Loop expansion of the average effective action in the functional renormalization group approach

Peter M. Lavrov[1,2,*] and Boris S. Merzlikin[1,3,†]

[1]Tomsk State Pedagogical University, Kievskaya St. 60, 634061 Tomsk, Russia
[2]National Research Tomsk State University, Lenin Av. 36, 634050 Tomsk, Russia
[3]National Research Tomsk Polytechnical University, Lenin Av. 30, 634050 Tomsk, Russia

(Received 11 August 2015; published 26 October 2015)

We formulate a perturbation expansion for the effective action in a new approach to the functional renormalization group method based on the concept of composite fields for regulator functions being their most essential ingredients. We demonstrate explicitly the principal difference between the properties of effective actions in these two approaches existing already on the one-loop level in a simple gauge model.

DOI: 10.1103/PhysRevD.92.085038 PACS numbers: 11.10.Ef, 11.15.Bt

I. INTRODUCTION

The functional renormalization group (FRG) approach [1–5] is a very popular method (see the recent review in Ref. [6] and references therein) to study quantum properties of physical models beyond the perturbation theory. The application of this method to gauge systems meets essential difficulties which are connected with gauge dependence of the average effective action even on shell [7,8]. It happens due to the presence of regulator functions which improve the behavior of propagators in IR and UV regions but destroy the gauge invariance of the initial classical action. It was the main reason in Refs. [7,8] to reformulate the standard FRG approach preserving its attractive features with regulator functions in a way leading to gauge independence of the effective action on its extremals. This is achieved when regulator functions are considered as composite fields introducing on the quantum level with the help of additional sources. In quantum tield theory (QFT), the effective action with composite fields was introduced and studied within the perturbation theory by Cornwell, Jackiw, and Tomboulis [9]. Later, it was shown that the effective action with composite fields in Yang–Mills theories [10] as well as in general gauge theories [11] does not depend on the gauge on its extremals. This allows one to consider quantum methods based on the idea of composite fields as consistent ones. Namely, this fact was the basis for a new approach to FRG [7,8].

In the present article, we study the properties of average effective actions, both in the standard and new FRG approaches in a loop approximation. Here, it should be noted that the FRG approach has been proposed as a method to study nonperturbative quantum effects with the help of the so-called FRG flow equation for the average effective action. On the other hand, all renormalization procedures in QFT are known in the framework of perturbation theory only. In particular, this means that any approach to the quantum description of models in QFT might be tested on the level of perturbation theory to satisfy some physical requirements. Among such requirements, the gauge independence of the effective action on shell is very essential. Because of this circumstance, in the present paper, we restrict ourselves to the study of properties of the average effective action proposed in Refs. [7,8] in the loop approximation. We find by explicit calculations the difference existing between the one-loop average effective actions in the standard and new FRG approaches already in the case of a simple gauge model. Moreover, the average effective action found in this model is exact in the case of the standard FRG approach without referring to the perturbation theory and to the flow equation.

II. AVERAGE EFFECTIVE ACTION IN THE STANDARD FRG APPROACH

We consider a Yang–Mills theory of fields A^a_μ with the action $S_0 = S_0(A)$ and assume its invariance under the gauge transformations,

$$S_0(A)\frac{\overleftarrow{\delta}}{\delta A^a_\mu}D^{ab}_\mu = 0, \qquad \delta A^a_\mu = D^{ab}_\mu \xi^b, \tag{2.1}$$

where $D^{ab}_\mu = \delta^{ab}\partial_\mu + f^{acb}A^c_\mu$ is the covariant derivative, ξ^a is an arbitrary gauge function, and f^{abc} are structure constants of a Lie group. Quantization of the model via the Faddeev–Popov method [12] involves the configuration field space

$$\varphi^A = \{A^a_\mu, B^a, C^a, \bar{C}^a\}, \tag{2.2}$$

including the ghost (C^a) and antighost ($\bar{C}^a$) fields and auxiliary fields (B^a) with the following distribution of Grassmann parities:

*lavrov@tspu.edu.ru
†merzlikin@tspu.edu.ru

$$\varepsilon(\varphi^A) = \varepsilon_A, \qquad \varepsilon(A^a_\mu) = \varepsilon(B^a) = 0,$$
$$\varepsilon(C^a) = \varepsilon(\bar{C}^a) = 1. \tag{2.3}$$

The Faddeev–Popov action, $S_{\rm FP}(\Phi)$, can be presented in the form

$$S_{\rm FP}(\varphi) = S_0(A) + \Psi(\varphi)\overleftarrow{d}, \tag{2.4}$$

where the nilpotent differential $\overleftarrow{d}$,

$$\overleftarrow{d} = \frac{\overleftarrow{\delta}}{\delta A^a_\mu} D^{ab}_\mu C^b + \frac{\overleftarrow{\delta}}{\delta \bar{C}^a} B^a + \frac{\overleftarrow{\delta}}{\delta C^a}\frac{1}{2} f^{abc} C^c C^b,$$
$$\overleftarrow{d}^2 = 0, \tag{2.5}$$

generates the Becchi-Rouet-Stora-Tyutin (BRST) transformation [13,14]

$$\delta\varphi^A = \varphi^A \overleftarrow{d}\mu. \tag{2.6}$$

Here, $\Psi(\varphi)$ is a gauge-fixing Fermion functional, and μ is a constant Grassmann parameter. Usually, the

$$\Psi(\varphi) = \bar{C}^a \chi^a(\varphi) \tag{2.7}$$

form of $\Psi(\varphi)$ is used. One of the more popular choices of gauge functions, $\chi^a(\varphi) = \chi^a(A, B)$, reads

$$\chi^a(A, B) = \partial^\mu A^a_\mu + \frac{\alpha}{2} B^a, \tag{2.8}$$

where α is a gauge parameter. The action $S_{\rm FP}(\varphi)$ is BRST invariant,

$$S_{\rm FP}(\varphi)\overleftarrow{d} = 0. \tag{2.9}$$

The main idea of the standard formulation of the FRG approach is to modify from the very beginning propagators of vector fields as well as ghost and antighost fields by introducing the regulator Lagrangians with a momentum-shell parameter k,

