Quantized gauged massless Rarita-Schwinger fields

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We study the quantization of a minimally gauged massless Rarita–Schwinger field, by both the Dirac bracket and functional integral methods. The Dirac bracket approach in the covariant radiation gauge leads to an anticommutator that has a nonsingular limit as gauge fields approach zero, is manifestly positive semidefinite, and is Lorentz invariant. The constraints also have the form needed to apply the Faddeev–Popov method for deriving a functional integral, using the same constrained Hamiltonian and inverse constraint matrix that appear in the Dirac bracket approach.

I. INTRODUCTION

In this paper, we continue the study of gauging a massless Rarita–Schwinger field begun in the preceding paper [1], referred to henceforth as (I), which dealt principally with the classical case (with a small excursion into first quantization). Here we turn to a detailed examination of the quantization of a gauged massless Rarita–Schwinger field. Our main aim is to show that a consistent quantization is possible in the gauge covariant radiation gauge, avoiding the problem of non-positivity of the canonical anticommutator first noted by Johnson and Sudarshan [2] and later rederived by Velo and Zwanziger [3]. Other objections to gauging a massless Rarita–Schwinger field—the issue of superluminal signaling and various "on-shell no-go" theorems—have already been taken up in (I). In referring to a formula numbered "Eq. (#)" in the preceding paper, we shall use the notation "Eq. (I-#)," while nonhyphenated equation numbers refer to equations from this paper.

In Sec. II, we give the Hamiltonian form of the equations of motion and constraints and introduce the Dirac bracket. This can be done without imposing a gauge-fixing condition; in particular, we do not use the condition $\Psi_0 = 0$ that was imposed in an initial arXiv posting [4] of this paper. When a gauge-fixing condition is omitted, the equation of motion for $\vec{\Psi}$ computed from the Dirac bracket agrees with the equation of motion of Eq. (I-29), in the form obtained when Ψ_0 is eliminated by using the secondary constraint $\omega = 0$; this demonstrates that the Dirac formalism is working correctly in the Rarita–Schwinger equation context. However, in the absence of a gauge-fixing constraint, the Dirac bracket anticommutator of $\vec{\Psi}$ with $\vec{\Psi}^\dagger$ agrees with the anticommutator calculated in Refs. [2] and [3], which is singular in the limit of vanishing gauge fields and is not positive semidefinite.

In Sec. III, we study the Dirac bracket in its classical and quantum forms with the imposition of a covariant radiation gauge constraint. We show that now the quantum Dirac bracket has the requisite positivity properties to be an

anticommutator; related details are given in the Appendix. In Sec. IV, we give an alternative approach to proving positivity of the anticommutator in the covariant radiation gauge, based on writing a Lagrangian for the equation of motion for $\vec{\Psi}$ in which Ψ_0 has already been eliminated by the use of the secondary constraint. In Sec. V, we discuss the Lorentz covariance of the covariant radiation gauge and show the Lorentz invariance of the Dirac bracket. In Sec. VI, we turn to path integral quantization in the covariant radiation gauge, leading to a formalism closely resembling the Dirac bracket approach. A brief concluding discussion is given in Sec. VII.

Our conclusion from this paper and the preceding one is that one can consistently gauge a massless Rarita–Schwinger field, at both the classical and quantum levels. This opens the possibility of using gauged Rarita–Schwinger fields as part of the anomaly cancelation mechanism in grand unified models, with anomalies of the spin-$\frac{1}{2}$ fields canceling against the spin-$\frac{3}{2}$ anomaly.

II. HAMILTONIAN FORM OF THE EQUATIONS AND THE DIRAC BRACKET

The standard route to canonical quantization is to transform the Lagrangian equations to the Hamiltonian form and to take the constraints into account by replacing the classical brackets by the Dirac brackets. In carrying this out, we will simplify the formulas by making the gauge choice $A_0 = 0$ for the non-Abelian gauge fields. This gauge choice is always attainable and leaves a residual non-Abelian gauge invariance with a time-independent gauge parameter. The Hamiltonian will then be covariant with respect to this restricted gauge transformation. For the moment, in discussing the canonical Hamiltonian and bracket formalism, we will allow $\vec{A}$ to be time dependent, so that $\vec{E} \neq 0$. But when we turn to the Dirac bracket construction corresponding to a constrained Hamiltonian, which is simplest in the case of time-independent constraints, we will assume a time-independent $\vec{A}$, corresponding in the $A_0 = 0$ gauge to $\vec{E} = 0$. (If we carry along the A_0

term in the formulas, then time-independent fields would not require $\vec{E} = 0$. So this specialization can be avoided at the price of somewhat lengthier equations.)

