Cosmology with a stiff matter era

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We consider the possibility that the Universe is made of a dark fluid described by a quadratic equation of state $P = K\rho^2$, where ρ is the rest-mass density and K is a constant. The energy density $\epsilon = \rho c^2 + K\rho^2$ is the sum of two terms: a rest-mass term ρc^2 that mimics "dark matter" ($P = 0$) and an internal energy term $u = K\rho^2 = P$ that mimics a "stiff fluid" ($P = \epsilon$) in which the speed of sound is equal to the speed of light. In the early universe, the internal energy dominates and the dark fluid behaves as a stiff fluid ($P \sim \epsilon$, $\epsilon \propto a^{-6}$). In the late universe, the rest-mass energy dominates and the dark fluid behaves as pressureless dark matter ($P \simeq 0$, $\epsilon \propto a^{-3}$). We provide a simple analytical solution of the Friedmann equations for a universe undergoing a stiff matter era, a dark matter era, and a dark energy era due to the cosmological constant. This analytical solution generalizes the Einstein–de Sitter solution describing the dark matter era, and the ΛCDM model describing the dark matter era and the dark energy era. Historically, the possibility of a primordial stiff matter era first appeared in the cosmological model of Zel'dovich where the primordial universe is assumed to be made of a cold gas of baryons. A primordial stiff matter era also occurs in recent cosmological models where dark matter is made of relativistic self-gravitating Bose-Einstein condensates (BECs). When the internal energy of the dark fluid mimicking stiff matter is positive, the primordial universe is singular like in the standard big bang theory. It expands from an initial state with a vanishing scale factor and an infinite density. We consider the possibility that the internal energy of the dark fluid is negative (while, of course, its total energy density is positive), so that it mimics anti-stiff matter. This happens, for example, when the BECs have an attractive self-interaction with a negative scattering length. In that case, the primordial universe is nonsingular and bouncing like in loop quantum cosmology. At $t = 0$, the scale factor is finite and the energy density is equal to zero. The universe first has a phantom behavior where the energy density increases with the scale factor, then a normal behavior where the energy density decreases with the scale factor. For the sake of generality, we consider a cosmological constant of arbitrary sign. When the cosmological constant is positive, the Universe asymptotically reaches a de Sitter regime where the scale factor increases exponentially rapidly with time. This can account for the accelerating expansion of the Universe that we observe at present. When the cosmological constant is negative (anti–de Sitter), the evolution of the Universe is cyclic. Therefore, depending on the sign of the internal energy of the dark fluid and on the sign of the cosmological constant, we obtain analytical solutions of the Friedmann equations describing singular and nonsingular expanding, bouncing, or cyclic universes.

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I. INTRODUCTION

The nature of dark matter (DM) is still unknown and remains one of the greatest mysteries of modern cosmology. In the standard cold dark matter (ΛCDM) model, DM is modeled as a pressureless fluid ($P = 0$). This is appropriate if DM is made of weakly interacting massive particles with a mass in the GeV–TeV range. These particles freeze out from thermal equilibrium in the early universe and, as a consequence of this decoupling, cool off rapidly as the universe expands. The ΛCDM model works remarkably well at large scales [1] but it encounters serious problems at small scales of the order of a few kpc, the typical galactic scale. In particular, it predicts that DM halos should be cuspy (the central density should diverge as r^{-1}) [2] while observations reveal that they have a flat core [3]. On the other hand, since the Jeans length is equal to zero (or is very low), the ΛCDM model predicts an overabundance of small-scale structures (subhalos/satellites), much more than what is observed around the Milky Way [4]. These problems are referred to as the "cusp problem" and the "missing satellite problem." The expression "small-scale crisis of CDM" has been introduced.

In order to solve this crisis, other models of DM have been developed. For example, it has been proposed that DM may be in the form of Bose-Einstein condensates (BECs) [5–7]. The bosons could be QCD axions with a mass $m \sim 10^{-6}\ \mathrm{eV}/c^2$, but other types of bosons with a small mass have also been considered. The quantum properties of BECDM may solve the problems of CDM. Indeed, the Heisenberg principle (for noninteracting bosons) or the pressure due to the scattering (for self-interacting bosons) prevent gravitational collapse at small scales and lead to central density cores instead of cusps. On the other hand, the finite Jeans length set by quantum mechanics (or scattering) provides a sharp small-scale cutoff in the matter power spectrum which may solve the missing satellite problem. These problems may also be solved if DM is made of fermions, such as sterile neutrinos, with a mass in the keV range [8,9]. This corresponds to warm dark matter. In that case, gravitational collapse is prevented by the Pauli exclusion principle for dwarf halos and by thermal pressure for large halos.

Harko [10] and Chavanis [11] independently considered the evolution of a universe made of self-interacting BECs in the Thomas-Fermi (TF) approximation. The equation of state of a nonrelativistic BEC is $P = 2\pi\hbar^2 a_s \rho^2/m^3$ [12] where m is the mass of the bosons, a_s is their scattering length, and ρ is the rest-mass density. In the relativistic regime, one needs to express the pressure P in terms of the energy density ϵ in order to solve the Friedmann equations [13]. Harko [10] and Chavanis [11] assumed that the energy density is simply given by the relation $\epsilon = \rho c^2$, so that $P = 2\pi\hbar^2 a_s \epsilon^2/m^3c^4$. However, the relation $\epsilon = \rho c^2$ is true only at sufficiently late times, in the weakly relativistic and nonrelativistic regimes, so it is not possible to extrapolate their results in the early universe. Indeed, their approach neglects the internal energy of the BECs [14].

If we consider a relativistic fluid at $T = 0$, or an adiabatic fluid, described by a barotropic equation of state $P(\rho)$, the relation between the energy density and the rest-mass density is given by an equation of the form $\epsilon = \rho c^2 + u(\rho)$ [see Eq. (9) below] where ρc^2 is the rest-mass energy and $u(\rho)$ is the internal energy which is entirely determined by the equation of state $P(\rho)$. On the other hand, the rest-mass density decreases as $\rho \propto a^{-3}$ [see Eq. (12) below], where a is the scale factor. The decomposition $\epsilon = \rho c^2 + u(\rho)$ is very interesting because it shows that the energy density of a relativistic fluid is the sum of a rest-mass term that mimics pressureless "dark matter" and an internal energy term that mimics a "new fluid." If we consider a polytropic equation of state $P = K\rho^\gamma$ with $\gamma = 1 + 1/n$, we find that $\epsilon = \rho c^2 + nK\rho^\gamma = \rho c^2 + nP$. The rest-mass energy is given by $\rho c^2 \propto a^{-3}$ and the internal energy is given by $u = K\rho^\gamma/(\gamma - 1) = P/(\gamma - 1) \propto a^{-3\gamma}$. We have to consider two cases.

When $\gamma < 1$ ($n < 0$), the rest-mass energy dominates in the early universe (a low, ρ large) and the internal energy dominates in the late universe (a large, ρ low). In that case, the rest-mass term mimics dark matter and the internal energy term mimics dark energy (when $u > 0$ and $P < 0$) or anti-dark energy (when $u < 0$ and $P > 0$). This interpretation is particularly relevant when $\gamma \to 0$ ($n \to -1$) and $K < 0$ because a fluid with a constant negative pressure $P = -\epsilon_\Lambda$, corresponding to $\gamma = 0$, $n = -1$ and $K = -\epsilon_\Lambda$ behaves similarly to the ΛCDM model (conversely, a fluid with a constant positive pressure $P = \epsilon_\Lambda$, corresponding to $\gamma = 0$, $n = -1$ and $K = \epsilon_\Lambda$, behaves similarly to the anti-ΛCDM model) [15–17]. Therefore, a fluid with $\gamma \simeq 0$ and $K < 0$ can account for small deviations from the ΛCDM model and solve some of its small-scale problems. This idea was developed in Ref. [18] by means of a fully predictive cosmological model based on a dark fluid with a logotropic equation of state. The logotropic equation of state $P = A\ln(\rho/\rho_*)$ can be viewed as the limiting form of the polytropic equation of state $P = K\rho^\gamma$ with $\gamma \to 0$ and $K \to \infty$ in such a way that $A = K\gamma$ is finite [19]. Therefore, the logotropic equation of state is very appropriate to describe dark energy for which $\gamma \simeq 0$. The considerations of Ref. [18] show that ρ_* can be identified with the Planck density $\rho_P = c^5/G^2\hbar = 5.16 \times 10^{99}\ \mathrm{g\,m^{-3}}$. On the other hand, the logotropic temperature A is interpreted as a fundamental constant given by $A \simeq \rho_\Lambda c^2/\ln(\rho_P/\rho_\Lambda) \simeq \rho_\Lambda c^2/[123\ln(10)] = 2.13 \times 10^{-9}\ \mathrm{g\,m^{-1}\,s^{-2}}$ where $\rho_\Lambda = 6.72 \times 10^{-24}\ \mathrm{g\,m^{-3}}$ is the cosmological density. This model provides a unification of dark matter and dark energy and is able to explain, without any free parameter, observational results that were not explained previously.

When $\gamma > 1$ ($n > 0$), the rest-mass energy dominates in the late universe and the internal energy dominates in the early universe. In that case, the rest-mass term mimics dark matter and the internal energy term mimics a primordial cosmological fluid existing before the matter era. For a quadratic equation of state ($\gamma = 2$), corresponding to the BEC model, we find that $\epsilon = \rho c^2 + K\rho^2 = \rho c^2 + P$. In the late universe, where the density ρ is low, $\epsilon \sim \rho c^2 \propto a^{-3}$ so that $P \sim K\epsilon^2/c^4$ like in the works of Harko [10] and Chavanis [11]. In this limit, $P \ll \epsilon$, so the system behaves essentially as CDM ($P = 0$) with a small correction due to the BEC. In the early universe, where the density ρ is high, $\epsilon \sim K\rho^2 \propto a^{-6}$ so that $P \sim \epsilon$. In that limit, the system behaves as a stiff fluid in which the speed of sound $c_s = \sqrt{P'(\epsilon)}c$ equals the speed of light ($c_s = c$). Therefore, everything happens *as if* the Universe were made of two noninteracting fluids, a dark matter fluid with a pressureless equation of state $P = 0$ and a "new fluid" with a stiff equation of state $P = \epsilon$. Recalling that $K = 2\pi\hbar^2 a_s/m^3$ in the BEC model, we note that the stiff fluid has a positive energy density $\epsilon = K\rho^2$ when $a_s > 0$ and a negative energy density when $a_s < 0$.[1] There is, however, no contradiction with the fundamental laws of physics because our decomposition in two noninteracting fluids is purely formal, or effective. In reality, we have just *one* dark fluid and, of course, its total energy density $\epsilon = \rho c^2 + K\rho^2$ is positive. The rest-mass term ρc^2 is always positive while the internal energy term $K\rho^2$ can be positive ($K \geq 0$) or negative ($K \leq 0$).

[1]BECs in which the bosons have an attractive self-interaction, corresponding to a negative scattering length $a_s < 0$, exist in nature and have been observed in laboratory experiments [12]. It is therefore natural to consider the cosmological implications of a negative scattering length of the bosons.

The possibility of a primordial stiff matter era first appeared in the cosmological model of Zel'dovich [20] in which the very early universe is assumed to be made of a cold gas of baryons with an equation of state $P = \epsilon$. The aim of Zel'dovich was to investigate the cosmological implications of an equation of state for which the speed of sound is equal to the speed of light [21]. In that case, the energy density decreases as $\epsilon \propto 1/a^6$. The stiff matter era ($P = \epsilon$, $\epsilon \propto a^{-6}$) precedes the radiation era ($P = \epsilon/3$, $\epsilon \propto a^{-4}$), the dark matter era ($P = 0$, $\epsilon \propto a^{-3}$), and the dark energy era ($P = -\epsilon$, $\epsilon = \epsilon_\Lambda$). Zel'dovich's model has been abandoned after the success of the hot big bang theory, but it remains interesting from a historical perspective. In addition, his paper is still very much quoted, showing that there is a lot of activity on stiff matter in general. Indeed, a stiff equation of state has several interesting properties [22] that deserve to be better explored.

Stiff matter also occurs in the context of relativistic scalar fields (SFs). A SF behaves as a stiff fluid when its kinetic energy dominates its potential energy. In cosmology, this corresponds to the kination epoch of scalar field evolution. Therefore, a primordial stiff matter era is a fundamental feature of any model based on a SF. Recently, Li *et al.* [23] developed a fully relativistic treatment of SF/BEC dark matter and showed that the Universe undergoes successively an intrinsic stiff matter era, followed by a radiation era (existing only for a self-interacting SF), and a matter era. The stiff matter era occurs when the SF oscillations are slower than the Hubble expansion while the radiation and matter eras occur when the SF oscillations are faster than the Hubble expansion. The same model has been investigated by Suárez and Chavanis [24] using a hydrodynamical representation of the Klein-Gordon-Einstein (KGE) equations.

For all the reasons given previously, it is important to study a cosmology including a stiff matter era. In this paper, we provide a simple analytical solution of the Friedmann equations for a universe undergoing a stiff matter era, a dark matter era, and a dark energy era due to the cosmological constant.[2] This analytical solution provides a simple cosmological model generalizing the Einstein–de Sitter (EdS) model and the ΛCDM model by incorporating a stiff matter era. There are not many analytical solutions of the Friedmann equations, so this solution is valuable. Furthermore, it is of physical interest since stiff fluids have often been advocated in astrophysics and cosmology.

We consider the general case where the energy density of the stiff fluid (more precisely the internal energy of the dark fluid) is positive or negative. The case of a stiff fluid with a positive energy density is very similar to the standard model of cosmology in the sense that the Universe starts from a big bang singularity at $t = 0$ in which the scale factor is equal to zero while the energy density is infinite. Initially, the scale factor increases as $a(t) \propto t^{1/3}$ and the energy density decreases as $\epsilon(t) \propto t^{-2}$. Interestingly, a stiff fluid with a negative energy density (anti-stiff fluid) prevents the primordial singularity. In that case, we obtain a model of a bouncing universe like in loop quantum cosmology (LQC) [25]. At $t = 0$, the scale factor is finite and the energy density is equal to zero. The universe first has a phantom behavior where the energy density increases with the scale factor, then a normal behavior where the energy density decreases with the scale factor. This model is symmetric by time reversal $t \to -t$. For the sake of generality, we consider a positive or a negative cosmological constant. At late times, a dark fluid with a positive cosmological constant enters in a de Sitter era in which the scale factor increases exponentially rapidly with time. This can account for the present acceleration of the Universe. By contrast, when the dark fluid has a negative cosmological constant (anti–de Sitter or anti-dark energy) the evolution of the universe is cyclic. Therefore, depending on the sign of the internal energy of the dark fluid and on the sign of the cosmological constant, we obtain analytical solutions of the Friedmann equations describing singular and nonsingular expanding, bouncing, or cyclic universes.

The paper is organized as follows. In Sec. II we derive general theoretical results needed in our analysis. In Sec. III, we consider a perfect fluid at $T = 0$, or an adiabatic fluid, described by a quadratic equation of state of the form $P = K\rho^2$, where ρ is the rest-mass density and K is a constant. We determine the relation between the energy density ϵ and the rest-mass density ρ and obtain an explicit equation of state $P(\epsilon)$ [see Eqs. (17) and (22)] relating the pressure to the energy density. For $K \geq 0$, this equation of state reduces at high energies to a stiff equation of state $P \sim \epsilon$ for which the speed of sound is equal to the speed of light, and at low energies to a quadratic equation of state $P \sim K\epsilon^2/c^4$. For $K \leq 0$, the equation of state $P(\epsilon)$ has two branches, one corresponding to a phantom behavior (starting from an anti-stiff matter era) and one corresponding to a normal behavior (ending in the matter era). In Sec. IV, we give examples of systems described by a quadratic equation of state $P = K\rho^2$ leading, in the regime of high density, to a stiff equation of state $P = \epsilon$. In Sec. V, we apply this equation of state to cosmology. This leads to a cosmological model exhibiting a primordial (anti-)stiff matter era, followed by a radiation era, a dark matter era, and an ultimate (anti-)dark energy era. In Secs. VI–IX, we provide simple analytical solutions of the Friedmann equations for a universe exhibiting a primordial (anti-)stiff matter era, a dark matter era, and an ultimate (anti-)dark energy era. We consider four cases: a singular expanding universe with $K \geq 0$ and $\Lambda \geq 0$; a nonsingular bouncing universe with $K \leq 0$ and $\Lambda \geq 0$; a singular cyclic universe with $K \geq 0$ and $\Lambda \leq 0$; a nonsingular bouncing cyclic universe with $K \leq 0$ and $\Lambda \leq 0$. In Sec. X, we provide simple analytical solutions of the Friedmann equations

[2]The main limitation of our analytical solution is that it does not describe the radiation era that takes place between the stiff matter era and the dark matter era.

taking the radiation era into account. Finally, in Sec. XI, we discuss a simple analytical model based on a generalized polytropic equation of state [15–17] describing the transition from an inflation era to a stiff matter era.

II. THEORETICAL FRAMEWORK

A. The Friedmann equations

We assume that the Universe is homogeneous and isotropic, and contains a uniform perfect fluid of energy density $\epsilon(t)$ and isotropic pressure $P(t)$. The radius of curvature of the three-dimensional space, or scale factor, is denoted as $a(t)$ and the curvature of space is denoted as k. The universe is closed if $k>0$, flat if $k=0$, and open if $k<0$. We assume that the Universe is flat ($k=0$) in agreement with the observations of the cosmic microwave background [26]. In that case, the Einstein equations can be written as [13]

$$\frac{d\epsilon}{dt}+3\frac{\dot{a}}{a}(\epsilon+P)=0, \tag{1}$$

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3c^2}(\epsilon+3P)+\frac{\Lambda}{3}, \tag{2}$$

$$H^2=\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3c^2}\epsilon+\frac{\Lambda}{3}, \tag{3}$$

where we have introduced the Hubble parameter $H=\dot{a}/a$ and accounted for a possible nonzero cosmological constant Λ. The cosmological constant is equivalent to a dark energy fluid with a constant density

$$\epsilon_\Lambda=\rho_\Lambda c^2=\frac{\Lambda c^2}{8\pi G}, \tag{4}$$

and an equation of state $P=-\epsilon$. Equations (1)–(3) are the well-known Friedmann equations describing a nonstatic universe. Among these three equations, only two are independent. The first equation can be viewed as an equation of continuity. For a given barotropic equation of state $P=P(\epsilon)$, it determines the relation between the energy density ϵ and the scale factor a. Then, the evolution of the scale factor $a(t)$ is given by Eq. (3).

Introducing the equation of state parameter $w=P/\epsilon$, and assuming $\Lambda=0$, we see from Eq. (2) that the Universe is decelerating if $w>-1/3$ (strong energy condition) and accelerating if $w<-1/3$.[3] On the other hand, according to Eq. (1), the energy density decreases with the scale factor if $w>-1$ (null dominant energy condition) and increases with the scale factor if $w<-1$. The latter case corresponds to a "phantom" universe [28].

[3] According to general relativity, the source for the gravitational potential is $\epsilon+3P$. Indeed, the spatial part $\mathbf{g}$ of the geodesic acceleration satisfies the exact equation $\nabla\cdot\mathbf{g}=-4\pi G(\epsilon+3P)$ showing that the source of geodesic acceleration is $\epsilon+3P$ not ϵ [27].

B. Relativistic thermodynamics

The local form of the first law of thermodynamics can be written as [13]

$$d\left(\frac{\epsilon}{\rho}\right)=-Pd\left(\frac{1}{\rho}\right)+Td\left(\frac{s}{\rho}\right), \tag{5}$$

where $\rho=nm$ is the mass density, n is the number density, and s is the entropy density in the rest frame. We assume that the Universe is made of a dark fluid with an equation of state $P(\rho)$. We assume that this fluid is at $T=0$ or that the evolution is adiabatic (isentropic), $d(s/\rho)=0$, which is the case for a perfect fluid (one can show that the Friedmann equations imply the conservation of the entropy) [13]. In that case, the first law of thermodynamics (5) reduces to

$$d\epsilon=\frac{P+\epsilon}{\rho}d\rho. \tag{6}$$

For a given equation of state, Eq. (6) can be integrated to obtain the relation between the energy density ϵ and the rest-mass density ρ. If the equation of state is prescribed under the form $P=P(\rho)$, Eq. (6) reduces to the first-order linear differential equation

$$\frac{d\epsilon}{d\rho}-\frac{1}{\rho}\epsilon=\frac{P(\rho)}{\rho}. \tag{7}$$

Using the method of variation of constants, we obtain

$$\epsilon=A\rho c^2+\rho\int^\rho\frac{P(\rho')}{\rho'^2}d\rho', \tag{8}$$

where A is a constant of integration. For an equation of state $P(\rho)$ such that $P\sim\rho^\gamma$ with $\gamma>1$ when $\rho\to 0$, we determine the constant A in Eq. (8) by requiring that $\epsilon\sim\rho c^2$ when $\rho\to 0$. This gives

$$\epsilon=\rho c^2+\rho\int_0^\rho\frac{P(\rho')}{\rho'^2}d\rho'=\rho c^2+u(\rho). \tag{9}$$

The term

$$u(\rho)=\rho\int_0^\rho\frac{P(\rho')}{\rho'^2}d\rho' \tag{10}$$

can be interpreted as an internal energy density [29]. Therefore, the energy density ϵ is the sum of a rest-mass energy term ρc^2 and an internal energy term $u(\rho)$. The rest-mass energy is positive while the internal energy can be positive or negative. Of course, the total energy $\epsilon=\rho c^2+u(\rho)$ is always positive.