$$L^1_k(x) = \frac{1}{2} A^{a\mu}(x)(R_{k,A})^{ab}_{\mu\nu}(x) A^{b\nu}(x), \tag{2.10}$$

$$L^2_k(x) = \bar{C}^a(x)(R_{k,gh})^{ab}(x) C^b(x)$$
$$= \bar{C}^a(x)(\bar{R}_{k,gh})^{ab}(x) C^b(x), \tag{2.11}$$

$$(\bar{R}_{k,gh})^{ab}(x) = \frac{1}{2}((R_{k,gh})^{ab}(x) - R_{k,gh})^{ba}(x)), \tag{2.12}$$

where regulator functions $R_{k,A}$ and $R_{k,gh}$ do not depend on the fields and obey the properties

$$\lim_{k\to 0} (R_{k,A})^{ab}_{\mu\nu} = 0, \qquad \lim_{k\to 0} (R_{k,gh})^{ab} = 0. \tag{2.13}$$

The generating functional of the Green function is constructed in the form of a path integral,

$$Z_k(J) = \frac{1}{N}\int \mathcal{D}\varphi \exp\left\{\frac{i}{\hbar}[S_{\rm FP}(\varphi) + S_k(\varphi) + J_A\varphi^A)]\right\}$$
$$= \exp\left\{\frac{i}{\hbar} W_k(J)\right\}, \tag{2.14}$$

where $W_k(J)$ is the generating functional of the connected Green functions; $J_A = J_A(x)$; $\varepsilon(J_A) = \varepsilon_A$; N is a normalization constant,

$$N = \int \mathcal{D}\varphi \exp\left\{\frac{i}{2\hbar}\varphi^A(iD^{-1}_{AB})\varphi^B\right\} = ({\rm sDet}(iD^{-1}))^{-\frac{1}{2}}; \tag{2.15}$$

and

$$iD^{-1}_{AB} = \overrightarrow{\partial}_A(S_{\rm FP}(\varphi) + S_k(\varphi))\overleftarrow{\partial}_B|_{\varphi=0}, \qquad \partial_A = \frac{\delta}{\delta\varphi^A}. \tag{2.16}$$

In Eqs. (2.14) and (2.16), $S_k(\varphi)$ is the regulator action,

$$S_k(\varphi) \equiv \frac{1}{2}\varphi^A((L^{1\prime\prime}_k)_{AB} + (L^{2\prime\prime}_k)_{AB})\varphi^B$$
$$= \int dx [L^1_k(x) + L^2_k(x)], \tag{2.17}$$

where

$$(L^{i\prime\prime}_k)_{AB} = \overrightarrow{\partial}_A L^i_k(x)\overleftarrow{\partial}_B, \qquad i = 1, 2, \tag{2.18}$$

are constant supermatrices.

The effective action, $\Gamma_k(\Phi)$, is defined as the modified Legendre transform of $W_k(J)$ [6] with respect to J_A,

$$\Gamma_k(\Phi) = W_k(J) - J_A\Phi^A - S_k(\Phi), \qquad \frac{\overrightarrow{\delta}}{\delta J_A} W_k(J) = \Phi^A, \tag{2.19}$$

so that[1]

$$\Gamma_k(\Phi)\overleftarrow{\partial}_A = -J_A - S_k(\Phi)\overleftarrow{\partial}_A. \tag{2.20}$$

The $\Gamma_k(\Phi)$ satisfies the functional integrodifferential equation

[1]We use the same notation ∂_A, meaning the derivative over field Φ_A.

$$\exp\left\{\frac{i}{\hbar}\Gamma_k(\Phi)\right\} = \frac{1}{N}\int D\varphi \exp\left\{\frac{i}{\hbar}[S_{\rm FP}(\Phi+\varphi)+S_k(\Phi+\varphi)-S_k(\Phi) - (S_k(\Phi)\overleftarrow{\partial}_A)\varphi^A - (\Gamma_k(\Phi)\overleftarrow{\partial}_A)\varphi^A]\right\}. \tag{2.21}$$

It has been shown in Ref. [7] that the effective action (2.21) depends on the gauge even on its extremals, $\Gamma_k(\Phi)\overleftarrow{\partial}_A = 0$. This fact indicates a serious problem with the physical interpretation of the results obtained for gauge theories in the framework of the standard FRG method. And this was the main reason to reformulate the standard FRG approach in the form being free of gauge dependence on shell.

III. EFFECTIVE ACTION IN THE NEW FRG FORMULATION

The new FRG approach involves external scalar sources $\Sigma_1(x)$ and $\Sigma_2(x)$, $\varepsilon(\Sigma_1(x)) = \varepsilon(\Sigma_2(x)) = 0$. The generating functional of the Green functions for Yang–Mills theories with composite fields is introduced as

$$\mathcal{Z}_k(J;\Sigma) = \int \mathcal{D}\varphi \exp\left\{\frac{i}{\hbar}[S_{\rm FP}(\varphi)+J_A\varphi^A+\Sigma_i L^i_k(\varphi)]\right\} = \exp\left\{\frac{i}{\hbar}\mathcal{W}_k(J;\Sigma)\right\}, \tag{3.1}$$

where $\mathcal{W}_k(J;\Sigma)$ is the generating functional of the Green functions in the presence of composite fields. Here, we introduce the following notation:

$$\Sigma_i L^i_k(\varphi) = \int dx[\Sigma_1(x)L^1_k(x)+\Sigma_2(x)L^2_k(x)]. \tag{3.2}$$

