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Dark matter from spacetime nonlocality

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We propose that dark matter is not yet another new particle in nature, but that it is a remnant of quantum gravitational effects on known fields. We arrive at this possibility in an indirect and surprising manner: by considering retarded, nonlocal, and Lorentzian evolution for quantum fields. This is inspired by recent developments in causal set theory, where such an evolution shows up as the continuum limit of scalar field propagation on a background causal set. Concretely, we study the quantum theory of a massless scalar field whose evolution is given not by the the d'Alembertian $\Box$, but by an operator $\widetilde{\Box}$ which is Lorentz invariant, reduces to $\Box$ at low energies, and defines an explicitly retarded evolution: $(\widetilde{\Box}\phi)(x)$ only depends on $\phi(y)$, where y is in the causal past of x. This modification results in the existence of a continuum of massive particles, in addition to the usual massless ones, in the free theory. When interactions are introduced, these massive or off-shell quanta can be produced by the scattering of massless particles, but once produced, they no longer interact, which makes them a natural candidate for dark matter.

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I. INTRODUCTION

The nature of dark matter is one of the most important problems in modern physics. Almost a century after it was hypothesized, though, our understanding of it is still limited to its gravitational signature on luminous matter. It is often assumed that dark matter is a new weakly interacting particle which is just hard to detect. However, so far there has been no conclusive direct or indirect detection in accelerators or cosmological/astrophysical settings. In what follows, we propose that dark matter is not yet another new particle in nature, but that it is a remnant of quantum gravitational effects on known fields. We arrive at this possibility in an indirect and surprising manner: by considering retarded, nonlocal, and Lorentzian evolution for quantum fields. Concretely, we study the consequences of replacing the d'Alembertian $\Box$ with an operator $\widetilde{\Box}$ which is Lorentz invariant, reduces to $\Box$ at low energies, and defines a retarded evolution: $(\widetilde{\Box}\phi)(x)$ only depends on $\phi(y)$, where y is in the causal past of x. Why is this type of evolution interesting, what does it have to do with quantum gravity, and how does it lead to a proposal for the nature of dark matter?

The causal set theory approach to quantum gravity postulates that the fundamental structure of spacetime is that of a locally finite and partially ordered set [1]. Its marriage of discreteness with causal order implies that physics cannot remain local at all scales. This nonlocality manifests itself concretely, for instance, when one seeks to describe the wave propagation of a scalar field on a causal set. It has been shown in this case that coarse graining the quantum gravitational degrees of freedom leads to a nonlocal field theory described by an operator exactly of the type $\widetilde{\Box}$ [2–6]. There are reasons to suspect that this type of nonlocality is not necessarily confined to the Planck scale, and that it may have nontrivial implications for physics at energy scales accessible by current experiments (see Refs. [7,8] and references therein for implications of nonlocality in the context of cosmology). It is then only natural to wonder what a quantum field theory built upon $\widetilde{\Box}$ would look like, especially that it may contain information about the fundamental structure of spacetime.

Studying $\widetilde{\Box}$ is also interesting from a purely field-theoretic perspective, since it forces us to relax one of the core assumptions of quantum field theory: locality. Most nonlocal and Lorentzian quantum field theories studied in the literature consider modifications of the type $\Box \to f(\Box)$. In this paper, we consider explicitly retarded operators, which are more generic and have more interesting properties as a result. For instance, the Fourier transform of $\widetilde{\Box}$ is generically complex, which is a direct consequence of retarded evolution. In fact, this feature is at the heart of our proposal for the nature of dark matter. It is also worth mentioning that quantizing a field theory of the type described here is nontrivial due to the absence of a local action principle. This presents a technical challenge, from which one may gain deeper insight into quantization schemes.

What is the relation between a quantum field theory based on $\widetilde{\Box}$ and dark matter? Upon quantizing a free massless scalar field $\phi(x)$ with the classical equation of motion $\widetilde{\Box}\phi(x) = 0$, we find *off-shell modes* in the mode expansion of the quantized field operator $\hat{\phi}(x)$. These are modes which do not satisfy any dispersion relation, unlike

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in usual local quantum field theory (LQFT) where every Fourier mode with 4-momentum p is an on-shell quanta, i.e. it satisfies $p \cdot p = 0$.[1] This is equivalent to the statement that the quantized field operator does not generically satisfy the classical equation of motion: $\tilde{\Box}\hat{\phi}(x) \neq 0$. Note that an off-shell mode of a massless scalar field has an effective mass, and can be thought of as a massive quanta in itself. We show that the off-shell modes can exist in "in" and "out" states of scattering, and are different from virtual particles which exist as intermediate states in Feynman diagrams. When considering the interacting theory, we find an extremely surprising result: the cross section of any scattering process which contains one or more off-shell particle(s)[2] in the "in" state is zero. That is to say, *on-shell quanta can scatter and produce off-shell particles, but once produced, off-shell particles no longer interact*. It is this behavior that makes these off-shell particles a natural candidate for dark matter. The phenomenological story would be that dark matter particles were produced in the early Universe in this fashion: as off-shell modes of quantum fields. This feature of the theory can be traced back to the fact that $\tilde{\Box}$ defines an explicitly retarded evolution, which as mentioned previously, may be a remnant of quantum gravitational degrees of freedom.

Our paper is organized as follows. In Sec. II, we start by setting forth a series of axioms which any nonlocal, retarded, and Lorentzian modification of $\Box$ at high energies should satisfy. In Sec. III, we argue there is no action principle for the theory of interest, which forces us to carefully study, in Sec. IV, what quantization scheme should be used. There, we argue that canonical quantization and the Feynman path-integral approach do not work, and explain why the Schwinger-Keldysh (also known as the double path integral or in-in) formalism provides the appropriate framework. Sections V and VI describe the interacting theory, where we work out the modified Feynman rules, find S-matrix amplitudes, and compute cross sections for various examples and comment on the time reversibility of the theory. Although a *continuum superposition* of off-shell particles can in principle scatter into on-shell modes, we argue why this is unlikely to happen. The extension to massive scalar fields is discussed in Sec. VII. Section VIII concludes the paper.

II. MODIFIED D'ALEMBERTIAN: DEFINITION

In this section we study generic spectral properties of nonlocal and Lorentzian modifications of the d'Alembertian $\Box$. We focus on a class of operators $\tilde{\Box}$ which defines an explicitly retarded evolution: $(\tilde{\Box}\phi)(x)$ depends only on $\phi(y)$ with y in the causal past of x. As we will see, such operators have interesting features which are absent in modifications of the type $f(\Box)$. We start by setting forth a series of axioms which a nonlocal, retarded, and Lorentzian modification of $\Box$ at high energies should satisfy.

(1) *Linearity*:

$$\tilde{\Box}(a\phi + b\psi) = a\tilde{\Box}\phi + b\tilde{\Box}\psi, \qquad a, b \in \mathbb{C}, \quad (1)$$

where ϕ and ψ are complex scalar fields and $\mathbb{C}$ denotes the set of complex numbers.

(2) *Reality*: For any real scalar field ϕ, $\tilde{\Box}\phi$ is also real. Note that reality and linearity imply for any complex scalar field ϕ that

$$(\tilde{\Box}\phi^*) = (\tilde{\Box}\phi)^*, \quad (2)$$

where $*$ denotes complex conjugation.

(3) *Poincaré invariance*: Evolution defined by $\tilde{\Box}$ is Poincaré invariant. Consider a scalar field $\phi(x)$ which transforms to $\phi'(x) = \phi(\Lambda^{-1}x)$ under a Poincaré transformation $x \to \Lambda x$. We require $\tilde{\Box}$ to be invariant under the action of Λ:

$$(\tilde{\Box}\phi')(x) = (\tilde{\Box}\phi)(\Lambda^{-1}x). \quad (3)$$

Taking Λ to be a spacetime translation $\Lambda(x) = x + a$, one finds that the eigenfunctions of $\tilde{\Box}$ are plane waves. To see this, let $\phi(x) = e^{ip\cdot x}$ and define $\psi(x) \equiv (\tilde{\Box}\phi)(x)$. It then follows from Eq. (3) that

$$e^{-ip\cdot a}\psi(x) = \psi(x - a), \quad (4)$$

where we have used the linearity condition. Solutions to the above equation are plane waves:

$$\psi(x) = \tilde{\Box}e^{ip\cdot x} = B(p)e^{ip\cdot x}, \quad (5)$$

where $B(p)$ is any function of the wave vector p. Therefore, it follows from translational invariance that $e^{ip\cdot x}$ is an eigenfunction of $\tilde{\Box}$ with the corresponding eigenvalue $B(p)$. Taking Λ to be a Lorentz transformation, it can be shown that $B(p)$ can only depend on the the Lorentzian norm of p, i.e. $p \cdot p \equiv \eta_{\mu\nu}p^\mu p^\nu$, and whether or not p is future or past directed, i.e. $\mathrm{sgn}(p^0)$:

$$B(p) = B(\mathrm{sgn}(p^0), p \cdot p). \quad (6)$$

Combining Eqs. (5) and (2) we find $B(-p) = B^*(p)$, which using Eq. (6) is equivalent to

$$B(-\mathrm{sgn}(p^0), p \cdot p) = B(\mathrm{sgn}(p^0), p \cdot p)^*. \quad (7)$$

For a spacelike wave vector p^μ, it is always possible to find a coordinate system in which $p^0 = 0$. As a

[1] We use a signature of $-+++$ for the Minkowski metric $\eta_{\mu\nu}$. Also, $p_1 \cdot p_2 \equiv \eta_{\mu\nu}p_1^\mu p_2^\nu$.

[2] In the quantum theory, an off-shell particle is a one-particle quantum state with a well-defined (nonzero) mass and momentum, i.e. a massive eigenstate of the Hamiltonian and momentum operator.

result, $B(p)$ is real for spacelike p. For timelike momenta, however, $B(p)$ may be complex and its imaginary part changes sign when $p^0 \to -p^0$.

Most nonlocal modifications of $\Box$ considered in the literature are of the form $f(\Box)$, in which case $B(p)$ is only a function of $p \cdot p$. In this paper we focus on a class of nonlocal operators for which $B(p)$ does depend on $\mathrm{sgn}(p^0)$, and find many interesting consequences as a result.

(4) *Locality at low energies*: Since $\Box$ provides a good description of nature at low energies, we require $\tilde{\Box} \to \Box$ in this regime. In other words, expanding $B(\mathrm{sgn}(p^0), p \cdot p)$ for "small" values of $p \cdot p$, we require the leading-order behavior to be that of $\Box$:

$$B(p) \overset{p\cdot p \to 0}{\longrightarrow} -p \cdot p. \tag{8}$$

Note that by a "small" value of $p \cdot p$, we mean in comparison to a scale which can be interpreted as the nonlocality scale, implicitly defined through $\tilde{\Box}$.

(5) *Stability*: We require that evolution defined by $\tilde{\Box}$ is stable. This condition implies that $B(p)$, when analytically continued to the complex plane of p, only has a zero at $p \cdot p = 0$ [3].

(6) *Retardedness*: $(\tilde{\Box}\phi)(x)$ only depends on $\phi(y)$, where y is in the causal past of x.