C. Conservation of the rest-mass density

Combining the thermodynamic relation (6) with the continuity equation (1), we get

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\rho = 0. \quad (11)$$

We note that this equation is exact for a fluid at $T = 0$, or for a perfect fluid, and that it does not depend on the explicit form of the equation of state $P(\rho)$. It corresponds to the conservation of the rest-mass density ρ or number density $n = \rho/m$. It can be integrated into

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3, \quad (12)$$

where ρ_0 is the present value of the rest-mass density and a_0 is the present value of the scale factor. This relation shows that the rest-mass energy term ρc^2 in Eq. (9) behaves as pressureless dark matter. Consequently, the internal energy term $u(\rho)$ behaves as a "new fluid." When $P = 0$ we have $\epsilon = \rho c^2$ with $\rho = \rho_0(a_0/a)^3$, and we recover the CDM model.

III. QUADRATIC EQUATION OF STATE (POLYTROPE $n = 1$)

We assume that the Universe is filled with a dark fluid described by a quadratic equation of state

$$P = K\rho^2 \quad (13)$$

corresponding to a polytrope of index $n = 1$ [30]. This is a particular case of the general class of polytropic equations of state $P = K\rho^\gamma$, where ρ is the rest-mass density, studied by Tooper [31] in the context of relativistic stars. This equation of state is fundamentally different from the polytropic equation of state $P = K\epsilon^\gamma$, where ϵ is the energy density, studied by Tooper [32] in the context of relativistic stars and by Chavanis [15–17] in cosmology (see Sec. XI). For the equation of state (13), the relation between the energy density and the rest-mass density, Eq. (9), takes the form

$$\epsilon = \rho c^2 + P = \rho c^2 + K\rho^2. \quad (14)$$

Combining Eqs. (12) and (14), we obtain

$$\epsilon = \rho_0 c^2 \left(\frac{a_0}{a}\right)^3 + K\rho_0^2 \left(\frac{a_0}{a}\right)^6. \quad (15)$$

This relation can also be obtained by solving the continuity equation (1) with the equation of state (17) when $K \geq 0$ or with the equation of state (22) when $K \leq 0$ (see Appendix D of Ref. [14]). We require that $\rho \geq 0$ and $\epsilon \geq 0$. We have to distinguish two cases depending on the sign of K.

A. Positive pressure: $K \geq 0$

When $K \geq 0$, the pressure is positive. The universe starts at $a = 0$ with an infinite rest-mass density and an infinite energy density. The rest-mass density decreases as a increases; see Eq. (12). The energy density ϵ decreases as a increases (i.e. ρ decreases); see Fig. 1. Equation (14) can be reversed to give

$$\rho = \frac{c^2}{2K}\left(\sqrt{1 + \frac{4K\epsilon}{c^4}} - 1\right). \quad (16)$$

Combining Eqs. (13) and (16), we obtain the relation between the pressure and the energy density

$$P = \frac{c^4}{4K}\left(\sqrt{1 + \frac{4K\epsilon}{c^4}} - 1\right)^2. \quad (17)$$

The pressure P decreases as ρ and ϵ decrease; see Fig. 2.

In the early universe ($a \to 0$, $\rho \to +\infty$), the internal energy (new fluid) dominates, and we have

$$\epsilon \sim K\rho^2 \sim K\rho_0^2 \left(\frac{a_0}{a}\right)^6, \qquad P \sim \epsilon, \qquad P = K\rho^2. \quad (18)$$

These equations describe a fluid with a linear equation of state $P = \epsilon$ that is called "stiff" because the speed of sound c_s defined by $c_s^2/c^2 = P'(\epsilon)$ is equal to the speed of light ($c_s = c$).

In the late universe ($a \to +\infty$, $\rho \to 0$), the rest-mass energy (dark matter) dominates, and we have

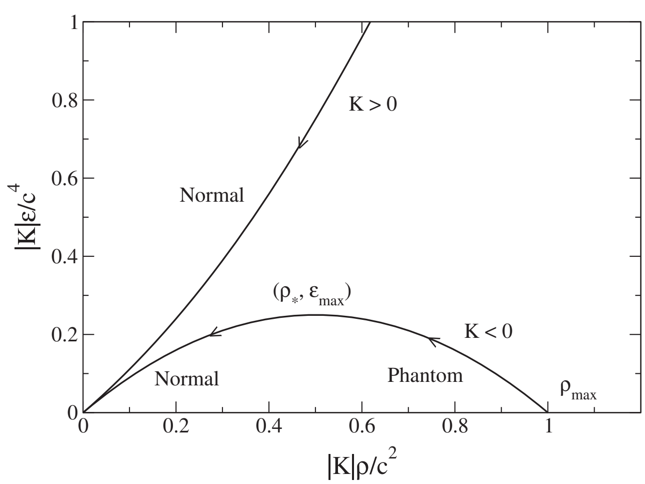

FIG. 1. Relation between the energy density and the rest-mass density.

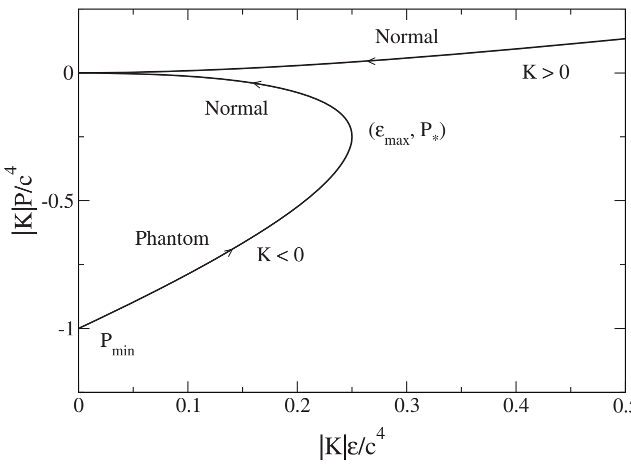

FIG. 2. Equation of state giving the pressure as a function of the energy density.

$$\epsilon \sim \rho c^2 \sim \rho_0 c^2 \left(\frac{a_0}{a}\right)^3, \qquad P \sim \frac{K}{c^4}\epsilon^2, \qquad P = K\rho^2. \quad (19)$$

These equations describe a fluid with a polytropic equation of state $P = K\epsilon^2/c^4$ of index $n = 1$. For very large values of the scale factor, we recover the results of the CDM model ($P = 0$) since $\epsilon \propto a^{-3}$. This is because $P \ll \epsilon$ in this limit.

For the equation of state (17), the speed of sound is given by

$$\frac{c_s^2}{c^2} = 1 - \frac{1}{\sqrt{1 + \frac{4K\epsilon}{c^4}}} = \frac{1}{1 + \frac{c^2}{2K\rho}}. \quad (20)$$

The speed of sound decreases as ρ and ϵ decrease; see Fig. 3. For $\rho \to +\infty$ (i.e. $\epsilon \to +\infty$), $c_s \to c$. For $\rho \to 0$ (i.e. $\epsilon \to 0$), $c_s \to 0$. We always have $c_s < c$.

Substituting Eq. (15) in the Friedmann equation (3), we see that the quadratic equation of state defined by Eq. (13)

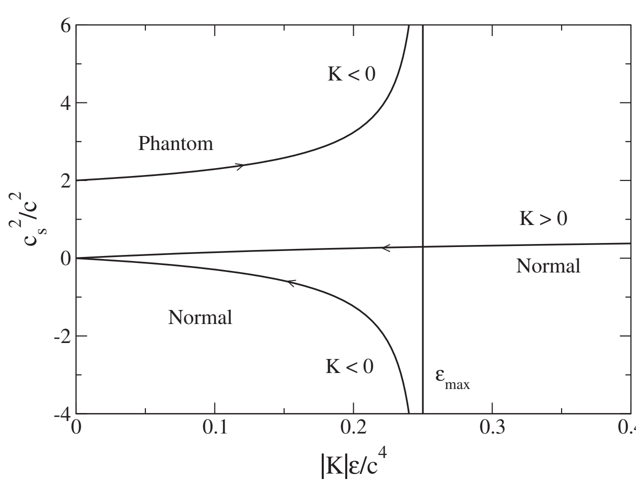

FIG. 3. Speed of sound as a function of the energy density.

with $K \geq 0$ describes a model of the Universe exhibiting a stiff matter era ($\epsilon \propto a^{-6}$), a dark matter era ($\epsilon \propto a^{-3}$), and a dark energy era ($\epsilon \sim \rho_\Lambda c^2$) due to the cosmological constant.

B. Negative pressure: $K \leq 0$

When $K \leq 0$, the pressure is negative. The energy density ϵ given by Eq. (15) is positive provided that $a \geq a_i$ with $a_i/a_0 = (|K|\rho_0/c^2)^{1/3}$. Therefore, the Universe starts with a finite scale factor a_i, a finite rest-mass density $\rho_{\rm max} = c^2/|K|$, and a vanishing energy density $\epsilon = 0$. The rest-mass density decreases as a increases; see Eq. (12). The energy density increases as a increases (i.e. ρ decreases), reaches a maximum $\epsilon_{\rm max} = c^4/4|K|$ at $a_*/a_0 = (2|K|\rho_0/c^2)^{1/3}$ (i.e. $\rho_* = c^2/2|K|$), decreases as a increases (i.e. ρ decreases) further, and tends to $\epsilon \to 0$ as $a \to +\infty$ (i.e. $\rho \to 0$); see Fig. 1. The branch $a_i \leq a \leq a_*$ (i.e. $\rho_* \leq \rho \leq \rho_{\rm max}$) corresponds to a phantom behavior in which the energy density increases as the scale factor increases. The branch $a \geq a_*$ (i.e. $\rho \leq \rho_*$) corresponds to a normal behavior in which the energy density decreases as the scale factor increases.

For $\epsilon \leq \epsilon_{\rm max}$, Eq. (14) can be reversed to give

$$\rho = \frac{c^2}{2|K|}\left(1 \pm \sqrt{1 - \frac{4|K|\epsilon}{c^4}}\right). \quad (21)$$

Combining Eqs. (13) and (21), we obtain the relation between the pressure and the energy density

$$P = -\frac{c^4}{4|K|}\left(1 \pm \sqrt{1 - \frac{4|K|\epsilon}{c^4}}\right)^2. \quad (22)$$

The pressure increases as a increases (i.e. ρ decreases). It starts from $P_{\rm min} = -c^4/|K|$ at $a = a_i$ (i.e. $\rho = \rho_{\rm max}$, $\epsilon = 0$), achieves the value $P_* = -c^4/4|K|$ at a_* (i.e. ρ_*, $\epsilon_{\rm max}$), and tends to $P \to 0^-$ as $a \to +\infty$ (i.e. $\rho \to 0, \epsilon \to 0$). The equation of state $P(\epsilon)$ is defined for $\epsilon \leq \epsilon_{\rm max}$ and is multivalued since it has two branches; see Fig. 2. The branch + corresponds to a phantom universe ($P < P_*$) and the branch − corresponds to a normal universe ($P > P_*$).

In the early universe ($a \to a_i$, $\rho \to \rho_{\rm max}$), the internal energy (new fluid) counteracts the rest-mass energy (dark matter), and we have

$$\epsilon \sim \frac{3c^4}{|K|}\left(\frac{a}{a_i} - 1\right), \qquad \epsilon \sim (\rho_{\rm max} - \rho)c^2,$$
$$P \simeq P_{\rm max} + 2\epsilon. \quad (23)$$

The new fluid formally behaves as a stiff fluid with a negative energy density that we shall call "anti-stiff" fluid (see Sec. V).

In the late universe ($a \to +\infty$, $\rho \to 0$), the rest-mass energy (dark matter) dominates, and we have

$$\epsilon \sim \rho c^2 \sim \rho_0 c^2 \left(\frac{a_0}{a}\right)^3, \qquad P \sim \frac{K}{c^4}\epsilon^2, \qquad P = K\rho^2. \tag{24}$$

These equations describe a fluid with a polytropic equation of state $P = K\epsilon^2/c^4$ of index $n = 1$. For very large values of the scale factor, we recover the results of the CDM model ($P = 0$) since $\epsilon \propto a^{-3}$. This is because $P \ll \epsilon$ in this limit.

For the equation of state (22), the speed of sound is given by

$$\frac{c_s^2}{c^2} = 1 \pm \frac{1}{\sqrt{1 - \frac{4|K|\epsilon}{c^4}}} = \frac{1}{1 - \frac{c^2}{2|K|\rho}}. \tag{25}$$

The ratio $c_s^2/c^2 = 2$ at $a = a_i$ (i.e. $\rho = \rho_{\rm max}$, $\epsilon = 0$), increases as a increases (i.e. ρ decreases, ϵ increases), tends to $+\infty$ as $a \to a_*^-$ (i.e. $\rho \to \rho_*^+$, $\epsilon \to \epsilon_{\rm max}$), tends to $-\infty$ as $a \to a_*^+$ (i.e. $\rho \to \rho_*^-$, $\epsilon \to \epsilon_{\rm max}$), increases as a increases further (ρ decreases further, ϵ decreases), and tends to 0^- as $a \to +\infty$ (i.e. $\rho \to 0$, $\epsilon \to 0$); see Fig. 3. The speed of sound is larger than the speed of light in the phantom era ($\rho > \rho_*$) and imaginary in the normal era ($\rho < \rho_*$).

Substituting Eq. (15) in the Friedmann equation (3), we see that the quadratic equation of state defined by Eq. (13) with $K \leq 0$ describes a model of the Universe exhibiting an anti-stiff matter era ($\epsilon \simeq 0$), a dark matter era ($\epsilon \propto a^{-3}$), and a dark energy era ($\epsilon \sim \rho_\Lambda c^2$) due to the cosmological constant.

IV. COSMOLOGY WITH A STIFF MATTER ERA

In this section, we give examples of systems described by a quadratic equation of state $P = K\rho^2$ leading, in the regime of high density, to a stiff equation of state $P = \epsilon$.

A. Gas of baryons interacting through a vector-meson field

In an early paper, Zel'dovich [20,21] introduced a cosmological model in which the primordial universe is made of a gas of baryons interacting through a vector-meson field and showed that the equation of state of this system is of the form of Eq. (13) with a polytropic constant

$$K = \frac{g^2 h^2}{2\pi m_m^2 m_b^2 c^2}, \tag{26}$$

where g is the baryon charge, m_m is the meson mass, and m_b is the baryon mass. Zel'dovich [21] introduced this equation of state as an example to show how the speed of sound could approach the speed of light at very high pressures and densities.

Zel'dovich [20,21] also mentioned that the complete equation of state of his model is of the form

$$P = K\rho^2 + K'\rho^{4/3}, \tag{27}$$

where the first term is the pressure coming from the self-interaction between the particles and the second term is the quantum (Fermi) pressure. For the equation of state (27), we find from Eq. (9) that the relation between the energy density and the rest-mass density is

$$\epsilon = \rho c^2 + K\rho^2 + 3K'\rho^{4/3}. \tag{28}$$

Using Eq. (12), we get

$$\epsilon = \rho_0 c^2 \left(\frac{a_0}{a}\right)^3 + K\rho_0^2 \left(\frac{a_0}{a}\right)^6 + 3K'\rho_0^{4/3}\left(\frac{a_0}{a}\right)^4. \tag{29}$$

Substituting Eq. (29) in the Friedmann equation (3), we see that the equation of state defined by Eq. (27) with $K \geq 0$ describes a model of the Universe exhibiting a stiff matter era ($\epsilon \propto a^{-6}$), a radiation era ($\epsilon \propto a^{-4}$), a dark matter era ($\epsilon \propto a^{-3}$), and a dark energy era ($\epsilon \sim \rho_\Lambda c^2$) due to the cosmological constant.

Although Zel'dovich's model was abandoned after the success of the hot big bang theory, it remains important from a historical point of view as it is the first suggestion that a stiff matter era may have occurred in the very early universe, before the matter and radiation eras.

B. Partially relativistic self-gravitating BECs

Some authors have proposed that dark matter may be made of self-gravitating BECs with short-range interactions [33–49]. In the TF approximation, a BEC is equivalent to a fluid with an equation of state of the form of Eq. (13) with a polytropic constant

$$K = \frac{2\pi\hbar^2 a_s}{m^3}, \tag{30}$$

where m is the mass of the bosons and a_s is their scattering length. This equation of state can be derived from the classical Gross-Pitaevskii equation [50,51] after writing it under the form of fluid equations by using the Madelung transformation [52] (see, e.g., Refs. [48,49]). This is a nonrelativistic equation of state that, in principle, is not valid in the relativistic regime. Nevertheless, we can consider a partially relativistic model in which we use the classical equation of state (13) with the relativistic relation (14) between the energy density and the rest-mass density. This approximation has been considered in Ref. [14].

An interesting feature of BECs is that the scattering length a_s can be positive or negative [12]. Positive values of a_s correspond to repulsive interactions and negative values of a_s correspond to attractive interactions. When a_s is positive, the pressure is positive ($K \geq 0$) and when a_s is negative the pressure is negative ($K \leq 0$). We shall consider the two possibilities in the following. As we have already explained, when applied to cosmology, Eq. (13) with $K \geq 0$ leads to a stiff matter era in the early universe while Eq. (13) with $K \leq 0$ leads to an anti-stiff matter era.

C. Fully relativistic scalar field

A stiff matter era can also be justified from a field theory for a relativistic SF. The phase of inflation in the very early universe is usually described by a real scalar field called the inflaton [53]. Similarly, in alternative theories to the cosmological constant, the dark energy responsible for the acceleration of the expansion of the Universe is described by a scalar field called quintessence [54].

A scalar field minimally coupled to gravity evolves according to the equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{31}$$

where $V(\phi)$ is the potential of the scalar field. The scalar field tends to run down the potential towards lower energies. The energy density and the pressure of the scalar field are given by

$$\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{32}$$

When the potential energy dominates the kinetic energy, $V(\phi) \gg \dot{\phi}^2/2$, we obtain the equation of state $P = -\epsilon$ of vacuum energy and dark energy. When the kinetic energy dominates the potential energy, $\dot{\phi}^2/2 \gg V(\phi)$, we obtain the equation of state $P = \epsilon$ of stiff matter. In cosmology, this corresponds to the kination epoch of scalar field evolution.

Recently, Li *et al.* [23] have used this argument to show that a universe filled with a relativistic SF/BEC undergoes a primordial stiff matter era preceding a radiation era (in the case where the SF is self-interacting) and a matter era. These results have been recovered by Suárez and Chavanis [24] from a hydrodynamical representation of the KGE equations. It is important to note that the justification of the stiff matter era in the fully relativistic SF/BEC model [23,24] is different from that given in the partially relativistic BEC model (see Ref. [14] and Sec. IV B). However, the present study, which provides analytical results for a cosmology including a stiff matter era, a radiation era, a matter era, and a dark energy era may be useful in the context of fully relativistic SF/BECs even if the equations determining the transition between these phases are different in the two models. For a SF without self-interaction (fuzzy dark matter), there is only a stiff matter era, a matter era, and a dark energy era (no radiation era). This situation corresponds to the analytical solutions considered in Secs. VI and VIII.

D. Analogy with loop quantum cosmology

Combining the Friedmann equation (3) with Eq. (14), and recalling Eq. (12), we obtain

$$H^2 = \frac{8\pi G}{3}\rho\left(1 + \frac{K\rho}{c^2}\right) + \frac{\Lambda}{3}, \qquad \rho = \rho_0\left(\frac{a_0}{a}\right)^3. \tag{33}$$

When $\Lambda = 0$ and $K < 0$, this equation is formally analogous to the modified Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right) \tag{34}$$

that appears in LQC [25].[4] In this analogy $\rho_{\max} = c^2/|K|$. This equation has a bouncing solution which prevents the big bang singularity. It is usually argued in LQC that everything happens as if the quantum gravity effects manifest themselves at the origin as a pressure which forbids the Universe from collapsing and then removes the original singularity [55]. Our approach is consistent with this interpretation since, in our case, Eq. (34) arises precisely from a pressure term with a quadratic equation of state [see Eq. (13)].

