

Path-integral approach to scale anomaly at finite temperature

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We derive the relativistic thermodynamic scale equation using imaginary-time path integrals, with complex scalar field theory taken as a concrete example. We use Fujikawa's method to derive the scaling anomaly for this system using a matrix regulator. We make a general scaling argument to show how for anomalous systems, the β function of the vacuum theory can be derived from measurement of macroscopic thermodynamic parameters.

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I. INTRODUCTION

In a series of seminal papers by Callan, Coleman, and Jackiw [1,2], it was noted that in general the trace of the Belifante stress-energy tensor θ_μ^μ for any renormalizable theory could be improved, so that classically for scale-invariant systems (systems invariant under the conformal group),

$$\theta_\mu^\mu = 0. \quad (1)$$

This improved tensor has a number of desirable properties over the canonical tensor (the one derived from Noether's theorem) such as having finite matrix elements in the quantum theory, and that the energy for bound states can be naturally expressed as the trace of this tensor. Shortly after these observations, it was noted that the same improvement program could be applied in the nonrelativistic case [3], so that for classical scale-invariant systems (systems invariant under the Schrödinger group)

$$2\theta^{00} - \sum_{i=1}^3 \theta^{ii} = 0, \quad (2)$$

where the 2 results from the fact that in nonrelativistic theories time must scale as twice the power of space.¹

Equations (1) and (2) fail to consider the trace anomaly. In general, the trace of the stress-energy tensor taken between bound states gives the energy of the bound state:

$$E_b = \int dV \langle \theta_\mu^\mu \rangle, \quad (3)$$

which derives from the fact that the time average of the field virial is zero for bound states [4]. With slight modification Eq. (3) holds in the nonrelativistic case too (see [5] for a

specific example). However, it is well known that even though $\theta_\mu^\mu = 0$ for a classically scale-invariant system, which would imply bound states can only have zero energy,² the quantization procedure can destroy this relationship. When this happens this is called a scale anomaly, and is the mechanism that allows the bound state energy to differ from zero.

As an example, in QCD with massless quarks (or no quarks at all), the Lagrangian is classically scale invariant so that $\theta_\mu^\mu = 0$. However, through the renormalization process, a scale appears as Λ_{QCD} . In general this makes $\langle \theta_\mu^\mu \rangle = \mathcal{A}$, where \mathcal{A} is the anomaly. The stress-energy tensor can then be further improved:

$$T^{\mu\nu} = \theta^{\mu\nu} + \frac{g^{\mu\nu}}{4} T_\eta^\eta, \quad (4)$$

so that $T^{\mu\nu}$ is no longer traceless. Then

$$E_b = \int dV \langle T^{00} \rangle = \int dV \langle \theta^{00} \rangle + \frac{E_b}{4}, \quad (5)$$

which implies that \mathcal{A} accounts for 1/4 of the energy of the hadron. This can explicitly be seen in the bag model where confinement of the quarks and gluons is the result of a cosmological constant term in the Lagrangian which contributes a positive energy and negative pressure $\Lambda g^{\mu\nu}$ to $\theta^{\mu\nu}$, which confines the system. Then from the tracelessness of $\theta^{\mu\nu}$, $\Lambda = \frac{1}{4} T_\mu^\mu$, so that confinement accounts for 1/4 of the hadron energy [4].

In this paper, we are interested in the thermal analogues of Eqs. (1) and (2). Both of these quantities are very important in their respective areas of physics. In the nonrelativistic sector, for an ultracold dilute gas, (2) would read

$$2\mathcal{E} - 3P = -\frac{\hbar^2}{3m} \lambda \langle (\psi^\dagger(x)\psi(x))^2 \rangle. \quad (6)$$

²This is also obvious from the fact that there are no scales to even form E_b .

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¹The Schrödinger equation has only one derivative of time, and two of space, so for scale invariance time must scale as twice the power of space.

The rhs is known as the Tan contact, and is extremely important in atomic physics. In terms of it, Tan derived a set of universal relations [6–8] that govern many relationships between the thermodynamics variables of the system and the behavior of the large momentum tails of correlation functions. These relationships hold even in the strongly interacting regime where perturbation theory becomes inadequate [9]. A field theoretic explanation of Tan’s result was later developed in terms of the operator product expansion [10].