From the action $S(\Psi_\mu) = \int dt L(\Psi_\mu)$ of Eq. (I-23) and the canonical momentum $\vec{P} = \frac{1}{2}\vec{\Psi}^\dagger \times \vec{\sigma}$, we find the canonical Hamiltonian to be

$$
\begin{aligned}
H &= \int d^3x \partial_0 \vec{\Psi} \cdot \vec{P} - L \\
&= -\frac{1}{2}\int d^3x[-\Psi_0^\dagger \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} + \vec{\Psi}^\dagger \cdot \vec{\sigma} \times \vec{D}\Psi_0 + \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi}] \\
&= -\frac{1}{2}\int d^3x[-\Psi_0^\dagger \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} + (i\vec{P} - \vec{P} \times \vec{\sigma}) \cdot (\vec{\sigma} \times \vec{D}\Psi_0 \\
&\quad + \vec{D} \times \vec{\Psi})],
\end{aligned}
\tag{1}
$$

where in the final line we have used the inversion formula $\vec{\Psi}^\dagger = i\vec{P} - \vec{P} \times \vec{\sigma}$.

We can now compute the classical brackets of various quantities with H. From

$$
\begin{aligned}
\frac{d\vec{\Psi}}{dt} &= [\vec{\Psi}, H]_C = \frac{1}{2}[i(\vec{\sigma} \times \vec{D}\Psi_0 + \vec{D} \times \vec{\Psi}) \\
&\quad - \vec{\sigma} \times (\vec{\sigma} \times \vec{D}\Psi_0 + \vec{D} \times \vec{\Psi})] \\
&= \vec{D}\Psi_0 + \frac{1}{2}[-\vec{\sigma} \times (\vec{D} \times \vec{\Psi}) + i\vec{D} \times \vec{\Psi}],
\end{aligned}
\tag{2}
$$

we obtain the $\vec{\Psi}$ equation of motion in the form given in Eq. (I-29). Similarly, from the bracket of $\vec{P}$ with H, we find the equation of motion for $\vec{\Psi}^\dagger$. Turning to brackets of the constraints with H, starting with $P_{\Psi_0^\dagger}$, we find

$$
\frac{dP_{\Psi_0^\dagger}}{dt} = [P_{\Psi_0^\dagger}, H]_C = -\frac{1}{2}\chi,
\tag{3}
$$

and so $P_{\Psi_0^\dagger} = 0$ for all times implies that $\chi = 0$. For the total time derivative of χ, we have

$$
\frac{d\chi}{dt} = \frac{\partial\chi}{\partial t} + [\chi, H]_C = \vec{\sigma} \times g\frac{\partial\vec{A}}{\partial t} \cdot \vec{\Psi} + [\chi, H]_C = -ig\omega,
\tag{4}
$$

and so $\chi = 0$ for all times implies that ω defined in Eq. (I-28) vanishes. Since ω contains a term proportional to Ψ_0, to continue this process by calculating the time derivative of ω, we must obtain $d\Psi_0/dt$ from a bracket of Ψ_0 with H (and similarly for $d\Psi_0^\dagger/dt$). This requires adding to H a term

$$
\Delta H = -\int d^3x\left[P_{\Psi_0}\frac{d\Psi_0}{dt} + P_{\Psi_0^\dagger}\frac{d\Psi_0^\dagger}{dt}\right].
\tag{5}
$$

Requiring ΔH to be self-adjoint then imposes the requirement

$$
P_{\Psi_0}^\dagger = -P_{\Psi_0^\dagger},
\tag{6}
$$

which was noted following Eq. (I-56). As noted in (I), the chain of successive brackets with H starting from $P_{\Psi_0^\dagger}$ and continuing to $\chi, \omega, \ldots$ leads only to constraints involving $\vec{\Psi}$ and Ψ_0 but never their adjoints. The doubling of the set of constraints, which turns the first class constraints into second class ones, comes from requiring that the adjoint of each fermionic constraint also be a constraint, not from taking successive brackets with H.