In LQC, $\rho_{\max}$ is of the order of the Planck density $\rho_P = c^5/G^2\hbar = 5.16 \times 10^{99}\ \mathrm{g/m^3}$ so that corrections manifest themselves only in the very early universe. If we take $\rho_{\max} = \rho_P$, and identify $\rho_{\max}$ with $c^2/|K|$, we obtain $|K| = c^2/\rho_P$. If we use the expression (30) of K for a BEC, and introduce the dimensionless self-interaction constant $\lambda/8\pi = a_s mc/\hbar$ so that $K = \lambda\hbar^3/4m^4c$, we obtain $|\lambda| = 4(m/M_P)^4$ where $M_P = (\hbar c/G)^{1/2} = 2.17 \times 10^{-5}$ g is the Planck mass. Other consequences of the assumption $|K| = c^2/\rho_P$ are discussed in Appendix A.

V. THE FRIEDMANN EQUATIONS FOR A UNIVERSE PRESENTING A STIFF MATTER ERA

We consider a universe made of several fluids each of them described by a linear equation of state $P = \alpha\epsilon$. The equation of continuity (1) implies that the energy density is related to the scale factor by $\epsilon = \epsilon_0(a_0/a)^{3(1+\alpha)}$, where the subscript 0 denotes present-day values of the quantities. A linear equation of state can describe dark matter ($\alpha = 0$, $\epsilon_m \propto a^{-3}$), radiation ($\alpha = 1/3$, $\epsilon_{\rm rad} \propto a^{-4}$), stiff matter

[4] We note, however, a crucial difference. In LQC, ρc^2 represents the energy density (that we denote as ϵ) so that Eq. (34) is a fundamental modification of the Friedmann equation (3). By contrast, in our study, ρc^2 is the rest-mass density and Eq. (34) is deduced from the usual Friedmann equation (3) by using Eq. (14).

($\alpha = 1$, $\epsilon_s \propto a^{-6}$), vacuum energy ($\alpha = -1$, $\epsilon = \epsilon_P$), and dark energy ($\alpha = -1$, $\epsilon = \epsilon_\Lambda$).

More specifically, we consider a universe made of stiff matter, radiation, dark matter and dark energy treated as noninteracting species. Summing the contribution of each species, the total energy density can be written as

$$\epsilon = \frac{\epsilon_{s,0}}{(a/a_0)^6} + \frac{\epsilon_{\mathrm{rad},0}}{(a/a_0)^4} + \frac{\epsilon_{m,0}}{(a/a_0)^3} + \epsilon_\Lambda. \tag{35}$$

In this model, the stiff matter dominates in the early universe. This is followed by the radiation era, by the dark matter era and, finally, by the dark energy era. Writing $\epsilon_{\alpha,0} = \Omega_{\alpha,0}\epsilon_0$ for each species, we get

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{s,0}}{(a/a_0)^6} + \frac{\Omega_{\mathrm{rad},0}}{(a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \tag{36}$$

In this model, we must require that the energy density of each individual fluid is positive so that $\Omega_{\alpha,0} \geq 0$ for each fluid. We now consider generalizations of this model.

The dark energy term [last term in Eq. (35)] can be interpreted in terms of a cosmological constant Λ by using the correspondence of Eq. (4). At a theoretical level, the cosmological constant may be positive or negative. The discovery of the present acceleration of the expansion of the universe favors a positive cosmological constant but the value of the cosmological constant, and its sign, may have changed in the course of time (it is also possible that the cosmological constant has a constant negative value and that the present acceleration of the Universe is due to another form of dark energy). For example, certain string theories assume that the cosmological constant was negative in the past, corresponding to an anti–de Sitter universe [56]. Therefore, in our theoretical analysis, for the sake of generality, we consider a cosmological constant of arbitrary sign. In terms of the correspondence of Eq. (4), this amounts to considering formally that the dark energy term ϵ_Λ is positive or negative. Therefore, we shall consider the two possibilities $\Omega_{\Lambda,0} \geq 0$ and $\Omega_{\Lambda,0} \leq 0$. The case $\Omega_{\Lambda,0} \leq 0$ will be referred to as anti-dark energy.

We now turn to the stiff matter term [first term in Eq. (35)]. When radiation is ignored, Eq. (36) is identical to Eq. (15) obtained from the equation of state (13) and (26) proposed by Zel'dovich [20,21] or from the equation of state (13) and (30) corresponding to a partially relativistic BEC [14]. On the other hand, Eq. (36) with radiation included is identical to Eq. (29) obtained from the more general equation of state (26) and (27) suggested by Zel'dovich [20,21]. It is very important to note that, in the point of view of Secs. III and IV, we do not have several noninteracting fluids as in the beginning of this section, but simply *one* dark fluid that can present different phases in the course of time. Therefore, the justification of Eq. (15), or Eq. (29), is different from the one given to obtain Eq. (36). Nevertheless, the two equations are formally the same. In this analogy, a positive value of K corresponds to a positive energy density of the stiff matter while a negative value of K corresponds to a negative energy density of the stiff matter. We shall therefore consider the two possibilities $\Omega_{s,0} \geq 0$ and $\Omega_{s,0} \leq 0$. The case $\Omega_{s,0} \leq 0$ will be referred to as anti-stiff matter. We again emphasize that, in the interpretation of Secs. III and IV, the term $\Omega_{s,0}/(a/a_0)^6$ represents the internal energy of the dark fluid, and the term $\Omega_{m,0}/(a/a_0)^3$ represents its rest-mass energy. The rest-mass energy is always positive ($\Omega_{m,0} \geq 0$) while the internal energy can be positive or negative. The laws of physics only require that the total energy density is positive. Therefore, in the approach developed in Secs. III and IV, it is possible to consider $\Omega_{s,0} \leq 0$.

Substituting Eq. (36) in the Friedmann equation (3), we obtain

$$\frac{H}{H_0} = \sqrt{\frac{\Omega_{s,0}}{(a/a_0)^6} + \frac{\Omega_{\mathrm{rad},0}}{(a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}} \tag{37}$$

with $\Omega_{s,0} + \Omega_{\mathrm{rad},0} + \Omega_{m,0} + \Omega_{\Lambda,0} = 1$ and $H_0 = (8\pi G\epsilon_0/3c^2)^{1/2}$. We note the relation

$$\frac{\epsilon}{\epsilon_0} = \left(\frac{H}{H_0}\right)^2 \tag{38}$$

that will be needed in the sequel. The evolution of the scale factor is given by

$$\int_{a_i/a_0}^{a/a_0} \frac{dx}{x\sqrt{\frac{\Omega_{s,0}}{x^6} + \frac{\Omega_{\mathrm{rad},0}}{x^4} + \frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = H_0 t, \tag{39}$$

where a_i is the initial value of the scale factor that has to be determined in each particular case.

In Secs. VI–IX, we ignore radiation ($\Omega_{\mathrm{rad},0} = 0$) and consider a universe made of (anti-)stiff matter, dark matter, and (anti-)dark energy. In that case, the Friedmann equation (39) reduces to

$$\int_{a_i/a_0}^{a/a_0} \frac{dx}{x\sqrt{\frac{\Omega_{s,0}}{x^6} + \frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = H_0 t. \tag{40}$$

It turns out that this equation can be integrated analytically. In Sec. X, we provide some particular analytical solutions of Eq. (39) in the case where the radiation is taken into account.

It is possible to develop a useful mechanical analogy to study the Friedmann equation (37). Defining $\tau = H_0 t$, $R = a/a_0$ and $V(R) = -(1/2)R^2\epsilon(R)/\epsilon_0$, Eq. (37) can be cast in the suggestive form

$$\frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 + V(R) = 0, \tag{41}$$

where

$$V(R) = -\frac{1}{2}\left[\frac{\Omega_{s,0}}{R^4} + \frac{\Omega_{\mathrm{rad},0}}{R^2} + \frac{\Omega_{m,0}}{R} + \Omega_{\Lambda,0}R^2\right]. \tag{42}$$

Equation (41) has the structure of the first integral for the one-dimensional motion of a particle with energy $E = 0$ in a potential $V(R)$. The potential $V(R)$ is plotted in Fig. 4 for the different cases studied in Secs. VI–IX. The condition $V(R) \leq 0$, i.e. $\epsilon \geq 0$, determines the range of accessible "radii" R. We must have

$$\Omega_{\Lambda,0}R^6 + \Omega_{m,0}R^3 + \Omega_{\mathrm{rad},0}R^2 + \Omega_{s,0} \geq 0. \tag{43}$$

When $\Omega_{\mathrm{rad},0} = 0$, Eq. (43) reduces to a second-degree equation for $x = R^3$. The range of accessible radii is specifically given in Secs. VI–IX. The phase portrait of the Universe, with the "velocity" $dR/d\tau = \sqrt{-2V(R)}$ plotted against the "position" R is represented in Fig. 5. These curves show the transition between a decelerating evolution ($\dot{R}$ decreases with R) and an accelerating evolution ($\dot{R}$ increases with R). The transition point R_c is determined by the condition $V'(R_c) = 0$ which is written as

$$2\Omega_{\Lambda,0}R^6 - \Omega_{m,0}R^3 - 2\Omega_{\mathrm{rad},0}R^2 - 4\Omega_{s,0} = 0. \tag{44}$$

When $\Omega_{\mathrm{rad},0} = 0$, Eq. (44) reduces to a second-degree equation for $x = R^3$. The transition point is specifically given in Secs. VI–IX.

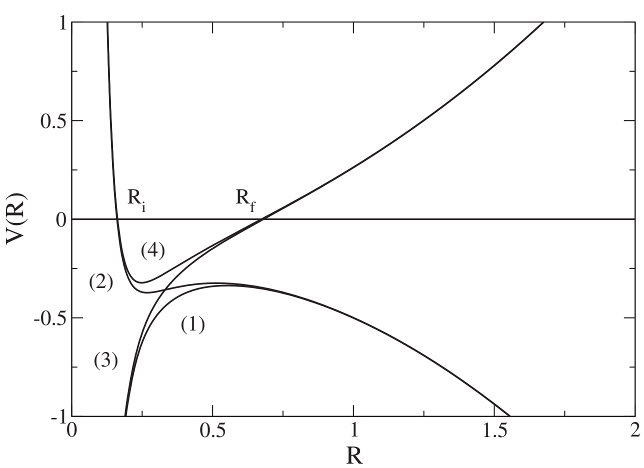

FIG. 4. Effective potential $V(R)$ in the case where (1) $\Omega_{s,0} > 0$ and $\Omega_{\Lambda,0} > 0$, (2) $\Omega_{s,0} < 0$ and $\Omega_{\Lambda,0} > 0$, (3) $\Omega_{s,0} > 0$ and $\Omega_{\Lambda,0} < 0$, (4) $\Omega_{s,0} < 0$ and $\Omega_{\Lambda,0} < 0$. We have taken $|\Omega_{s,0}| = 10^{-3}$, $\Omega_{\mathrm{rad},0} = 0$, $\Omega_{m,0} = 0.237$, and $|\Omega_{\Lambda,0}| = 0.763$.

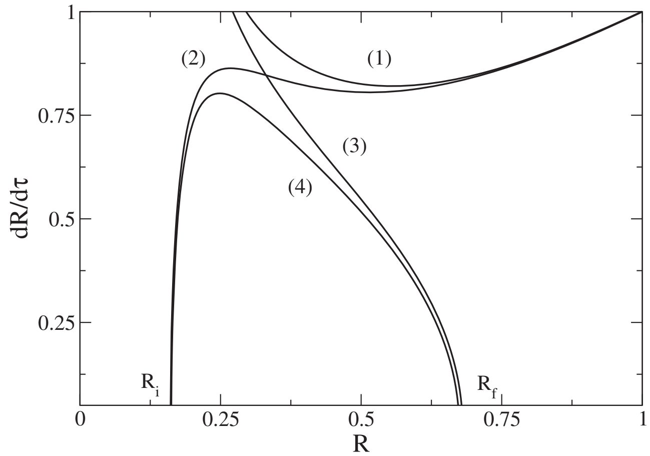

FIG. 5. Phase portrait of the Universe in the four cases considered in Fig. 4.

VI. THE CASE $\Omega_{s,0} \geq 0$ AND $\Omega_{\Lambda,0} \geq 0$

We first consider the case of a positive stiff energy density ($\Omega_{s,0} \geq 0$) and a positive cosmological constant ($\Omega_{\Lambda,0} \geq 0$). The total energy density is

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{s,0}}{(a/a_0)^6} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \tag{45}$$

The energy density starts from $\epsilon = +\infty$ at $a = a_i = 0$, decreases, and tends to ϵ_Λ for $a \to +\infty$. The relation between the energy density and the scale factor is shown in Fig. 6. The proportions of stiff matter, dark matter and dark energy as a function of the scale factor are shown in Fig. 7.

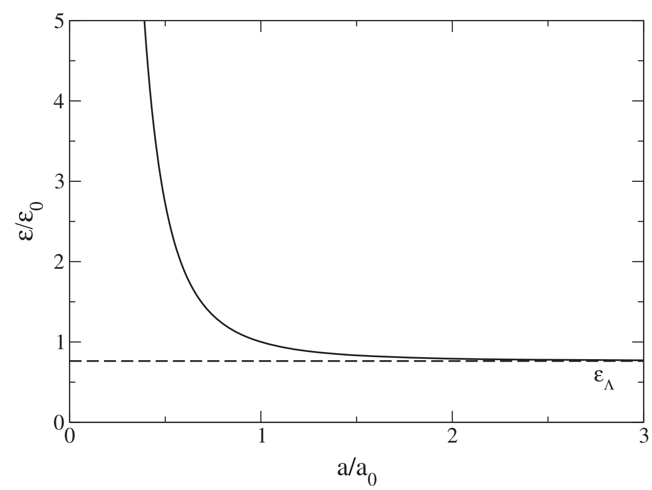

FIG. 6. Energy density as a function of the scale factor. We have taken $\Omega_{m,0} = 0.237$, $\Omega_{\Lambda,0} = 0.763$, and $\Omega_{s,0} = 10^{-3}$ (here and in the following figures, we have chosen a relatively large value of the density of stiff matter $\Omega_{s,0}$ for a better illustration of the results).

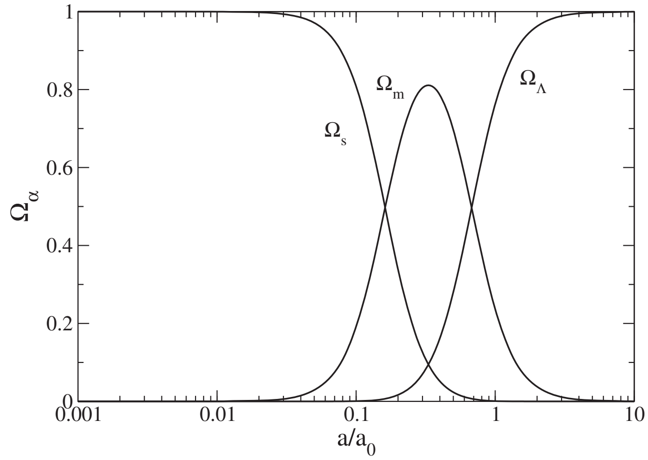

FIG. 7. Evolution of the proportion $\Omega_\alpha = \epsilon_\alpha/\epsilon$ of the different components of the universe with the scale factor.

A. Stiff matter, dark matter, and dark energy

We consider a universe made of stiff matter, dark matter, and dark energy. Using the identity

$$\int \frac{dx}{x\sqrt{\frac{a}{x^3}+\frac{b}{x^6}+c}} = \frac{1}{3\sqrt{c}}\ln\left[a+2cx^3+2\sqrt{c}\sqrt{b+ax^3+cx^6}\right], \quad (46)$$

Eq. (40) can be solved analytically to give

$$\frac{a}{a_0} = \left[\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} + 2\sqrt{\frac{\Omega_{s,0}}{\Omega_{\Lambda,0}}}\right)\sinh^2\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t\right) + \sqrt{\frac{\Omega_{s,0}}{\Omega_{\Lambda,0}}}\left(1-e^{-3\sqrt{\Omega_{\Lambda,0}}H_0 t}\right)\right]^{1/3}. \quad (47)$$

From Eq. (47), we can compute $H = \dot{a}/a$ leading to

$$\left(\frac{a}{a_0}\right)^3 \frac{H}{H_0} = \left(\frac{\Omega_{m,0}}{2\sqrt{\Omega_{\Lambda,0}}} + \sqrt{\Omega_{s,0}}\right)\sinh\left(3\sqrt{\Omega_{\Lambda,0}}H_0 t\right) + \sqrt{\Omega_{s,0}}e^{-3\sqrt{\Omega_{\Lambda,0}}H_0 t}. \quad (48)$$

The energy density ϵ/ϵ_0 is then given by Eq. (38) where H/H_0 can be obtained from Eq. (48) with Eq. (47).

At $t = 0$, the Universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$ (big bang). The scale factor increases with time. For $t \to +\infty$, we obtain

$$\frac{a}{a_0} \sim \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} + 2\sqrt{\frac{\Omega_{s,0}}{\Omega_{\Lambda,0}}}\right)^{1/3} \frac{1}{2^{2/3}} e^{\sqrt{\Omega_{\Lambda,0}}H_0 t}. \quad (49)$$

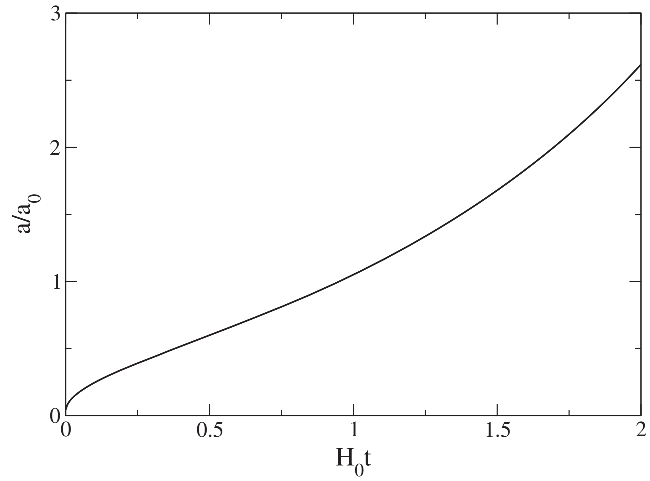

FIG. 8. Evolution of the scale factor as a function of time.

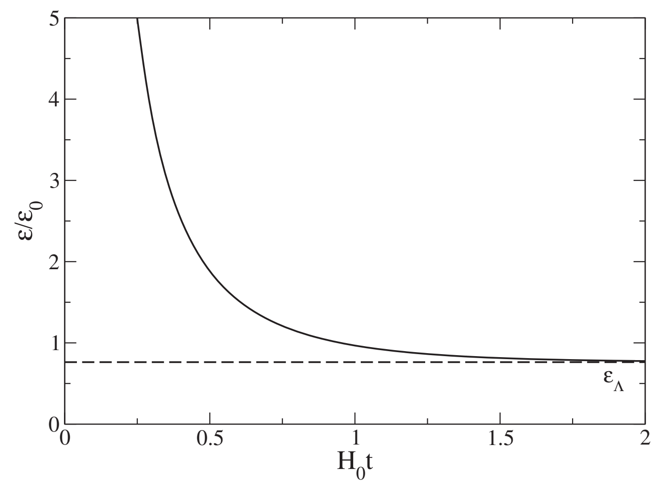

FIG. 9. Evolution of the energy density as a function of time.

The energy density decreases with time and tends to ϵ_Λ for $t \to +\infty$. The expansion is decelerating during the stiff matter era and the dark matter era while it is accelerating during the dark energy era. Using Eq. (44), we find that the transition takes place at

$$\frac{a_c}{a_0} = \left(\frac{\Omega_{m,0} + \sqrt{\Omega_{m,0}^2 + 32\Omega_{\Lambda,0}\Omega_{s,0}}}{4\Omega_{\Lambda,0}}\right)^{1/3}. \quad (50)$$

The temporal evolution of the scale factor and energy density is shown in Figs. 8 and 9.

B. Stiff matter and dark matter

We consider a universe made of stiff matter and dark matter. In the absence of dark energy ($\Omega_{\Lambda,0} = 0$), using the identity

$$\int \frac{dx}{x\sqrt{\frac{a}{x^3}+\frac{b}{x^6}}} = \frac{2}{3a}\sqrt{b+ax^3}, \qquad (51)$$

we obtain

$$\frac{a}{a_0} = \left(\frac{9}{4}\Omega_{m,0}H_0^2t^2 + 3\sqrt{\Omega_{s,0}}H_0t\right)^{1/3}, \qquad (52)$$

$$\frac{\epsilon}{\epsilon_0} = \frac{4}{9H_0^2t^2}\left(\frac{1+\frac{2\sqrt{\Omega_{s,0}}}{3\Omega_{m,0}H_0t}}{1+\frac{4\sqrt{\Omega_{s,0}}}{3\Omega_{m,0}H_0t}}\right)^2. \qquad (53)$$

C. Stiff matter and dark energy

We consider a universe made of stiff matter and dark energy. In the absence of matter ($\Omega_{m,0}=0$), using the identity

$$\int \frac{dx}{x\sqrt{\frac{b}{x^6}+c}} = \frac{1}{3\sqrt{c}}\ln\left[2cx^3+2\sqrt{c}\sqrt{b+cx^6}\right], \qquad (54)$$

or setting $X=b/cx^6$ and using the identity

$$\int \frac{dX}{X\sqrt{X+1}} = \ln\left(\frac{\sqrt{1+X}-1}{\sqrt{1+X}+1}\right), \qquad (55)$$

we get

$$\frac{a}{a_0} = \left(\frac{\Omega_{s,0}}{\Omega_{\Lambda,0}}\right)^{1/6}\sinh^{1/3}(3\sqrt{\Omega_{\Lambda,0}}H_0t), \qquad (56)$$

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\Lambda,0}}{\tanh^2(3\sqrt{\Omega_{\Lambda,0}}H_0t)}. \qquad (57)$$

The universe starts accelerating at $a_c/a_0=(2\Omega_{s,0}/\Omega_{\Lambda,0})^{1/6}$.