In QCD, the analog would be [11]

$$\mathcal{E} - 3P = \sum_{i=1}^{n_f} m_i \langle \bar{\psi}_i \psi_i \rangle + \frac{2}{g} \beta(g) \frac{1}{4} \langle F_{\mu\nu}^a F^{\mu\nu a} \rangle. \quad (7)$$

In the low temperature regime where the coupling g is strong, the trace anomaly of the rhs is calculated by calculating the lhs of Eq. (7) using a lattice action. The goal is to calculate the QCD equation of state $P = P(T, \mu, V)$ rather than the anomaly itself. However, for technical reasons [12], $\mathcal{E} - 3P$ is important as an intermediate step in lattice QCD for calculating $P(T, \mu, V)$, where it is given by

$$\mathcal{A} = \mathcal{E} - 3P = -\frac{T d \ln Z}{V d \ln a}, \quad (8)$$

and plugging into Eq. (7) gives after using thermodynamic identities

$$\frac{\partial}{\partial \ln T} \left(\frac{P}{T^4} \right) = \frac{\mathcal{A}}{T^4}, \quad (9)$$

which can then be integrated to get $P(T, \mu, V)$. a is the lattice spacing and Z is the partition function with lattice action.

In this paper, following the approach initiated in [13–15] for nonrelativistic systems, we provide a continuum/non-lattice path-integral approach to deriving the thermodynamic trace equation $\mathcal{E} - 3P$, where anomalies naturally appear as a result of a change of variables of the path-integral measure, the thermal analog of Fujikawa’s method. This is in contrast to an operator approach, where one takes the thermal quantum statistical expectation values of both sides of equations like (1) and (2), and identifying $\langle T^{00} \rangle = \mathcal{E}$ and $\langle T^{ii} \rangle = \mathcal{P}_H$, where \mathcal{P}_H is the hydrodynamic pressure [16]. Within this path-integral approach, no reference needs to be made about improvement of the stress-energy tensor, or the validity of equating the hydrodynamic pressure \mathcal{P}_H with the thermodynamic pressure P derived from the grand partition function, which is nontrivial, especially in the presence of anomalies [17,18]. For concreteness, we will take as our system a complex scalar field theory, but the results can be extended for other systems. The Lagrangian is given by

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2 \quad (10)$$

and has a $U(1)$ symmetry

$$\begin{aligned} \phi &\rightarrow e^{i\theta} \phi, \\ \phi^\dagger &\rightarrow e^{-i\theta} \phi^\dagger, \end{aligned} \quad (11)$$

leading to a conserved charge:

$$\begin{aligned} j_0 &= i \phi^\dagger \overleftrightarrow{\partial}_0 \phi, \\ Q &= i \int d^3x \phi^\dagger \overleftrightarrow{\partial}_0 \phi. \end{aligned} \quad (12)$$

Under scale transformation

$$\begin{aligned} x'^\mu &= e^\rho x^\mu, \\ \phi'(x') &= e^{-\rho} \phi(x), \\ \phi'^\dagger(x') &= e^{-\rho} \phi^\dagger(x). \end{aligned} \quad (13)$$

II. THERMODYNAMIC DILATION EQUATION

For a homogeneous system the grand potential $\Omega = \Omega(\beta, \mu, V)$ in the large volume limit equals $-PV$, so that the partition function is $Z = e^{-\beta\Omega} = e^{\beta PV}$, and can be expressed via a path integral:

$$\begin{aligned} Z &= e^{\beta PV} = \sum_i \langle i | e^{-\beta(H-\mu Q)} | i \rangle \\ &= \int [d\phi][d\phi^*] e^{-S_E + \mu \int_0^\beta \int_V d^3x d\tau j_0}, \end{aligned} \quad (14)$$

with³

$$\begin{aligned} S_E &= \int_0^\beta \int_V d^3x d\tau \left(\partial_\mu \phi^* \partial_\mu \phi + (m^2 - \mu^2) \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 \right), \\ j_0 &= -\phi^* \overleftrightarrow{\partial}_\tau \phi. \end{aligned} \quad (15)$$

Now consider an infinitesimal “relativistic thermodynamic scaling”

$$\begin{aligned} \beta' &= e^\rho \beta = \beta + \rho \beta = \beta + \delta\beta, \\ L'_i &= e^\rho L_i = L_i + \rho L_i = L_i + \delta L_i, \\ \mu' &= \mu. \end{aligned} \quad (16)$$

³Due to the dependence of j_0 on conjugate momenta, when integrating out conjugate momenta to pass into the Lagrangian formulation of the path integral, \mathcal{L}_E acquires an additional $\mu^2 \phi^* \phi$ term: see [19].

where L_i is the length of the box in the i direction and ρ is a dimensionless infinitesimal parameter.