Using the explicit structure of the regulator Lagrangians (2.10), (2.11) and the definition (3.1), we deduce the relations

$$\frac{\vec{\delta}}{\delta\Sigma_i}\mathcal{Z}_k = \frac{\hbar}{2i}\left(\frac{\vec{\delta}^2}{\delta J_B\delta J_A}\mathcal{Z}_k\right)(L^{i\prime\prime}_k)_{AB}(-1)^{\varepsilon_B}, \tag{3.3}$$

or, in terms of $\mathcal{W}_k$,

$$\frac{\vec{\delta}}{\delta\Sigma_i}\mathcal{W}_k = \frac{\hbar}{2i}\left[\left(\frac{\vec{\delta}^2}{\delta J_B\delta J_A}\mathcal{W}_k\right)+\frac{i}{\hbar}\left(\frac{\vec{\delta}}{\delta J_B}\mathcal{W}_k\right)\left(\frac{\vec{\delta}}{\delta J_A}\mathcal{W}_k\right)\right] \times (L^{i\prime\prime}_k)_{AB}(-1)^{\varepsilon_B}. \tag{3.4}$$

The effective action with composite fields, $\Gamma_k = \Gamma_k(\Phi;F)$, can be introduced by means of the double Legendre transformations

$$\Gamma_k(\Phi;F) = \mathcal{W}_k(J;\Sigma) - J_A\Phi^A - \Sigma_i\left[L^i_k(\Phi)+\frac{1}{2}\hbar F^i\right], \tag{3.5}$$

where

$$\frac{\vec{\delta}}{\delta J_A}\mathcal{W}_k(J;\Sigma) = \Phi^A, \qquad \frac{\vec{\delta}}{\delta\Sigma_i}\mathcal{W}_k(J;\Sigma) = L^i_k(\Phi)+\frac{1}{2}\hbar F^i, \qquad i=1,2. \tag{3.6}$$

From Eqs. (3.5) and (3.6), it follows that

$$\Gamma_k(\Phi;F)\overleftarrow{\partial}_A = -J_A - \Sigma_i(L^i_k(\Phi)\overleftarrow{\partial}_A), \qquad \Gamma_{k,i}(\Phi;F) = -\frac{1}{2}\hbar\Sigma_i, \qquad \Gamma_{k,i} = \Gamma_k\frac{\overleftarrow{\delta}}{\delta F^i}. \tag{3.7}$$

Let us introduce the full sets of fields $\mathcal{F}^{\mathcal{A}}$ and sources $\mathcal{J}_{\mathcal{A}}$ according to

$$\mathcal{F}^{\mathcal{A}} = (\Phi^A, F^i), \qquad \mathcal{J}_{\mathcal{A}} = (J_A, \Sigma_i). \tag{3.8}$$

From the condition of the solvability of Eqs. (3.7) with respect to the sources J and Σ, it follows that

$$\left(\frac{\vec{\delta}}{\delta\mathcal{J}_{\mathcal{B}}}\mathcal{F}^{\mathcal{C}}(\mathcal{J})\right)\left(\frac{\vec{\delta}}{\delta\mathcal{F}^{\mathcal{C}}}\mathcal{J}_{\mathcal{A}}(\mathcal{F})\right) = \delta^{\mathcal{B}}_{\mathcal{A}}. \tag{3.9}$$

One can express $\mathcal{J}_{\mathcal{A}}$ as a function of the fields in the form

$$\mathcal{J}_{\mathcal{A}} = \left(-(\Gamma_k\overleftarrow{\partial}_A)+\frac{2}{\hbar}\Gamma_{k,i}(L^i_k(\Phi)\overleftarrow{\partial}_A), -\frac{2}{\hbar}\Gamma_{k,i}\right), \tag{3.10}$$

and therefore

$$\frac{\vec{\delta}}{\delta\mathcal{F}^{\mathcal{A}}}\mathcal{J}_{\mathcal{B}}(\mathcal{F}) = -(G''_k)_{\mathcal{AB}}, \qquad \frac{\vec{\delta}}{\delta\mathcal{J}_{\mathcal{A}}}\mathcal{F}^{\mathcal{B}}(\mathcal{J}) = -(G''^{-1}_k)^{\mathcal{AB}}. \tag{3.11}$$

Here,

$$(G''_k)_{\mathcal{AB}} = \begin{pmatrix} (\Gamma''_k)_{AB} - \frac{2}{\hbar}\Gamma_{k,i}(L^{i\prime\prime}_k)_{AB} - \frac{2}{\hbar}(\vec{\partial}_A\Gamma_{k,i})(L^i_k(\Phi)\overleftarrow{\partial}_B) & \frac{2}{\hbar}(\vec{\partial}_A\Gamma_{k,j}) \\ (\Gamma_{k,i}\overleftarrow{\partial}_B) - \frac{2}{\hbar}(\Gamma''_k)_{ij}(L^j_k(\Phi)\overleftarrow{\partial}_B) & \frac{2}{\hbar}(\Gamma''_k)_{ij} \end{pmatrix}, \tag{3.12}$$

and

$$(\Gamma''_k)_{AB} = \vec{\partial}_A \Gamma_k \overleftarrow{\partial}_B, \qquad (\Gamma''_k)_{ij} = \frac{\vec{\delta}}{\delta F^i}\Gamma_k \frac{\overleftarrow{\delta}}{\delta F^j}, \tag{3.13}$$

$$(G''_k)_{\mathcal{AC}}(G''^{-1}_k)^{\mathcal{CB}} = \delta^{\mathcal{B}}_{\mathcal{A}}, \quad (G''^{-1}_k)^{\mathcal{AC}}(G''_k)_{\mathcal{CB}} = \delta^{\mathcal{A}}_{\mathcal{B}}. \tag{3.14}$$

Let us introduce the supermatrix

$$\mathcal{W}^{\mathcal{AB}}_k = \frac{\vec{\delta}}{\delta \mathcal{J}_{\mathcal{A}}}\mathcal{F}^{\mathcal{B}}(\mathcal{J}). \tag{3.15}$$