D. Stiff matter

We consider a universe made of stiff matter. In the absence of dark matter and dark energy ($\Omega_{m,0}=\Omega_{\Lambda,0}=0$), we find that

$$\frac{a}{a_0} = (3\sqrt{\Omega_{s,0}}H_0t)^{1/3}, \qquad \frac{\epsilon}{\epsilon_0} = \frac{1}{9H_0^2t^2}. \qquad (58)$$

E. Dark matter and dark energy

We consider a universe made of dark matter and dark energy. In the absence of stiff matter ($\Omega_{s,0}=0$), using the identity

$$\int \frac{dx}{x\sqrt{\frac{a}{x^3}+c}} = \frac{1}{3\sqrt{c}}\ln\left[a+2cx^3+2\sqrt{c}\sqrt{ax^3+cx^6}\right], \qquad (59)$$

or setting $X=a/cx^3$ and using the identity (55), we obtain

$$\frac{a}{a_0} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}\sinh^{2/3}\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0t\right), \qquad (60)$$

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\Lambda,0}}{\tanh^2(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0t)}. \qquad (61)$$

This is the ΛCDM model. The universe starts accelerating at $a_c/a_0=(\Omega_{m,0}/2\Omega_{\Lambda,0})^{1/3}$.

F. Dark energy

We consider a universe made of dark energy. In the absence of stiff matter and dark matter ($\Omega_{s,0}=\Omega_{m,0}=0$), we obtain

$$a(t) = a(0)e^{\sqrt{\frac{\Lambda}{3}}t}, \qquad \epsilon=\epsilon_\Lambda. \qquad (62)$$

This is de Sitter's solution.

G. Dark matter

We consider a universe made of dark matter. In the absence of stiff matter and dark energy ($\Omega_{s,0}=\Omega_{\Lambda,0}=0$), we obtain

$$\frac{a}{a_0} = \left(\frac{9}{4}\Omega_{m,0}H_0^2t^2\right)^{1/3}, \qquad \frac{\epsilon}{\epsilon_0} = \frac{4}{9H_0^2t^2}. \qquad (63)$$

This is the EdS solution.

VII. THE CASE $\Omega_{s,0}\le 0$ AND $\Omega_{\Lambda,0}\ge 0$

We consider the case of a negative stiff energy density ($\Omega_{s,0}\le 0$) and a positive cosmological constant ($\Omega_{\Lambda,0}\ge 0$). The total energy density is

$$\frac{\epsilon}{\epsilon_0} = -\frac{|\Omega_{s,0}|}{(a/a_0)^6} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \qquad (64)$$

The energy density is positive for $a\ge a_i$ with

$$\frac{a_i}{a_0} = \left(\frac{-\Omega_{m,0}+\sqrt{\Delta}}{2\Omega_{\Lambda,0}}\right)^{1/3}, \qquad (65)$$

where we have defined

$$\Delta = \Omega_{m,0}^2 + 4\Omega_{\Lambda,0}|\Omega_{s,0}|. \qquad (66)$$

The energy density starts from $\epsilon=0$ at $a=a_i$, increases, reaches a maximum at

$$\frac{a_*}{a_0} = \left(\frac{2|\Omega_{s,0}|}{\Omega_{m,0}}\right)^{1/3}, \qquad \frac{\epsilon_*}{\epsilon_0} = \frac{\Delta}{4|\Omega_{s,0}|}, \qquad (67)$$

decreases, and tends to ϵ_Λ for $a \to +\infty$. The relation between the energy density and the scale factor is shown in Fig. 10. The proportions of stiff matter, dark matter and dark energy as a function of the scale factor are shown in Fig. 11.

A. Anti-stiff matter, dark matter, and dark energy

We consider a universe made of anti-stiff matter, dark matter, and dark energy. Using the identity (46), Eq. (40) with a_i given by Eq. (65) can be solved analytically to give

$$\frac{a}{a_0} = \left[\frac{\sqrt{\Delta}}{2\Omega_{\Lambda,0}}\cosh\left(3\sqrt{\Omega_{\Lambda,0}}H_0 t\right) - \frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right]^{1/3}. \qquad (68)$$

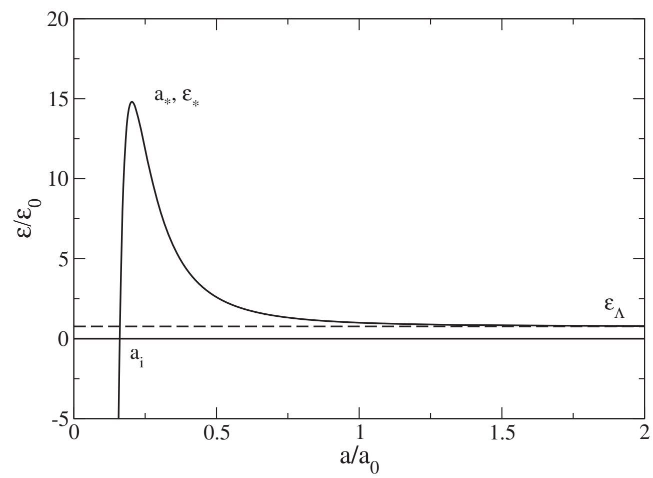

FIG. 10. Energy density as a function of the scale factor. We have taken $\Omega_{m,0} = 0.237$, $\Omega_{\Lambda,0} = 0.763$, and $\Omega_{s,0} = -10^{-3}$.

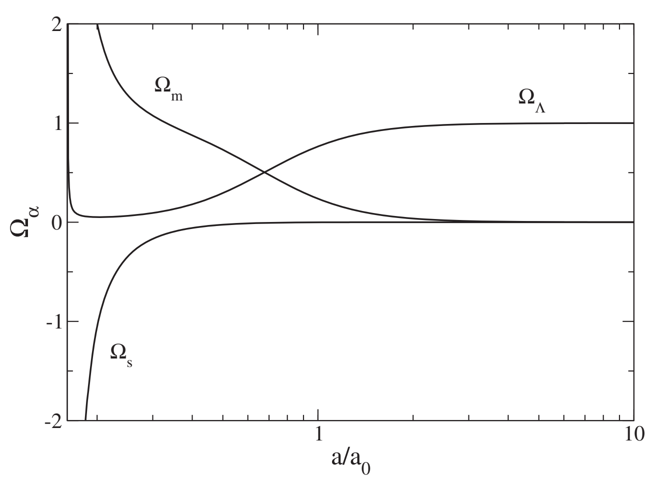

FIG. 11. Evolution of the proportion $\Omega_\alpha = \epsilon_\alpha/\epsilon$ of the different components of the Universe with the scale factor.

From Eq. (68), we can compute $H = \dot{a}/a$ leading to

$$\left(\frac{a}{a_0}\right)^3 \frac{H}{H_0} = \frac{\sqrt{\Delta}}{2\sqrt{\Omega_{\Lambda,0}}}\sinh\left(3\sqrt{\Omega_{\Lambda,0}}H_0 t\right). \qquad (69)$$

The energy density ϵ/ϵ_0 is then given by Eq. (38) where H/H_0 can be obtained from Eq. (69) with Eq. (68).

At $t = 0$ the Universe starts from a nonsingular state at which the scale factor $a = a_i$ and the energy density $\epsilon = 0$. The scale factor increases with time. For $t \to +\infty$, we obtain

$$\frac{a}{a_0} \sim \left[\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^2 + 4\frac{|\Omega_{s,0}|}{\Omega_{\Lambda,0}}\right]^{1/6} \frac{1}{2^{2/3}} e^{\sqrt{\Omega_{\Lambda,0}}H_0 t}. \qquad (70)$$

The energy density increases, reaches its maximum value ϵ_* at $t = t_*$ where

$$t_* = \frac{1}{3\sqrt{\Omega_{\Lambda,0}}H_0}\ln\left(\frac{\sqrt{\Delta} + \sqrt{4\Omega_{\Lambda,0}|\Omega_{s,0}|}}{\Omega_{m,0}}\right), \qquad (71)$$

decreases and tends to ϵ_Λ for $t \to +\infty$. The universe is accelerating during the anti-stiff matter era, decelerating during the dark matter era, and accelerating during the dark energy era. Using Eq. (44), we find that the transitions take place at

$$\frac{a_c^{(\pm)}}{a_0} = \left(\frac{\Omega_{m,0} \pm \sqrt{\Omega_{m,0}^2 - 32\Omega_{\Lambda,0}|\Omega_{s,0}|}}{4\Omega_{\Lambda,0}}\right)^{1/3}. \qquad (72)$$

The temporal evolution of the scale factor and energy density is shown in Figs. 12 and 13.

This model of the Universe is mathematically interesting for the following reasons. For $0 \le t \le t_*$, the evolution of the Universe is phantom-like because the energy density

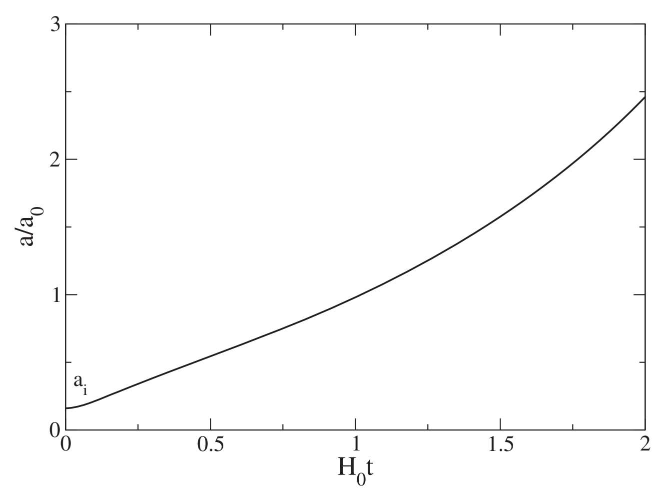

FIG. 12. Evolution of the scale factor as a function of time.

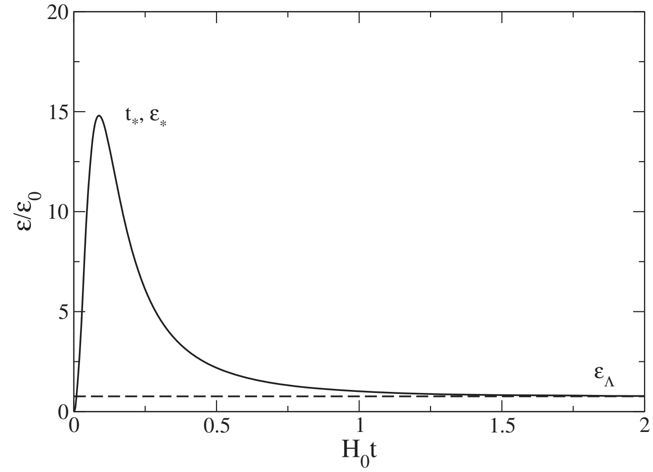

FIG. 13. Evolution of the energy density as a function of time.

increases as the scale factor increases. For $t \geq t_*$, the evolution of the Universe is normal because the energy density decreases as the scale factor increases. On the other hand, this model does not present a big bang singularity at $t = 0$ since the scale factor has a finite value a_i and the energy density vanishes ($\epsilon = 0$). At that moment the Universe is globally empty (it "disappears"). Actually, this model is time symmetric (with respect to the transformation $t \to -t$) so it describes a bouncing universe. The solution valid for $t \geq 0$ can be extended to $t \leq 0$ by symmetry. Therefore, this universe exists eternally in the past and in the future. The scale factor starts from $+\infty$ at $t \to -\infty$, decreases, reaches a minimum a_i at $t = 0$, increases, and tends to $+\infty$ as $t \to +\infty$. In parallel, the energy density starts from ϵ_Λ at $t \to -\infty$, increases, reaches a maximum ϵ_* at $-t_*$, decreases, vanishes at $t = 0$, increases, reaches a maximum ϵ_* at t_*, decreases, and tends to ϵ_Λ at $t \to +\infty$. As mentioned in Sec. IV D, this model shares similarities with the bouncing universe predicted by LQC.

B. Anti-stiff matter and dark matter

We consider a universe made of anti-stiff matter and dark matter. In the absence of dark energy ($\Omega_{\Lambda,0} = 0$), using the identity (51), we obtain

$$\frac{a}{a_0} = \left(\frac{9}{4}\Omega_{m,0} H_0^2 t^2 + \frac{|\Omega_{s,0}|}{\Omega_{m,0}}\right)^{1/3}, \tag{73}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{4}{9H_0^2 t^2} \frac{1}{(1 + \frac{4|\Omega_{s,0}|}{9\Omega_{m,0}^2 H_0^2 t^2})^2}. \tag{74}$$

At $t = 0$ the Universe starts from a nonsingular state at which the scale factor $a = a_i$ with

$$\frac{a_i}{a_0} = \left(\frac{|\Omega_{s,0}|}{\Omega_{m,0}}\right)^{1/3}, \tag{75}$$

and the energy density $\epsilon = 0$. The scale factor increases with time. The energy density increases, reaches its maximum value

$$\frac{a_*}{a_0} = \left(\frac{2|\Omega_{s,0}|}{\Omega_{m,0}}\right)^{1/3}, \qquad \frac{\epsilon_*}{\epsilon_0} = \frac{\Omega_{m,0}^2}{4|\Omega_{s,0}|}, \tag{76}$$

at

$$t_* = \frac{2\sqrt{|\Omega_{s,0}|}}{3\Omega_{m,0} H_0}, \tag{77}$$

and decreases to zero. The universe starts decelerating at $a_c/a_0 = (4|\Omega_{s,0}|/\Omega_{m,0})^{1/3}$.

C. Anti-stiff matter and dark energy

We consider a universe made of anti-stiff matter and dark energy. In the absence of matter ($\Omega_{m,0} = 0$), using the identities (54) and (55), we obtain

$$\frac{a}{a_0} = \left(\frac{|\Omega_{s,0}|}{\Omega_{\Lambda,0}}\right)^{1/6} \cosh^{1/3}(3\sqrt{\Omega_{\Lambda,0}} H_0 t), \tag{78}$$

$$\frac{\epsilon}{\epsilon_0} = \Omega_{\Lambda,0} \tanh^2(3\sqrt{\Omega_{\Lambda,0}} H_0 t). \tag{79}$$

At $t = 0$ the Universe starts from a nonsingular state at which the scale factor $a = a_i$ with

$$\frac{a_i}{a_0} = \left(\frac{|\Omega_{s,0}|}{\Omega_{\Lambda,0}}\right)^{1/6}, \tag{80}$$

and the energy density $\epsilon = 0$. The scale factor increases with time. The energy density increases with time and tends to ϵ_Λ for $t \to +\infty$.

VIII. THE CASE $\Omega_{s,0} \geq 0$ AND $\Omega_{\Lambda,0} \leq 0$

We consider the case of a positive stiff energy density ($\Omega_{s,0} \geq 0$) and a negative cosmological constant ($\Omega_{\Lambda,0} \leq 0$). The total energy density is

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{s,0}}{(a/a_0)^6} + \frac{\Omega_{m,0}}{(a/a_0)^3} - |\Omega_{\Lambda,0}|. \tag{81}$$

The energy density is positive for $a \leq a_f$ with

$$\frac{a_f}{a_0} = \left(\frac{\Omega_{m,0} + \sqrt{\Delta}}{2|\Omega_{\Lambda,0}|}\right)^{1/3}, \tag{82}$$

where we have defined

$$\Delta = \Omega_{m,0}^2 + 4|\Omega_{\Lambda,0}|\Omega_{s,0}. \tag{83}$$

The energy density starts from $\epsilon = +\infty$ at $a = 0$, decreases, and reaches $\epsilon = 0$ at $a = a_f$. The relation

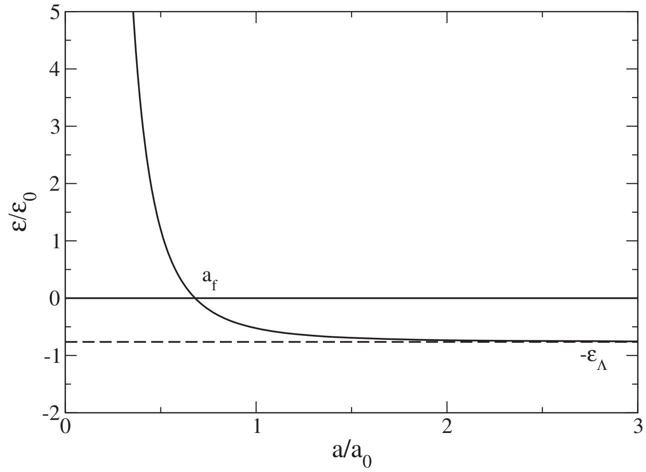

FIG. 14. Energy density as a function of the scale factor. We have taken $\Omega_{m,0} = 0.237$, $\Omega_{\Lambda,0} = -0.763$, and $\Omega_{s,0} = 10^{-3}$.

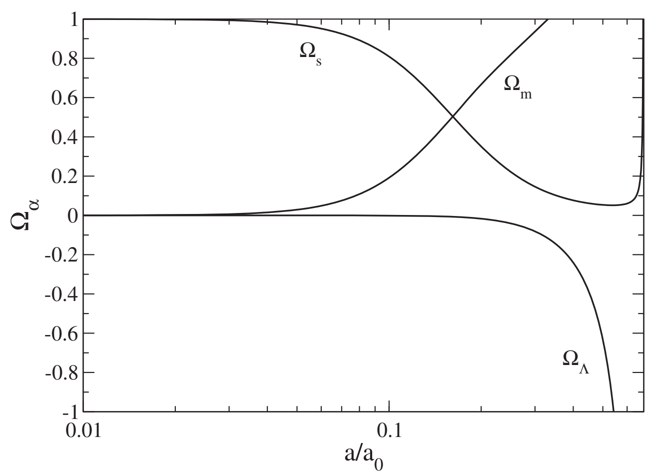

FIG. 15. Evolution of the proportion $\Omega_\alpha = \epsilon_\alpha/\epsilon$ of the different components of the Universe with the scale factor.

between the energy density and the scale factor is shown in Fig. 14. The proportions of stiff matter, dark matter and dark energy as a function of the scale factor are shown in Fig. 15.

A. Stiff matter, dark matter, and anti-dark energy

We consider a universe made of stiff matter, dark matter, and anti-dark energy. From Eqs. (47) and (48) with $\Omega_{\Lambda,0} < 0$, we get

$$\frac{a}{a_0} = \left[\frac{\Omega_{m,0}}{|\Omega_{\Lambda,0}|}\sin^2\left(\frac{3}{2}\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right) + \sqrt{\frac{\Omega_{s,0}}{|\Omega_{\Lambda,0}|}}\sin\left(3\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right)\right]^{1/3} \tag{84}$$

and

$$\left(\frac{a}{a_0}\right)^3 \frac{H}{H_0} = \frac{\Omega_{m,0}}{2\sqrt{|\Omega_{\Lambda,0}|}}\sin\left(3\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right) + \sqrt{\Omega_{s,0}}\cos\left(3\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right). \tag{85}$$

The energy density is then given by Eq. (38) where H/H_0 can be obtained from Eq. (85) with Eq. (84).

At $t = 0$ the Universe starts from a singularity at which the scale factor $a = 0$ and the energy density $\epsilon \to +\infty$ (big bang). Between $t = 0$ and $t = t_1$ where

$$t_1 = \frac{\pi - \tan^{-1}(2\sqrt{\Omega_{s,0}|\Omega_{\Lambda,0}|}/\Omega_{m,0})}{3\sqrt{|\Omega_{\Lambda,0}|}H_0}, \tag{86}$$

the energy density decreases from $\epsilon \to +\infty$ to $\epsilon = 0$ and the scale factor increases from $a = 0$ to $a = a_f$ (at $t = t_1$, the Universe is globally empty—it disappears—since $\epsilon = 0$). During this period, the universe is always decelerating. Between $t = t_1$ and $t = t_2 = 2t_1$ the energy density increases from $\epsilon = 0$ to $\epsilon \to +\infty$ and the scale factor decreases from $a = a_f$ to $a = 0$ (big crunch). This process continues periodically in time with a period t_2. However, it may not be possible to cross the singularity at $t = t_2$, so the physical solution may be restricted to the interval $0 \leq t \leq t_2$. The temporal evolution of the scale factor and energy density is shown in Figs. 16 and 17.