In the large volume limit it is assumed that $P(\beta, \mu, V) = P(\beta, \mu)$,⁴ so under the transformation of Eq. (16)

$$\begin{aligned} \delta(\beta PV) &= (\delta\beta)PV + \beta(\delta P)V + \beta P(\delta V) \\ &= \rho \left(\beta PV + \beta \left(\frac{\partial P}{\partial \beta} \beta \right) V + \beta P(3V) \right). \end{aligned} \quad (17)$$

Now using the identity $\beta V \frac{\partial P}{\partial \beta} = -PV - E + \mu Q$, we get

$$\delta(\beta PV) = \rho(-\beta E + \beta P(3V) + \beta \mu Q), \quad (18)$$

and therefore

$$\delta(e^{\beta PV}) = \delta(\beta PV)e^{\beta PV} = \rho\beta(-E + 3PV + \mu Q)e^{\beta PV}. \quad (19)$$

Equation (19) represents the effect of the scaling in Eq. (16) on the lhs of Eq. (14). Now we analyze the effect of

this scaling to the rhs of Eq. (14), the path-integral part, from which anomalies originate, and eventually equate the two expressions.

The scaling in Eq. (16) represents a dilation of the system:

$$\begin{aligned} x'^\mu &= e^\rho x^\mu, \\ \phi'(x') &= e^{-\rho} \phi(x), \\ \phi'^*(x') &= e^{-\rho} \phi^*(x). \end{aligned} \quad (20)$$

The dilated system has

$$e^{\beta' P' V'} = \int [d\phi'] [d\phi'^*] e^{-S'_E + \mu \int_0^{\beta'} \int_V d^D x' dt' j'_0}, \quad (21)$$

where

$$\begin{aligned} S'_E &= \int_0^{e^\rho \beta} \int_{e^\rho V} d^3 x' dt' \left(\partial'_\mu \phi'^* \partial'_\mu \phi' + (m^2 - \mu^2) \phi'^* \phi' + \frac{\lambda}{4} (\phi'^* \phi')^2 \right), \\ \mu \int_0^{\beta'} \int_{V'} d^3 x' dt' j'_0 &= \mu \int_0^{e^\rho \beta} \int_{e^\rho V} d^3 x' dt' (-\phi'^* \overleftrightarrow{\partial}'_\tau \phi'). \end{aligned} \quad (22)$$

To compare to the undilated system, we “pull back” to unprimed variables by substituting Eq. (20) into Eq. (21) and Eq. (22). Equation (22) becomes

$$\begin{aligned} S'_E &= \int_0^{e^\rho \beta} \int_{e^\rho V} d^3 x' dt' \left(\partial'_\mu \phi'^* \partial'_\mu \phi' + (m^2 - \mu^2) \phi'^* \phi' + \frac{\lambda}{4} (\phi'^* \phi')^2 \right) \\ &= \int_0^\beta \int_V e^{4\rho} d^3 x dt \left(e^{-2\rho} \frac{\partial \phi^*}{\partial (e^\rho x_\mu)} \frac{\partial \phi}{\partial (e^\rho x_\mu)} + (m^2 - \mu^2) e^{-2\rho} \phi^* \phi + \frac{\lambda}{4} (e^{-2\rho} \phi^* \phi)^2 \right) \\ &= S_E + 2\rho \int_0^\beta \int_V d^3 x dt (m^2 - \mu^2) \phi^* \phi. \end{aligned} \quad (23)$$

Similarly,

$$\mu \int_0^{\beta'} \int_{V'} d^3 x' dt' j'_0 = \mu \int_0^{e^\rho \beta} \int_{e^\rho V} d^3 x' dt' (-\phi'^* \overleftrightarrow{\partial}'_\tau \phi') = \mu \int_0^\beta \int_V d^3 x dt j_0 + \rho \mu \int_0^\beta \int_V d^3 x dt j_0. \quad (24)$$