We are now ready to implement the Dirac bracket procedure. The basic idea is to change the canonical bracket $[F, G]_C$ to a modified bracket $[F, G]_D$, which projects F and G onto the subspace obeying the constraints, so that the constraints are built into the brackets, or after quantization, into the canonical anticommutators. The constraints can then be "strongly" implemented in the Hamiltonian by setting terms proportional to the constraints to zero. After an integration by parts, the second line of Eq. (1) takes the form

$$
H = -\frac{1}{2}\int d^3x[-\Psi_0^\dagger\chi - \chi^\dagger\Psi_0 + \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi}],
\tag{7}
$$

so setting the constraints $\chi^\dagger, \chi$ equal to zero in Eq. (7), we see that the constrained Hamiltonian is just

$$
\begin{aligned}
H &= -\frac{1}{2}\int d^3x\vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi} \\
&= -\frac{1}{2}\int d^3x(i\vec{P} - \vec{P} \times \vec{\sigma}) \cdot \vec{D} \times \vec{\Psi},
\end{aligned}
\tag{8}
$$

which coincides with the energy integral computed in Eq. (I-35) from the stress-energy tensor.

We proceed now to calculate the Dirac bracket for the case when $F = F(\vec{\Psi})$ and $G = G(\vec{\Psi}, \vec{\Psi}^\dagger)$; the case when $F = F(\vec{\Psi}^\dagger)$ can then be obtained by taking the adjoint, and the case when $F = F(\vec{\Psi}, \vec{\Psi}^\dagger)$ can be obtained by combining the extra bracket terms from both calculations. When F has no dependence on $\vec{\Psi}^\dagger$, it has vanishing brackets with the constraints ϕ_a of Eq. (I-55) and nonvanishing brackets with the constraints χ_a of Eq. (I-56). The Dirac bracket then has the form [see Eqs. (I-A20) and (I-A21) for why M^{-1} appears]

$$
[F, G]_D = [F, G]_C - \sum_a \sum_b [F, \chi_a]_C M_{ab}^{-1}[\phi_b, G],
\tag{9}
$$

where $M_{ab}(\vec{x}, \vec{y}) = [\phi_a(\vec{x}), \chi_b(\vec{y})]_C$ is the matrix defined in Eqs. (I-58) and (I-59). We recall that this matrix has the form

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & \mathcal{U} & \mathcal{S} & \mathcal{T} \\ 0 & \mathcal{V} & \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{W} & \mathcal{C} & \mathcal{D} \end{pmatrix}, \qquad (10)$$

where in the $SU(n)$ gauge field case each entry in M is a $2n \times 2n$ matrix. Using the block inversion method given in Eqs. (I-A18) and (I-A19), we find that M^{-1} is given by

$$M^{-1} = \begin{pmatrix} \Sigma & 1 & -(\mathcal{S}\mathcal{F}+\mathcal{T}\mathcal{H}) & -(\mathcal{S}\mathcal{G}+\mathcal{T}\mathcal{I}) \\ -1 & 0 & 0 & 0 \\ \mathcal{F}\mathcal{V}+\mathcal{G}\mathcal{W} & 0 & \mathcal{F} & \mathcal{G} \\ \mathcal{H}\mathcal{V}+\mathcal{I}\mathcal{W} & 0 & \mathcal{H} & \mathcal{I} \end{pmatrix},$$

$$(11)$$

where

$$\Sigma = \mathcal{U} - \mathcal{S}(\mathcal{F}\mathcal{V}+\mathcal{G}\mathcal{W}) - \mathcal{T}(\mathcal{H}\mathcal{V}+\mathcal{I}\mathcal{W}), \quad (12)$$

and where $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}$ are the elements of the block inversion of the matrix N of Eq. (I-60),

$$\begin{pmatrix} \mathcal{F} & \mathcal{G} \\ \mathcal{H} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (13)$$