Remark: Other types of cyclic universes have been studied in the past, the most famous ones (corresponding to $k = \pm 1$, $\Lambda < 0$ and $P = 0$, or $k = +1$, $\Lambda = 0$ and $P = 0$) being due to Friedmann [57,58], Einstein [59], and Lemaître [60]. They were called "phoenix universes" by Lemaître because, in these models, the Universe undergoes regular periods of expansion and contraction during which it "dies" (big crunch) and "rises again" (big bang).

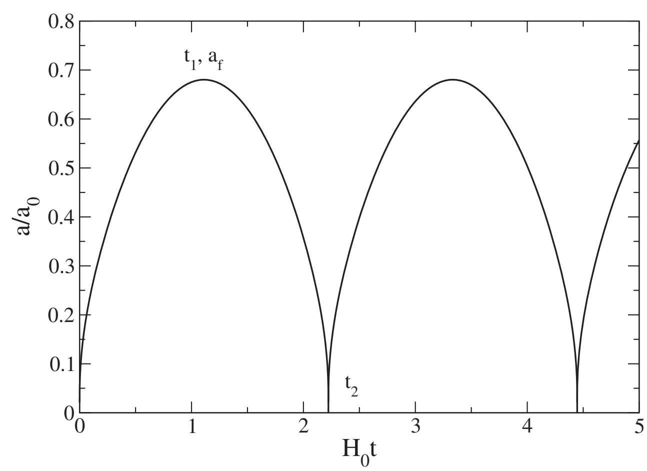

FIG. 16. Evolution of the scale factor as a function of time.

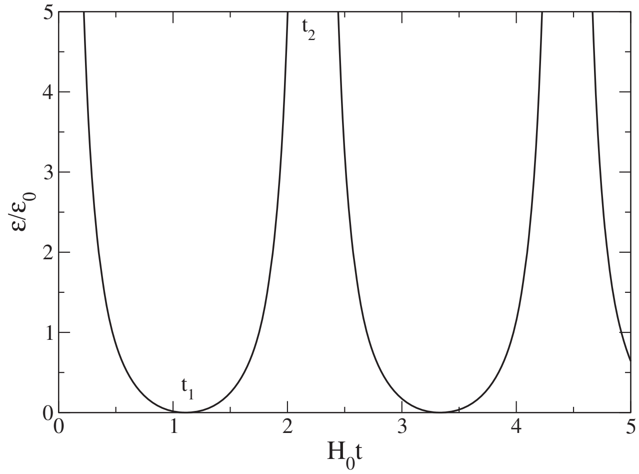

FIG. 17. Evolution of the energy density as a function of time.

B. Stiff matter and anti-dark energy

We consider a universe made of stiff matter and anti-dark energy. In the absence of matter ($\Omega_{m,0} = 0$), we get

$$\frac{a}{a_0} = \left(\frac{\Omega_{s,0}}{|\Omega_{\Lambda,0}|}\right)^{1/6} \sin^{1/3}\left(3\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right), \tag{87}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{|\Omega_{\Lambda,0}|}{\tan^2(3\sqrt{|\Omega_{\Lambda,0}|}H_0 t)}. \tag{88}$$

At $t = 0$ the Universe starts from a singularity at which the scale factor $a = 0$ and the energy density $\epsilon \to +\infty$. Between $t = 0$ and $t = t_1$ where

$$t_1 = \frac{\pi}{6\sqrt{|\Omega_{\Lambda,0}|}H_0}, \tag{89}$$

the energy density decreases from $\epsilon \to +\infty$ to $\epsilon = 0$ and the scale factor increases from $a = 0$ to $a = a_f$ where

$$\frac{a_f}{a_0} = \left(\frac{\Omega_{s,0}}{|\Omega_{\Lambda,0}|}\right)^{1/6}. \tag{90}$$

Between $t = t_1$ and $t = t_2 = 2t_1$ the energy density increases from $\epsilon = 0$ to $\epsilon \to +\infty$ and the scale factor decreases from $a = a_f$ to $a = 0$. This process continues periodically in time with a period t_2.

C. Dark matter and anti-dark energy

We consider a universe made of dark matter and anti-dark energy. In the absence of stiff matter ($\Omega_{s,0} = 0$), we obtain

$$\frac{a}{a_0} = \left(\frac{\Omega_{m,0}}{|\Omega_{\Lambda,0}|}\right)^{1/3} \sin^{2/3}\left(\frac{3}{2}\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right), \tag{91}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{|\Omega_{\Lambda,0}|}{\tan^2(\frac{3}{2}\sqrt{|\Omega_{\Lambda,0}|}H_0 t)}. \tag{92}$$

At $t = 0$ the Universe starts from a singularity at which the scale factor $a = 0$ and the energy density $\epsilon \to +\infty$. Between $t = 0$ and $t = t_1$ where

$$t_1 = \frac{\pi}{3\sqrt{|\Omega_{\Lambda,0}|}H_0}, \tag{93}$$

the energy density decreases from $\epsilon \to +\infty$ to $\epsilon = 0$ and the scale factor increases from $a = 0$ to $a = a_f$ where

$$\frac{a_f}{a_0} = \left(\frac{\Omega_{m,0}}{|\Omega_{\Lambda,0}|}\right)^{1/3}. \tag{94}$$

Between $t = t_1$ and $t = t_2 = 2t_1$ the energy density increases from $\epsilon = 0$ to $\epsilon \to +\infty$ and the scale factor decreases from $a = a_f$ to $a = 0$. This process continues periodically in time with a period t_2. This solution corresponds to the anti-ΛCDM model (see Sec. 6 of Ref. [16]).

IX. THE CASE $\Omega_{s,0} \leq 0$ AND $\Omega_{\Lambda,0} \leq 0$

We consider the case of a negative stiff energy density ($\Omega_{s,0} \leq 0$) and a negative cosmological constant ($\Omega_{\Lambda,0} \leq 0$). The total energy density is

$$\frac{\epsilon}{\epsilon_0} = -\frac{|\Omega_{s,0}|}{(a/a_0)^6} + \frac{\Omega_{m,0}}{(a/a_0)^3} - |\Omega_{\Lambda,0}|. \tag{95}$$

If $|\Omega_{s,0}| > \Omega_{m,0}^2/(4|\Omega_{\Lambda,0}|)$ the energy density is always negative so this situation is not possible. Therefore, we assume $|\Omega_{s,0}| \leq \Omega_{m,0}^2/(4|\Omega_{\Lambda,0}|)$. In that case, the energy density is positive for $a_i \leq a \leq a_f$ with

$$\frac{a_i}{a_0} = \left(\frac{\Omega_{m,0} - \sqrt{\Delta}}{2|\Omega_{\Lambda,0}|}\right)^{1/3} \tag{96}$$

and

$$\frac{a_f}{a_0} = \left(\frac{\Omega_{m,0} + \sqrt{\Delta}}{2|\Omega_{\Lambda,0}|}\right)^{1/3}, \tag{97}$$

where we have defined

$$\Delta = \Omega_{m,0}^2 - 4|\Omega_{\Lambda,0}||\Omega_{s,0}|. \tag{98}$$

The energy density starts from $\epsilon = 0$ at $a = a_i$, increases, reaches a maximum at

$$\frac{a_*}{a_0} = \left(\frac{2|\Omega_{s,0}|}{\Omega_{m,0}}\right)^{1/3}, \qquad \frac{\epsilon_*}{\epsilon_0} = \frac{\Delta}{4|\Omega_{s,0}|}, \tag{99}$$

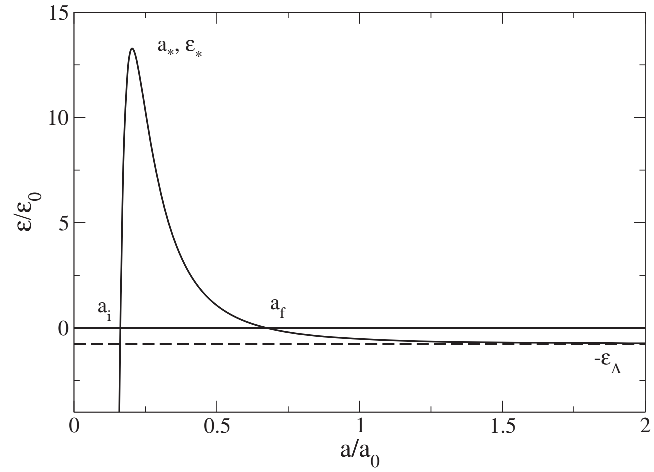

FIG. 18. Energy density as a function of the scale factor. We have taken $\Omega_{m,0} = 0.237$, $\Omega_{\Lambda,0} = -0.763$, and $\Omega_{s,0} = -10^{-3}$.

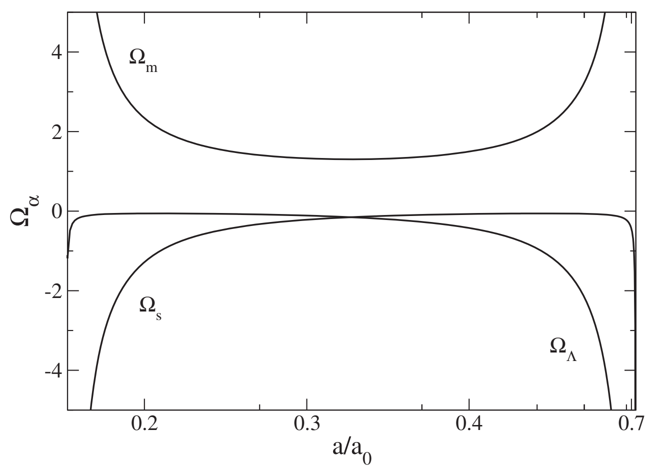

FIG. 19. Evolution of the proportion $\Omega_\alpha = \epsilon_\alpha/\epsilon$ of the different components of the Universe with the scale factor.

decreases, and reaches $\epsilon = 0$ at $a = a_f$. The relation between the energy density and the scale factor is shown in Fig. 18. The proportions of stiff matter, dark matter and dark energy as a function of the scale factor are shown in Fig. 19.

A. Anti-stiff matter, dark matter, and anti-dark energy

We consider a universe made of anti-stiff matter, dark matter, and anti-dark energy. From Eqs. (68) and (69) with $\Omega_{\Lambda,0} < 0$, we get

$$\frac{a}{a_0} = \left[\frac{\Omega_{m,0}}{2|\Omega_{\Lambda,0}|} - \frac{\sqrt{\Delta}}{2|\Omega_{\Lambda,0}|}\cos\left(3\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right)\right]^{1/3} \tag{100}$$

and

$$\left(\frac{a}{a_0}\right)^3 \frac{H}{H_0} = \frac{\sqrt{\Delta}}{2\sqrt{|\Omega_{\Lambda,0}|}}\sin\left(3\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right). \tag{101}$$

The energy density is then given by Eq. (38) where H/H_0 can be obtained from Eq. (101) with Eq. (100).

At $t = 0$ the Universe starts from a nonsingular state at which the scale factor $a = a_i$ and the energy density $\epsilon = 0$. Between $t = 0$ and $t = t_*$ where

$$t_* = \frac{\cos^{-1}(\sqrt{\Delta}/\Omega_{m,0})}{3\sqrt{|\Omega_{\Lambda,0}|}H_0}, \tag{102}$$

the energy density increases from $\epsilon = 0$ to its maximum value $\epsilon = \epsilon_*$ and the scale factor increases from $a = a_i$ to $a = a_*$ (phantom behavior). Between $t = t_*$ and $t = t_1$ where

$$t_1 = \frac{\pi}{3\sqrt{|\Omega_{\Lambda,0}|}H_0}, \tag{103}$$

the energy density decreases from $\epsilon = \epsilon_*$ to $\epsilon = 0$ and the scale factor increases from $a = a_*$ to $a = a_f$. The universe is accelerating during the anti-stiff matter era and decelerating during the dark matter and anti-dark energy eras. Using Eq. (44), we find that the transition takes place at

$$\frac{a_c}{a_0} = \left(\frac{-\Omega_{m,0} + \sqrt{\Omega_{m,0}^2 + 32|\Omega_{\Lambda,0}||\Omega_{s,0}|}}{4|\Omega_{\Lambda,0}|}\right)^{1/3}. \tag{104}$$

Between $t = t_1$ and t'_* where

$$t'_* = \frac{2\pi - \cos^{-1}(\sqrt{\Delta}/\Omega_{m,0})}{3\sqrt{|\Omega_{\Lambda,0}|}H_0}, \tag{105}$$

the energy density increases from $\epsilon = 0$ to $\epsilon = \epsilon_*$ and the scale factor decreases from $a = a_f$ to $a = a_*$. Between t'_* and $t_2 = 2t_1$ the energy density decreases from $\epsilon = \epsilon_*$ to $\epsilon = 0$ and the scale factor decreases from $a = a_*$ to $a = a_i$ (phantom behavior). This process continues periodically in time with a period t_2. The temporal evolution of the scale factor and energy density is shown in Figs. 20 and 21.

This model of universe is mathematically interesting because it combines the bouncing properties of the model of Sec. VII (it can be extended to $t \leq 0$ by symmetry) together with the oscillatory properties of the model of Sec. VIII. This model is never singular. It exists eternally in the past and in the future, and is perpetually oscillating. It displays phases of expansion and phases of contraction. It behaves in certain periods of time as a phantom universe, and in other periods of time normally. At some moments, it is globally empty ($\epsilon = 0$ with $a > 0$). Therefore, this model of universe is mathematically very rich.

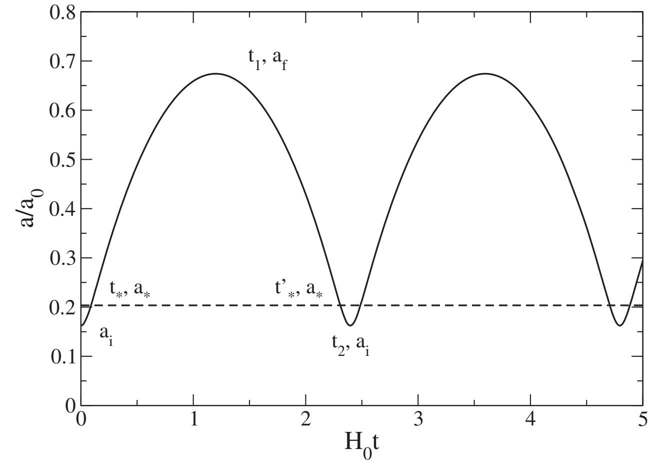

FIG. 20. Evolution of the scale factor as a function of time.

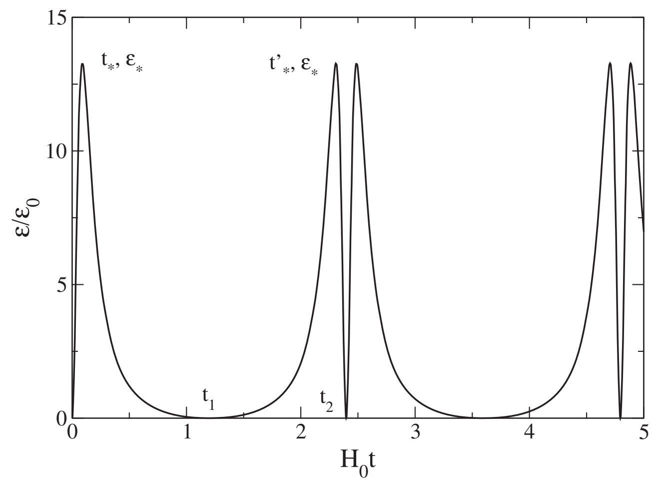

FIG. 21. Evolution of the energy density as a function of time.

X. SOME ANALYTICAL SOLUTIONS INCLUDING THE RADIATION ERA

We now come back to the general equation (39) including the contribution of radiation and provide some particular analytical solutions.

A. Stiff matter and radiation

We consider a universe made of stiff matter and radiation. The total energy is

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{s,0}}{(a/a_0)^6} + \frac{\Omega_{\mathrm{rad},0}}{(a/a_0)^4}. \tag{106}$$

The energy density starts from $\epsilon \to +\infty$ at $a = a_i = 0$ and decreases to zero as a increases to $+\infty$. In the absence of dark matter and dark energy ($\Omega_{m,0} = \Omega_{\Lambda,0} = 0$) the integral in Eq. (39) can be performed analytically giving

$$2\sqrt{\Omega_{\mathrm{rad},0}}\frac{a}{a_0}\sqrt{\Omega_{s,0} + \Omega_{\mathrm{rad},0}\left(\frac{a}{a_0}\right)^2} - 2\Omega_{s,0} \times \ln\left[\Omega_{\mathrm{rad},0}\frac{a}{a_0} + \sqrt{\Omega_{\mathrm{rad},0}}\sqrt{\Omega_{s,0} + \Omega_{\mathrm{rad},0}\left(\frac{a}{a_0}\right)^2}\right] + \Omega_{s,0}\ln(\Omega_{s,0}\Omega_{\mathrm{rad},0}) = 4(\Omega_{\mathrm{rad},0})^{3/2}H_0 t. \tag{107}$$

Using the identity $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$, we can rewrite the foregoing expression as

$$\sqrt{\frac{\Omega_{\mathrm{rad},0}}{\Omega_{s,0}}}\frac{a}{a_0}\sqrt{1 + \frac{\Omega_{\mathrm{rad},0}}{\Omega_{s,0}}\left(\frac{a}{a_0}\right)^2} - \sinh^{-1}\left(\sqrt{\frac{\Omega_{\mathrm{rad},0}}{\Omega_{s,0}}}\frac{a}{a_0}\right) = \frac{2(\Omega_{\mathrm{rad},0})^{3/2}}{\Omega_{s,0}}H_0 t. \tag{108}$$

At $t = 0$ the Universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$. The scale factor increases with time while the energy density decreases with time. The universe is always decelerating. The temporal evolution of the scale factor and energy density is shown in Fig. 22.