Plugging in these expressions into Eq. (21),

$$e^{\beta' P' V'} = \int J[d\phi][d\phi^*] e^{-S_E + \mu \int_0^\beta \int_V d^3 x dt j_0 - 2\rho \int_0^\beta \int_V d^3 x dt (m^2 - \mu^2) \phi^* \phi + \rho \mu \int_0^\beta \int_V d^3 x dt j_0}, \quad (25)$$

where J is the Jacobian of the transformation $(\phi', \phi'^*) \rightarrow (\phi, \phi^*)$. Expressing $J = 1 - \rho A$ and using Eq. (19)⁵

$$\delta(e^{\beta PV}) = \rho\beta(-E + 3PV + \mu Q)e^{\beta PV} = \rho \left(-A - 2 \left\langle \int_0^\beta \int_V d^3 x dt (m^2 - \mu^2) \phi^\dagger \phi \right\rangle + \left\langle \mu \int_0^\beta \int_V d^3 x dt j_0 \right\rangle \right) e^{\beta PV}. \quad (26)$$

⁴This can be shown via cluster decomposition; e.g., see [14].

⁵ $\langle F(\phi, \phi^\dagger) \rangle \equiv \frac{1}{2} \int [d\phi][d\phi^*] F(\phi, \phi^*) e^{-S_E + \mu \int_0^\beta \int_V d^3 x dt j_0}$.

The chemical potential terms drop out on both sides⁶ and we get

$$\mathcal{E} - 3P = 2m^2 \langle \phi^\dagger \phi \rangle + \mathcal{A}, \quad (27)$$

where

$$\begin{aligned} J &= \left[\frac{\partial \phi' \partial \phi'^*}{\partial \phi \partial \phi^*} \right] = e^{\text{Tr} \log (I_2 (\delta^4(x-y) + \rho(-1-x_\mu \partial_\mu) \delta^4(x-y)))} \\ &= e^\rho \int d^4x \text{tr} [(-1-x_\mu \partial_\mu) \delta^4(x-y) I_2] |_{x=y} \\ &= 1 + \rho \int d^4x \text{tr} [(-1-x_\mu \partial_\mu) \delta^4(x-y) I_2] |_{x=y}, \end{aligned} \quad (28)$$

so that

$$\mathcal{A} = \text{tr} [(1+x_\mu \partial_\mu) \delta^4(x-y) I_2] |_{x=y}. \quad (29)$$

I_2 is the two dimensional identity matrix which results from having two fields, ϕ and ϕ^* .⁷ $\mathcal{A} = \frac{\mathcal{A}}{\beta V}$ is the anomaly, a divergent quantity that requires regularization.

III. FUJIKAWA CALCULATION

In Euclidean space, $\mathcal{L}_E = \partial_\mu \phi^\dagger \partial_\mu \phi + m^2 \phi^\dagger \phi + \frac{\lambda}{4} (\phi^\dagger \phi)^2$. A saddle point expansion about a constant classical background ϕ produces the quadratic piece \mathcal{L}_2 :

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2} \begin{pmatrix} \eta^\dagger & \eta \end{pmatrix} \begin{pmatrix} -\partial^2 + m^2 + \lambda \phi^* \phi & \frac{\lambda}{2} \phi \phi \\ \frac{\lambda}{2} \phi^* \phi^* & -\partial^2 + m^2 + \lambda \phi^* \phi \end{pmatrix} \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix} \\ &\equiv \frac{1}{2} \begin{pmatrix} \eta^\dagger & \eta \end{pmatrix} \begin{pmatrix} -\partial^2 + C & \frac{\lambda}{2} \phi \phi \\ \frac{\lambda}{2} \phi^* \phi^* & -\partial^2 + C \end{pmatrix} \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} \eta^\dagger & \eta \end{pmatrix} M \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix}, \end{aligned} \quad (30)$$

where $C = m^2 + \lambda \phi^* \phi$, η is the fluctuating field around ϕ , and M is a Hermitian matrix. Following Fujikawa [20], we use M , the bilinear matrix, as the Hermitian matrix that goes in our regulator.⁸ Choose a regulator of the form $R = R(\frac{M}{\Lambda^2})$ with the property that $R(0) = I_2$. The expression to be regulated is

$$\mathcal{A} = \text{tr} \begin{pmatrix} \theta \delta(x-y) & 0 \\ 0 & \theta \delta(x-y) \end{pmatrix} \Big|_{x=y} \quad (31)$$

where $\theta = 1 + x_\mu \partial_\mu$, so that

$$\mathcal{A}_R = \text{tr} \left[R \left(\frac{M}{\Lambda^2} \right) \theta \delta(x-y) I_2 \right] \Big|_{x=y}. \quad (32)$$