Let us briefly consider a class of operators which satisfy all the aforementioned axioms. We shall let Λ denote the nonlocality energy scale and define

$$\Lambda^{-2}(\tilde{\Box}\phi)(x) = a\phi(x) + \Lambda^4 \int_{J^-(x)} f(\Lambda^2 \tau_{xy}^2)\phi(y) d^4y, \tag{9}$$

where a is a dimensionless real number, $J^-(x)$ denotes the causal past of x, and τ_{xy} is the Lorentzian distance between x and y:

$$\tau_{xy}^2 = (x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2. \tag{10}$$

Examples of such operators have arisen in the causal set theory program [2–6]. This operator is clearly linear, real, Poincaré invariant and retarded. It is shown in Appendix A that there are choices of a and f for which $\tilde{\Box}$ is also stable and has the desired infrared behavior (8). One such choice is

$$f(s) = \frac{4}{\pi}\delta(s - \epsilon) - \frac{e^{-s/2}}{4\pi}(24 - 12s + s^2),$$
$$a = -2, \tag{11}$$

where ϵ is an infinitesimally small positive number.

The eigenvalues $B(p)$ of $\tilde{\Box}$ take the form (see Ref. [3])

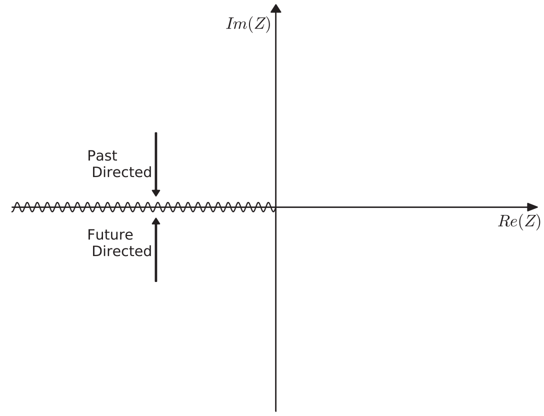

FIG. 1. Analytic structure of $B(p)$ in the complex plane of $Z = p \cdot p/\Lambda^2$.

$$\Lambda^{-2}B(p) = \lim_{\epsilon \to 0^+} g((p + ip_\epsilon) \cdot (p + ip_\epsilon)/\Lambda^2), \tag{12}$$

$$g(Z) = a + 4\pi Z^{-\frac{1}{2}} \int_0^\infty f(s^2)s^2 K_1(Z^{1/2}s)ds, \tag{13}$$

where p_ϵ is an infinitesimally small ($p_\epsilon \cdot p_\epsilon = -\epsilon^2$), timelike, and future-directed ($p_\epsilon^0 > 0$) wave vector. The analytic structure of $B(p)$ is shown in Fig. 1. Figure 2 shows the behavior of $B(p)$ as a function of $p \cdot p$ and $\mathrm{sgn}(p^0)$ for the choice of f and a given in Eq. (11).

III. CLASSICAL THEORY

How would such nonlocal and retarded evolution manifest itself? To get a start on answering this question, we modify the evolution of a massless scalar field ϕ coupled to a source $J(x)$ via $\Box \to \tilde{\Box}$:

$$\Box\phi(x) = J(x) \to \tilde{\Box}\phi(x) = J(x). \tag{14}$$

It is worth noting that the solutions of $\tilde{\Box}\phi(x) = 0$ are identical to those of $\Box\phi(x) = 0$. This follows from requiring a stable evolution for $\tilde{\Box}$ (see Ref. [3]). As we will see in Sec. III B, however, the story changes when $J(x) \neq 0$.

A. Absence of an action principle

It is natural to ask whether an action principle exists for ϕ, whose variation would produce the nonlocal equation of motion $\tilde{\Box}\phi(x) = J(x)$. One might propose to substitute $\Box$ with $\tilde{\Box}$ in the action of a massless scalar field:

$$S[\phi] = \int d^4x \left(\frac{1}{2}\phi(x)\tilde{\Box}\phi(x) - J(x)\phi(x)\right). \tag{15}$$

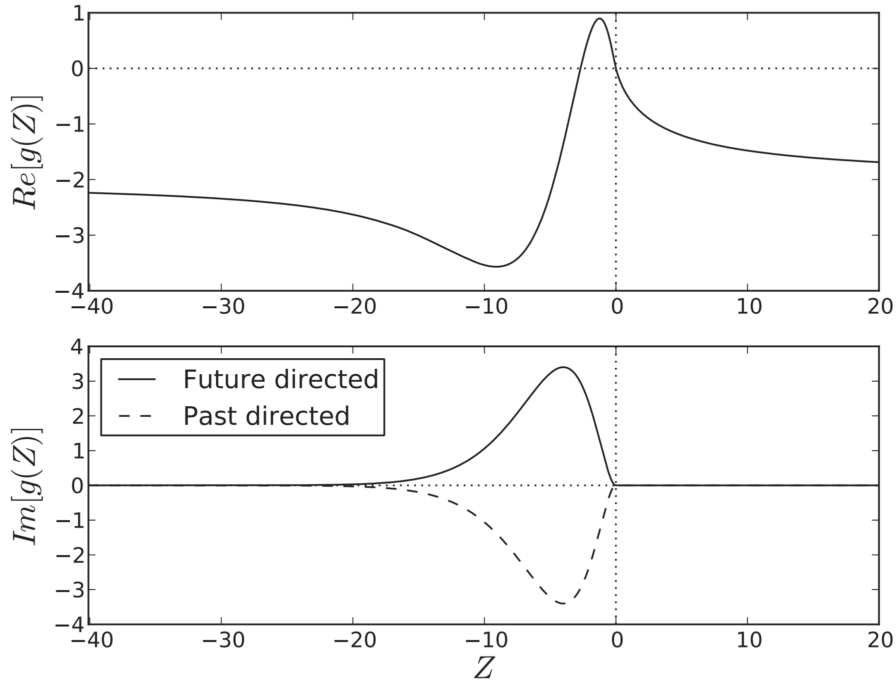

FIG. 2. The Fourier transform $B(p) = g(p \cdot p/\Lambda^2)$ of $\widetilde{\Box}$ defined in Eq. (9), where a and f are given by Eq. (11).

Requiring $S[\phi]$ to be stationary with respect to first-order variations in ϕ we find[3]

$$\frac{1}{2}(\widetilde{\Box} + \widetilde{\Box}^T)\phi(x) = J(x), \tag{19}$$

where $\widetilde{\Box}^T$ is defined in Fourier space via

$$\widetilde{\Box}^T e^{ip\cdot x} = B(p)^* e^{ip\cdot x}. \tag{20}$$

In the case of the retarded operator (9), for instance, $\widetilde{\Box}^T\phi(x)$ is the right-hand side of Eq. (9) with the domain of integration changed to the causal *future* of point x. Therefore, Eq. (15) does not lead to a retarded equation of motion.

Due to the absence of a local Lagrangian description, quantizing a massless scalar field theory built upon $\widetilde{\Box}$ is nontrivial. We shall address this problem in Sec. IV, where we argue that the Schwinger-Keldysh quantization scheme can still be used to obtain the desired nonlocal quantum field theory.

B. Green's function

The Green's functions of $\Box$ and $\widetilde{\Box}$ are quite different, especially in the ultraviolet where their spectra differ. One important difference is that $\widetilde{\Box}$, unlike $\Box$, has a unique inverse. Since $\widetilde{\Box}$ is a retarded operator by definition, it only has a retarded Green's function. Recall that $\Box$ has both a retarded $G^R(x,y)$ and advanced $G^A(x,y)$ Green's function

$$\Box_x G^{R,A}(x,y) = \delta^{(4)}(x-y), \tag{21}$$

which satisfy the following "boundary conditions": $G^R(x,y)$ vanishes unless $x \succ y$ (x is in the causal future of y), and $G^A(x,y)$ vanishes unless $y \succ x$. The two Green's functions are related to one another via $G^A(x,y) = G^R(y,x)$. In the case of $\widetilde{\Box}$, the Green's function is unique (just the retarded one) and switching the arguments of the retarded Green's function *does not* produce another Green's function. Let us show why this is.

Let $\widetilde{G}(x,y)$ denote the Green's function associated with $\widetilde{\Box}$:

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[3]To see this, it is instructive to express the action in Fourier space. Define the Fourier transform $f(p)$ of $f(x)$ via

$$f(x) = \int \frac{d^4p}{(2\pi)^4} f(p) e^{ip\cdot x}. \tag{16}$$

Then, it can be shown that

$$S = \int \frac{d^4p}{(2\pi)^4}\left[\phi(p)^* \frac{1}{4}(B(p) + B(p)^*)\phi(p) - \phi(p)^* J(p)\right]. \tag{17}$$

Requiring S to be stationary with respect to first-order variations $\phi(p)$ we find

$$\frac{1}{2}(B(p) + B(p)^*)\phi(p) = J(p). \tag{18}$$

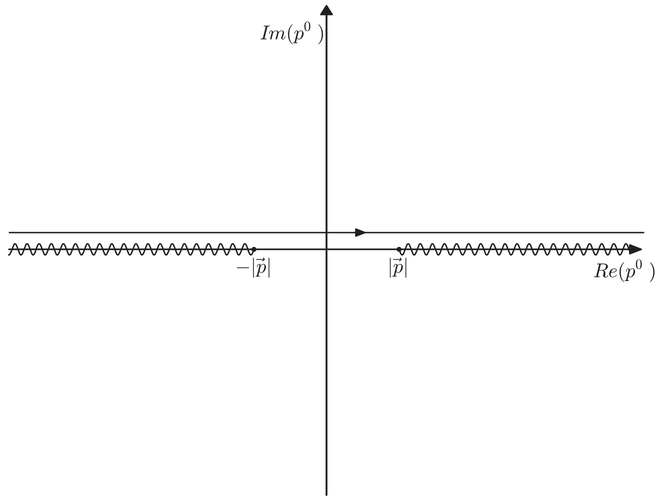

FIG. 3. The integration path in the complex p^0 plane which defines the retarded Green's function associated with $\widetilde{\Box}$.

$$\widetilde{\Box}_x \widetilde{G}(x,y) = \delta^{(4)}(x-y). \tag{22}$$

Note that $\widetilde{G}(x,y)$ can be expressed as

$$\widetilde{G}(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{B(p)} e^{ip\cdot(x-y)}. \tag{23}$$

The path of integration in the complex p^0 plane is shown in Fig. 3. This comes from the fact that $\widetilde{\Box}$ is a retarded operator, so $B(p)$ analytically continued to the complex p^0 plane takes its value above the cut. When $B(p)$ has no zeros in the complex plane apart from at $p\cdot p = 0$, which is guaranteed by the stability requirement, this choice of contour ensures that $\widetilde{G}(x,y) \equiv \widetilde{G}^R(x,y)$ is indeed retarded. Switching the arguments of $\widetilde{G}^R(x,y)$, we find

$$\widetilde{G}^R(y,x) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{B(p)} e^{ip\cdot(y-x)} \tag{24}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{B(-p)} e^{ip\cdot(x-y)} \tag{25}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{B(p)^*} e^{ip\cdot(x-y)}, \tag{26}$$

where in the second line we have changed integration variables from p to $-p$. Then

$$\widetilde{\Box}_x \widetilde{G}^R(y,x) = \int \frac{d^4p}{(2\pi)^4} \frac{B(p)}{B(p)^*} e^{ip\cdot(x-y)} \neq \delta^{(4)}(x-y), \tag{27}$$

since $B(p)$ is generically complex. As we will see in the sections to come, the fact that $\widetilde{\Box}$ has a unique inverse plays a crucial role in the quantum theory of $\widetilde{\Box}$.