B. Anti-stiff matter and radiation

We consider a universe made of anti-stiff matter and radiation. The total energy is

$$\frac{\epsilon}{\epsilon_0} = -\frac{|\Omega_{s,0}|}{(a/a_0)^6} + \frac{\Omega_{\mathrm{rad},0}}{(a/a_0)^4}. \tag{109}$$

The energy density is positive for $a \geq a_i$ with

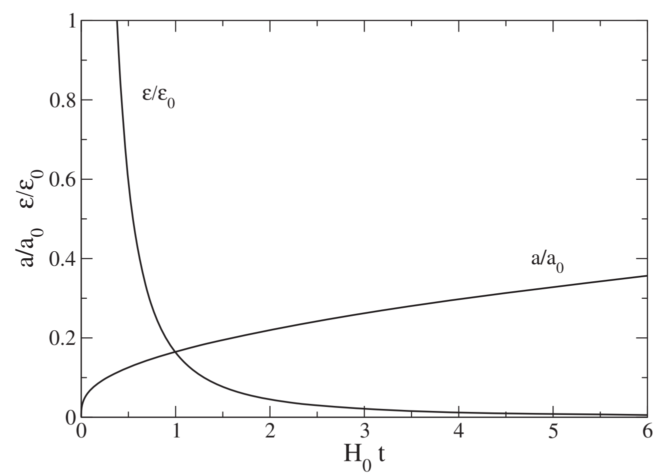

FIG. 22. Evolution of the scale factor and energy density as a function of time. We have taken $\Omega_{s,0} = 10^{-6}$ and $\Omega_{\mathrm{rad},0} = 8.4810^{-5}$.

$$\frac{a_i}{a_0} = \left(\frac{|\Omega_{s,0}|}{\Omega_{\mathrm{rad},0}}\right)^{1/2}. \quad (110)$$

The energy density starts from $\epsilon = 0$ at $a = a_i$, increases, reaches a maximum at

$$\frac{a_*}{a_0} = \left(\frac{3|\Omega_{s,0}|}{2\Omega_{\mathrm{rad},0}}\right)^{1/2}, \qquad \frac{\epsilon_*}{\epsilon_0} = \frac{4\Omega_{\mathrm{rad},0}^3}{27|\Omega_{s,0}|^2}, \quad (111)$$

and decreases to zero as a increases to $+\infty$. In the absence of dark matter and dark energy ($\Omega_{m,0} = \Omega_{\Lambda,0} = 0$) the integral in Eq. (39) can be performed analytically giving

$$2\sqrt{\Omega_{\mathrm{rad},0}}\frac{a}{a_0}\sqrt{-|\Omega_{s,0}| + \Omega_{\mathrm{rad},0}\left(\frac{a}{a_0}\right)^2} + 2|\Omega_{s,0}| \times \ln\left[\Omega_{\mathrm{rad},0}\frac{a}{a_0} + \sqrt{\Omega_{\mathrm{rad},0}}\sqrt{-|\Omega_{s,0}| + \Omega_{\mathrm{rad},0}\left(\frac{a}{a_0}\right)^2}\right] - |\Omega_{s,0}|\ln(|\Omega_{s,0}|\Omega_{\mathrm{rad},0}) = 4(\Omega_{\mathrm{rad},0})^{3/2}H_0 t. \quad (112)$$

Using the identity $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$, we can rewrite the foregoing expression as

$$\sqrt{\frac{\Omega_{\mathrm{rad},0}}{|\Omega_{s,0}|}}\frac{a}{a_0}\sqrt{\frac{\Omega_{\mathrm{rad},0}}{|\Omega_{s,0}|}\left(\frac{a}{a_0}\right)^2 - 1} + \cosh^{-1}\left(\sqrt{\frac{\Omega_{\mathrm{rad},0}}{|\Omega_{s,0}|}}\frac{a}{a_0}\right) = \frac{2(\Omega_{\mathrm{rad},0})^{3/2}}{|\Omega_{s,0}|}H_0 t. \quad (113)$$

At $t = 0$ the Universe starts from a nonsingular state at which the scale factor $a = a_i$ and the energy density $\epsilon = 0$. The scale factor increases with time. The energy density starts from $\epsilon = 0$, increases, reaches its maximum value ϵ_* at $t = t_*$ where

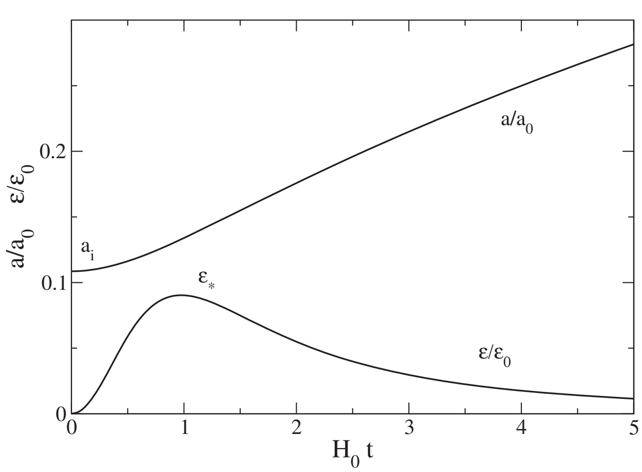

FIG. 23. Evolution of the scale factor and energy density as a function of time. We have taken $\Omega_{s,0} = -10^{-6}$ and $\Omega_{\mathrm{rad},0} = 8.4810^{-5}$.

$$t_* = \frac{|\Omega_{s,0}|}{4\Omega_{\mathrm{rad},0}^{3/2}H_0}[\sqrt{3} + 2\ln(\sqrt{3}+1) - \ln 2], \quad (114)$$

and decreases to 0 as $t \to +\infty$. The universe is accelerating during the anti-stiff matter era and decelerating during the radiation era. Using Eq. (44), we find that the transition takes place at $a_c/a_0 = (2|\Omega_{s,0}|/\Omega_{\mathrm{rad},0})^{1/2}$. The temporal evolution of the scale factor and energy density is shown in Fig. 23. This solution can be extended by symmetry to $t \leq 0$ so it describes a bouncing universe. There is no big bang singularity. The universe is successively normal, phantom, and normal again.

C. Radiation

We consider a universe made of radiation. In the absence of stiff matter, dark matter, and dark energy ($\Omega_{s,0} = \Omega_{m,0} = \Omega_{\Lambda,0} = 0$) we get

$$\frac{a}{a_0} = \Omega_{\mathrm{rad},0}^{1/4}\sqrt{2H_0 t}, \qquad \frac{\epsilon}{\epsilon_0} = \frac{1}{(2H_0 t)^2}. \quad (115)$$

D. Radiation and dark matter

We consider a universe made of radiation and dark matter. The total energy is

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\mathrm{rad},0}}{(a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3}. \quad (116)$$

The energy density starts from $\epsilon \to +\infty$ at $a = a_i = 0$ and decreases to zero as a increases to $+\infty$. In the absence of stiff matter and dark energy ($\Omega_{s,0} = \Omega_{\Lambda,0} = 0$) the integral in Eq. (39) can be performed analytically leading to

$$H_0 t = -\frac{2}{3}\frac{1}{(\Omega_{m,0})^{1/2}}\left(\frac{2\Omega_{\mathrm{rad},0}}{\Omega_{m,0}} - \frac{a}{a_0}\right)\sqrt{\frac{\Omega_{\mathrm{rad},0}}{\Omega_{m,0}} + \frac{a}{a_0}} + \frac{4}{3}\frac{(\Omega_{\mathrm{rad},0})^{3/2}}{(\Omega_{m,0})^2}. \quad (117)$$

Equation (117) can also be written as

$$\left(\frac{a}{a_0}\right)^3 - 3\frac{\Omega_{\mathrm{rad},0}}{\Omega_{m,0}}\left(\frac{a}{a_0}\right)^2 = \frac{9}{4}\Omega_{m,0}H_0^2 t^2 - 6\frac{\Omega_{\mathrm{rad},0}^{3/2}}{\Omega_{m,0}}H_0 t. \quad (118)$$

This is a cubic equation for a/a_0. At $t = 0$ the Universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$. The scale factor increases with time while the energy density decreases with time. The universe is always decelerating.

E. Radiation and dark energy

We consider a universe made of radiation and dark energy. The total energy is

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\mathrm{rad},0}}{(a/a_0)^4} + \Omega_{\Lambda,0}. \tag{119}$$

The energy density starts from $\epsilon \to +\infty$ at $a = a_i = 0$ and tends to ϵ_Λ as $a \to +\infty$. In the absence of stiff matter and dark matter ($\Omega_{s,0} = \Omega_{m,0} = 0$), we get

$$\frac{a}{a_0} = \left(\frac{\Omega_{\mathrm{rad},0}}{\Omega_{\Lambda,0}}\right)^{1/4} \sinh^{1/2}(2\sqrt{\Omega_{\Lambda,0}}H_0 t), \tag{120}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\Lambda,0}}{\tanh^2(2\sqrt{\Omega_{\Lambda,0}}H_0 t)}. \tag{121}$$

At $t = 0$ the Universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$. The scale factor increases with time while the energy density decreases with time and tends to ϵ_Λ for $t \to +\infty$. The universe is decelerating during the radiation era and accelerating during the dark energy era. Using Eq. (44), we find that the transition takes place at $a_c/a_0 = (\Omega_{\mathrm{rad},0}/\Omega_{\Lambda,0})^{1/4}$.

F. Radiation and anti-dark energy

We consider a universe made of radiation and anti-dark energy. The total energy is

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\mathrm{rad},0}}{(a/a_0)^4} - |\Omega_{\Lambda,0}|. \tag{122}$$

The energy density is positive for $a \le a_f$ with

$$\frac{a_f}{a_0} = \left(\frac{\Omega_{\mathrm{rad},0}}{|\Omega_{\Lambda,0}|}\right)^{1/4}. \tag{123}$$

The energy density starts from $\epsilon \to +\infty$ at $a = a_i = 0$, decreases, and reaches $\epsilon = 0$ at $a = a_f$. In the absence of stiff matter and dark matter ($\Omega_{s,0} = \Omega_{m,0} = 0$), we get

$$\frac{a}{a_0} = \left(\frac{\Omega_{\mathrm{rad},0}}{|\Omega_{\Lambda,0}|}\right)^{1/4} \sin^{1/2}\left(2\sqrt{|\Omega_{\Lambda,0}|}H_0 t\right), \tag{124}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{|\Omega_{\Lambda,0}|}{\tan^2(2\sqrt{|\Omega_{\Lambda,0}|}H_0 t)}. \tag{125}$$

At $t = 0$ the Universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$. Between $t = 0$ and $t = t_1$ where

$$t_1 = \frac{\pi}{4\sqrt{|\Omega_{\Lambda,0}|}H_0}, \tag{126}$$

the energy density decreases from $\epsilon \to +\infty$ to $\epsilon = 0$ and the scale factor increases from $a = 0$ to $a = a_f$. The universe is always decelerating. Between t_1 and $t_2 = 2t_1$ the energy density increases from $\epsilon = 0$ to $\epsilon \to +\infty$ and the scale factor decreases from $a = a_f$ to $a = 0$. This process continues periodically in time with a period t_2.

XI. TRANSITION BETWEEN THE INFLATION AND THE STIFF MATTER ERA

It is believed that the very early universe underwent an inflationary era, driven by the vacuum energy with a constant density of the order of the Planck density $\rho_P = c^5/G^2\hbar = 5.16 \times 10^{99}$ g/m^3, during which the scale factor increased exponentially rapidly. To complete our description of a cosmology with a stiff matter era, it is important to describe the transition between the inflation era and the stiff matter era.

An interpolation formula describing the transition between a phase of inflation where the energy density is constant ($\epsilon = \rho_P c^2$) and a phase described by a linear equation of state $P = \alpha\epsilon$ can be obtained by considering an equation of state of the form [15–17]

$$P = \alpha\epsilon - (\alpha + 1)\epsilon\left(\frac{\epsilon}{\rho_P c^2}\right)^{1/n}, \tag{127}$$

with $n > 0$. This is the sum of a linear equation of state $P = \alpha\epsilon$ and a polytropic equation of state $P = K\epsilon^\gamma$ with $\gamma = 1 + 1/n$ and $K = -(\alpha + 1)/(\rho_P c^2)^{1/n}$. Therefore, by taking $\alpha = 1$, we can describe the transition between the inflation and the stiff matter era. However, to be more general, we leave the parameter α unspecified (but assume $0 \le \alpha \le 1$ to simplify the discussion) so that we can also describe the transition between the inflation and the radiation era ($\alpha = 1/3$) or the transition between the inflation and the matter era ($\alpha = 0$). It is also possible to introduce a generalized polytropic model based on a quadratic equation of state that describes simultaneously the early inflation, the intermediate decelerating expansion, and the late acceleration of the Universe [61,62].

For the equation of state (127), assuming $w = P/\epsilon \ge -1$ (nonphantom), the continuity equation (1) can be integrated into

$$\epsilon = \frac{\rho_P c^2}{[1 + (a/a_*)^{3(1+\alpha)/n}]^n}, \tag{128}$$

where a_* is a constant of integration. The speed of sound is given by

$$\frac{c_s^2}{c^2} = \alpha - (\alpha + 1)\frac{n+1}{n}\left(\frac{\epsilon}{\rho_P c^2}\right)^{1/n}. \tag{129}$$

For $a \ll a_*$, we obtain $\epsilon \to \rho_P c^2$, $P \to -\rho_P c^2$ ($P \sim -\epsilon$) and $c_s^2/c^2 \to -(\alpha + n + 1)/n$, leading to a phase of inflation.

For $a \gg a_*$, we obtain $\epsilon \sim \rho_P c^2/(a/a_*)^{3(1+\alpha)}$, $P \sim \alpha\epsilon$, and $c_s^2/c^2 \simeq \alpha$ corresponding to a linear equation of state. Therefore, a_* marks the transition between the inflation era and the α era. Using the asymptotic result $\epsilon_\alpha \sim \rho_P c^2/(a/a_*)^{3(1+\alpha)}$, the transition scale factor is determined by $a_*/a_0 = (\Omega_{\alpha,0}\epsilon_0/\rho_P c^2)^{1/[3(1+\alpha)]}$ where $\Omega_{\alpha,0}$ is the present fraction of the α fluid in the Universe.

Substituting Eq. (128) in the Friedmann equation (3) with $\Lambda = 0$, we obtain

$$H = \frac{\dot{a}}{a} = \frac{(8\pi G\rho_P/3)^{1/2}}{[1+(a/a_*)^{3(1+\alpha)/n}]^{n/2}}. \tag{130}$$

The general solution of this equation is given by [15]

$$\frac{3}{2}(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P} = \left(\frac{a}{a_*}\right)^{3(\alpha+1)/2} \times {}_2F_1\left[-\frac{n}{2},-\frac{n}{2},1-\frac{n}{2},-\left(\frac{a_*}{a}\right)^{3(\alpha+1)/n}\right]+C, \tag{131}$$

where $t_P = (G\rho_P)^{-1/2} = (\hbar G/c^5)^{1/2} = 5.39\times 10^{-44}$ s is the Planck time and C is an integration constant determined such that $a = l_P$ at $t = 0$, where $l_P = (G\hbar/c^3)^{1/2} = 1.62\times 10^{-35}$ m is the Planck length. Some explicit solutions of Eq. (130) are given in Ref. [15]. For example, for $n = 1$, one has

$$\sqrt{(a/a_*)^{3(1+\alpha)}+1} - \ln\left(\frac{1+\sqrt{(a/a_*)^{3(1+\alpha)}+1}}{(a/a_*)^{3(1+\alpha)}}\right) = \frac{3}{2}(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P}+C. \tag{132}$$

The time evolution of the scale factor and energy density is represented in Figs. 24 and 25.

For $a \ll a_*$,

$$\epsilon \simeq \rho_P c^2, \qquad a \simeq l_P e^{(\frac{8\pi}{3})^{1/2} t/t_P}, \tag{133}$$

so the energy density is constant (equal to the Planck density) and the scale factor increases exponentially rapidly with time (inflation era). For $a \gg a_*$,

$$\frac{\epsilon}{\rho_P c^2} \sim \left[\frac{3}{2}(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P}\right]^{-2}, \tag{134}$$

$$\frac{a}{a_*} \sim \left[\frac{3}{2}(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P}\right]^{2/[3(1+\alpha)]}, \tag{135}$$

so the energy density and the scale factor evolve algebraically with time (α-era).

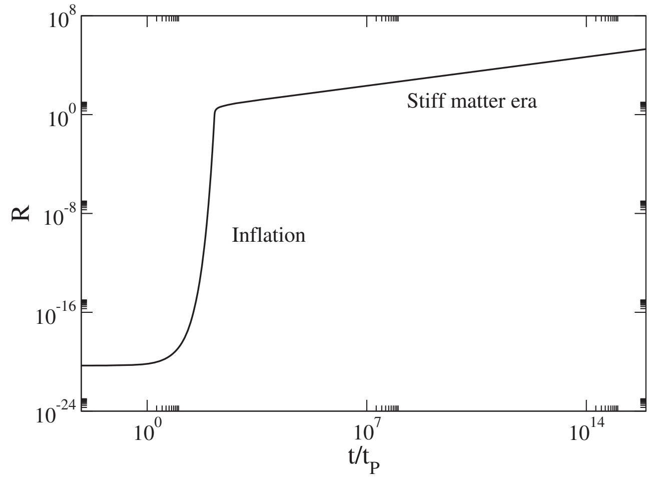

FIG. 24. Evolution of the scale factor as a function of time. For illustration, in this figure and in the following ones, we have taken $\alpha = 1$ and $n = 1$.

If we consider the transition between the inflation era and the radiation era ($\alpha = 1/3$), we find that the temperature is given by a generalized Stefan-Boltzmann law [see Eq. (84a) in Ref. [15]]. The evolution of the temperature is discussed in detail in Refs. [15–17,61]. In our model, the temperature is initially very low, increases exponentially rapidly during the inflation up to a fraction (~0.523) of the Planck temperature $T_P = 1.42\times 10^{32}$ K which is of the order of the grand unified theory (GUT) scale, then decreases algebraically during the radiation era. On the other hand, our model generates a value of the entropy as large as $S/k_B = 5.04\times 10^{87}$ [15–17,61]. This is very different from the standard inflationary scenario [53,63–65]. In that scenario, the Universe is radiation dominated up to $t_i = 10^{-35}$ s and expands exponentially rapidly by a factor 10^{30} in the interval $t_i < t < t_f$ with $t_f = 10^{-33}$ s. For $t > t_f$, the evolution is again radiation

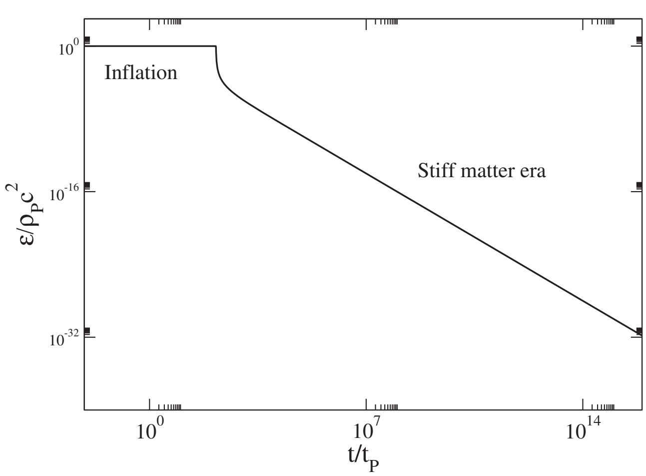

FIG. 25. Evolution of the energy density as a function of time.

dominated. At $t = t_i$, the temperature is about 10^{27} K (this corresponds to the epoch at which most GUTs have a significant influence on the evolution of the Universe). During the exponential inflation, the temperature drops drastically and one must advocate a phase of reheating by various high-energy processes (not very well understood) to restore the initial temperature.

Our model of inflation has the following properties. The universe is accelerating when $a < a_c$ and decelerating when $a > a_c$ where

$$\frac{a_c}{a_*} = \left(\frac{2}{1+3\alpha}\right)^{n/[3(\alpha+1)]}. \qquad (136)$$

The speed of sound is imaginary ($c_s^2 < 0$) when $a < a_e$ and real ($c_s^2 > 0$) when $a > a_e$ where

$$\frac{a_e}{a_*} = \left(\frac{\alpha+n+1}{\alpha n}\right)^{n/[3(\alpha+1)]}. \qquad (137)$$

The speed of sound is always less than the speed of light. The pressure is negative when $a < a_w$ and positive when $a > a_w$ where

$$\frac{a_w}{a_*} = \left(\frac{1}{\alpha}\right)^{n/[3(\alpha+1)]}. \qquad (138)$$

It has a maximum value

$$\frac{P_e}{\rho_P c^2} = \frac{\alpha}{n+1}\left[\frac{\alpha n}{(\alpha+1)(n+1)}\right]^n \qquad (139)$$

at $a = a_e$.

The phase of inflation in the very early universe is usually described by a scalar field, called the inflaton, which evolves according to Eq. (31). We can show [16] that the equation of state (127) is equivalent to a scalar field with a potential (see Fig. 26):

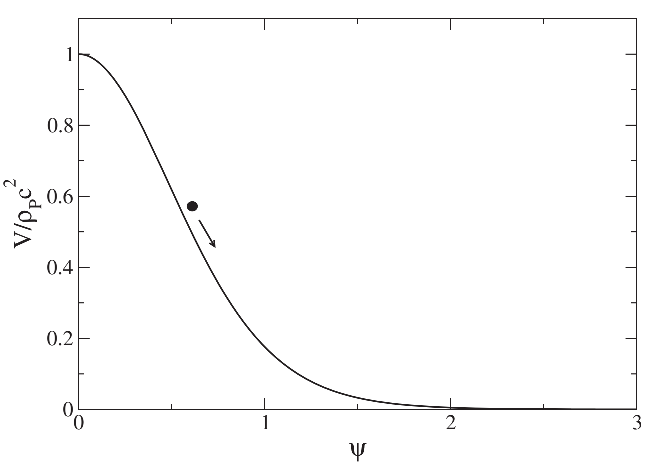

FIG. 26. Potential of the scalar field (inflaton).

$$V(\psi) = \frac{1}{2}\rho_P c^2 \frac{(1-\alpha)\cosh^2\psi + \alpha + 1}{\cosh^{2(n+1)}\psi}, \qquad (140)$$

where we have defined

$$\psi = \left(\frac{8\pi G}{3c^2}\right)^{1/2} \frac{3\sqrt{\alpha+1}}{2n}\phi. \qquad (141)$$

For $\psi \to 0$,

$$\frac{V(\psi)}{\rho_P c^2} \simeq 1 - \frac{1+\alpha+2n}{2}\psi^2 + \frac{2+2\alpha+4n+3\alpha n+3n^2}{6}\psi^4, \qquad (142)$$

which is consistent with the symmetry-breaking scalar field potential used to describe inflation.[5] For $\psi \to +\infty$,

$$\frac{V(\psi)}{\rho_P c^2} \sim 2^{2n-1}(1-\alpha)e^{-2n\psi}, \qquad (\alpha \neq 1), \qquad (143)$$

$$\frac{V(\psi)}{\rho_P c^2} \sim 2^{2(n+1)}e^{-2(n+1)\psi}, \qquad (\alpha = 1). \qquad (144)$$

We can also show [16] that the relation between the scalar field and the scale factor is

$$\sinh\psi = \left(\frac{a}{a_*}\right)^{3(\alpha+1)/2n}. \qquad (145)$$

The end of the inflation, and the beginning of the α era, corresponds to $a = a_*$, and hence to $\psi = \psi_* = \sinh^{-1}(1) = \ln(1+\sqrt{2}) = 0.881374$. Combining Eqs. (127), (128) and (145), we find that the energy density and the pressure of the Universe are related to the scalar field by

$$\epsilon = \frac{\rho_P c^2}{\cosh^{2n}\psi}, \qquad (146)$$

$$P = \frac{\rho_P c^2}{\cosh^{2n}\psi}\left[\alpha - (\alpha+1)\frac{1}{\cosh^2\psi}\right]. \qquad (147)$$

On the other hand, using Eqs. (131) and (145), we find that the temporal evolution of the scalar field is given by

[5]Our model of inflation [15–17] has been recently used in Ref. [66] to study the primordial quantum fluctuations in the very early universe.