Substituting these into Eq. (9), we find for the Dirac bracket a lengthy expression, which simplifies considerably after noting that $[F(\vec{\Psi}),\chi_1]_C = [F(\vec{\Psi}),-P_{\Psi_0}]_C = 0$ and $[\phi_1, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C = [P_{\Psi_0^\dagger}, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C = 0$, leaving the relatively simple formula

$$\begin{aligned}
[F(\vec{\Psi}), G(\vec{\Psi}, \vec{\Psi}^\dagger)]_D &= [F(\vec{\Psi}), G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C \\
&- [F(\vec{\Psi}),\chi_3]_C(\mathcal{F}[\phi_3, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C + \mathcal{G}[\phi_4, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C) \\
&- [F(\vec{\Psi}),\chi_4]_C(\mathcal{H}[\phi_3, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C + \mathcal{I}[\phi_4, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C).
\end{aligned} \qquad (14)$$

We note that only the matrix N enters, in this case through its inverse, rather than the full matrix of constraint brackets M. The final step is to evaluate the inverse block matrix elements $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}$ from the expressions for $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, again by using the block inversion formulas of Eqs. (I-A18) and (I-A19). Let us define the Green's function $\mathcal{D}^{-1}(\vec{x} - \vec{y})$ by

$$(i(\vec{L}_{\vec{x}})^2 + \vec{\sigma} \cdot \vec{L}_{\vec{x}} \times \vec{L}_{\vec{x}})\mathcal{D}^{-1}(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y}) \qquad (15)$$

and a second Green's function $\mathcal{Z}(\vec{x} - \vec{y})$ by

$$\begin{aligned}
\mathcal{Z}(\vec{x}-\vec{y}) &= \mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C} \\
&= -2ig\vec{\sigma} \cdot \vec{B}\delta^3(\vec{x}-\vec{y}) - 4\vec{D}_{\vec{x}} \cdot \vec{L}_{\vec{x}}\mathcal{D}^{-1}(\vec{x}-\vec{y})\vec{L}_{\vec{y}} \cdot \vec{D}_{\vec{y}},
\end{aligned} \qquad (16)$$

where in the covariant radiation gauge $\vec{L} = \vec{D}$. Then the needed inverse block matrices are

$$\begin{aligned}
\mathcal{F} &= \mathcal{Z}^{-1}, \\
\mathcal{G} &= -\mathcal{Z}^{-1}\mathcal{B}\mathcal{D}^{-1}, \\
\mathcal{H} &= -\mathcal{D}^{-1}\mathcal{C}\mathcal{Z}^{-1}, \\
\mathcal{I} &= \mathcal{D}^{-1} + \mathcal{D}^{-1}\mathcal{C}\mathcal{Z}^{-1}\mathcal{B}\mathcal{D}^{-1}.
\end{aligned} \qquad (17)$$

We wish now to apply the Dirac bracket formula to the cases (i) $F(\vec{\Psi}) = \vec{\Psi}$ and $G(\vec{\Psi}, \vec{\Psi}^\dagger) = \vec{\Psi}^\dagger$ and (ii) $F(\vec{\Psi}) = \vec{\Psi}$ and $G(\vec{\Psi}, \vec{\Psi}^\dagger) = H$, with H the constrained Hamiltonian of Eq. (8). The following canonical brackets are needed for this:

$$\begin{aligned}
[\vec{\Psi}(\vec{x}),\chi_3(\vec{y})]_C &= 2\vec{D}_{\vec{x}}\delta^3(\vec{x} - \vec{y}), \\
[\vec{\Psi}(\vec{x}),\chi_4(\vec{y})]_C &= (i\vec{L}_{\vec{x}} - \vec{\sigma} \times \vec{L}_{\vec{x}})\delta^3(\vec{x} - \vec{y}), \\
[\phi_3(\vec{x}), \vec{\Psi}^\dagger(\vec{y})]_C &= 2\vec{D}_{\vec{x}}\delta^3(\vec{x} - \vec{y}) = -2\delta^3(\vec{x} - \vec{y})\vec{D}_{\vec{y}}, \\
[\phi_4(\vec{x}), \vec{\Psi}^\dagger(\vec{y})]_C &= -(i\vec{L}_{\vec{x}} - \vec{L}_{\vec{x}} \times \vec{\sigma})\delta^3(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y})(i\vec{L}_{\vec{y}} - \vec{L}_{\vec{y}} \times \vec{\sigma}), \\
[\phi_3(\vec{x}), H]_C &= ig\vec{B}(\vec{x}) \cdot \vec{\Psi}(\vec{x}), \\
[\phi_4(\vec{x}), H]_C &= \frac{1}{2}(i\vec{L}_{\vec{x}} - \vec{L}_{\vec{x}} \times \vec{\sigma}) \times \vec{D}_{\vec{x}} \cdot \vec{\Psi}(\vec{x}).
\end{aligned} \qquad (18)$$