IV. QUANTUM THEORY

We wish to construct a quantum theory of a massless scalar field ϕ whose classical limit reproduces the retarded evolution induced by $\widetilde{\Box}$. The quantization scheme which we believe is most suited in this case is the Schwinger-Keldysh (or double path integral) formalism. In what follows, we will first review the usual paths to quantization (i.e. canonical quantization and the Feynman path integral) and show why they fail in the case of a nonlocal and retarded operator like $\widetilde{\Box}$. The goal of these discussions is to make clear why we choose the Schwinger-Keldysh formalism to construct a quantum field theory based on $\widetilde{\Box}$.

A. Canonical quantization

Let us consider the canonical quantization of a free massless scalar field ϕ. The typical route to quantization is as follows: start from an action principal for ϕ, derive the Hamiltonian in terms of ϕ and its conjugate momentum, impose equal-time commutation relations, and finally specify the dynamics via the Heisenberg equation. There is an equivalent approach, however, which defines the theory with no reference to an action principle, using the Klein-Gordon equation supplemented by the so-called Peierls form of the commutation relations:

$$\Box\hat{\phi}(x) = 0, \tag{28}$$

$$[\hat{\phi}(x), \hat{\phi}(y)] = i\Delta(x,y), \tag{29}$$

where $\Delta(x,y)$ is the Pauli-Jordan function

$$\begin{aligned}\Delta(x,y) &= G^R(x,y) - G^A(x,y)\\ &= G^R(x,y) - G^R(y,x).\end{aligned} \tag{30}$$

It is well known that Eq. (29) is entirely equivalent to, but more explicitly covariant than, the more commonly seen equal-time commutation relations (see e.g. Sec. C.2 of Ref. [9]). Since $\Delta(x,y)$ is the difference of two Green's functions, it satisfies the equation of motion:

$$\Box_x \Delta(x,y) = 0. \tag{31}$$

This is why Eqs. (28) and (29) are consistent with one another: both the left- and right-hand sides of Eq. (29) vanish when $\Box_x$ is applied.

It is tempting to build the quantum theory of $\widetilde{\Box}$ in a similar fashion:

$$\widetilde{\Box}\hat{\phi}(x) = 0, \tag{32}$$

$$[\hat{\phi}(x), \hat{\phi}(y)] = i\widetilde{\Delta}(x,y) \equiv i(\widetilde{G}^R(x,y) - \widetilde{G}^R(y,x)). \tag{33}$$

In this case, however, $\widetilde{\Delta}(x,y)$ does not satisfy the equation of motion $[\widetilde{\Box}_x \widetilde{\Delta}(x,y) \neq 0]$ because $\widetilde{G}^R(y,x)$ is not a

Green's function of $\widetilde{\Box}$ [see Sec. III and Eq. (27)]. Therefore, the equation of motion (32) is not consistent with the commutation relations (33).

It is worth noting that the root of this inconsistency is that the Fourier transform $B(p)$ of $\widetilde{\Box}$ is complex, which in turn follows from the fact that $\widetilde{\Box}$ is retarded by definition. In Sec. IV C we will arrive at a consistent quantum theory via the Schwinger-Keldysh formalism, using which we also build a Hilbert-space representation of the theory. There we will see that the equation of motion (32) is given up in favor of the commutation relations (33). As it turns out, the degree to which Eq. (32) is violated depends on the imaginary part of $B(p)$.

B. Feynman path integral

The Feynman path integral formalism requires a local Lagrangian description for the scalar field ϕ. As was argued in Sec. III A, however, this is not viable if one requires a retarded equation of motion. Therefore, the Feynman path integral formalism is also not suitable for quantizing this theory.

C. Schwinger-Keldysh formalism

The Schwinger-Keldysh formalism has a natural way of incorporating a retarded operator. In this approach an amplitude [called the decoherence functional $\mathcal{D}(\phi^+,\phi^-)$] is assigned to a pair of paths (ϕ^+,ϕ^-), which are constrained to meet at the final time $[\phi^+(t_f,\mathbf{x}) = \phi^-(t_f,\mathbf{x})]$. The decoherence functional for a free massless scalar field takes the form

$$\mathcal{D}(\phi^+,\phi^-) = \mathrm{Exp}\left[i\int d^4x \frac{1}{2}\phi^q\Box^R\phi^{cl} + \frac{1}{2}\phi^{cl}\Box^A\phi^q + \frac{1}{2}\phi^q\Box^K\phi^q\right], \tag{34}$$

where

$$\phi^{cl} \equiv \frac{1}{\sqrt{2}}(\phi^+ + \phi^-), \tag{35}$$

$$\phi^{q} \equiv \frac{1}{\sqrt{2}}(\phi^+ - \phi^-). \tag{36}$$

In Eq. (34), $\Box^R$ is the retarded d'Alembertian, $\Box^A = (\Box^R)^\dagger$ is the advanced d'Alembertian, and $\Box^K$ is an anti-Hermitian operator which contains information about the initial wave function [10].[4] Any source term $J(x)$ can be included by adding $-J\phi^+ + J\phi^- = -\sqrt{2}J\phi^q$ to the integrand.

Any n-point function in this theory is given by

$$\langle\phi^{(\alpha_1)}(x_1)\ldots\phi^{(\alpha_n)}(x_n)\rangle = \int D\phi^+ D\phi^- \phi^{(\alpha_1)}(x_1)\ldots\phi^{(\alpha_n)}(x_n)\mathcal{D}(\phi^+,\phi^-), \tag{37}$$

where $\alpha_i \in \{+,-,q,cl\}$. These correlation functions are related to the correlation functions in Hilbert-space representation by the following rule:

$$\langle\phi^+(x_1)\ldots\phi^+(x_n)\phi^-(y_1)\ldots\phi^-(y_m)\rangle = \langle 0|\widetilde{T}[\hat{\phi}(y_1)\ldots\hat{\phi}(y_m)]T[\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)]|0\rangle \tag{38}$$

where $T(\widetilde{T})$ is the (anti-)time-ordered operator, and $|0\rangle$ is the vacuum state of the free theory.

In order to come up with a quantum theory for a nonlocal retarded operator, we replace $\Box^R$ with $\widetilde{\Box}$ in Eq. (34) (and $\Box^K$ with $\widetilde{\Box}^K$[5]).

1. Classical limit

Before going any further, let us take a look at the classical limit of this theory. Performing Gaussian integrals (in the presence of a source term), we get

$$\langle\phi^{cl}(x)\rangle = \frac{1}{\sqrt{2}}\int d^4y \widetilde{G}^R(x,y)J(y), \tag{39}$$

$$\langle\phi^q(x)\rangle = 0, \tag{40}$$

resulting in

$$\langle\phi^+(x)\rangle = \langle\phi^-(x)\rangle = \int d^4y \widetilde{G}^R(x,y)J(y). \tag{41}$$

It shows that in the classical limit where the field is represented by its expectation value, there is no difference between ϕ^+ and ϕ^- and both satisfy the retarded equation of motion $\widetilde{\Box}\phi = J$.

2. Green's functions

Let us consider the two-point correlation functions of this theory in the absence of any source:

$$-i\langle\phi^{cl}(x)\phi^q(y)\rangle = \widetilde{G}^R(x,y), \tag{42}$$

$$-i\langle\phi^{q}(x)\phi^{cl}(y)\rangle \equiv \widetilde{G}^A(x,y) = \widetilde{G}^R(y,x), \tag{43}$$

$$-i\langle\phi^{cl}(x)\phi^{cl}(y)\rangle \equiv \widetilde{G}^K(x,y) = -\int d^4z d^4w \widetilde{G}^R(x,z)\widetilde{B}^K(z,w)\widetilde{G}^A(w,y), \tag{44}$$

[4] The retarded and advanced d'Alembertians are defined via $G^{R,A}(\Box^{R,A}f) = f$ for all suitable test functions f, where $G^{R,A}$ are the integral operators associated with the retarded and advanced Green's functions $G^{R,A}(x,y)$.

[5] We still need to determine $\widetilde{\Box}^K$. This has been done in Sec. IV C 3.

$$-i\langle\phi^q(x)\phi^q(y)\rangle = 0 \tag{45}$$

where $\widetilde{B}^K(x,y)$ is the kernel of $\widetilde{\Box}^K$.[6] Using the definition of ϕ^q and ϕ^{cl}, we get

$$-i\langle\phi^+(x)\phi^+(y)\rangle = \frac{1}{2}[\widetilde{G}^K(x,y) + \widetilde{G}^R(x,y) + \widetilde{G}^A(x,y)], \tag{46}$$

$$-i\langle\phi^-(x)\phi^-(y)\rangle = \frac{1}{2}[\widetilde{G}^K(x,y) - \widetilde{G}^R(x,y) - \widetilde{G}^A(x,y)], \tag{47}$$

$$-i\langle\phi^-(x)\phi^+(y)\rangle = \frac{1}{2}[\widetilde{G}^K(x,y) + \widetilde{G}^R(x,y) - \widetilde{G}^A(x,y)]. \tag{48}$$

Note that if this theory has an equivalent representation in terms of field operator in a Hilbert space, then the abovementioned terms correspond to the time-ordered two-point function, anti-time-ordered two-point function and two-point function respectively [see Eq. (38)].

We require that the theory describes a free scalar field in flat spacetime at its ground state. As a result, all n-point correlation functions of this theory must be translation invariant,

$$\langle\phi^{(\alpha_1)}(x_1)\ldots\phi^{(\alpha_n)}(x_n)\rangle = \langle\phi^{(\alpha_1)}(x_1+y)\ldots\phi^{(\alpha_n)}(x_n+y)\rangle. \tag{49}$$

This condition requires that all operators $\widetilde{\Box}$, $\widetilde{\Box}^\dagger$ and $\widetilde{\Box}^K$ must be translation invariant. Consequently, we get

$$\widetilde{\Box}^K e^{ip\cdot x} = \widetilde{B}^K(p)e^{ip\cdot x}, \tag{50}$$

$$\widetilde{G}^K(x,y) = -\int\frac{d^4p}{(2\pi)^4}\widetilde{G}^R(p)\widetilde{B}^K(p)\widetilde{G}^A(p)e^{ip\cdot(x-y)}. \tag{51}$$

Note that $\widetilde{\Box}^K$ is an anti-Hermitian operator. It means $\widetilde{B}^K(p)$ is a totally imaginary number [and $\widetilde{G}^K(p) \equiv -\widetilde{G}^R(p)\widetilde{B}^K(p)\widetilde{G}^A(p)$ is also totally imaginary since $\widetilde{G}^R(p)\widetilde{G}^A(p)$ is real].