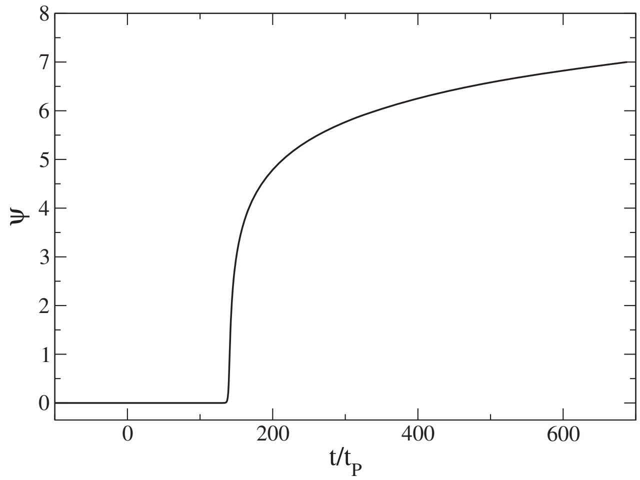

FIG. 27. Evolution of the scalar field as a function of time.

$$\frac{3}{2}(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P}=\sinh^n\psi \times {}_2F_1\left[-\frac{n}{2},-\frac{n}{2},1-\frac{n}{2},-\frac{1}{\sinh^2\psi}\right]+C. \quad (148)$$

It is represented in Fig. 27. For $t\to-\infty$ (i.e. $a\to 0$), we find that

$$\psi\sim\left(\frac{l_P}{a_*}\right)^{3(\alpha+1)/2n}e^{\frac{3(\alpha+1)}{2n}(\frac{8\pi}{3})^{1/2}t/t_P}\to 0. \quad (149)$$

For $t\to+\infty$ (i.e. $a\to+\infty$), we find that

$$\psi\simeq\frac{1}{n}\left\{\ln[3(\alpha+1)]+(n-1)\ln 2+\frac{1}{2}\ln\left(\frac{8\pi}{3}\right)+\ln\left(\frac{t}{t_P}\right)\right\}\to+\infty. \quad (150)$$

For $n=1$, using Eq. (132), we obtain for all times

$$\cosh\psi-\ln\left(\frac{1+\cosh\psi}{\sinh\psi}\right)=\frac{3}{2}(\alpha+1)\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P}+C. \quad (151)$$

Combining the results of this section with those of Sec. V, we can propose a general cosmological model of the form

$$\left(\frac{H}{H_0}\right)^2=\frac{\Omega_{\alpha,0}}{[(a/a_0)^{3(1+\alpha)/n}+(a_*/a_0)^{3(1+\alpha)/n}]^n}+\frac{\Omega_{s,0}}{(a/a_0)^6}+\frac{\Omega_{\mathrm{rad},0}}{(a/a_0)^4}+\frac{\Omega_{m,0}}{(a/a_0)^3}+\Omega_{\Lambda,0} \quad (152)$$

which describes an inflation era, followed by an α era, a stiff matter era, a radiation era, a matter era, and a dark energy era. The α era can represent the stiff matter era ($\alpha=1$), the radiation era ($\alpha=1/3$), or the matter era ($\alpha=0$).

The transition between the α era and the dark energy era is described by the analytical solution [16,62]

$$\frac{a}{a_0}=\left(\frac{\Omega_{\alpha,0}}{\Omega_{\Lambda,0}}\right)^{\frac{1}{3(1+\alpha)}}\sinh^{\frac{2}{3(1+\alpha)}}\left[\frac{3}{2}(1+\alpha)\sqrt{\Omega_{\Lambda,0}}H_0t\right], \quad (153)$$

$$\frac{\epsilon}{\epsilon_0}=\frac{\Omega_{\Lambda,0}}{\tanh^2[\frac{3}{2}(1+\alpha)\sqrt{\Omega_{\Lambda,0}}H_0t]}. \quad (154)$$

The transition between the α era and the anti-dark energy era ($\Omega_{\Lambda,0}<0$) is described by the analytical solution

$$\frac{a}{a_0}=\left(\frac{\Omega_{\alpha,0}}{|\Omega_{\Lambda,0}|}\right)^{\frac{1}{3(1+\alpha)}}\sin^{\frac{2}{3(1+\alpha)}}\left[\frac{3}{2}(1+\alpha)\sqrt{|\Omega_{\Lambda,0}|}H_0t\right], \quad (155)$$

$$\frac{\epsilon}{\epsilon_0}=\frac{|\Omega_{\Lambda,0}|}{\tan^2[\frac{3}{2}(1+\alpha)\sqrt{|\Omega_{\Lambda,0}|}H_0t]}. \quad (156)$$

XII. CONCLUSION

We have obtained analytical solutions of the Friedmann equations for a universe undergoing a primordial stiff matter era in which the speed of sound is equal to the speed of light. The idea of a stiff matter era preceding the radiation and matter eras first appeared in the cosmological model of Zel'dovich [20,21] in which the very early universe is assumed to be made of a cold gas of baryons. This idea reappeared recently in certain models of relativistic SF/BEC cosmologies [14,23,24] that are presently considered with great interest. In this paper, we have studied the evolution of the homogeneous background. For the sake of generality, we have considered a positive and a negative energy density of the stiff matter (leading to singular or nonsingular bouncing models of the Universe) and a positive or a negative value of the cosmological constant (leading to expanding or oscillating models of the Universe). In a future work, we shall consider the evolution of the perturbations in these different models.

As mentioned long ago by Barrow [22], a stiff equation of state has several interesting properties in astrophysics and cosmology that deserve to be better explored. Some works have shown that the presence of stiff matter in cosmological models produces an abundance of relic species of particles after the big bang due to the expansion and cooling of the universe [67]. The presence of stiff matter in the early universe may also help explain the baryon asymmetry and the density perturbation of the right amplitude for the formation of large-scale structures in our universe [68]. It may also affect the spectrum of relic gravitational waves created during inflation [69]. These important problems should also be considered in future works.

APPENDIX A: AN ESTIMATE OF $\Omega_{s,0}$

Following the considerations of Sec. IV D, we propose to determine the polytropic constant K in Eq. (13) by

$$K = \pm \frac{c^2}{\rho_P}, \tag{A1}$$

where $\rho_P = 5.16 \times 10^{99}$ g/m^3 is the Planck density. The quadratic equation of state (13) can then be written as

$$P = \pm \rho_P c^2 \left(\frac{\rho}{\rho_P}\right)^2. \tag{A2}$$

This equation of state leads to a fully predictive model, without free parameters, including a stiff matter era (+) or an anti-stiff matter era (−) that manifests itself when the rest-mass density ρ is of the order of the Planck density ρ_P.

Comparing Eq. (15) with Eq. (36), we find that $\rho_0 c^2 = \Omega_{m,0}\epsilon_0$ and $K\rho_0^2 = \Omega_{s,0}\epsilon_0$. Using Eq. (A1), we obtain

$$\Omega_{s,0} = \frac{\Omega_{m,0}^2 \epsilon_0}{\rho_P c^2}. \tag{A3}$$

Taking $\epsilon_0/c^2 = 9.26 \times 10^{-24}$ g m^{-3}, $\Omega_{m,0} = 0.274$ and $\rho_P = 5.16 \times 10^{99}$ g m^{-3}, we find $\Omega_{s,0} = 1.35 \times 10^{-124}$ ($\sim \rho_\Lambda/\rho_P$) so the present fraction of stiff matter is extremely small. This comes from our assumption that stiff matter manifests itself only at the Planck scale (see Sec. IV D).

APPENDIX B: A GENERALIZATION OF THE ANALYTICAL SOLUTIONS

We consider a universe containing three noninteracting fluids, each of them described by a linear equation of state $p_i = \alpha_i \epsilon_i$ with $\alpha_1 = \alpha$, $\alpha_2 = (\alpha - 1)/2$, and $\alpha_3 = -1$ (dark energy). Since $\alpha_1 \le 1$, we have $\alpha_2 \le 0$. Therefore, the second fluid necessarily has a negative (or a vanishing) pressure. On the other hand, the two fluids are either both normal ($\alpha_1 > -1$ and $\alpha_2 > -1$) or both phantom ($\alpha_1 < -1$ and $\alpha_2 < -1$). Some triplets $(\alpha_1, \alpha_2, \alpha_3)$ of physical interest are $(1, 0, -1)$, $(0, -1/2, -1)$, $(1/3, -1/3, -1)$, and $(-1, -1, -1)$. The first case $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, -1)$ corresponds to a universe made of stiff matter, dark matter and dark energy as studied in the main part of this paper. The total energy density is

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{s,0}}{(a/a_0)^{3(1+\alpha)}} + \frac{\Omega_{m,0}}{(a/a_0)^{3(1+\alpha)/2}} + \Omega_{\Lambda,0}. \tag{B1}$$

The Friedmann equation (3) takes the form

$$\int_{a_i/a_0}^{a/a_0} \frac{dx}{x\sqrt{\frac{\Omega_{s,0}}{x^{3(1+\alpha)}} + \frac{\Omega_{m,0}}{x^{3(1+\alpha)/2}} + \Omega_{\Lambda,0}}} = H_0 t. \tag{B2}$$

With the change of variables $X = x^{(1+\alpha)/2}$, we obtain

$$\int_{(a_i/a_0)^{(1+\alpha)/2}}^{(a/a_0)^{(1+\alpha)/2}} \frac{dX}{X\sqrt{\frac{\Omega_{s,0}}{X^6} + \frac{\Omega_{m,0}}{X^3} + \Omega_{\Lambda,0}}} = \frac{1+\alpha}{2} H_0 t, \tag{B3}$$

where we recognize the integral in Eq. (40). As a result, the evolution of the scale factor and energy density of a universe containing three noninteracting fluids with linear coefficients $\alpha_1 = \alpha$, $\alpha_2 = (\alpha - 1)/2$, and $\alpha_3 = -1$ is given by the equations of this paper with the substitutions $a/a_0 \to (a/a_0)^{(1+\alpha)/2}$ and $H_0 t \to (1+\alpha)H_0 t/2$.

APPENDIX C: GENERAL POLYTROPIC EQUATION OF STATE

In this paper (see also Ref. [14]), we have considered a relativistic fluid described by a polytropic equation of state $P = K\rho^2$ of index $n = 1$ (i.e. $\gamma = 2$), where ρ is the rest-mass density. This quadratic equation of state appears in the model of Zel'dovich [20,21]. This is also the standard equation of state of a self-interacting BEC at $T = 0$ [12].

More generally, we can consider a polytropic equation of state [30]:

$$P = K\rho^\gamma, \qquad \gamma = 1 + \frac{1}{n}, \tag{C1}$$

with an arbitrary index γ. For $\gamma = 1$, it reduces to the isothermal equation of state

$$P = \rho \frac{k_B T_{\rm eff}}{m}, \tag{C2}$$

with an effective temperature $T_{\rm eff}$. These equations of state can be derived from the Gross-Pitaevskii (GP) equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi + \frac{K\gamma m}{\gamma - 1}|\psi|^{2(\gamma-1)}\psi, \tag{C3}$$

or

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi + 2k_B T_{\rm eff} \ln|\psi|\psi, \tag{C4}$$

describing a BEC at $T = 0$ with a power-law ($\gamma \ne 1$) or a logarithmic ($\gamma = 1$) self-interaction potential [11]. The usual GP equation is recovered for $\gamma = 2$ and $K = 2\pi\hbar^2 a_s/m^3$, where a_s is the scattering length of the bosons. Combining Eqs. (C3) and (C4), we can obtain a generalized GP equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi + \frac{4\pi a_s \hbar^2}{m^2}|\psi|^2\psi + 2k_B T_{\rm eff} \ln|\psi|\psi, \tag{C5}$$

that incorporates (effective) thermal effects and self-interaction.

We assume that the Universe is filled with a single relativistic dark fluid at $T = 0$ described by the polytropic equation of state (C1). From Eq. (9), we find that the energy density is related to the rest-mass density by [14,31]

$$\epsilon = \rho c^2 + K\rho \ln(\rho/\rho_*), \qquad (\gamma = 1), \tag{C6}$$

$$\epsilon = \rho c^2 + \frac{K}{\gamma - 1}\rho^\gamma = \rho c^2 + nP, \qquad (\gamma \neq 1). \tag{C7}$$

Combining Eq. (12) with Eqs. (C6) and (C7), we obtain for $\gamma = 1$

$$\epsilon = \rho_0 c^2 \left(\frac{a_0}{a}\right)^3 + K\rho_0 \left(\frac{a_0}{a}\right)^3 \ln\left[\frac{\rho_0}{\rho_*}\left(\frac{a_0}{a}\right)^3\right] \tag{C8}$$

and for $\gamma \neq 1$

$$\epsilon = \rho_0 c^2 \left(\frac{a_0}{a}\right)^3 + \frac{K}{\gamma - 1}\rho_0^\gamma \left(\frac{a_0}{a}\right)^{3\gamma}. \tag{C9}$$

When $\gamma > 1$ (i.e. $n > 0$), we find that $P \sim \epsilon/n$ and $\epsilon \sim nK\rho^\gamma \propto a^{-3\gamma}$ in the early universe (a small, ρ large) and that $P \sim K(\epsilon/c^2)^\gamma$ and $\epsilon \sim \rho c^2 \propto a^{-3}$ in the late universe (a large, ρ small). When $\gamma < 1$ (i.e. $n < 0$), we find that $P \sim K(\epsilon/c^2)^\gamma$ and $\epsilon \sim \rho c^2 \propto a^{-3}$ in the early universe (a small, ρ large) and that $P \sim \epsilon/n$ and $\epsilon \sim nK\rho^\gamma \propto a^{-3\gamma}$ in the late universe (a large, ρ small).

Introducing relevant notations $\Omega_{m,0} = \rho_0 c^2/\epsilon_0$, $\Omega'_{m,0} = (K\Omega_{m,0}/c^2)\ln(\Omega_{m,0}\epsilon_0/\rho_* c^2)$ and $\Omega_{\gamma,0} = K\rho_0^\gamma/[(\gamma - 1)\epsilon_0]$ (see below), the relation between the energy density and the scale factor can be rewritten as

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m,0}}{(a/a_0)^3} + \frac{\Omega'_{m,0}\ln[(\Omega_{m,0}\epsilon_0/\rho_* c^2)(a_0/a)^3]}{(a/a_0)^3 \ln(\Omega_{m,0}\epsilon_0/\rho_* c^2)} \tag{C10}$$

for $\gamma = 1$, and as

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m,0}}{(a/a_0)^3} + \frac{\Omega_{\gamma,0}}{(a/a_0)^{3\gamma}} \tag{C11}$$

for $\gamma \neq 1$. By construction, $\Omega_{m,0} + \Omega'_{m,0} = 1$ and $\Omega_{m,0} + \Omega_{\gamma,0} = 1$.

For the polytropic equation of state (C1) with $\gamma \neq 1$, our treatment shows that the energy density (C9) of the dark fluid is the sum of two terms: an ordinary term $\epsilon_m \equiv \rho c^2 \propto a^{-3}$ (rest-mass energy) equivalent to dark matter and a new[6] term $\epsilon_\gamma \equiv u = nK\rho^\gamma = nP \propto a^{-3\gamma}$ (internal energy) depending on the polytropic index γ. Therefore, everything happens *as if* we had two fluids: a "dark matter fluid" $(a/a_0)^{-3}$ with a proportion Ω_m and a "new fluid" $(a/a_0)^{-3\gamma}$ with a proportion Ω_γ. We recall, however, that in our approach we intrinsically have just *one* dark fluid. What we call "dark matter" corresponds to its rest-mass energy density and what we call "new fluid" corresponds to its internal energy. When $\gamma > 1$, the new fluid (internal energy) dominates in the early universe and dark matter (rest-mass energy) dominates in the late universe. When $\gamma < 1$, dark matter (rest-mass energy) dominates in the early universe and the new fluid (internal energy) dominates in the late universe. For $\gamma = 2$ (i.e. $n = 1$), the new fluid mimics stiff matter ($\epsilon_s \propto a^{-6}$) as discussed in this paper. For $\gamma = 0$ (i.e. $n = -1$), the new fluid mimics dark energy ($\epsilon_\Lambda = \mathrm{cst}$) as discussed in Refs. [15–17] when the pressure is a negative constant ($P = -\epsilon_\Lambda$). For $\gamma = 4/3$ (i.e. $n = 3$) the new term mimics the radiation of an ultra-relativistic gas ($\epsilon_{\rm rad} \propto a^{-4}$).

More generally, according to Eq. (8), the new term that appears in the energy equation is equal to the internal energy of the dark fluid

$$\epsilon_{\rm new} = u(\rho) = \rho \int^\rho \frac{P(\rho')}{\rho'^2}\, d\rho'. \tag{C12}$$

This relation clearly shows that the new term is related to pressure effects (collisions). When $P(\rho)$ is close to a negative constant, corresponding to $\gamma \to 0$ and $K < 0$ in the polytropic model, we suggest that the new term $\epsilon_{\rm new}$ represents dark energy. Our procedure may be used to obtain generalized models of dark energy by considering different equations of state $P(\rho)$, that do not change too much with the density ($\gamma \simeq 0$). In our approach, we have a single dark fluid described by an equation of state $P(\rho)$ unifying dark matter and dark energy: in Eq. (9), the rest-mass term ρc^2 mimics "dark matter" and the internal energy term $u(\rho)$ mimics "dark energy" (or, more generally, a new fluid).[7] We suggest that the dark fluid may be in the form of relativistic self-interacting BECs at $T = 0$ although other

[6]We call it "new" because in the CDM model where $P = 0$, the energy density is just equal to $\epsilon = \rho_0 c^2 (a_0/a)^3$. However, as soon as $P \neq 0$, an additional term appears in the energy density as a direct consequence of relativistic thermodynamics. In this sense, there is nothing really new or surprising.

[7]Let us be more precise by considering the polytropic model. For the internal energy to mimic dark energy, we need $\gamma < 1$ so that the internal energy $u = K\rho^\gamma/(\gamma - 1) = P/(\gamma - 1)$ dominates the rest-mass ρc^2 in the late universe where the density is low. We also need $u > 0$ since the observations reveal that some energy is *missing* with respect to the pressureless dark matter model (EdS model), i.e. $\epsilon > \rho c^2$. These two conditions ($\gamma < 1$ and $u > 0$) imply that the pressure must be negative ($P < 0$). When $\gamma < 1$ and $P > 0$, the internal energy is negative ($u < 0$), so it mimics anti-dark energy (like a negative cosmological constant). On the other hand, when $\gamma > 1$, the internal energy u dominates the rest-mass ρc^2 in the early universe where the density is high. In that case, the internal energy mimics a new primordial fluid like the stiff fluid considered in this paper (when $P = K\rho^2$). We have $u > 0$ when $P > 0$ and $u < 0$ when $P < 0$.

possibilities may be contemplated. In the BEC model, the pressure is negative when the self-interaction is attractive ($K < 0$). This may justify equations of state with a negative pressure (such as the equation of state of dark energy) as suggested in Ref. [11].

Since dark matter and dark energy can be unified by a single dark fluid with a polytropic equation of state $P = K\rho^\gamma$ with an almost vanishing index $\gamma \simeq 0$, it is relevant to consider the logotropic equation of state [19,70]:

$$P = A\ln(\rho/\rho_*), \tag{C13}$$

which can be viewed as the limiting form of a polytrope of index $\gamma \to 0$ ($n \to -1$) with $K \to \infty$ such that $A = K\gamma$ is finite [19]. If the dark fluid is made of BECs, using the formalism of Ref. [11], the logotropic equation of state can be derived from a GP equation of the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\left(\Phi - \frac{A}{Nm|\psi|^2}\right)\psi. \tag{C14}$$

This corresponds to a GP equation with an inverted quadratic potential, i.e. with the exponent -2 instead of $+2$ in the usual GP equation [12]. This equation can also be obtained as the limiting form of Eq. (C3) when $\gamma \to 0$ and $K \to \infty$ with $A = K\gamma$ finite. For the logotropic equation of state (C13), the energy relation (9) becomes

$$\epsilon = \rho c^2 - A\ln\left(\frac{\rho}{\rho_*}\right) - A. \tag{C15}$$

Combining Eq. (12) and Eq. (C15), we obtain

$$\epsilon = \rho_0 c^2\left(\frac{a_0}{a}\right)^3 - A\ln\left[\frac{\rho_0}{\rho_*}\left(\frac{a_0}{a}\right)^3\right] - A. \tag{C16}$$

For $a \to 0$, $\epsilon \propto a^{-3}$ like in a pressureless universe. For $a \to +\infty$, $\epsilon \sim 3A\ln a$ (assuming $A > 0$). Integrating the Friedmann equation (3), we obtain the "super–de Sitter" asymptotic behavior

$$a \propto e^{\frac{2\pi GA}{c^2}t^2}, \qquad \epsilon \sim \frac{6\pi GA^2}{c^2}t^2, \qquad (t \to +\infty). \tag{C17}$$

The evolution of the scale factor is given by a stretched exponential. It exhibits a typical time scale $t_L = c/\sqrt{2\pi GA}$. Since the energy density increases as the scale factor increases, the Universe is phantom [28] (this is a particular case of the models of a phantom universe studied in Ref. [71]). However, it is nonsingular since the scale factor and the energy density become infinite in infinite time. This is called a little rip [72]. Furthermore, we have $\dot H \to 4\pi GA/c^2 = 2/t_L^2$ as $t \to +\infty$. The logotropic model was studied in detail in Ref. [18]. It is able to account for cosmological and galactic (dark matter) observations remarkably well. This may be a hint that dark matter and dark energy are the manifestation of a single dark fluid with a logotropic equation of state.