Additionally, for case i, we need the canonical bracket

$$[\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_C = [\Psi_i(\vec{x}), iP_j(\vec{y}) - \epsilon_{jkl}P_k(\vec{y})\sigma_l]_C$$

$$= -i(\delta_{ij} + i\epsilon_{jil}\sigma_l)\delta^3(\vec{x} - \vec{y}) = -i\sigma_j\sigma_i\delta^3(\vec{x} - \vec{y}) = -2i\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)\delta^3(\vec{x} - \vec{y}), \qquad (19)$$

and for case ii, we need the canonical bracket

$$[\Psi_i(\vec{x}), H]_C = \frac{1}{2}(i\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}) - \vec{\sigma} \times (\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x})))_i. \qquad (20)$$

Up to this point, we have not specialized $\vec{L}$ so as to make it easy to ascertain what the formulas become when the gauge fixing is omitted (as in Refs. [2] and [3]). When $\vec{L} = 0$, the matrix N degenerates to its upper left element $\mathcal{A}$. This is reflected in the fact that $\mathcal{Z}$ of Eq. (16) simplifies to

$$\mathcal{Z}(\vec{x} - \vec{y}) = \mathcal{A} = -2ig\vec{\sigma} \cdot \vec{B}\delta^3(\vec{x} - \vec{y}), \qquad (21)$$

which is a local function of $\vec{x}$ and so is algebraically invertible. The Dirac bracket of $\vec{\Psi}(\vec{x})$ with the constrained Hamiltonian now simplifies to

$$\frac{d\vec{\Psi}(\vec{x})}{dt} = [\vec{\Psi}(\vec{x}), H]_D = \frac{1}{2}[i\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}) - \vec{\sigma} \times (\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}))] - \int d^3y\{2\vec{D}_{\vec{x}}[\mathcal{Z}^{-1}(\vec{x} - \vec{y})ig\vec{B}(\vec{y}) \cdot \vec{\Psi}(\vec{y})]\}$$

$$= \frac{1}{2}[i\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}) - \vec{\sigma} \times (\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}))] + \vec{D}_{\vec{x}}\frac{1}{\vec{\sigma} \cdot \vec{B}(\vec{x})}\vec{B}(\vec{x}) \cdot \vec{\Psi}(\vec{x}). \qquad (22)$$

The second line of this equation is just the $\vec{\Psi}$ equation of motion in the form of Eq. (I-29) (when $A_0 = 0$), with Ψ_0 eliminated by using the secondary constraint, which when $\vec{E} = 0$ reads $\vec{\sigma} \cdot \vec{B}\Psi_0 = \vec{B} \cdot \vec{\Psi}$. This shows that the Dirac bracket formalism correctly incorporates the Ψ_0 term of Eq. (I-29). The reason a local result is obtained from this calculation is that, in the absence of gauge fixing, the Dirac bracket only projects into the subspace that preserves the primary constraint $\chi = 0$, and since the equation of motion of Eq. (I-29) preserves this constraint, it already resides in the subspace projected into by the $\vec{L} = 0$ Dirac bracket.

When $\vec{L} = 0$, for the Dirac bracket of $\vec{\Psi}_i(\vec{x})$ with $\vec{\Psi}_j^\dagger(\vec{y})$, we find

$$[\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_D = [\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_C - \int d^3w d^3z[\Psi_i(\vec{x}), \chi_3(\vec{w})]_C \mathcal{Z}^{-1}(\vec{w} - \vec{z})[\phi_3(\vec{z}), \Psi_j^\dagger(\vec{y})]_C$$

$$= -2i\left[\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)\delta^3(\vec{x} - \vec{y}) - D_{\vec{x}i}\frac{\delta^3(\vec{x} - \vec{y})}{g\vec{\sigma} \cdot \vec{B}(\vec{x})}\vec{D}_{\vec{y}j}\right]$$

$$= -2i\langle\vec{x}|\left[\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)1 + \Pi_i\frac{1}{g\vec{\sigma} \cdot \vec{B}}\Pi_j\right]|\vec{y}\rangle, \qquad (23)$$