This expression equals

$$\begin{aligned} \mathcal{A}_R &= \int \frac{d^4k}{(2\pi)^4} \text{tr} R \left(\frac{-\partial^2 + C}{\Lambda^2} \quad \frac{\frac{\lambda \phi \phi}{2\Lambda^2}}{\frac{-\partial^2 + C}{\Lambda^2}} \right) \theta e^{-ik(x-y)} \Big|_{x=y} \\ &= \int \frac{d^4k}{(2\pi)^4} \text{tr} R \left(\frac{k^2 + C}{\Lambda^2} \quad \frac{\frac{\lambda \phi \phi}{2\Lambda^2}}{\frac{k^2 + C}{\Lambda^2}} \right) (1 - ix_\mu k_\mu) \\ &= \Lambda^4 \int \frac{d^4k}{(2\pi)^4} \text{tr} R \left(k^2 + \frac{C}{\Lambda^2} \quad \frac{\frac{\lambda \phi \phi}{2\Lambda^2}}{k^2 + \frac{C}{\Lambda^2}} \right) (1 - i\Lambda x_\mu k_\mu) \\ &= \Lambda^4 \int \frac{d^4k}{(2\pi)^4} \text{tr} R \left(k^2 + \frac{C}{\Lambda^2} \quad \frac{\frac{\lambda \phi \phi}{2\Lambda^2}}{k^2 + \frac{C}{\Lambda^2}} \right), \end{aligned} \quad (33)$$

where the k_μ term is odd so vanishes over the integral when multiplied by the even function $R(-k) = R(k) = f(k^2)$. Next we define

$$\begin{aligned} D &= k^2 I_2, \\ B &= \frac{1}{\Lambda^2} \begin{pmatrix} C & \frac{\lambda \phi \phi}{2} \\ \frac{\lambda \phi^* \phi^*}{2} & C \end{pmatrix}, \end{aligned} \quad (34)$$

so that the equation can be written succinctly:

$$\mathcal{A}_R = \Lambda^4 \int \frac{d^4k}{(2\pi)^4} \text{tr} R(D+B). \quad (35)$$

We then Taylor expand about D (note that $[D, B] = 0$ so the Taylor expansion is valid):

⁶Using the identity $Q = \frac{\partial P}{\partial \mu}$ and Eq. (14), $Q = \frac{\partial P}{\partial \mu} = \langle j_0 \rangle + 2\mu \langle \phi^\dagger \phi \rangle$.

⁷Note that Tr in Eq. (28) refers to both discrete (2×2) and continuous variables, whereas tr in Eq. (29) refers to only (2×2).

⁸E.g., for the chiral anomaly with $\mathcal{L} = \bar{\psi} i \mathcal{D} \psi$, the matrix $i\mathcal{D}$ is to be used as the argument of the regulator. M , the quadratic piece of the quantum action, naturally captures the 1-loop effects of interactions which are responsible for anomalies.

$$\mathcal{A}_R = \Lambda^4 \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left(R(D) + R'(D)B + \frac{1}{2}R''(D)B^2 + \dots \right). \quad (36)$$

The first term is the same as in the noninteracting case, which is taken to be anomaly free [21], so we neglect it. The second term can be absorbed by a mass counterterm. Terms higher order than the third term fall faster than $\frac{1}{\Lambda^4}$ so the Λ^4 prefactor in Eq. (36) cannot keep them from going to zero. Only the 3rd term is independent of the cutoff. Therefore,

$$\begin{aligned} \mathcal{A}_R &= \Lambda^4 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2} \text{tr}(R''(D)B^2) \\ &= \Lambda^4 \int \frac{k^2 dk^2}{16\pi^2} \frac{1}{2} \text{tr}(R''(D)B^2), \end{aligned} \quad (37)$$

where the solid angle $\Omega = 2\pi^2$ was used. Now

$$B^2 = \frac{1}{\Lambda^2} \begin{pmatrix} C^2 + \frac{\lambda^2(\phi^*\phi)^2}{4} & \lambda C\phi\phi \\ \lambda C\phi^*\phi^* & C^2 + \frac{\lambda^2(\phi^*\phi)^2}{4} \end{pmatrix} \equiv \frac{1}{\Lambda^2} \begin{pmatrix} B_1 & B_2 \\ B_2^* & B_1 \end{pmatrix}, \quad (38)$$