Remark: At the background level, a single dark fluid with an equation of state $P(\rho)$ is equivalent to two noninteracting fluids [73]: a dark matter fluid with an equation of state $P_m = 0$ and a new fluid with an equation of state $P_{\rm new}(\epsilon_{\rm new})$ defined in implicit form by $P_{\rm new} = P(\rho)$ and $\epsilon_{\rm new} = u(\rho)$ where $u(\rho)$ is defined by Eq. (C12) and ρ is a running parameter (in the two-fluids model, it does not have a physical interpretation). For the polytropic equation of state (C1), we find $P_{\rm new} = (\gamma - 1)\epsilon_{\rm new}$ and for the logotropic equation of state (C13) we get $P_{\rm new} = -\epsilon_{\rm new} - A$ [73].

APPENDIX D: EVOLUTION OF THE UNIVERSE AS A FUNCTION OF THE POLYTROPIC INDEX

We consider a universe filled with a single relativistic dark fluid at $T = 0$ described by the polytropic equation of state (C1). We first assume $\gamma \neq 1$ (the case $\gamma = 1$ is treated in Appendix D 3). The relation between the energy density and the rest-mass density is given by Eq. (9) or, equivalently, by Eq. (C7). The rest-mass density evolves with the scale factor according to Eq. (12) and the energy density evolves with the scale factor according to Eq. (C9) or, equivalently, according to Eq. (C11). The Friedmann equation (3) with $\Lambda = 0$ can be written as

$$\frac{\dot a}{a} = H_0\sqrt{\frac{\Omega_{m,0}}{(a/a_0)^3} + \frac{\Omega_{\gamma,0}}{(a/a_0)^{3\gamma}}} \tag{D1}$$

with $\Omega_{m,0} + \Omega_{\gamma,0} = 1$. Its formal solution is

$$\int_0^{a/a_0}\frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \frac{\Omega_{\gamma,0}}{x^{3\gamma}}}} = H_0 t. \tag{D2}$$

This integral can be expressed in terms of hypergeometric functions, but we shall not need this result here.

1. The case $\gamma < 1$

When $\gamma < 1$ the rest-mass energy (dark matter) dominates in the early universe where ρ is large and the internal energy (new fluid) dominates in the late universe where ρ is small.

a. The early universe (dark matter)

In the early universe, for $\rho \to +\infty$, the relations between the energy density, the rest-mass energy and the pressure reduce to

$$\epsilon \sim \rho c^2, \qquad P = K\rho^\gamma, \qquad P \sim K\left(\frac{\epsilon}{c^2}\right)^\gamma. \tag{D3}$$

The dark fluid has a polytropic equation of state with index γ. However, since $P \ll \epsilon$ when $\epsilon \to +\infty$, everything happens in the Friedmann equations *as if* the fluid were pressureless ($P = 0$). The relation between the energy density and the scale factor reduces to

$$\frac{\epsilon}{\epsilon_0} \sim \frac{\Omega_{m,0}}{(a/a_0)^3}. \tag{D4}$$

The early universe is normal since the energy density decreases as the scale factor increases. The Friedmann equation

$$\frac{\dot{a}}{a} \sim H_0 \sqrt{\frac{\Omega_{m,0}}{(a/a_0)^3}} \tag{D5}$$

can be integrated easily. For $t \to 0$, we have

$$\frac{a}{a_0} \sim \left(\frac{3}{2}\sqrt{\Omega_{m,0}} H_0 t\right)^{2/3}, \tag{D6}$$

$$\frac{\epsilon}{\epsilon_0} \sim \frac{1}{(\frac{3}{2} H_0 t)^2}, \qquad \epsilon \sim \frac{c^2}{6\pi G t^2}. \tag{D7}$$

The scale factor starts from $a = 0$ and increases with time. The energy density starts from $+\infty$ and decreases with time. The early universe is decelerating. This corresponds to the EdS solution (63) that is usually derived for a pressureless fluid ($P = 0$).

b. The late universe with $K < 0$ (dark energy)

We now consider the late universe. We first assume $K < 0$ so that the internal energy is positive ($u = K\rho^\gamma/(\gamma - 1) > 0$) and the pressure is negative ($P = K\rho^\gamma < 0$). This implies $\Omega_{\gamma,0} = K\rho_0^\gamma/[(\gamma - 1)\epsilon_0] > 0$. In that case, the new fluid mimics dark energy. For $\rho \to 0$, the relations between the energy density, the rest-mass energy and the pressure reduce to

$$\epsilon \sim \frac{K}{\gamma - 1}\rho^\gamma, \qquad P = K\rho^\gamma, \qquad P \sim (\gamma - 1)\epsilon. \tag{D8}$$

The dark fluid has a linear equation of state with a negative coefficient $\alpha = \gamma - 1 < 0$. For $a \to +\infty$, the relation between the energy density and the scale factor reduces to

$$\frac{\epsilon}{\epsilon_0} \sim \frac{\Omega_{\gamma,0}}{(a/a_0)^{3\gamma}}. \tag{D9}$$

In order to integrate the Friedmann equation

$$\frac{\dot{a}}{a} \sim H_0 \sqrt{\frac{\Omega_{\gamma,0}}{(a/a_0)^{3\gamma}}}, \tag{D10}$$

we have to consider different cases.

We first assume $0 < \gamma < 1$ (i.e. $-1 < \alpha < 0$). In that case, the late universe is normal since the energy density decreases as the scale factor increases. For $t \to +\infty$, we have

$$\frac{a}{a_0} \sim \left(\frac{3\gamma}{2}\sqrt{\Omega_{m,0}} H_0 t\right)^{2/3\gamma}, \tag{D11}$$

$$\frac{\epsilon}{\epsilon_0} \sim \frac{1}{(\frac{3\gamma}{2} H_0 t)^2}, \qquad \epsilon \sim \frac{c^2}{6\pi G \gamma^2 t^2}. \tag{D12}$$

The scale factor increases algebraically with time and tends to infinity as $t \to +\infty$. The energy density decreases algebraically with time and tends to zero as $t \to +\infty$. The late universe is decelerating if $\gamma > 2/3$ (i.e. $\alpha > -1/3$) and accelerating if $\gamma < 2/3$ (i.e. $\alpha < -1/3$). Coming back to the general equations, the Universe starts accelerating when $w = P/\epsilon < -1/3$, corresponding to $a > a_c$ with

$$\frac{a_c}{a_0} = \left[\frac{\Omega_{m,0}}{(2 - 3\gamma)\Omega_{\gamma,0}}\right]^{1/[3(1-\gamma)]}. \tag{D13}$$

We now assume $\gamma = 0$ (i.e. $\alpha = -1$) corresponding to a constant negative pressure $P = -\epsilon_\Lambda$. In that case, the energy density tends to a constant in the late universe: $\epsilon \to \Omega_{\Lambda,0}\epsilon_0 = \epsilon_\Lambda$. For $t \to +\infty$, we have

$$\frac{a}{a_0} \propto e^{\sqrt{\Omega_{\Lambda,0}} H_0 t}. \tag{D14}$$

The scale factor increases exponentially rapidly with time and tends to infinity as $t \to +\infty$. This corresponds to the de Sitter solution of Eq. (62). The late universe is accelerating. If we come back to the general equations, the polytropic model with $\gamma = 0$ corresponds to the ΛCDM model for which we have the analytical solution of Eqs. (60) and (61).

We finally assume $\gamma < 0$ (i.e. $\alpha < -1$). In that case, the late universe is phantom since the energy density increases as the scale factor increases. The universe is accelerating and undergoes a future finite time singularity called a big rip [74]. The scale factor and the energy density become infinite in a finite time $t_{\rm BR}$. Close to the big rip time, we have

$$\frac{a}{a_0} \sim \frac{1}{[\frac{3|\gamma|}{2}\sqrt{\Omega_{m,0}} H_0 (t_{\rm BR} - t)]^{2/3|\gamma|}}, \tag{D15}$$

$$\frac{\epsilon}{\epsilon_0} \sim \frac{1}{[\frac{3|\gamma|}{2} H_0 (t_{\rm BR} - t)]^2}. \tag{D16}$$

Coming back to the general equations, the Universe is accelerating for $a \geq a_c$ given by Eq. (D13) and becomes phantom for $a \geq a_M$ with

$$\frac{a_M}{a_0} = \left(\frac{\Omega_{m,0}}{|\gamma|\Omega_{\gamma,0}}\right)^{1/[3(|\gamma|+1)]}. \quad \text{(D17)}$$

At that point, the energy density reaches its minimum value ($\dot{H} = 0$):

$$\frac{\epsilon_M}{\epsilon_0} = \Omega_{m,0}\left(1 + \frac{1}{|\gamma|}\right)\left(\frac{|\gamma|\Omega_{\gamma,0}}{\Omega_{m,0}}\right)^{1/(1+|\gamma|)}. \quad \text{(D18)}$$

The big rip occurs at

$$H_0 t_{\rm BR} = \int_0^{+\infty} \frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3} + \Omega_{\gamma,0} x^{3|\gamma|}}}. \quad \text{(D19)}$$

For completeness, we consider the logotropic equation of state (C13) which is intermediate between a constant equation of state ($\gamma = 0$) and a polytropic equation of state with $\gamma = 0^-$. We assume $A > 0$. In that case, the late universe is phantom since the energy density increases as the scale factor increases [see Eq. (C16)]. However, there is no future singularity. The scale factor increases super-exponentially rapidly with time [see Eq. (C17)] and tends to infinity as $t \to +\infty$. The energy density increases quadratically with time [see Eq. (C17)] and tends to infinity as $t \to +\infty$. This corresponds to a little rip [72]. We note that the logotropic model is the only model of the polytropic family that is phantom but nonsingular. The phantom models without future singularity are attractive from the physical viewpoint because the occurrence of a finite time singularity may lead to some inconsistencies. More details on the logotropic universe can be found in Ref. [18].

c. The late universe with $K > 0$ (anti-dark energy)

We now assume $K > 0$ so that the internal energy is negative ($u = K\rho^\gamma/(\gamma - 1) < 0$) and the pressure is positive ($P = K\rho^\gamma > 0$). This implies $\Omega_{\gamma,0} = K\rho_0^\gamma/[(\gamma - 1)\epsilon_0] < 0$. In that case, the new fluid mimics anti-dark energy. The energy density vanishes at

$$\frac{a_f}{a_0} = \left(\frac{|\Omega_{\gamma,0}|}{\Omega_{m,0}}\right)^{1/[3(\gamma-1)]}. \quad \text{(D20)}$$

At that moment $H = \dot{a}/a = 0$. The scale factor starts from $a = 0$ at $t = 0$ (big bang), increases, reaches its maximum value a_f at a time

$$H_0 t_1 = \int_0^{a_f/a_0} \frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3} - \frac{|\Omega_{\gamma,0}|}{x^{3\gamma}}}}, \quad \text{(D21)}$$

decreases, and vanishes at $t = t_2 = 2t_1$ (big crunch). During the phase of expansion, the Universe is decelerating. The energy density starts from infinity at $t = 0$, decreases, vanishes at $t = t_1$ (the Universe disappears), increases, and tends to infinity at $t = t_2 = 2t_1$. This process continues periodically in time. Close to t_1, we have the behaviors

$$\frac{a}{a_f} \simeq 1 - \frac{3}{4}\frac{\Omega_{m,0}}{(a_f/a_0)^3}(1-\gamma)H_0^2(t_1 - t)^2, \quad \text{(D22)}$$

$$\frac{\epsilon}{\epsilon_0} \simeq \frac{9}{4}\frac{\Omega_{m,0}^2}{(a_f/a_0)^6}(1-\gamma)^2 H_0^2(t_1 - t)^2. \quad \text{(D23)}$$

For $\gamma = 0$, we recover the anti-ΛCDM model of Sec. VIII C.

The logotropic equation of state with $A < 0$ behaves similarly.

2. The case $\gamma > 1$

When $\gamma > 1$ the rest-mass energy (dark matter) dominates in the late universe where ρ is small and the internal energy (new fluid) dominates in the early universe where ρ is large.

a. The late universe (dark matter)

In the late universe, for $\rho \to 0$, the relations between the energy density, the rest-mass energy and the pressure reduce to Eq. (D3). Therefore, Eqs. (D4)–(D7) remain valid except that they now apply to the late universe, for $t \to +\infty$. The scale factor increases algebraically with time and tends to infinity as $t \to +\infty$. The energy density decreases algebraically with time and tends to zero as $t \to +\infty$. The late universe is decelerating. This corresponds to the EdS solution (63) that is usually derived for a pressureless fluid ($P = 0$).

b. The early universe with $K > 0$ (primordial fluid)

We now consider the early universe. We first assume $K > 0$ so that the internal energy is positive ($u = K\rho^\gamma/(\gamma - 1) > 0$) and the pressure is positive ($P = K\rho^\gamma > 0$). This implies $\Omega_{\gamma,0} = K\rho_0^\gamma/[(\gamma - 1)\epsilon_0] > 0$. In that case, the new fluid mimics a primordial fluid. For $\rho \to +\infty$, the relations between the energy density, the rest-mass energy and the pressure reduce to Eq. (D8). Therefore, Eqs. (D9)–(D12) remain valid except that they now apply to the early universe, for $t \to 0$. The scale factor starts from $a = 0$ at $t = 0$ and increases algebraically with time. The energy density starts from infinity at $t = 0$ and decreases algebraically with time. The early universe is always decelerating. For $\gamma = 2$, we recover the stiff matter model of Sec. VI B.

c. The early universe with $K < 0$ (anti-primordial fluid)

We now assume $K < 0$ so that the internal energy is negative ($u = K\rho^\gamma/(\gamma - 1) < 0$) and the pressure is negative ($P = K\rho^\gamma < 0$). This implies $\Omega_{\gamma,0} = K\rho_0^\gamma/[(\gamma - 1)\epsilon_0] < 0$. In that case, the new fluid mimics an anti-primordial fluid. The energy density vanishes at

$$\frac{a_i}{a_0}=\left(\frac{|\Omega_{\gamma,0}|}{\Omega_{m,0}}\right)^{1/[3(\gamma-1)]}. \tag{D24}$$

At that moment $H=\dot{a}/a=0$. The energy density reaches its maximum value

$$\frac{\epsilon_*}{\epsilon_0}=\frac{(\gamma-1)\Omega_{m,0}}{\gamma(\frac{\gamma|\Omega_{\gamma,0}|}{\Omega_{m,0}})^{1/(\gamma-1)}} \tag{D25}$$

at

$$\frac{a_*}{a_0}=\left(\frac{\gamma|\Omega_{\gamma,0}|}{\Omega_{m,0}}\right)^{1/[3(\gamma-1)]}. \tag{D26}$$

At that moment $\dot{H}=0$. The scale factor starts from $a=a_i$ at $t=0$ and increases with time. The energy density starts from $\epsilon=0$ at $t=0$, increases, reaches its maximum value ϵ_* at a time

$$H_0t_*=\int_0^{a_*/a_0}\frac{dx}{x\sqrt{\frac{\Omega_{m,0}}{x^3}-\frac{|\Omega_{\gamma,0}|}{x^{3\gamma}}}}, \tag{D27}$$

and decreases. The universe is phantom for $t<t_*$ and normal for $t>t_*$. It is accelerating for $a<a_c$ and decelerating for $a>a_c$ with

$$\frac{a_c}{a_0}=\left[\frac{(3\gamma-2)|\Omega_{\gamma,0}|}{\Omega_{m,0}}\right]^{1/[3(\gamma-1)]}. \tag{D28}$$

Close to $t=0$, we have the behaviors

$$\frac{a}{a_i}\simeq 1+\frac{3}{4}\frac{\Omega_{m,0}}{(a_i/a_0)^3}(\gamma-1)H_0^2t^2, \tag{D29}$$

$$\frac{\epsilon}{\epsilon_0}\simeq\frac{9}{4}\frac{\Omega_{m,0}^2}{(a_i/a_0)^6}(\gamma-1)^2H_0^2t^2. \tag{D30}$$

For $\gamma=2$, we recover the anti-stiff matter model of Sec. VII B.

3. The case $\gamma=1$

For $\gamma=1$, the relation between the energy density and the rest-mass density is given by Eq. (9) or, equivalently, by Eq. (C6). The rest-mass density evolves with the scale factor according to Eq. (12) and the energy density evolves with the scale factor according to Eq. (C8) or, equivalently, according to Eq. (C10). The energy density starts from infinity at $a=0$, decreases and vanishes at

$$\frac{a_f}{a_0}=\left(\frac{\Omega_{m,0}\epsilon_0}{\rho_*c^2}\right)^{1/[3(1-\Omega_{m,0})]}, \tag{D31}$$

where we have used $\Omega_{m,0}+\Omega'_{m,0}=1$. At that moment $H=\dot{a}/a=0$. The relation between the energy density and the scale factor can be rewritten as

$$\frac{\epsilon}{\epsilon_0}=\frac{1}{(a/a_0)^3}\frac{\ln(a_f/a)}{\ln(a_f/a_0)}. \tag{D32}$$

The Friedmann equation (3) with $\Lambda=0$ takes the form

$$\frac{\dot{a}}{a}=H_0\sqrt{\frac{1}{(a/a_0)^3}\frac{\ln(a_f/a)}{\ln(a_f/a_0)}}. \tag{D33}$$

Its solution is

$$\mathrm{erfc}\left[\sqrt{\frac{3}{2}\ln\left(\frac{a_f}{a}\right)}\right]=t/t_1, \tag{D34}$$

where

$$H_0t_1=\left(\frac{2\pi}{3}\right)^{1/2}\left(\frac{a_f}{a_0}\right)^{3/2}\sqrt{\ln\left(\frac{a_f}{a_0}\right)} \tag{D35}$$

is the time at which the energy density vanishes ($a=a_f$). The scale factor starts from $a=0$ at $t=0$ (big bang), increases until its maximum value a_f (corresponding to $\epsilon=0$) is reached at $t=t_1$, decreases, and vanishes at $t=t_2=2t_1$ (big crunch). During the phase of expansion, the universe is decelerating. The energy density starts from infinity at $t=0$, decreases, vanishes at $t=t_1$ (the Universe disappears), increases and tends to infinity at $t=t_2=2t_1$. This process continues periodically in time. Using the equivalent $\mathrm{erfc}(x)\sim e^{-x^2}/x\sqrt{\pi}$ for $x\to+\infty$, we obtain for $t\to 0$ the modified EdS solution

$$\left(\frac{a}{a_0}\right)^{3/2}\sqrt{\frac{\ln(a_f/a_0)}{\ln(a_f/a)}}=\frac{3}{2}H_0t, \tag{D36}$$

$$\frac{\epsilon}{\epsilon_0}\sim\frac{1}{(\frac{3}{2}H_0t)^2}. \tag{D37}$$

Using the expansion $\mathrm{erfc}(x)\simeq 1-2x/\sqrt{\pi}+\ldots$ for $x\to 0$, we find for $t\to t_1$

$$\frac{a}{a_f}\simeq 1-\frac{\pi}{6}\left(1-\frac{t}{t_1}\right)^2, \tag{D38}$$

$$\frac{\epsilon}{\epsilon_0} \simeq \frac{\pi}{6} \frac{1}{(a_f/a_0)^3 \ln(a_f/a_0)} \left(1 - \frac{t}{t_1}\right)^2. \quad \text{(D39)}$$

Final remark: the models of the Universe just described that are based on the polytropic equation of state of type II defined by $P = K\rho^\gamma$, where ρ is the rest-mass density, are very different from the models of the Universe based on the polytropic equation of state of type I defined by $P = K\epsilon^\gamma$, where ϵ is the energy density [15,16,71]. The evolution of the polytropic universe of type I as a function of the polytropic index γ is reviewed in the Appendix B of Ref. [71].

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