3. Fixing $\widetilde{G}^K$

From here on, we assume that there is a Hilbert-space representation of this theory with a Hamiltonian evolution. We will justify this assumption later by finding the representation itself. In Appendix B we show that this assumption leads to the following relation, when the quantum system is in its ground state:

$$\widetilde{G}^K(p) = \text{sgn}(p^0)[\widetilde{G}^R(p) - \widetilde{G}^A(p)]. \tag{52}$$

Note that Eq. (52) is nothing but the fluctuation dissipation theorem (FDT) at zero temperature. This fixes the eigenvalues of $\widetilde{\Box}^K$ as follows:

$$\widetilde{B}^K(p) = 2i\text{Im}B(p)\text{sgn}(p^0). \tag{53}$$

4. Hilbert-space representation

We wish to find an equivalent Hilbert-space representation in terms of a field operator $\hat{\phi}(x)$ for this theory. As we mentioned earlier, Eq. (48) is the two-point function of such a representation,

$$W(x,y) \equiv \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \langle\phi^-(x)\phi^+(y)\rangle, \tag{54}$$

where $|0\rangle$ is the ground state. If we use Eqs. (48) and (52), we arrive at

$$W(x,y) = \int\frac{d^4p}{(2\pi)^4}\frac{2\text{Im}[B(p)]\theta(p^0)}{|B(p)|^2}e^{ip\cdot(x-y)}, \tag{55}$$

where we call $\widetilde{W}(p) \equiv \frac{2\text{Im}[B(p)]\theta(p^0)}{|B(p)|^2}$. Since $W(x,y)$ is a positive operator, $\text{Im}[B(p)]\theta(p^0)$ must be a non-negative number. So, we further *assume*

$$\text{sgn}(\text{Im}[B(p)]) = \text{sgn}(p^0). \tag{56}$$

Once this condition is satisfied, the field operator $\hat{\phi}(x)$ and ground state $|0\rangle$, defined to be

$$\hat{\phi}(x) = \int\frac{d^4p}{(2\pi)^2}\sqrt{\widetilde{W}(p)}(\hat{a}_p e^{ip\cdot x} + \hat{a}_p^\dagger e^{-ip\cdot x}), \tag{57}$$

$$[\hat{a}_p, \hat{a}_q] = \delta^{(4)}(p-q), \tag{58}$$

$$\hat{a}_p|0\rangle = 0 \quad \forall\ p, \tag{59}$$

yield the desired correlation functions.

Note that a_p is only defined for timelike future-directed p, because otherwise $\widetilde{W}(p)$ is zero in the field expansion. It means that all timelike future-directed (positive-energy) momenta contribute to the field expansion (57).

5. Hamiltonian

By definition, the time evolution operator is the operator that evolves $\hat{\phi}(x)$ in time,

$$\hat{\phi}(t,\mathbf{x}) = \hat{U}(t,t_0)\hat{\phi}(t_0,\mathbf{x})\hat{U}^\dagger(t,t_0). \tag{60}$$

It can be directly checked that

$$\hat{U}(t,t_0) = e^{-i\hat{H}_0(t-t_0)}, \tag{61}$$

[6] If $\delta_y(x) \equiv \delta^{(4)}(x-y)$, then $\tilde{B}^K(x,y) \equiv (\widetilde{\Box}^K\delta_y)(x)$. With this definition, $(\widetilde{\Box}^K\phi)(x) = \int d^4y\tilde{B}^K(x,y)\phi(y)$.

$$\hat{H}_0 = \int d^4p\, p^0 \hat{a}_p^\dagger \hat{a}_p, \tag{62}$$

gives the right time evolution.

State $|0\rangle$ defined in Eq. (59) is the *ground state* of this Hamiltonian. Excited states (n-particle states) can be built by acting $a^\dagger$'s on $|0\rangle$,

$$|p_1 \ldots p_n\rangle = \hat{a}_{p_1}^\dagger \ldots \hat{a}_{p_n}^\dagger |0\rangle. \tag{63}$$

The excited state $|p\rangle$ represents a particle with energy p^0 and momentum $\mathbf{p}$[7] where p^0 is independent of $\mathbf{p}$.[8] This shows that the theory contains a continuum of massive particles with positive energy. The existence of a continuum of massive particles in the context of causal set theory also has been pointed out in Ref. [11], although their result is rather different in some other aspects.

6. Comparison to local evolution

At this point, it would be illustrative to consider the result of this formalism for LQFT. In this case

$$B(p) = B_{\text{local}}(p) = (p^0 + i\epsilon)^2 - |\mathbf{p}|^2, \tag{64}$$

where ϵ is a small positive number taken to zero at the end of the calculation. The two-point function is given by

$$\widetilde{W}(p) = 2\frac{\epsilon p^0}{(p^2)^2 + (\epsilon p^0)^2}\theta(p^0) = 2\pi\delta(p^2)\theta(p^0). \tag{65}$$

As a result,

$$W(x,y) = \int \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2)\theta(p^0) e^{ip\cdot(x-y)}, \tag{66}$$

$$\hat{\phi}(x) = \int \frac{d^4p}{(2\pi)^2}\sqrt{2\pi\delta(p^2)\theta(p^0)}(\hat{a}_p e^{ip\cdot x} + \hat{a}_p^\dagger e^{-ip\cdot x}). \tag{67}$$

The two-point function and field expansion are exactly the ones we expected. Only on-shell particles ($p\cdot p = 0$) contribute to the field expansion.

Here, we see one important difference between local and retarded nonlocal evolution. In the local case, only on-shell modes ($p\cdot p = 0$) contribute to the field expansion. As a result, excited states of the theory consist of all *on-shell* particles. In the nonlocal retarded case (where generically $\text{Im}[B(p)] \neq 0$), off-shell modes ($p\cdot p \neq 0$) also contribute to the field expansion. Consequently, one expects the existence of off-shell modes in "in" and "out" states of scatterings in the interacting theory.

Let us investigate properties of $\widetilde{W}(p)$ for a generic nonlocal retarded operator. First of all, it is only nonzero for timelike future-directed momenta. This means that only timelike future-directed momenta contribute to the field expansion and can exist in "in" and "out" states (particles with timelike momentum and positive energy).

Considering that $B(p)$ is only zero at $p\cdot p = 0$, $\widetilde{W}(p)$ is a finite number for all $p\cdot p \neq 0$ (we will see the significance of this result in Sec. VI B). On the other hand, since in the subspace of on-shell modes the $\widetilde{\Box}$ operator is exactly the same as $\Box$, we conclude that $\widetilde{W}(p) = 2\pi\delta(p^2)\theta(p^0)$ for $p\cdot p = 0$. Therefore, $\widetilde{W}(p)$ consists of a divergent part at $p\cdot p = 0$ and a finite part for $p\cdot p \neq 0$. This means that there are two different contributions to the field expansion (57), one from on-shell modes that is the same as Eq. (67) and one from off-shell modes which only exists in the case of nonlocal retarded evolution

$$\hat{\phi}(x) = \int \frac{d^4p}{(2\pi)^2}\sqrt{2\pi\delta(p^2)\theta(p^0)}(\hat{a}_p e^{ip\cdot x} + \hat{a}_p^\dagger e^{-ip\cdot x}) + \int_{p^2\neq 0} \frac{d^4p}{(2\pi)^2}\sqrt{\widetilde{W}(p)}(\hat{a}_p e^{ip\cdot x} + \hat{a}_p^\dagger e^{-ip\cdot x}). \tag{68}$$

D. Sorkin-Johnston quantization

The Sorkin-Johnston (SJ) proposal defines a unique vacuum state for a free massive scalar field in an arbitrarily curved spacetime [12]. This proposal is a continuum generalization of Johnston's formulation of a free quantum scalar field theory on a background causal set [13]. As is the case for $\widetilde{\Box}$, canonical quantization does not admit an obvious generalization for a causal set. The SJ quantization scheme uses only the retarded Green's function $G_R(x,y)$ to arrive at the quantum theory. Since $\widetilde{\Box}$ also admits a retarded Green's function, one can apply the SJ prescription to arrive at a free quantum field theory of the massless scalar field we have been considering. In what follows, we will show that the SJ proposal applied to $\widetilde{\Box}$ produces the same free quantum theory as the Schwinger-Keldysh formalism, provided condition (56) is met.

Consider the corresponding integral operator of the kernel $i\Delta(x,y) = G_R(x,y) - G_R(y,x)$:

$$(i\Delta f)(x) = \int i\Delta(x,y) f(y) d^4y. \tag{69}$$

It can be shown that $i\Delta$ is Hermitian, which implies it has real eigenvalues, and that its nonzero eigenvalues come in positive and negative pairs:

$$(i\Delta T_{\mathbf{p}})(x) = \lambda_{\mathbf{p}}^2 T_{\mathbf{p}}(x) \to (i\Delta T_{\mathbf{p}}^*)(x) = -\lambda_{\mathbf{p}}^2 T_{\mathbf{p}}^*(x). \tag{70}$$

[7]The momentum operator $\hat{\mathbf{P}} \equiv \int d^4p\, \mathbf{p}\hat{a}_p^\dagger \hat{a}_p$ is the generator of spacial translation.

[8]Note that these states are different from the usual states $|\mathbf{p}\rangle$ used in LQFT which describe a particle with momentum $\mathbf{p}$ and energy $|\mathbf{p}|$.

We have assumed here that the eigenfunctions $T_{\mathbf{p}}$ form an orthonormal basis of L^2, which can always be achieved since $i\Delta$ is Hermitian. The Sorkin-Johnston proposal is then to define the two-point function to be the positive part of $i\Delta(x,y)$ in the following sense:

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \sum_{\mathbf{p}} \lambda_{\mathbf{p}}^2 T_{\mathbf{p}}(x) T_{\mathbf{p}}^*(y). \tag{71}$$

Taking $G_R(x,y)$ to be the retarded Green's function of $\widetilde{\Box}$ [see Eqs. (23) and (26)], we find

$$i\Delta e^{ip\cdot x} = \frac{2\mathrm{Im}(B(p))}{|B(p)|^2} e^{ip\cdot x}, \tag{72}$$

which using the SJ formalism then leads to the two-point function

$$\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{2\mathrm{Im}(B(p))}{|B(p)|^2} \theta(\mathrm{Im}(B(p))) e^{ip\cdot x}. \tag{73}$$

If condition (56) is satisfied, this two-point function is that derived from the Schwinger-Keldysh formalism [see Eqs. (55) and (56)]. It is reassuring that two different paths to quantization, at least at the free level, lead to the same theory.

V. INTERACTING FIELD THEORY

Let us now consider the interacting theory. We introduce the interaction in the Hilbert-space representation by adding a potential term to the free Hamiltonian as follows:

$$\hat{H}(t) = \hat{H}_0 + \int d^3\mathbf{x} V(\hat{\phi}(t,\mathbf{x})). \tag{74}$$

Starting with a general initial wave function, one is able to find the final state of the system by solving the Heisenberg equation of motion in principle. However, in practice this is a very hard task to do. So, we try to find the S-matrix amplitudes perturbatively.

In order to do so, we can use the available machinery of LQFT, and move to the interaction picture. Time evolution in the interaction picture is given by

$$\hat{U}_I = Te^{-i\int d^4x V(\hat{\phi}_I)} \tag{75}$$

where $\hat{\phi}_I$ is the field in the interaction picture given by Eq. (57). The perturbative expansion of $\hat{U}_I$ yields S-matrix amplitudes. Performing the calculations to find the S-matrix, we come up with modified Feynman rules for this theory. We explain these modifications in the following two examples.

A. Example 1: 2-2 scattering $p_1p_2 \to q_1q_2$ in $\frac{\lambda}{4!}\phi^4$ theory

The scattering amplitude $S_{q_1q_2,p_1p_2}$ is given by

$$S_{q_1q_2,p_1p_2} = \langle q_1q_2|Te^{-i\int d^4x \frac{\lambda}{4!}\hat{\phi}_I^4}|p_1p_2\rangle. \tag{76}$$

To first order in λ, it yields

$$\begin{aligned} S_{q_1q_2,p_1p_2} &= -i\frac{\lambda}{4!}\int d^4x \langle q_1q_2|\hat{\phi}_I^4(x)|p_1p_2\rangle \\ &= \frac{-i\lambda}{(2\pi)^4}\sqrt{\widetilde{W}(p_1)\widetilde{W}(p_2)\widetilde{W}(q_1)\widetilde{W}(q_2)}\delta^{(4)} \\ &\quad \times\left(\sum p - \sum q\right), \end{aligned} \tag{77}$$

where we have substituted for $\hat{\phi}_I$ from Eq. (57). It is interesting to note that Eq. (77) is time-reversal invariant.