Then, we have

$$\mathcal{W}^{\mathcal{AB}}_k = \begin{pmatrix} \mathcal{W}^{AB}_k & \frac{2}{\hbar}(\mathcal{W}^{Aj}_k - \mathcal{W}^{AC}_k(\vec{\partial}_C L^j_k(\Phi))) \\ \mathcal{W}^{iB}_k & \frac{2}{\hbar}(\mathcal{W}^{ij}_k - \mathcal{W}^{iC}_k(\vec{\partial}_C L^j_k(\Phi))) \end{pmatrix}, \tag{3.16}$$

where

$$\mathcal{W}^{AB}_k = \frac{\vec{\delta}^2 \mathcal{W}_k}{\delta J_A \delta J_B}, \qquad \mathcal{W}^{Ai}_k = \frac{\vec{\delta}^2 \mathcal{W}_k}{\delta J_A \delta \Sigma_i},$$
$$\mathcal{W}^{iA}_k = \frac{\vec{\delta}^2 \mathcal{W}_k}{\delta \Sigma_i \delta J_A}, \qquad \mathcal{W}^{ij}_k = \frac{\vec{\delta}^2 \mathcal{W}_k}{\delta \Sigma_i \delta \Sigma_j}. \tag{3.17}$$

From Eqs. (3.9) and (3.16), the following relations hold:

$$\left[(\Gamma''_k)_{AC} - \frac{2}{\hbar}\Gamma_{k,i}(L^{i\prime\prime}_k)_{AC} - \frac{2}{\hbar}(\vec{\partial}_A \Gamma_{k,i})(L^i_k(\Phi)\overleftarrow{\partial}_C)\right]\mathcal{W}^{CB}_k + \frac{2}{\hbar}(\vec{\partial}_A\Gamma_{k,j})\mathcal{W}^{jB} = -\delta_A{}^B, \tag{3.18}$$

$$\left[(\Gamma''_k)_{AC} - \frac{2}{\hbar}\Gamma_{k,i}(L^{i\prime\prime}_k)_{AC} - \frac{2}{\hbar}(\vec{\partial}_A \Gamma_{k,i})(L^i_k(\Phi)\overleftarrow{\partial}_C)\right](\mathcal{W}^{Cj}_k - \mathcal{W}^{CD}_k(\vec{\partial}_D L^j_k(\Phi))) + \frac{2}{\hbar}(\vec{\partial}_A\Gamma_{k,i})(\mathcal{W}^{ij}_k - \mathcal{W}^{iC}_k(\vec{\partial}_C L^j_k(\Phi))) = 0, \tag{3.19}$$

$$\left[(\Gamma_{k,i}\overleftarrow{\partial}_C) - \frac{2}{\hbar}(\Gamma''_k)_{ij}(L^j_k(\Phi)\overleftarrow{\partial}_C)\right]\mathcal{W}^{CB}_k + \frac{2}{\hbar}(\Gamma''_k)_{ij}\mathcal{W}^{jB}_k = 0, \tag{3.20}$$

$$\frac{2}{\hbar}\left[(\Gamma_{k,i}\overleftarrow{\partial}_C) - \frac{2}{\hbar}(\Gamma''_k)_{ij}(L^j_k(\Phi)\overleftarrow{\partial}_C)\right](\mathcal{W}^{Cj}_k - \mathcal{W}^{CD}_k(\vec{\partial}_D L^j_k(\Phi))) + \left(\frac{2}{\hbar}\right)^2(\Gamma''_k)_{il}(\mathcal{W}^{lj}_k - \mathcal{W}^{lC}_k(\vec{\partial}_C L^j_k(\Phi))) = -\delta_i{}^j. \tag{3.21}$$

In particular, from Eqs. (3.18) and (3.20), we deduce the presentation for $\mathcal{W}^{AB}$ in terms of the effective action $\Gamma_k(\Phi; F)$,

$$\mathcal{W}^{AB}_k = -\left((\Gamma''_k)_{AB} - \frac{2}{\hbar}\Gamma_{k,i}(L^{i\prime\prime}_k)_{AB} - (\vec{\partial}_A\Gamma_{k,i})(\Gamma^{-1}_k)^{ij}(\Gamma_{k,j}\overleftarrow{\partial}_B)\right)^{-1}. \tag{3.22}$$

This allows us to present the relation (3.4) on the level of the effective action in a closed form:

$$-iF^i = \mathcal{W}^{AB}_k(L^{i\prime\prime}_k)_{BA}(-1)^{\varepsilon_A} = \mathrm{sTr}\mathcal{W}^{AC}_k(L^{i\prime\prime}_k)_{CB}. \tag{3.23}$$

Finally, we discuss the structure of supermatrices $(L^{i\prime\prime}_k)_{AB}$ and the inverse one. According to Eqs. (2.10) and (2.11), we have

$$(L^{1\prime\prime}_k)_{AB} = \begin{pmatrix} (R_{k,A})^{ab}_{\mu\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.24}$$

$$(L^{2\prime\prime}_k)_{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (\bar{R}_{k,gh})^{ba} \\ 0 & (\bar{R}_{k,gh})^{ab} & 0 \end{pmatrix}. \tag{3.25}$$

Then,

$$\Sigma_i(L^{i\prime\prime}_k)_{AB} = \begin{pmatrix} \Sigma_1(R_{k,A})^{ab}_{\mu\nu} & 0 & 0 \\ 0 & 0 & \Sigma_2(\bar{R}_{k,gh})^{ba} \\ 0 & \Sigma_2(\bar{R}_{k,gh})^{ab} & 0 \end{pmatrix}. \tag{3.26}$$

It is useful to introduce the supermatrix

$$(L_k{}''^{-1})^{AB} = \begin{pmatrix} (R^{-1}_{k,A})^{\mu\nu}_{ab} & 0 & 0 \\ 0 & 0 & (\bar{R}^{-1}_{k,gh})_{ba} \\ 0 & (\bar{R}^{-1}_{k,gh})_{ab} & 0 \end{pmatrix}, \tag{3.27}$$

where

$$(R_{k,A})^{ac}_{\mu\alpha}(R^{-1}_{k,A})^{\alpha\nu}_{cb} = \delta^a_b\delta^\nu_\mu, \qquad (\bar{R}_{k,gh})^{ac}(\bar{R}^{-1}_{k,gh})_{cb} = \delta^a_b. \tag{3.28}$$

We obtain a useful relation,

$$\Sigma_i(L^{i\prime\prime}_k)_{AC}(L_k{}^{\prime\prime-1})^{CB} = \begin{pmatrix} \Sigma_1\delta^a_b\delta^\nu_\mu & 0 & 0 \\ 0 & \Sigma_2\delta^a_b & 0 \\ 0 & 0 & \Sigma_2\delta^a_b \end{pmatrix}. \tag{3.29}$$