and since $R(D)$ is diagonal, we can define

$$R(D) = f(k^2)I_2. \quad (39)$$

Note that the derivative in Eq. (37) is with respect to k^2 . Therefore,

$$\begin{aligned} \mathcal{A}_R &= \Lambda^4 \int \frac{k^2 dk^2}{16\pi^2} \frac{1}{2} \text{tr}(R''(D)B^2) \\ &= B_1 \int \frac{k^2 dk^2}{16\pi^2} f''(k^2), \end{aligned} \quad (40)$$

where we have safely taken $\Lambda \rightarrow \infty$. Integrating by parts,

$$\begin{aligned} \mathcal{A}_R &= \frac{B_1}{16\pi^2} [k^2 f'(k^2)] \Big|_0^\infty - \frac{B_1}{16\pi^2} \int dk^2 f'(k^2) \\ &= \frac{B_1}{16\pi^2} [k^2 f'(k^2)] \Big|_0^\infty - \frac{B_1}{16\pi^2} f(k^2) \Big|_0^\infty \\ &= \frac{B_1}{16\pi^2}, \end{aligned} \quad (41)$$

where we require

$$\begin{aligned} f(0) &= 1 \\ f(\infty) &= 0 \\ [k^2 f'(k^2)] \Big|_0^\infty &= 0, \end{aligned} \quad (42)$$

which are the same conditions on the regulator for the chiral case [22].

Plugging in B_1 from Eq. (38) into Eq. (41), we get

$$\mathcal{A}_R = \frac{C^2 + \frac{\lambda^2(\phi^*\phi)^2}{4}}{16\pi^2} = \frac{5\lambda^2(\phi^*\phi)^2}{64\pi^2} + \frac{m^4}{16\pi^2} + \frac{\lambda m^2(\phi^*\phi)}{8\pi^2}. \quad (43)$$

The second term is independent of the coupling, and since the free theory is taken to be nonanomalous, we can subtract it. The third term can be absorbed into the mass term of Eq. (27), leaving only the 1st term as the anomaly [23]. Therefore

$$\mathcal{E} - 3P = \frac{5\lambda^2}{64\pi^2} \langle (\phi^\dagger \phi)^2 \rangle. \quad (44)$$

Note that the anomaly \mathcal{A}_R occurs inside the path integral, and $\frac{1}{2} \int [d\phi d\phi^*] f(\phi, \phi^*) e^{-S_E + \dots} = \langle f(\phi, \phi^\dagger) \rangle$, so that in Eq. (44) there are expectation values. This replacement is valid up to 1-loop [23].

IV. DIMENSIONAL ANALYSIS FOR RELATIVISTIC SYSTEMS

In relativistic theories we set $\hbar = c = k_B = 1$. The units for all quantities can then be written as $\hbar^i c^j k_B^k L^\ell$, where L is a variable in the problem with units of length. Suppose the system has microscopic parameters g_k , which can be coupling constants or dimensionally transmuted quantities. We define $[g_k] = \ell$ as the power of L when g_k is written in units of $\hbar^i c^j k_B^k L^\ell$. So for example $[m] = [E] = -1$. The grand potential $\Omega = \Omega(\beta, \mu_i, V, g_i)$ has $[\Omega] = -1$ and can be written as

$$\Omega(\beta, z_i, V, g_i) = V\beta^{-1-D} f(z_i, g_i\beta^{-[g_i]}), \quad (45)$$

where $f(z_i, g_i\beta^{-[g_i]})$ is a dimensionless function of dimensionless variables, z_i is the fugacity corresponding to μ_i ($z_i = e^{\beta\mu_i}$), and D is the number of spatial dimensions.⁹ Ω has this form because β and μ_i do not depend on the absolute size of the system (they are intensive variables). If one doubles the system keeping β and μ_i constant, then Ω , being an extensive quantity, should double. So Ω must be proportional to V .¹⁰ To make up for the remaining dimension ($[\Omega] = -1$), we are free to pull out one of the dimensionful arguments of Ω , and the rest of the arguments must be ratios with the argument we pulled out. We will pull out β . This is equivalent to choosing our scale as β and measuring all other quantities in units of β .