In the transition from local to retarded nonlocal propagation, here we see the first change in the scattering amplitudes. The values assigned to each external line have changed from $\sqrt{2\pi\delta(p^2)\theta(p^0)}$ to $\sqrt{\widetilde{W}(p)}$. Note that here the scattering amplitude is computed in the basis of the 4-momentum $|p\rangle$ which is different from the 3-momentum basis $|\mathbf{p}\rangle$ of LQFT.

B. Example 2: 2-2 scattering $p_1p_2 \to q_1q_2$ in $\frac{\lambda}{3!}\phi^3$ theory

In this case, $S_{q_1q_2,p_1p_2}$ is given by

$$S_{q_1q_2,p_1p_2} = \langle q_1q_2|Te^{-i\int d^4x \frac{\lambda}{3!}\hat{\phi}_I^3}|p_1p_2\rangle. \tag{78}$$

To second order in λ, it yields

$$\begin{aligned} S_{q_1q_2,p_1p_2} &= \frac{1}{2}\left(\frac{-i\lambda}{3!}\right)^2 \int d^4x d^4y \langle q_1q_2|T\hat{\phi}_I^3(x)\hat{\phi}_I^3(y)|p_1p_2\rangle \\ &= \frac{-i\lambda^2}{(2\pi)^8}\sqrt{\widetilde{W}(p_1)\widetilde{W}(p_2)\widetilde{W}(q_1)\widetilde{W}(q_2)}\delta^{(4)} \\ &\quad \times\left(\sum p - \sum q\right) \\ &\quad \times[\widetilde{G}^F(p_1+p_2)+\widetilde{G}^F(p_1-q_1)+\widetilde{G}^F(p_1-q_2)]. \end{aligned}$$

$\widetilde{G}^F(p) = \frac{\theta(p^0)}{B(p)} + \frac{\theta(-p^0)}{B^*(p)}$ is the time-ordered two-point function (46) in Fourier space. In the transition from the local to the nonlocal operator, here we see another change in the scattering amplitude. The values assigned to each internal line have changed to the new value for the Feynman propagator $\widetilde{G}_F(p)$.

From these examples, it is obvious how scattering amplitudes can be computed in this theory. For any Feynman diagram only the values assigned to external lines and internal lines have changed. Note that the amplitude of some diagrams in LQFT is zero, as a result

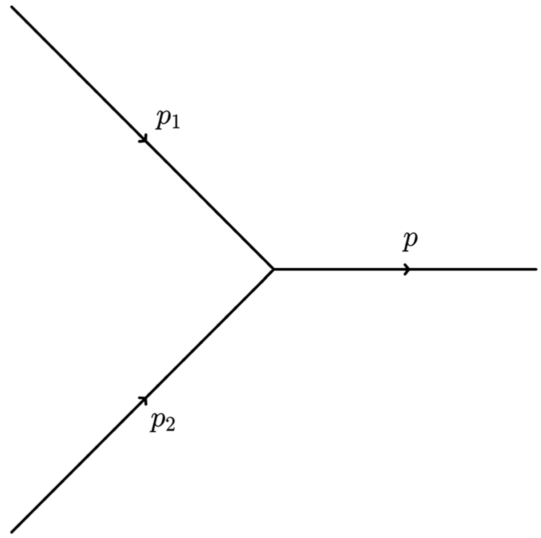

FIG. 4. The amplitude of this diagram in LQFT is zero, because of the energy-momentum conservation; two massless particles cannot produce a massless particle. However, in our theory there is a continuum of massive particles and the amplitude of this scattering is generically nonzero.

of energy-momentum conservation, while in this theory they are not. For example in LQFT $\lambda\phi^3$ theory, the amplitude assigned to Fig. 4 is zero, because the sum of two (nonparallel) null vectors cannot be a null vector. However, in this theory there is a continuum of massive particles, and for example two on-shell particles can interact and produce one off-shell particle.

VI. FROM SCATTERING AMPLITUDE TO TRANSITION RATE

At this point, we want to find the rate of a process using the S-matrix amplitudes. In Sec. VI B we have shown that if one (or more) of the incoming particles is off shell, then the differential transition rate of such scattering is zero. It means that in order to have a nonzero transition rate (and cross section), all of the incoming particles must be on shell. This is the most distinctive property of off-shell particles: the cross section of any scattering with off-shell particles is zero.

For now consider the scattering from state $|\alpha\rangle = |p_1\ldots p_{N_\alpha}\rangle$ to $|\beta\rangle = |q_1\ldots q_{N_\beta}\rangle$ where all the incoming particles are on shell, $p_i^2 = 0$. Assuming that the interactions happen inside a box with volume V (see Ref. [14]), the differential transition rate is given by

$$d\Gamma = 2\pi^{N_\alpha+1}\left[\frac{(2\pi)^3}{V}\right]^{N_\alpha-1}\frac{1}{E_{\mathbf{p_1}}\ldots E_{\mathbf{p_{N_\alpha}}}}\delta^{(4)}\left(\sum p_i - \sum q_i\right) \times |\widetilde{M}_{\beta\alpha}|^2 d^4q_1\ldots d^4q_{N_\beta}, \tag{79}$$

where $E_{\mathbf{p_i}} = |\mathbf{p_i}|$ and

$$S_{\beta\alpha} = -2\pi i\delta^{(4)}\left(\sum p_i - \sum q_i\right)\sqrt{\widetilde{W}(p_1)\ldots\widetilde{W}(p_{N_\alpha})}\widetilde{M}_{\beta\alpha}. \tag{80}$$

In the case of 2-2 scattering, the differential cross section is given by

$$d\sigma = \frac{d\Gamma}{\frac{u}{V}} = \frac{\pi^2(2\pi)^4}{E_{\mathbf{p_1}}E_{\mathbf{p_2}}u}\delta^{(4)}\left(\sum p_i - \sum q_i\right)|\widetilde{M}_{\beta\alpha}|^2 d^4q_1 d^4q_2, \tag{81}$$

where

$$u = \frac{\sqrt{(p_1.p_2)^2 - p_1^2p_2^2}}{p_1^0p_2^0} \tag{82}$$

is the speed of particle 1 in the frame of reference of particle 2 (and vice versa) and $\frac{u}{V}$ is the flux of incoming particles.

A. $p_1p_2 \to q_1q_2$ cross section in $\frac{\lambda}{4!}\phi^4$

As an example, we will find the cross section of $p_1p_2 \to q_1q_2$ where $p_i^2 = 0$. Using Eq. (77) and the definition (80), to first order in λ

$$\widetilde{M} = \frac{\lambda}{(2\pi)^5}\sqrt{\widetilde{W}(q_1)\widetilde{W}(q_2)}. \tag{83}$$

As a result, the cross section is given by

$$d\sigma = \frac{\lambda^2}{4(2\pi)^4|p_1\cdot p_2|}\widetilde{W}(q_1)\widetilde{W}(q_2)\delta^{(4)} \times (p_1 + p_2 - q_1 - q_2)d^4q_1d^4q_2. \tag{84}$$

Let us constrain the outgoing particles to be only on shell $q_i^2 = 0$. In this case $\widetilde{W}$ functions in Eq. (84) pick up a delta function and one can check that Eq. (84) for outgoing on-shell particles results in the usual cross section of $\lambda\phi^4$ in LQFT. However, if we constrain (at least) one of the outgoing particles to be off shell with a *fixed* mass, the cross section becomes zero. The cross section over outgoing off-shell particles is only nonzero when the integration over the continuum mass is also performed. We see the significance of this in the next section when considering the scattering of off-shell particles. Due to the contribution of off-shell states, the total cross section (84) is increased compared to the local theory.

B. Off-shell particles and cross section

In order to calculate the cross section of any scattering involving incoming off-shell particles, we make use of the fact that off-shell particles can be thought as a continuum of massive particles.

This can be done by expressing the two-point function as a sum over massive two-point functions:

$$W(x,y) = \int_0^\infty d\mu^2 \rho(\mu^2) \times \int \frac{d^4p}{(2\pi)^4} 2\pi\theta(p^0)\delta(p^2+\mu^2)e^{ip\cdot(x-y)}, \tag{85}$$

where $\rho(-p^2) = \frac{\widetilde{W}(p)}{2\pi}$ for $p^0 > 0$. Note from Eq. (68) that $\rho(\mu^2) = \delta(\mu^2) + \widetilde{\rho}(\mu^2)$ where $\widetilde{\rho}$ is a finite function. In other words,

$$W(x,y) = \int \frac{d^4p}{(2\pi)^4} 2\pi\theta(p^0)\delta(p^2)e^{ip\cdot(x-y)} + \int_0^\infty d\mu^2 \widetilde{\rho}(\mu^2) \int \frac{d^4p}{(2\pi)^4} 2\pi\theta(p^0)\delta \times (p^2+\mu^2)e^{ip\cdot(x-y)}. \tag{86}$$

In order to make everything more similar to LQFT, we discretize the mass parameter to get

$$W(x,y) = \int \frac{d^4p}{(2\pi)^4} 2\pi\theta(p^0)\delta(p^2)e^{ip\cdot(x-y)} + \sum_{j=1}^{\infty} \Delta\mu^2 \widetilde{\rho}(\mu_j^2) \times \int \frac{d^4p}{(2\pi)^4} 2\pi\theta(p^0)\delta(p^2+\mu_j^2)e^{ip\cdot(x-y)}, \tag{87}$$

where $\mu_j^2 = j\Delta\mu^2$. Equation (87) is the same as Eq. (86) in the limit $\Delta\mu^2 \to 0$.

The following field operator will yield the above two-point function:

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2|\mathbf{p}|}} (\hat{a}_{\mathbf{p},0} e^{ip\cdot x} + \text{c.c.})|_{p^0=|\mathbf{p}|} + \sum_{j=1}^{\infty} \sqrt{\Delta\mu^2 \widetilde{\rho}(\mu_j^2)} \times \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2E_{\mathbf{p},\mu_j}}} (\hat{a}_{\mathbf{p},\mu_j} e^{ip\cdot x} + \text{c.c.})$$

where

$$E_{\mathbf{p},\mu} = \sqrt{\mathbf{p}^2 + \mu^2}, \tag{88}$$

$$[\hat{a}_{\mathbf{p},\mu_i}, \hat{a}^\dagger_{\mathbf{q},\mu_j}] = \delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{\mu_i,\mu_j}, \tag{89}$$

$$\hat{a}_{\mathbf{p},\mu}|0\rangle = 0 \tag{90}$$

and the state $|\mathbf{p},\mu\rangle \equiv \hat{a}^\dagger_{\mathbf{p},\mu}|0\rangle$ is a one-particle state with momentum $\mathbf{p}$, mass μ and energy $E_{\mathbf{p},\mu}$.