It has been proven in Ref. [7] that the functional $\Gamma_k(\Phi;F)$ does not depend on the gauge on its extremals,

$$\Gamma_k(\Phi;F)\overleftarrow{\partial}_A = 0, \qquad \Gamma_{k,i}(\Phi;F) = 0. \tag{3.30}$$

IV. LOOP APPROXIMATION

In this section, we consider the procedure of loop expansions for $\Gamma_k(\Phi;F)$, following mainly Ref. [9]. Our starting point is the relation,

$$\exp\left\{\frac{i}{\hbar}\Gamma_k(\Phi;F)\right\} = \exp\left\{-\frac{i}{2}\Sigma_iF^i\right\}\int\mathcal{D}\varphi\exp\left\{\frac{i}{\hbar}[S_{\mathrm{FP}}(\varphi) + J_A(\varphi^A-\Phi^A) + \Sigma_i(L^i_k(\varphi) - L^i_k(\Phi))]\right\}, \tag{4.1}$$

which follows from (3.1), (3.5), and (3.7). Making the background-quantum splitting

$$\varphi \to \varphi + \Phi, \tag{4.2}$$

we present Eq. (4.1) in the form

$$\exp\left\{\frac{i}{\hbar}\bar{\Gamma}_k(\Phi;F)\right\} = \exp\left\{-\frac{i}{2}\Sigma_iF^i\right\}\int\mathcal{D}\varphi\exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}\varphi^A((S''_{\mathrm{FP}})_{AB} + \Sigma_i(L^{i\prime\prime}_k)_{AB})\varphi^B - (\bar{\Gamma}_k(\Phi;F)\overleftarrow{\partial}_A)\varphi^A + S_{\mathrm{int}}(\Phi,\varphi)\right]\right\}, \tag{4.3}$$

where the notations

$$\bar{\Gamma}_k(\Phi;F) = \Gamma_k(\Phi;F) - S_{\mathrm{FP}}(\Phi), \tag{4.4}$$

$$S_{\mathrm{int}}(\Phi,\varphi) = S_{\mathrm{FP}}(\Phi+\varphi) - S_{\mathrm{FP}}(\Phi) - (S_{\mathrm{FP}}(\Phi)\overleftarrow{\partial}_A)\varphi^A - \frac{1}{2}\varphi^A(S''_{\mathrm{FP}})_{AB}\varphi^B, \tag{4.5}$$

$$i\mathcal{D}^{-1}_{AB}(\Phi) = \overrightarrow{\partial}_A S_{\mathrm{FP}}(\Phi)\overleftarrow{\partial}_B \equiv (S''_{\mathrm{FP}})_{AB} \tag{4.6}$$

and the relations (3.7) are used.

Then, we assume the average effective action in the form

$$\bar{\Gamma}_k(\Phi;F) = \hbar\Gamma^{(1)}_k(\Phi;F) + \Gamma_{k2}(\Phi;F). \tag{4.7}$$

Here, $\Gamma^{(1)}_k(\Phi;F)$ is the one-loop effective action for the set of fields Φ_A, taking into account composite fields F^i. The term $\Gamma_{k2}(\Phi;F)$ includes all the two-particle-irreducible vacuum graphs in a theory with vertices determined by $S_{\mathrm{int}}(\Phi,\varphi)$ and propagators set equal to F^i. Note that $\Gamma_{k2}(\Phi;F)$ by itself is of order $\hbar^2$ [9].

To calculate the one-loop contribution $\Gamma^{(1)}_k(\Phi,F)$, we have to omit in the functional integral (4.3) all terms of order more than φ^2. Then, we have

$$\exp\{i\Gamma^{(1)}_k(\Phi;F)\} = \exp\left\{-\frac{i}{2}\Sigma_iF^i\right\}\int\mathcal{D}\varphi\exp\left\{\frac{i}{2\hbar}\varphi^A((S''_{\mathrm{FP}})_{AB} + \Sigma_i(L^{i\prime\prime}_k)_{AB})\varphi^B - i(\Gamma^{(1)}_k(\Phi;F)\overleftarrow{\partial}_A)\varphi^A\right\}. \tag{4.8}$$

The last term in the exponent of the functional integral reproduces one-particle-reducible diagrams and should be omitted in calculating the vertex functions. More systematically, we have the representation of the exponent

$$\exp\left\{-i(\Gamma^{(1)}_k(\Phi;F)\overleftarrow{\partial}_A)\varphi^A\right\} = 1 - i(\Gamma^{(1)}_k(\Phi;F)\overleftarrow{\partial}_A)\varphi^A - \frac{1}{2}(\Gamma^{(1)}_k(\Phi;F)\overleftarrow{\partial}_A)(\Gamma^{(1)}_k(\Phi;F)\overleftarrow{\partial}_B)\varphi^B\varphi^A + \cdots. \tag{4.9}$$

After integration over φ^A, the first term on the right-hand side (4.9) takes the one-loop contribution to the average effective action; the second term vanishes as the Gaussian integral of an odd function; the third term is responsible for the cancelation of tadpole diagrams. As a result, we arrive at the relation

$$\Gamma^{(1)}_k(\Phi;F) - \Gamma^{(1)}_{k,i}F^i = \frac{i}{2}\mathrm{sTr}\ln(i\mathcal{D}^{-1}_{AB}(\Phi) - 2\Gamma^{(1)}_{k,i}(L^{i\prime\prime}_k)_{AB}). \tag{4.10}$$

At the lower order in $\hbar$, the relation (3.23) reads

$$\{(i\mathcal{D}^{-1}_{AB}(\Phi) - 2\Gamma^{(1)}_{k,j}(L^{j\prime\prime}_k)_{AB})^{-1}(L^{i\prime\prime}_k)_{BA}\}(-1)^{\varepsilon_A} = -iF^i. \tag{4.11}$$