Take the derivative of Eq. (45) with respect to β at constant fugacity z_i and volume V , and multiply times β :

⁹For example, if the coupling g_1 has dimensions of length, the corresponding dimensionless variable is $g_1\beta^{-1} = g_1T$ which is dimensionless. If the coupling g_2 has dimensions of energy, $g_2\beta^{-(-1)} = g_2\beta = \frac{g_2}{T}$.
¹⁰ $\Omega = -PV$, so Eq. (45) is consistent with the statement that $P(\beta, \mu, V) = P(\beta, \mu)$.

$$\begin{aligned}
\beta \frac{\partial \Omega}{\partial \beta} \Big|_{z_i, V} &= (-1 - D)\Omega + V\beta^{-1-D} \beta \frac{\partial f(z_i, g_i \beta^{-[g_i]})}{\partial \beta} \Big|_{z_i} \\
&= (-1 - D)\Omega \\
&\quad + V\beta^{-1-D} \beta \left[\sum_k \frac{-[g_k]g_k}{\beta} \frac{\partial f(z_i, g_i \beta^{-[g_i]})}{\partial g_k} \right] \Big|_{z_i} \\
&= (-1 - D)\Omega - \sum_k [g_k]g_k \frac{\partial \Omega}{\partial g_k}. \tag{46}
\end{aligned}$$

Now, we use the thermodynamic identity $E = \frac{\partial(\beta\Omega)}{\partial \beta} \Big|_{z_i, V} = \Omega + \beta \frac{\partial \Omega}{\partial \beta} \Big|_{z_i, V}$.

$$\begin{aligned}
E - DPV &= \left(\Omega + \beta \frac{\partial \Omega}{\partial \beta} \Big|_{z_i, V} \right) - DPV \\
&= \left(\Omega + (-1 - D)\Omega - \sum_k [g_k]g_k \frac{\partial \Omega}{\partial g_k} \right) - DPV \\
&= - \left(P + (-1 - D)P - \sum_k [g_k]g_k \frac{\partial P}{\partial g_k} \right) V - DPV \\
&= \sum_k [g_k]g_k \frac{\partial P}{\partial g_k} V \\
\mathcal{E} - DP &= \sum_k [g_k]g_k \frac{\partial P}{\partial g_k}, \tag{47}
\end{aligned}$$

where the derivatives are at constant β , μ , and V .

V. β FUNCTION

For a system that develops a microscopic scale M through dimensional transmutation via renormalization of the coupling constant

$$\begin{aligned}
\mathcal{E} - DP &= [M]M \frac{d\lambda}{dM} \frac{\partial P}{\partial \lambda} = -M \frac{d\lambda}{dM} \frac{\partial P}{\partial \lambda} \\
&= -\beta(\lambda) \frac{\partial P}{\partial \lambda} = \beta(\lambda) \left\langle \frac{\partial \mathcal{H}_I}{\partial \lambda} \right\rangle, \tag{48}
\end{aligned}$$

since $\frac{\partial P}{\partial \lambda} = \frac{1}{\beta V} \frac{\partial}{\partial \lambda} \ln \left\{ \int [d\phi][d\phi^*] e^{-S_E + \mu \int d^D x d\tau j^p} \right\}$ pulls down the interaction term in the path integral, creating a thermal average.

Comparison of Eq. (10), Eq. (27), Eq. (44), and Eq. (48) gives

$$\beta(\lambda) = \frac{5\lambda^2}{16\pi^2}, \tag{49}$$

as

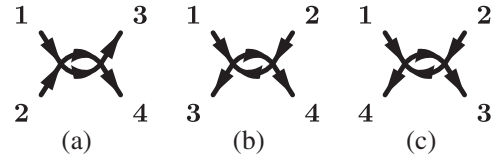


FIG. 1. Diagrams contributing to the β function for complex scalar field theory. 1 and 2 refer to incoming particles, 3 and 4 to outgoing particles.

$$\mathcal{E} - 3P = \frac{5\lambda^2}{64\pi^2} \langle (\phi^\dagger \phi)^2 \rangle = \beta(\lambda) \left\langle \frac{(\phi^\dagger \phi)^2}{4} \right\rangle \tag{50}$$

would give Eq. (49).