From now on, consider a concrete example of 2-2 scattering with $\frac{\lambda}{4!}\hat{\phi}^4$ interaction and incoming particles with definite mass and momentum. The idea behind this proof can be generalized to more complicated examples. Up to first order in λ

$$\langle \mathbf{p_1}, m_1; \mathbf{p_2}, m_2|\hat{S}|\mathbf{q_1}, \mu_1; \mathbf{q_2}, \mu_2\rangle = -\frac{i\lambda}{(2\pi)^2}\delta^{(4)}\left(\sum p - \sum q\right) \times \sqrt{\prod_{i=1}^{2} \frac{(\Delta\mu^2)^2 \rho(\mu_i^2)\rho(m_i^2)}{4E_{\mathbf{q_i},\mu_i}E_{\mathbf{p_i},m_i}}}. \tag{91}$$

In Eq. (91), if any of the particles were on shell (say $\mu_1 = 0$), we should set $\Delta\mu^2\rho(\mu_1^2) = 1$; otherwise ρ is replaced by $\widetilde{\rho}$.

The differential cross section is given by

$$d\sigma = (2\pi)^{-2}\lambda^2 \frac{(\Delta\mu^2)^4 \rho(\mu_1^2)\rho(\mu_2^2)\rho(m_1^2)\rho(m_2^2)}{16uE_{\mathbf{p_1},m_1}E_{\mathbf{p_2},m_2}E_{\mathbf{q_1},\mu_1}E_{\mathbf{q_2},\mu_2}} \times \delta^{(4)}(p_1+p_2-q_1-q_2)d^3\mathbf{p_1}d^3\mathbf{p_2}. \tag{92}$$

In order to get the total cross section, we should also sum over the mass parameter in the phase space of outgoing particles. In the (mass) continuum limit this means

$$\sum \Delta\mu^2 \rho(m_i^2) \to \int dm_i^2 \rho(m_i^2) \tag{93}$$

which absorbs two factors of $\Delta\mu^2$ in Eq. (92); however, there are two remaining factors of $\Delta\mu^2$. If the incoming particles (even one of them) are off shell, since $\rho(\mu^2)$ is a finite number, in the limit $\Delta\mu^2 \to 0$, the cross section becomes zero. This means that the (total) transition rate of scattering with off-shell particles with *fixed* mass is zero. The cross section is only nonzero when both of the incoming particles are on shell.

This is, in fact, consistent with what we have found in the previous section. There, we have shown that the transition rate of on-shell → off-shell is nonzero, only when the integration over mass of the off-shell particles is performed. In fact, the scattering transition rate of on-shell particles to off-shell particles with fixed masses is zero. Since the theory is time-reversal invariant, this suggests that the scattering transition rate of off-shell particles with fixed masses must be zero too, consistent with what we have found here.

This also means that an initial state with a suitable continuum superposition of off-shell masses can scatter into on-shell modes (time reverse of the process of on-shell scattering into off-shell). However, as we argue in the next section, these states are fine-tuned and generally we do not expect to find the system in these superpositions.

C. Off-shell→on-shell scattering: continued

In the previous section, we showed that the transition rate of scattering with off-shell particle(s) is zero. However, a suitable continuum superposition of off-shell particles can scatter nontrivially. In this section, we want to explain this point to a greater extent and argue that it is unlikely to find the system in these superpositions. We will not go through the details of the calculations since they are not essential to our argument in this section.

We make use of the following toy model theory that mimics many properties of the proposed nonlocal theory:

$$\mathcal{L} = \frac{1}{2}\psi_0 \Box \psi_0 + \sum_{i=1}^{N} \frac{1}{2}\psi_{m_i}(\Box - m_i^2)\psi_{m_i} - \lambda\psi^4,$$
$$\psi \equiv \psi_0 + \sum_{i=1}^{N} \frac{g_i}{\sqrt{N}}\psi_{m_i}. \tag{94}$$

This is a theory of one massless scalar field (playing the role of on-shell modes) in addition to N massive scalar fields (playing the role of off-shell modes) and we are interested in the $N \to \infty$ limit of the theory (λ and g_i's are coupling constants and do not scale with N). The advantage of working with this theory is that while its behavior is very similar to the nonlocal theory, Eq. (94) is a local quantum field theory and possibly more comprehensible to the reader. The interaction term in Eq. (94) is designed in a way that interactions with massive (off-shell) fields are suppressed by a factor of $\sqrt{N}$ and in the $N \to \infty$ limit their interactions become negligible. On the other hand, the number of off-shell fields goes to infinity. In what follows, we explain that this theory imitates many properties of off-shell and on-shell particles in the nonlocal theory.

First, let us define the following quantities: $\sigma^{\vec{p}_1\vec{p}_2}_{m_1m_2\to\mu_1\mu_2}$ is the scattering cross section of two particles with masses and momenta $m_1, \vec{p}_1$ and $m_2, \vec{p}_2$ into two particles with masses μ_1 and μ_2 (ψ_{μ_1} and ψ_{μ_2}) and $\sigma^{\vec{p}_1\vec{p}_2}_{m_1m_2}$ is the total scattering cross section of two particles with masses and momenta $m_1, \vec{p}_1$ and $m_2, \vec{p}_2$.

Consider the scattering of two ψ_0 particles into two final particles. If we restrict the two final particles to be massive (off-shell fields with *fixed* masses), then the scattering cross section in the $N \to \infty$ limit goes to zero. However, if we sum over all massive final states (all off-shell particles), the total cross section is nonzero. In fact, for different final states the corresponding cross sections scales with N as follows:

$$\sigma^{\vec{p}_1\vec{p}_2}_{00\to 00} \propto N^0,$$
$$\sigma^{\vec{p}_1\vec{p}_2}_{00\to 0m} \propto \frac{1}{N}, \qquad m \neq 0,$$
$$\sigma^{\vec{p}_1\vec{p}_2}_{00\to m_1m_2} \propto \frac{1}{N^2}, \qquad m_1, m_2 \neq 0.$$

While the interactions with individual massive fields are suppressed, the number of massive states scales with N. In this way, the *total* scattering cross section of two initial massless particles into two massive final states, summed over all masses, is finite and nonzero (the same scaling works for scattering into one massless and one massive particle).

On the other hand, any scattering with (at least) one massive initial state results in a zero cross section. For example, the following total scattering cross sections (summed over all final states) scale with N as

$$\sigma^{\vec{p}_1\vec{p}_2}_{0m} \propto \frac{1}{N}, \qquad m \neq 0, \tag{95}$$

$$\sigma^{\vec{p}_1\vec{p}_2}_{m_1m_2} \propto \frac{1}{N^2}, \qquad m_1, m_2 \neq 0, \tag{96}$$

and they vanish in the $N \to \infty$ limit.

As we showed, massive particles in this theory [Eq. (94)] mimic the properties of off-shell states in the nonlocal theory; they can be produced by the scattering of massless states, while the reverse process (scattering of massive states into massless) does not happen.

However, the theory is (obviously) time-reversal invariant and massive → massless scatterings must take place. This is indeed true, but as we demonstrate here the initial massive state that scatters nontrivially must be a superposition of different masses. Consider state γ, a superposition of M different masses, scattering off a massless particle. Then, the total transition probability $\Gamma_{0\gamma}$ scales as

$$\Gamma_{0\gamma} < A\frac{M}{N} \tag{97}$$

where A has no dependence on M and N (see Appendix C for proof). This transition probability is nonzero in the $N \to \infty$ limit, only when M also scales with N.

So, massive → massless scattering indeed happens. However, the massive state that scatters nontrivially must be a superposition of (infinitely) many different masses and in this sense is fine-tuned. It is similar to an egg that smashes into pieces upon falling on the ground; the reverse process of pieces assembling into an egg can in principle happen, but it is very unlikely.

In this sense, we expect the off-shell to on-shell scattering in the nonlocal theory to be negligible. In principle this transition can happen, but it is very implausible. The essence of our reasoning in this section is based on thermodynamical arguments and although it is not a complete proof, we hope that we have provided enough evidence to show that off-shell → on-shell scattering is very unlikely. Definitely, further quantitative studies are needed to augment (or disprove) our claim. Perhaps, a good starting point is to consider the toy model theory (94), since it shares a lot of properties with the nonlocal theory.

VII. EXTENSION TO MASSIVE SCALAR FIELDS

Throughout this paper, we only considered the modification of a massless scalar field. But what about massive scalar fields? One may suggest replacing $\Box$ with $\tilde{\Box}$ in the equation of motion of a massive scalar field as follows:

$$(\tilde{\Box} - M^2)\phi(x) = J(x) \tag{98}$$

and follow similar steps of quantization. However, this method does not work. If M is a real number, then there is no mode satisfying Eq. (98) in the absence of J. In other words, there are no on-shell modes.

Another way is to choose M to be a complex number such that for a timelike future-directed momentum $p, B(p) = M^2$. In this case, the mass of an on-shell mode is given by $m^2 \equiv -p^2$. However, $\tilde{\Box} - M^2$ is no longer a real operator and the solution to Eq. (98) generically cannot be real.

The extension to massive scalar fields can be done by considering the following observation. All of the properties in the massless case can be read from the analytic structure of $B(p)$ in Fig. 1. Massless modes are on shell because there is a simple zero at $p^2 = 0$ and there are off-shell modes for timelike momenta because there is a cut for timelike momenta in Fig. 1.

In this way, the extension to the massive case seems much simpler. $\Box - m^2$ must be replaced with $\tilde{\Box}_m$ whose eigenvalues $B_m(p)$ satisfy the following:

(1) There is only one simple zero at $p^2 = -m^2$. Also $\lim_{p^2+m^2\to 0} \frac{B_m(p)}{p^2+m^2} = -1$ to get the correct local limit.

(2) The cut must be only on momenta with higher masses $p^2 < -m^2$. Otherwise, in the quantum theory, there are off-shell modes with masses smaller than m which makes the on-shell mode unstable (on-shell modes can always decay into off-shell modes with less mass).

(3) $\mathrm{Im}B_m(p) \geq 0$ for $p^0 > 0$.

Conditions 4 and 5 in Sec. II and Eq. (56) must be replaced by the above-mentioned conditions. One easy way to come up with such an operator is to make use of the existing operator $B(p)$ in the massless case, and consider it as a function of p^2 and $\mathrm{sgn}(p^0)$. Then,

$$B_m(p) = B(p^2 + m^2, \mathrm{sgn}(p^0)) \tag{99}$$

has all the desired properties (this also has been shown in Ref. [11]).

VIII. CONCLUSION

In this paper, we studied the physical consequences of a causal nonlocal evolution of a massless scalar field. We started by modifying the d'Alembertian to a causal nonlocal operator at high energies. Quantization of a free field showed that the field represents a continuum of massive particles. In fact, there were two sets of modes: on-shell modes (massless particles) and off-shell modes (massive particles).

The Feynman rules for the perturbative calculation of S-matrix amplitudes were discussed. The most important result (in our opinion) is the fact that the cross section of any scattering with off-shell particles is zero. This suggests that although these modes exist and probably can be detected by other means, there is no way of detecting them through scattering experiments. This property opens up the possibility that dark matter particles might be just the off-shell modes of known matter. Finally, we extended this formalism to massive scalar fields.

Throughout this paper we only considered scalar field theories, but how about other types of fields? The extension to other types of fields, such as a vector field, is not as straightforward as for scalar fields. Incorporating gauge symmetry in the theory is another important issue. Whether causal Lorentzian evolution can be extended to vector fields (and other types of fields rather than scalars) can be the subject of future studies.