From Eq. (4.11), it follows that

$$\Gamma^{(1)}_{k,j}(L^{j\prime\prime}_k)_{AB} = -\frac{i}{2}n_j(F^j)^{-1}(L^{j\prime\prime}_k)_{AB} + \frac{i}{2}\mathcal{D}^{-1}_{AB}(\Phi), \tag{4.12}$$

where[2]

$$n_1 = \mathrm{Tr}\delta^\mu_\nu\delta^b_a, \qquad n_2 = -2\mathrm{Tr}\delta^b_a. \tag{4.13}$$

Then, we find

$$\Gamma^{(1)}_{k,1} = -\frac{i}{2}n_1(F^1)^{-1} + \frac{1}{2n_1}\mathrm{Tr}(i\mathcal{D}^{-1})^{ac}_{\mu\alpha}(R^{-1}_{k,A})^{\alpha\nu}_{cb}, \tag{4.14}$$

$$\Gamma^{(1)}_{k,2} = \frac{i}{4}n_2(F^2)^{-1} + \frac{1}{2n_2}\mathrm{Tr}(i\mathcal{D}^{-1})^{ac}(\bar{R}^{-1}_{k,gh})_{cb}. \tag{4.15}$$

These relations can be presented in the form

$$\Gamma^{(1)}_{k,j} = -\frac{i}{2}m_j(F^j)^{-1} + \frac{1}{2m_j}\mathrm{sTr}i\mathcal{D}^{-1}_{AC}(L_k{}^{\prime\prime-1})^{CB}, \tag{4.16}$$

where in the first term on the left-hand side of (4.16) there is no summation over index j and $n_1 = m_1$, $n_2 = -2m_2$,

$$\Gamma^{(1)}_k(\Phi;F) = \frac{1}{2m_j}\mathrm{sTr}i\mathcal{D}^{-1}_{AC}(L_k{}^{\prime\prime-1})^{CB}F^j + \frac{i}{2}\mathrm{sTr}\ln(in_j(F^j)^{-1}(L^{j\prime\prime}_k)_{AB}) + \mathrm{const}, \tag{4.17}$$

where "const" is used to collect all terms independent on the background fields.

Let us consider the equation for $\Gamma_{k2}(\Phi;F)$,

$$\begin{aligned}\Gamma_{k2}(\Phi;F) - \Gamma_{k2,j}(\Phi;F)F^j = {}& -\hbar(\Gamma^{(1)}_k(\Phi;F) - \Gamma^{(1)}_{k,j}(\Phi;F)F^j)\\ & - i\hbar\ln\int\mathcal{D}\varphi\exp\left[\frac{i}{2\hbar}\varphi^A\left(i\mathcal{D}^{-1}_{AB}(\Phi) + (-2\Gamma^{(1)}_{k,j}(\Phi;F) - \frac{2}{\hbar}\Gamma_{k2,j}(\Phi;F))(L^{j\prime\prime}_k)_{AB}\right)\varphi^B\right.\\ & \left. - \frac{i}{\hbar}(\bar{\Gamma}_k(\Phi;F)\overleftarrow{\partial}_A)\varphi^A + \frac{i}{\hbar}S_{\mathrm{int}}(\Phi,\varphi)\right],\end{aligned} \tag{4.18}$$

or, taking into account Eqs. (4.12), (4.16), (4.17), one can rewrite the last equation in the form

$$\begin{aligned}\Gamma_{k2}(\Phi;F) - \Gamma_{k2,j}(\Phi;F)F^j = {}& -\frac{i\hbar}{2}\mathrm{sTr}\ln[in_j(F^j)^{-1}(L^{j\prime\prime}_k)_{AB}] - i\hbar\ln\int\mathcal{D}\varphi\exp\left[\frac{i}{2\hbar}\varphi^A(in_j(F^j)^{-1} - \frac{2}{\hbar}\Gamma_{k2,j}(\Phi;F))(L^{j\prime\prime}_k)_{AB}\varphi^B\right.\\ & \left. - \frac{i}{\hbar}(\bar{\Gamma}_k(\Phi;F)\overleftarrow{\partial}_A)\varphi^A + \frac{i}{\hbar}S_{\mathrm{int}}(\Phi,\varphi)\right].\end{aligned} \tag{4.19}$$

Further analysis of this equation requires the explicit form of $S_{\mathrm{int}}(\Phi,\varphi)$, supported by the additional restriction on $\Gamma_{k2,j}(\Phi;F)$, which comes from the consistency condition (3.23). We are going to study in the future these equations and their solutions using some special field models.

V. GAUGE (IN)DEPENDENCE: A SIMPLE EXAMPLE

In this section, we illustrate the problem of gauge dependence using a simple example. To this end, we consider the average effective action $\bar{\Gamma}_k(\Phi;F)$ up to first order in $\hbar$,

$$\bar{\Gamma}_k(\Phi;F) = \hbar\Gamma^{(1)}_k(\Phi;F), \tag{5.1}$$

where $\Gamma^{(1)}_k(\Phi;F)$ is defined in Eq. (4.17). Note that in consistent gauge theories the effective action does not depend on the gauge on its extremals. First, we check the gauge dependence of the effective action (5.1). Consider the quantum equations of motion $\Gamma^{(1)}_{k,j}(\Phi;F) = 0$. Because of Eqs. (4.12) and (4.16), we have

[2] Here, we do not discuss a suitable definition of the functional traces, but we assume their existence only.

$$-\frac{i}{2}n_j(F^j)^{-1}(L_k^{j\prime\prime})_{AB}+\frac{1}{2}i\mathcal{D}^{-1}_{AB}(\Phi)=0,$$
$$-\frac{i}{2}m_j(F^j)^{-1}+\frac{1}{2m_j}\mathrm{sTr}i\mathcal{D}^{-1}_{AC}(L_k{}^{\prime\prime-1})^{CB}=0. \tag{5.2}$$

Substituting (5.2) into (4.17) and keeping in mind the definition (4.6), we obtain

$$\Gamma_k^{(1)}(\Phi;F)=\frac{i}{2}\mathrm{sTr}\ln S''_{\mathrm{FP}}(\Phi). \tag{5.3}$$

