole and a Kerr superextreme object. The absence of the equilibrium between two underextreme Kerr constituents with positive Komar masses [the systems (a) in Fig. 1] was strictly proved in our paper [13], and the nonexistence proof was later extended to the case of two extreme Kerr constituents [29], thus ruling out the two-black-hole equilibrium states in the EDK solution.

Remarkably, the expressions for the areas of the horizons calculated for the equilibrium configurations of type (a) with the aid of Tomimatsu's formulas [30]

$$\begin{aligned}
A_u &= 2\pi(\alpha_1-\alpha_2)\sqrt{-\omega_u e^{2\gamma_u}},\\
A_d &= 2\pi(\alpha_3-\alpha_4)\sqrt{-\omega_d e^{2\gamma_d}},
\end{aligned} \tag{30}$$

where ω_u, ω_d, γ_u, γ_d are constant values of the functions ω and γ on the respective horizons, are able to provide us with a simple demonstration that the individual Komar masses m_u and m_d cannot simultaneously take on positive values in such configurations. Taking into account that $\delta = +1$ in the (a)-type equilibrium states, one can arrive at the following final expressions for A_u and A_d:

$$A_u = -\frac{4\pi m_u[(s+m_d)(M+s+\epsilon a_d) - \sigma_u(M+s)]^2}{s(M+s)(M+s+\epsilon a_d)},$$
$$A_d = -\frac{4\pi m_d[(s+m_u)(M+s+\epsilon a_u) - \sigma_d(M+s)]^2}{s(M+s)(M+s+\epsilon a_u)}, \tag{31}$$

whence it follows immediately that in order for the masses of the black-hole constituents and areas of the horizons to take positive values simultaneously, the following two conditions must be satisfied:

$$M+s+\epsilon a_d < 0, \qquad M+s+\epsilon a_u < 0. \tag{32}$$

However, after rewriting the equilibrium condition (14) in the form ($\delta = +1$)

$$s(M+s) - (m_u+s)(M+s+\epsilon a_u) - (m_d+s)(M+s+\epsilon a_d) = 0, \tag{33}$$

we see that, under the suppositions made, the inequalities (32) convert the left-hand side of (33) into a strictly positive quantity, which signifies the absence of equilibrium configurations of two Kerr black holes. Note, however, that in the systems (b) and (c) the black-hole component has the horizon area defined by one of the expressions (31), with ϵ substituted by $\epsilon\delta$, so that the balance condition (33) may have physically meaningful solutions because in such systems only one of the inequalities (32) has to be satisfied.

It would certainly be of interest to briefly discuss a direct mathematical interrelation existing between the equilibrium configurations of the EDK solution and the analogous configurations of the double-Reissner-Nordström (DRN) solution [24,31]. While the former configurations are defined by the condition (15), the latter equilibrium states of two electrically charged Reissner-Nordström sources [32,33] are defined by the balance condition

$$m_u m_d - \left(q_u + \frac{m_u q_d - m_d q_u}{m_u+m_d+s}\right) \times \left(q_d + \frac{m_d q_u - m_u q_d}{m_u+m_d+s}\right) = 0 \tag{34}$$

(the reader is referred to [24,31,34] for the details of its derivation), where m_u and m_d are Komar masses of the upper and lower constituents, q_u and q_d are the corresponding charges, while s is the relative coordinate distance. The connection between Eqs. (15) and (34) is described by the following two theorems.

Theorem I.—If m_u, m_d, a_u, a_d, s is an equilibrium configuration of the EDK solution, then the substitution

$$a_u = \frac{\epsilon\delta q_u(m_u q_d - m_d q_u)}{m_u m_d - q_u q_d},$$
$$a_d = \frac{\epsilon\delta q_d(m_d q_u - m_u q_d)}{m_u m_d - q_u q_d},$$
$$m_d q_u - m_u q_d \neq 0, \tag{35}$$

into Eq. (15) defines an equilibrium configuration of the DRN solution.

Theorem II.—Given an equilibrium configuration of the DRN solution, m_u, m_d, q_u, q_d, s, the substitution

$$q_u^2 = -\frac{m_u m_d a_u^2}{\Delta_0}, \qquad q_d^2 = -\frac{m_u m_d a_d^2}{\Delta_0},$$
$$q_u q_d = \frac{m_u m_d a_u a_d}{\Delta_0}, \tag{36}$$

with $\Delta_0 \equiv a_u a_d + \epsilon\delta(m_u a_d + m_d a_u) \neq 0$, $\Delta_0 m_u m_d < 0$, converts Eq. (34) into condition (15).

The proof of these theorems is straightforward and consists in the substitution of (35) and (36) into Eqs. (15) and (34), respectively.

VI. CONCLUSION

We hope that the physical representation of the general family of equilibrium configurations of two Kerr sources obtained in the present paper will make this family more accessible for concrete applications and will simplify the analysis of particular cases which exhibit interesting physical properties. Although two Kerr black holes cannot be in the gravitational equilibrium, this fact does not diminish the importance of the EDK solution because there are other physically meaningful equilibrium configurations it offers—those between a black hole and a superextreme source, and between two superextreme Kerr constituents, both types of the configurations permitting their components to have exclusively positive Komar masses. It is probably worth remarking that for many years the superextreme solutions had been largely underestimated compared to the black-hole ones in spite of the theoretical evidence that they may arise from the gravitational collapse [35,36], or are able to open new horizons for the gravitational experiment (an important prediction made four decades ago by Penrose [37]). In relation with the latter aspect we would like to emphasize that the discovery of the physically relevant equilibrium states between a black hole and a superextreme Kerr constituents (for particular examples we refer the reader to [10]) is highly important from the physical point of view, mainly because the balance in such two-body systems might signify that the spin-spin repulsive force is actually by far stronger than was thought in the 1980s when only the equilibrium configurations composed of two superextreme objects were found, and in our opinion this could have relevance to the experimental detection of the spin-spin interaction. It also appears that the recent paper of Jacobson and Sotiriou [38] on destroying black holes with test bodies establishes an interesting physical bridge between the two types of exact solutions, and we expect that the binary equilibrium configurations described by the EDK solution will be able to shed additional light on the physical interaction of black holes and superextreme sources.

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