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APPENDIX A: EXISTENCE AND EXAMPLES OF $\tilde{\Box}$

Here we will show there are operators $\tilde{\Box}$ which satisfy all the axioms introduced in Sec. II. In fact, we will outline a procedure for constructing such operators.

We shall consider the following operator:

$$\Lambda^{-2}(\tilde{\Box}\phi)(x) = a\phi(x) + \Lambda^4 \int_{J^-(x)} f(\Lambda^2 \tau_{xy}^2)\phi(y) d^4y, \tag{A1}$$

where Λ denotes the nonlocality energy scale, a is a dimensionless real number, $J^-(x)$ denotes the causal past of x, and τ_{xy} is the Lorentzian distance between x and y

$$\tau_{xy}^2 = (x^0 - y^0)^2 - |\mathbf{x} - \mathbf{y}|^2. \tag{A2}$$

It may be shown that

$$\widetilde{\Box} e^{ip\cdot x} = B(p) e^{ip\cdot x}, \tag{A3}$$

$$B(p) = \Lambda^2 \widetilde{g}(p/\Lambda), \tag{A4}$$

$$\widetilde{g}(z) = a + \int_{J^+(0)} f((y^0)^2 - |\mathbf{y}|^2) e^{-iz\cdot y} d^4 y, \tag{A5}$$

where as usual $x \cdot y = \eta_{\mu\nu} x^\mu y^\nu$. Evaluating $\widetilde{g}(z)$ amounts to computing the Laplace transform of a retarded, Lorentz-invariant function, which has been done in Ref. [15]. It follows from their result that

$$\widetilde{g}(z) = g(z \cdot z), \tag{A6}$$

$$g(Z) = a + 4\pi Z^{-\frac{1}{2}} \int_0^\infty f(s^2) s^2 K_1(Z^{1/2} s) ds, \tag{A7}$$

where an infinitesimal timelike and future-directed imaginary part ought to be added to z on the right-hand side of Eq. (A6) (see Ref. [3] for more details).

1. IR conditions

The infrared condition (8) is equivalent to satisfying

$$g(Z) \xrightarrow{Z\to 0} -Z. \tag{A8}$$

In Ref. [3], a framework was developed to determine what constraints Eq. (A8) places on a and f, for some specific choices of f which arise in causal set theory. Generalizing that methodology in a straightforward manner, we find that Eq. (A8) is true if and only if the following conditions are satisfied:

$$\int_0^\infty f(s^2) s^{2k+1} ds = 0, \qquad k = 0, 1, 2 \tag{A9}$$

$$\int_0^\infty f(s^2) s^5 \ln s ds = -\frac{4}{\pi}, \tag{A10}$$

$$a + 2\pi \int_0^\infty f(s^2) s^3 \ln s ds = 0. \tag{A11}$$

2. From $B(p)$ to $\widetilde{\Box}$

It is often desirable to constrain the behavior of $B(p)$, as opposed to $\widetilde{\Box}$ directly. For instance, as is argued in Sec. IV C 4, the quantum theory is well behaved only when the imaginary part of $B(p)$ (for timelike and future-directed p) is always positive. The question then becomes: are there any choices of a and f which allow for this possibility, provided the IR conditions (A9)–(A11) are satisfied? To answer this question, we turn the problem around. Given a choice of $B(p)$, we reconstruct a and f and then ask if the IR conditions are met.

It can be shown that for $x > 0$ (see e.g. 10.27.9 and 10.27.10 of Ref. [15])

$$g(-x^2 - i\epsilon) = g_R(-x^2 - i\epsilon) + i g_I(-x^2 - i\epsilon), \tag{A12}$$

$$g_R(-x^2 - i\epsilon) = a + \frac{2\pi}{x} \int_0^\infty f(s^2) s^2 Y_1(xs) ds, \tag{A13}$$

$$g_I(-x^2 - i\epsilon) = -\frac{2\pi^2}{x} \int_0^\infty f(s^2) s^2 J_1(xs) ds. \tag{A14}$$

We can now use the following orthonormality conditions of Bessel functions (see e.g. 1.17.13 of Ref. [15]) to express f in terms of $\widetilde{g}_I$:

$$\delta(x - \tilde{x}) = x \int_0^\infty t J_1(xt) J_1(\tilde{x} t) dt. \tag{A15}$$

Doing so yields

$$f(s^2) = f_g(s^2) + h(s^2), \tag{A16}$$

$$f_g(s^2) = -\frac{1}{2\pi^2 s} \int_0^\infty g_I(-x^2 - i\epsilon) x^2 J_1(sx) dx, \tag{A17}$$

where h satisfies for all x

$$\int_0^\infty h(s^2) s^2 J_1(xs) ds = 0. \tag{A18}$$

This means that specifying $\widetilde{g}_I(-x^2 - i\epsilon)$ fixes f up to any part for which the right-hand side of Eq. (A14) vanishes. One example of a nontrivial function which satisfies Eq. (A18) is the delta function: $h(x) = \delta^+(x) \equiv \delta(x - \epsilon)$, where ϵ is an arbitrarily small positive real number.

We can now express the IR conditions in terms of g_I and h:

$$\int_0^\infty h(s^2) s^{2k+1} ds - \frac{1}{2\pi^2} \int_0^\infty g_I(-x^2 - i\epsilon) x^2 \times \int_0^\infty ds s^{2k} J_1(xs) = 0, \tag{A19}$$

$$\int_0^\infty h(s^2) s^5 \ln s ds - \frac{1}{2\pi^2} \int_0^\infty g_I(-x^2 - i\epsilon) x^2 \times \int_0^\infty ds s^4 J_1(xs) \ln s = -\frac{4}{\pi}, \tag{A20}$$

$$a + 2\pi \int_0^\infty h(s^2) s^3 \ln s ds - \frac{1}{\pi} \int_0^\infty g_I(-x^2 - i\epsilon) x^2 \times \int_0^\infty ds s^2 J_1(xs) \ln s = 0. \tag{A21}$$

The above integrals over s are not absolutely convergent, so we use the usual trick:

$$\int_0^\infty ds J_1(xs)e^{-\delta s} \xrightarrow{\delta\to 0} \frac{1}{x}, \tag{A22}$$

$$\int_0^\infty ds s^2 J_1(xs)e^{-\delta s} \xrightarrow{\delta\to 0} \frac{3\delta}{x^4}, \tag{A23}$$

$$\int_0^\infty ds s^4 J_1(xs)e^{-\delta s} \xrightarrow{\delta\to 0} \frac{-45\delta}{x^6}, \tag{A24}$$

$$\int_0^\infty ds s^2 J_1(xs)\ln s e^{-\delta s} \xrightarrow{\delta\to 0} -2x^{-3}, \tag{A25}$$

$$\int_0^\infty ds s^4 J_1(xs)\ln s e^{-\delta s} \xrightarrow{\delta\to 0} 16x^{-5}. \tag{A26}$$

Having the delta function example in mind, we shall require h to satisfy for all $k = 1, 2$

$$\int_0^\infty h(s^2)s^{2k+1}ds = 0, \qquad \int_0^\infty h(s^2)s^{2k+1}\ln s ds = 0, \tag{A27}$$

Also, we assume that the following integrals converge:

$$\left|\int_0^\infty g_I(-x^2 - i\epsilon)x^{-k}dx\right| < \infty, \qquad k = 1,2,3,4 \tag{A28}$$

$$\left|\int_0^\infty g_I(-x^2 - i\epsilon)x^{-k}\ln x dx\right| < \infty \qquad k = 2,4. \tag{A29}$$

The IR conditions then reduce to

$$\int_0^\infty g_I(-x^2 - i\epsilon)x dx = \pi^2\int_0^\infty h(u)du, \tag{A30}$$

$$\int_0^\infty g_I(-x^2 - i\epsilon)x^{-3}dx = \frac{\pi}{2}, \tag{A31}$$

$$\int_0^\infty g_I(-x^2 - i\epsilon)x^{-1}dx = -\frac{\pi}{2}a. \tag{A32}$$

Note that the only nontrivial condition to satisfy is Eq. (A30), since Eq. (A31) just fixes the normalization of g_I and Eq. (A32) determines a. Note that for positive $g_I(-x^2 - i\epsilon)$ which is required by consistent quantum theory, a must be a negative number.

If h is taken to be zero, then g_I ought to change sign, which leads to a quantum theory with an unbounded Hamiltonian. We note that the class of operators which arise in causal set theory in Ref. [3] all have $h = 0$, and therefore this feature.

Let us work out a complete example in four dimensions. Let

$$g_I(-x - i\epsilon) = Ax^2e^{-x/2}, \qquad h(x) = \alpha\delta^+(x) \tag{A33}$$

where A and α are real constants. It can then be shown using Eqs. (A30)–(A32)

$$A = \frac{\pi}{2}, \qquad \alpha = \frac{4}{\pi}, \qquad a = -2. \tag{A34}$$

It then follows from Eq. (A17) that

$$f_g(s) = -\frac{e^{-s/2}}{4\pi}(24 - 12s + s^2). \tag{A35}$$

Therefore

$$f(s) = \frac{4}{\pi}\delta^+(s) - \frac{e^{-s/2}}{4\pi}(24 - 12s + s^2). \tag{A36}$$

3. Stability from positivity of g_I

We have required that evolution defined by $\widetilde{\Box}$ should be stable. Instabilities are in general associated with "unstable modes," and in line with Ref. [3], we shall use this as our criterion of instability. More specifically, we take such a mode to be a plane wave $e^{ip\cdot x}$ satisfying the equation of motion $\widetilde{\Box}e^{ip\cdot x} = 0$, with the wave vector p possessing a future-directed timelike imaginary part (i.e. $p = p_R + ip_I$ where $p_I \cdot p_I < 0$ and $p_I^0 > 0$). It was shown in Ref. [3] that the necessary and sufficient condition for avoiding unstable modes is

$$g(Z) \neq 0, \qquad \forall\ Z \neq 0 \quad \text{and} \quad Z \in \mathbb{C}. \tag{A37}$$

On the other hand, we argued in Sec. IV C 4 that for consistency reasons we need to assume $\mathrm{Im}B(p) > 0$ for $p^0 > 0$ which implies $g(Z)$ has a positive (negative) imaginary part under (above) the cut in Fig. 1.

Here, we show that the stability condition and positivity of $g_I(-x^2 - i\epsilon)$ (see Appendix A 2) are consistent, and additionally the latter is a sufficient condition for stability. In order to prove it, we make the following assumptions:

(1) $g(Z)$ has a simple zero at $Z = 0$. IR conditions on $g(Z)$ [Eq. (A8)] guarantee this assumption.
(2) $g(Z)$ has a positive (negative) imaginary part under (above) the cut.

We prove this by counting the number of zeros of g inside contour $C = C_1 + C_2 + C_3 + C_4$ in Fig. 5.