In this approximation, the average effective action (5.3) coincides with the one-loop answer for effective action in a given Yang–Mills theory. It is well-known fact (see, for example, Ref. [15]) that it does not depend on the gauge when the fields Φ^A satisfy the quantum equations of motion. The one-loop contribution to the average effective action, $\Gamma_k^{(1)}(\Phi)$, in the standard FRG approach reads

$$\Gamma_k^{(1)}(\Phi)=\frac{i}{2}\mathrm{sTr}\ln(S''_{\mathrm{FP}}(\Phi)+S''_k(\Phi)). \tag{5.4}$$

This action depends on the gauge even on its extremals. To illustrate this feature explicitly, we restrict ourselves to the case of the electromagnetic field in flat space-time. The classical action of the model is

$$S_0(A)=-\frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu},\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{5.5}$$

We choose the gauge-fixing function in the form

$$\chi(A,B)=\frac{1}{\sqrt{1+\lambda}}\partial^\alpha A_\alpha+B. \tag{5.6}$$

Integrating over field B yields the gauge fixing action

$$S_{gf}(A)=-\frac{1}{2(1+\lambda)}\int d^4x(\partial^\alpha A_\alpha)^2. \tag{5.7}$$

The action for ghosts reads

$$S_{gh}(\bar{C},C)=\frac{1}{\sqrt{1+\lambda}}\int d^4x\bar{C}(\partial^\alpha\partial_\alpha)C. \tag{5.8}$$

The effective action of the model in the standard approach [12] to gauge theories is[3]

$$\Gamma(\Phi)=S(\Phi)+i\hbar\Gamma^{(1)}(\lambda),$$
$$S(\Phi)=S_0(A)+S_{gf}(A)+S_{gh}(\bar{C},C), \tag{5.9}$$

where

$$\Gamma^{(1)}(\lambda)=\frac{1}{2}\mathrm{Tr}\ln\left(\Box\delta^\alpha_\beta-\frac{\lambda}{1+\lambda}\partial^\alpha\partial_\beta\right)-\mathrm{Tr}\ln\left(\frac{1}{\sqrt{1+\lambda}}\Box\right). \tag{5.10}$$

The dependence of the effective action $\Gamma(\Phi)$ (5.9) on the gauge parameter λ is described by the relation

$$\delta\Gamma(\Phi)=\frac{\delta S(\Phi)}{\delta\Phi}\delta\Phi+i\hbar\frac{\partial\Gamma^{(1)}(\lambda)}{\partial\lambda}\delta\lambda. \tag{5.11}$$

Using the quantum equations of motion, which in our case coincide with classical ones,

$$\frac{\delta\Gamma(\Phi)}{\delta\Phi}=\frac{\delta S(\Phi)}{\delta\Phi}=0, \tag{5.12}$$

we see that all dependence on λ comes from $\Gamma^{(1)}(\lambda)$. In turn,

$$\begin{aligned}\Gamma^{(1)}(\lambda)&=\Gamma^{(1)}(0)+\frac{1}{2}\mathrm{Tr}\ln\left(\delta^\alpha_\beta-\frac{\lambda}{1+\lambda}\frac{\partial^\alpha\partial_\beta}{\Box}\right)\\&\quad-\ln\frac{1}{\sqrt{1+\lambda}}\mathrm{Tr}\mathbf{1}\\&=\Gamma^{(1)}(0)+\frac{1}{2}\ln\frac{1}{1+\lambda}\mathrm{Tr}\frac{\partial^\alpha\partial_\beta}{\Box}\\&\quad-\ln\frac{1}{\sqrt{1+\lambda}}\mathrm{Tr}\mathbf{1}=\Gamma^{(1)}(0),\end{aligned} \tag{5.13}$$

where the relation $\mathrm{Tr}(\frac{\partial^\alpha\partial_\beta}{\Box})=\mathrm{Tr}\mathbf{1}$ is used. Therefore,

$$\delta\Gamma(\Phi)\Big|_{\frac{\delta\Gamma(\Phi)}{\delta\Phi}=0}=0. \tag{5.14}$$

According to Eq. (5.3), the same result is valid for the average effective action in the new FRG approach [7,8].

Calculation of the one-loop effective action of the model within the standard FRG method gives

$$\Gamma_k(\Phi)=S(\Phi)+i\hbar\Gamma_k^{(1)}(\lambda), \tag{5.15}$$

where the action $S(\Phi)$ is defined in Eq. (5.9). The regulator action $S_k(A,\bar{C},C)$ for the model under consideration has the form

$$S_k(\Phi)=\frac{1}{2}\int d^4xA^\alpha(R_{k,A})_{\alpha\beta}A^\beta+\int d^4x\bar{C}R_{k,gh}C, \tag{5.16}$$

and the one-loop contribution (5.4), $\Gamma_k^{(1)}(\lambda)$, reads

[3]In this case, $\Phi^A=(A,\bar{C},C)$.

$$\Gamma_k^{(1)}(\lambda) = \frac{1}{2}\mathrm{Tr}\ln\left(\Box\delta^\alpha_\beta - \frac{\lambda}{1+\lambda}\partial^\alpha\partial_\beta + (R_{k,A})^\alpha_\beta\right) - \mathrm{Tr}\ln\left(\frac{1}{\sqrt{1+\lambda}}\Box + R_{k,gh}\right). \tag{5.17}$$

As in the previous case, the quantum equations of motion,

$$\frac{\delta\Gamma_k(\Phi)}{\delta\Phi} = \frac{\delta S(\Phi)}{\delta\Phi} = 0, \tag{5.18}$$

coincide with the classical ones, and the gauge dependence of the effective action $\Gamma_k(A)$ (5.15) on its extremals comes essentially from $\Gamma_k^{(1)}(\lambda)$, which can be presented in the form

$$\Gamma_k^{(1)}(\lambda) = \Gamma^{(1)}(\lambda) + \frac{1}{2}\mathrm{Tr}\ln(1 - G^\alpha_\gamma(\lambda)(R_{k,A})^\gamma_\beta) - \mathrm{Tr}\ln\left(1 + \sqrt{1+\lambda}\frac{R_{k,gh}}{\Box}\right). \tag{5.19}$$