The β function of Eq. (49) can be gotten from setting $e = 0$ for the charge e in the calculation for the four-scalar vertex in scalar electrodynamics [24]. A diagrammatic calculation requires the identification of 3 diagrams (see Fig 1). Diagram (a) contains a symmetry factor of $1/2$ due to the swapping of internal propagators. Modulo the symmetry factor, each diagram contributes the same amount to the β function, giving $1/2 + 1 + 1 = 5(1/2)$, or the first diagram's contribution multiplied by 5. The matrix M used for regularization automatically mixes the interactions, giving the factor of 5. Using the definition of the beta function $M \frac{d\lambda}{dM} = \frac{5\lambda^2}{16\pi^2}$ and setting the renormalization scale $M = T$, one can solve the differential equation for the coupling $\lambda(T) = \frac{16\pi^2}{5 \ln(\frac{T}{\Lambda})}$, where Λ is the Landau pole. As $\frac{T}{\Lambda} \rightarrow 0$ the coupling is small and the system behaves like a gas of noninteracting bosons, while as $T \rightarrow \Lambda$ the coupling blows up and perturbation theory fails.

VI. CONCLUSIONS

In this paper we have extended to relativistic systems the path-integral approach to the study of quantum anomalies for many-body systems initiated in [13–15]. A notable difference is that in the relativistic case we have a very wide class of regulators characterized by the function $f(k^2)$ of Eq. (39), which, other than satisfying Eq. (42), is of a very general nature. An interesting result of this paper is the extraction of the leading order result for the beta function for complex fields, Eq. (49), obtained here by comparing Eqs. (10), (27), (44) and (48), without resorting to graphical methods [24,25]. This result gives further support to the importance of Fujikawa's approach in the description of quantum anomalies for systems at finite temperature and density. We are currently pursuing further studies and extensions of this method, as well as applications to other systems with classical scale symmetry.

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- [1] C. G. Callan, S. Coleman, and R. Jackiw, *Ann. Phys. (N.Y.)* **59**, 42 (1970).
- [2] S. Coleman and R. Jackiw, *Ann. Phys. (N.Y.)* **67**, 552 (1971).
- [3] C. Hagen, *Phys. Rev. D* **5**, 377 (1972).
- [4] J. Gaiete, *Phys. Usp.* **56**, 919 (2013).
- [5] G. N. J. Añaños, H. E. Camblong, C. Gorrichátegui, E. Hernández, and C. R. Ordóñez, *Phys. Rev. D* **67**, 045018 (2003).
- [6] S. Tan, *Ann. Phys. (Amsterdam)* **323**, 2952 (2008).
- [7] S. Tan, *Ann. Phys. (Amsterdam)* **323**, 2971 (2008).
- [8] S. Tan, *Ann. Phys. (Amsterdam)* **323**, 2987 (2008).
- [9] E. Braaten, in *The BCS-BEC Crossover and the Unitary Fermi Gas*, Lecture Notes in Physics, Vol. 836, edited by W. Zwerger (Springer, Berlin, 2012), pp. 193–231.
- [10] E. Braaten and L. Platter, *Phys. Rev. Lett.* **100**, 205301 (2008).
- [11] J. Preskill, California Institute of Technology, Pasadena CA, 1983 (unpublished).
- [12] C. DeTar and U. Heller, *Eur. Phys. J. A* **41**, 405 (2009).
- [13] C. R. Ordóñez, [arXiv:1503.01384](https://arxiv.org/abs/1503.01384).
- [14] C. L. Lin and C. R. Ordóñez, *Adv. High Energy Phys.* **2015**, 796275 (2015).
- [15] C. L. Lin and C. R. Ordóñez, *Phys. Rev. D* **91**, 085023 (2015).
- [16] P. Jizba, *Phys. Rev. D* **69**, 085011 (2004).
- [17] N. P. Landsman and C. G. van Weert, *Phys. Rep.* **145**, 141 (1987).
- [18] I. Drummond, R. Horgan, P. Landshoff, and A. Rebhan, *Phys. Lett. B* **460**, 197 (1999).
- [19] J. I. Kapusta and C. Gale, *Finite-Temperature Field Theory: Principles and Applications* (Cambridge University Press, Cambridge, 2006).
- [20] K. Fujikawa, *Phys. Rev. Lett.* **42**, 1195 (1979).
- [21] M. Umezawa, *Phys. Rev. D* **39**, 3672 (1989).
- [22] S. Weinberg, *The Quantum Theory of Fields: Vol. 2* (Cambridge University Press, Cambridge, 1996).
- [23] K. Shizuya and H. Tsukahara, *Z. Phys. C* **31**, 553 (1986).
- [24] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, Cambridge, 2007).
- [25] S. Coleman and E. Weinberg, *Phys. Rev. D* **7**, 1888 (1973).