If N and P are the number of zeros and poles of g, respectively, inside the contour C (taken to be counterclockwise), then

$$\int_C dZ\frac{g'(Z)}{g(Z)} = -2\pi i(N - P). \tag{A38}$$

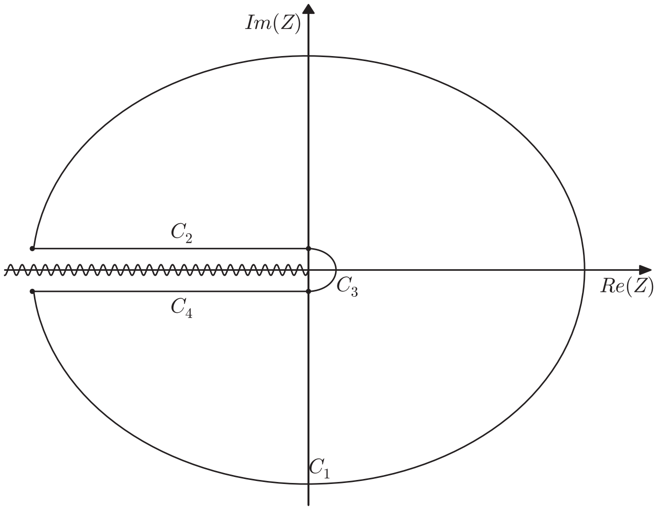

FIG. 5. The integration path in the complex Z plane. The closed contour is taken to be counterclockwise.

Let us evaluate the left-hand side of Eq. (A38) for each contour separately.

(1) C_1: According to Eq. (A7), $g(Z)$ approaches the constant value of $a < 0$ (see Appendix A 2) for large Z. In fact, $g(Z) \to a + \mathcal{O}(\frac{1}{Z^n})$ for some positive value of n (which depends on the function f). This means for $a \neq 0$,

$$\int_{C_1} dZ \frac{g'(Z)}{g(Z)} = 0. \tag{A39}$$

(2) C_2 & C_4: Since the values of g above and under the cut are complex conjugate of each other, the contribution from these diagrams can be added together to get

$$\begin{aligned}\int_{C_2+C_4} dZ \frac{g'(Z)}{g(Z)} &= 2i\mathrm{Im} \int_{-\infty}^{0} dx \frac{g'(x+i\epsilon)}{g(x+i\epsilon)} \\ &= 2i\mathrm{Im} \ln\left[\frac{g(0+i\epsilon)}{g(-\infty+i\epsilon)}\right],\end{aligned} \tag{A40}$$

where ϵ is an infinitesimal positive number.

If we define $g(Z) = r_g(Z) e^{i\varphi_g(Z)}$, the right-hand side of Eq. (A40) (apart from the factor of $2i$) measures how much φ_g rotates from $Z = -\infty + i\epsilon$ to $Z = 0 + i\epsilon$. Since $\mathrm{Im} g(x + i\epsilon) < 0$ on the whole negative real line, $\ln[g(x + i\epsilon)]$ is definable on one Riemann sheet. Combining this result with $g(-\infty + i\epsilon) = a < 0$ and $g(0 + i\epsilon) = -i\epsilon$, we get

$$\int_{C_2+C_4} dZ \frac{g'(Z)}{g(Z)} = i\pi. \tag{A41}$$

(3) C_3: IR conditions require that close to $Z = 0$, $g(Z) = -Z$. This means

$$\int_{C_3} dZ \frac{g'(Z)}{g(Z)} = \int_{C_3} \frac{1}{Z} = -i\pi. \tag{A42}$$

Adding the values of all the contours and considering the fact that $g(Z)$ is finite everywhere ($P = 0$), we conclude that the number of zeros of g in the complex plane of Z (inside contour C) is zero. Since there is no zero on the negative real line [$\mathrm{Im} g(x + i\epsilon) \neq 0$], there is no zero of g in the complex plane of Z except the one at $Z = 0$. Therefore, stability has been proven.

APPENDIX B: FDT

Here, we present the proof of Eq. (52).[9] Let us start with the following definitions:

$$i\Delta(x, y) \equiv [\hat{\phi}(x), \hat{\phi}(y)], \tag{B1}$$

$$G^{(1)}(x, y) \equiv \langle\{\hat{\phi}(x), \hat{\phi}(y)\}\rangle, \tag{B2}$$

$$W^{+}(x, y) \equiv \langle\hat{\phi}(x)\hat{\phi}(y)\rangle, \tag{B3}$$

$$W^{-}(x, y) \equiv \langle\hat{\phi}(y)\hat{\phi}(x)\rangle, \tag{B4}$$

$$G^{F}(x, y) \equiv -i\langle T\hat{\phi}(x)\hat{\phi}(y)\rangle, \tag{B5}$$

where $\{\}$ is the anticommutator and $\langle\rangle$ shows the expectation value in a quantum state. If we define

$$G^{R}(x, y) \equiv \Delta(x, y) H(x \succ y), \tag{B6}$$

$$G^{A}(x, y) \equiv -\Delta(x, y) H(x \prec y), \tag{B7}$$

where H is the Heaviside function: $H(x \succ y) = 1$ if $x \succ y$ and otherwise 0. We get the following relations:

$$\begin{aligned} i\Delta(x, y) &= W^{+}(x, y) - W^{-}(x, y) \\ &= i[G^{R}(x, y) - G^{A}(x, y)], \end{aligned} \tag{B8}$$

$$G^{(1)}(x, y) = W^{+}(x, y) + W^{-}(x, y), \tag{B9}$$

$$G^{A}(x, y) = G^{R}(y, x), \tag{B10}$$

$$G^{F}(x, y) = \frac{1}{2}[G^{R}(x, y) + G^{A}(x, y)] - \frac{i}{2} G^{(1)}(x, y). \tag{B11}$$

For a translational invariant system, the value of all the two-point functions depend only on spacetime separation. This will allow us to define the following Fourier transform with respect to time:

[9] Most of the content of this appendix is taken from Ref. [16].

$$\overline{A}(\omega, \mathbf{x}, \mathbf{x}') \equiv \int dt A(t, \mathbf{x}; t', \mathbf{x}') e^{-i\omega(t-t')}. \tag{B12}$$

Now, let us assume that the quantum system is in a thermal state with temperature $T = \frac{1}{\beta}$. It requires that

$$W^{\pm}(t, \mathbf{x}; t', \mathbf{x}') = W^{\mp}(t + i\beta, \mathbf{x}; t', \mathbf{x}'), \tag{B13}$$

resulting in the following relation in Fourier space:

$$\overline{W}^{+}(\omega, \mathbf{x}, \mathbf{y}) = e^{\beta\omega}\overline{W}^{-}(\omega, \mathbf{x}, \mathbf{y}). \tag{B14}$$

Using Eq. (B8), we get

$$\overline{W}^{+}(\omega, \mathbf{x}, \mathbf{y}) = \frac{i\overline{\Delta}(\omega, \mathbf{x}, \mathbf{y})}{1 - e^{-\beta\omega}}, \tag{B15}$$

$$\overline{W}^{-}(\omega, \mathbf{x}, \mathbf{y}) = -\frac{i\overline{\Delta}(\omega, \mathbf{x}, \mathbf{y})}{1 - e^{\beta\omega}}. \tag{B16}$$

On the other hand since $G^R(t, x; t', x') = G^A(t', x; t, x')$, in Fourier space they are complex conjugate. As a result,

$$\begin{aligned} \mathrm{Im}\overline{G}^F(\omega, \mathbf{x}, \mathbf{y}) &= -\frac{1}{2}\,\mathrm{Re}\overline{G}^{(1)}(\omega, \mathbf{x}, \mathbf{y}) \\ &= -\frac{1}{2}[\overline{W}^{+}(\omega, \mathbf{x}, \mathbf{y}) + \overline{W}^{-}(\omega, \mathbf{x}, \mathbf{y})] \\ &= -\frac{1}{2}i\overline{\Delta}(\omega, \mathbf{x}, \mathbf{y}) \coth\left(\frac{\beta\omega}{2}\right) \end{aligned} \tag{B17}$$

where Im and Re are the imaginary part and real part respectively and in the second line we have used the positivity of the two-point function W^+ [giving that $\overline{W}^{+}(\omega, \mathbf{x}, \mathbf{y})$ and $\overline{W}^{-}(\omega, \mathbf{x}, \mathbf{y})$ are real].

With the assumption that this field theory in the Hilbert-space representation has an equivalent representation in terms of a double path integral, the time-ordered two-point function is given by Eq. (46). In Fourier space, it reads

$$\overline{G}^F(\omega, \mathbf{x}, \mathbf{y}) = \frac{1}{2}[\overline{G}^K(\omega, \mathbf{x}, \mathbf{y}) + \overline{G}^R(\omega, \mathbf{x}, \mathbf{y}) + \overline{G}^A(\omega, \mathbf{x}, \mathbf{y})]. \tag{B18}$$

$\overline{G}^K(\omega, \mathbf{x}, \mathbf{y})$ is a totally imaginary number and $\overline{G}^R(\omega, \mathbf{x}, \mathbf{y}) + \overline{G}^A(\omega, \mathbf{x}, \mathbf{y})$ is a real number. As a result,

$$\overline{G}^K(\omega, \mathbf{x}, \mathbf{y}) = 2i\mathrm{Im}\overline{G}^F(\omega, \mathbf{x}, \mathbf{y}). \tag{B19}$$

Combining Eqs. (B8), (B17), and (B19) we arrive at

$$\overline{G}^K(\omega, \mathbf{x}, \mathbf{y}) = \coth\left(\frac{\beta\omega}{2}\right)[\overline{G}^R(\omega, \mathbf{x}, \mathbf{y}) - \overline{G}^A(\omega, \mathbf{x}, \mathbf{y})], \tag{B20}$$

which reduces to Eq. (52) at zero temperature.

APPENDIX C: QUANTUM TRANSITION

We start by proving a simple theorem for any quantum system. Consider a quantum-mechanical system in the (normalized) initial state $|\alpha\rangle$ that evolves in time and the probability of finding the system at a later time t_f in the state $|\beta_i\rangle$ is called P_i, and assume $|\beta_i\rangle$'s are orthonormal:

$$P_i = |\langle\beta_i|U|\alpha\rangle|^2 \tag{C1}$$

where U is the time evolution operator.

Now, consider a (normalized) state $|\beta\rangle$ as a superposition of $|\beta_i\rangle$ states:

$$\begin{aligned} |\beta\rangle &= \sum_i c_i|\beta_i\rangle, \\ \sum_i |c_i|^2 &= 1. \end{aligned} \tag{C2}$$

The probability P of measuring the system at time t_f in the state $|\beta\rangle$ is given by

$$P = |\langle\beta|U|\alpha\rangle|^2. \tag{C3}$$

Then,

$$\begin{aligned} P &= |\langle\beta|U|\alpha\rangle|^2 \\ &= \left|\sum\nolimits_i c_i^*\langle\beta_i|U|\alpha\rangle\right|^2 \\ &\leq \left(\sum_i |c_i|^2\right)\left(\sum_i |\langle\beta_i|U|\alpha\rangle|^2\right) \\ &= \sum_i P_i \end{aligned} \tag{C4}$$

where we have used the triangular inequality in the second line. So P is bounded from above by $\sum_i P_i$.

Now, let us get back to the scattering of a massless particle with state γ, a superposition of M different masses, in Sec. VI C. We already have shown [see Eq. (95)] that Γ_{0m_i} defined as the transition probability of a massless particle scattering with a massive particle (mass m_i) scales with N as

$$\Gamma_{0m_i} = \frac{A_i}{N} \tag{C5}$$

where A_i depends on the momentum of the particles but is independent of N. Using Eq. (C4) for the transition probabilities, we conclude that

$$\Gamma_{0\gamma} \leq \sum_i \frac{A_i}{N} \leq A\frac{M}{N} \tag{C6}$$

where A is the maximum of the A_i's.

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