Here, $\eta_{\alpha\beta} = \mathrm{diag}(1,\text{-}1,\text{-}1,\text{-}1)$ is the Minkowski metric, $\Gamma^{(1)}(\lambda) = \Gamma^{(1)}(0)$ is defined in Eq. (5.10), and $G^\alpha_\gamma(\lambda)$ is the Green function

$$\left(\Box\delta^\alpha_\gamma - \frac{\lambda}{1+\lambda}\partial^\alpha\partial_\gamma\right)G^\gamma_\beta(\lambda) = -\delta^\alpha_\beta,$$
$$G^\gamma_\beta(\lambda) = -\frac{\delta^\gamma_\beta}{\Box} - \lambda\frac{\partial^\gamma\partial_\beta}{\Box^2}. \tag{5.20}$$

The last two terms on the right-hand side (5.19) explicitly depend on the gauge-fixing parameter λ. Using the following property of cutoff functions $R_k(p) \to 0$ when $k \to 0$, we can approximate the trace of the logarithm by a linear term:

$$\Gamma_k^{(1)}(\lambda) \approx \Gamma^{(1)}(0) + \frac{1}{2}\mathrm{Tr}\left(\frac{(R_{k,A})^\alpha_\beta}{\Box} + \lambda\frac{\partial^\alpha\partial_\gamma(R_{k,A})^\gamma_\beta}{\Box^2}\right) - \sqrt{1+\lambda}\,\mathrm{Tr}\left(\frac{R_{k,gh}}{\Box}\right). \tag{5.21}$$

It is clear that

$$\frac{\partial\Gamma_k^{(1)}(\lambda)}{\partial\lambda} \neq 0, \tag{5.22}$$

and one meets the gauge dependence of the average effective action within the standard FRG approach even on shell.

VI. DISCUSSIONS

In this paper, we have studied the procedure of loop expansion in the new FRG approach based on the idea to consider regulator functions being main ingredients of standard FRG method as composite fields [7,8]. We have derived an explicit formula at leading order in $\hbar$ for the average effective action. We have explicitly demonstrated the gauge dependence of the average effective actions constructed within the standard and new FRG methods, using a simple gauge model of Abelian vector fields. This example confirmed the general statement of Refs. [7,8] concerning the gauge dependence of the standard average effective action even on shell. It is very important to note that, in fact, the average effective action for the model (5.5)–(5.8) is exact in the case of the standard FRG approach without referring to perturbation theory and to solutions of the flow equation. In our opinion, this result indicates at least that the gauge dependence problem within the standard FRG approach remains open up to now. Perhaps not all the hidden features of the modified Slavnov–Taylor identities (among recent studies, see, for example, Ref. [16]) and the FRG flow equation are used to respect the BRST symmetry.

The main feature of the FRG approach is its nonperturbative character, encoding into the FRG flow equation for the average effective action. This equation is a very complicated nonlinear functional differential equation for which exact solutions are not known and different approximations have been developed (for details, see Ref. [6]). In turn, the structure of the FRG flow equation in the new approach [7,8] is also very complicated but differs from the standard one. This means that solutions to the new FRG equation require serious efforts to develop new approximation methods. We plan in the future to present our study of the problem.

ACKNOWLEDGMENTS

The authors would like to thank I. L. Shapiro for interesting discussions. P. M. Lavrov is grateful to the Mainz Institute for Theoretical Physics for its hospitality and its partial support during the completion of this work. The authors are thankful to the grant of Russian Ministry of Education and Science, Project No. 2014/387/122 for support. The work is also supported in part by Presidential Grant No. 88.2014.2 for LRSS and DFG Grant No. LE 838/12-2.

[1] J. Polchinski, Renormalization and effective Lagrangians, Nucl. Phys. **B231**, 269 (1984).
[2] C. Wetterich, Average action and the renormalization group equations, Nucl. Phys. **B352**, 529 (1991).
[3] C. Wetterich, Exact evolution equation for the effective potential, Phys. Lett. B **301**, 90 (1993).
[4] M. Reuter and C. Wetterich, Average action for the Higgs model with abelian gauge symmetry, Nucl. Phys. **B391**, 147 (1993).
[5] M. Reuter and C. Wetterich, Effective average action for gauge theories and exact evolution equations, Nucl. Phys. **B417**, 181 (1994).
[6] H. Gies, Introduction to the functional RG and applications to gauge theories, Lect. Notes Phys. **852**, 287 (2012).
[7] P. M. Lavrov and I. L. Shapiro, On the Functional Renormalization Group approach for Yang-Mills fields, J. High Energy Phys. 06 (2013) 086.
[8] P. M. Lavrov and I. L. Shapiro, New method of the Functional Renormalization Group approach for Yang-Mills fields, Phys. Part. Nucl. Lett. **11**, 886 (2014).
[9] J. M. Cornwell, R. Jackiw, and E. Tomboulis, Effective action for composite operators, Phys. Rev. D **10**, 2428 (1974).
[10] P. M. Lavrov, Effective action for composite fields in gauge theories, Theor. Math. Phys. **82**, 282 (1990).
[11] P. M. Lavrov and S. D. Odintsov, The gauge dependence of the effective action of composite fields in general gauge theories, Int. J. Mod. Phys. A **04**, 5205 (1989).
[12] L. D. Faddeev and V. N. Popov, Feynman diagrams for the Yang-Mills field, Phys. Lett. B **25**, 29 (1967).
[13] C. Becchi, A. Rouet, and R. Stora, The Abelian Higgs Kibble Model, unitarity of the *S*-operator, Phys. Lett. B **52**, 344 (1974).
[14] I. V. Tyutin, Gauge invariance in field theory and statistical physics in operator formalism, arXiv:0812.0580.
[15] A. O. Barvinsky and G. A. Vilkovisky, The generalized Schwinger-DeWitt technique in gauge theories and quantum gravity, Phys. Rep. **119**, 1 (1985).
[16] M. Safari, Splitting Ward identity, arXiv:1508.06244.