where in the final line we have written $iD_{\vec{x}i} = \Pi_i$ to relate to the abstract operator notation of Velo and Zwanziger [3]. Multiplying the final line by i to convert the Dirac bracket to an anticommutator and by a factor $1/2$ reflecting our different field normalization, Eq. (23) becomes the expression for the anticommutator given in the zero mass limit of Eq. (4.12) of Ref. [3]. Using identities in Appendix A of (I), one can verify (as in Appendix C of Ref. [3]) that

$$(\vec{\sigma} \times \vec{D}_{\vec{x}})_i\left[\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)\delta^3(\vec{x} - \vec{y}) - D_{\vec{x}i}\frac{\delta^3(\vec{x} - \vec{y})}{g\vec{\sigma} \cdot \vec{B}(\vec{x})}\vec{D}_{\vec{y}j}\right] = 0; \qquad (24)$$

that is, the constraint χ is explicitly projected to zero. However, as noted in the Introduction to (I), the anticommutator of Eq. (23) becomes singular as $\vec{B} \to 0$, rather than

limiting to the free Rarita–Scwhinger anticommutator. This problem is a direct consequence of omitting a gauge-fixing constraint, by taking $\vec{L} = 0$ in calculating the matrix N.

Now setting $\vec{L} = \vec{D}$ for the covariant radiation gauge, we find for the Dirac bracket of $\Psi_i(\vec{x})$ with the constrained Hamiltonian

$$\frac{d\vec{\Psi}(\vec{x})}{dt} = [\vec{\Psi}(\vec{x}), H]_D = \frac{1}{2}[i\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}) - \vec{\sigma} \times (\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}))]$$

$$- \int d^3y \{ 2\vec{D}_{\vec{x}}[\mathcal{F}(\vec{x}-\vec{y})ig\vec{B}(\vec{y}) \cdot \vec{\Psi}(\vec{y})$$

$$+ \mathcal{G}(\vec{x}-\vec{y})\frac{1}{2}(i\vec{D}_{\vec{y}} - \vec{D}_{\vec{y}} \times \vec{\sigma}) \times \vec{D}_{\vec{y}} \cdot \vec{\Psi}(\vec{y})]$$

$$+ (i\vec{D}_{\vec{x}} - \vec{\sigma} \times \vec{L}_{\vec{x}})[\mathcal{H}(\vec{x}-\vec{y})ig\vec{B}(\vec{y}) \cdot \vec{\Psi}(\vec{y})$$

$$+ \mathcal{I}(\vec{x}-\vec{y})\frac{1}{2}(i\vec{D}_{\vec{y}} - \vec{D}_{\vec{y}} \times \vec{\sigma}) \times \vec{D}_{\vec{y}} \cdot \vec{\Psi}(\vec{y})] \}. \quad (25)$$

The first line of this equation gives the second term of the unconstrained equation of motion in the form of Eq. (I-29), while the remaining terms replace the first term of Eq. (I-29) to guarantee that

$$\frac{d\phi_3}{dt} = \frac{d\chi}{dt} = \frac{d(\vec{\sigma} \times \vec{D} \cdot \vec{\Psi})}{dt} = \vec{\sigma} \times \vec{D} \cdot \frac{d\vec{\Psi}}{dt} = 0,$$

$$\frac{d\phi_4}{dt} = d\frac{\vec{D} \cdot \vec{\Psi}}{dt} = \vec{D} \cdot \frac{d\vec{\Psi}}{dt} = 0, \quad (26)$$

where we have used the fact that we are assuming that $\vec{D}$ is time independent. That is, the Dirac bracket simultaneously projects the equation of motion into the subspace where both $\chi = 0$ and $\vec{D} \cdot \vec{\Psi} = 0$. The restriction to $\vec{D}$ being time independent can be avoided by treating the gauge fields as dynamical variables, taking into account their own constraint structure, and noting that the radiation gauge-fixing constraint $\vec{\nabla} \cdot \vec{P}_{\vec{A}} = 0$, with $\vec{P}_{\vec{A}}$ the canonical momentum conjugate to $\vec{A}$, has nonvanishing fermionic brackets with all Rarita–Schwinger constraints involving $\vec{D} = \vec{\nabla} + g\vec{A}$. This requires an extension of the Dirac bracket construction to take the new, Grassmann-odd, brackets into account, and the extended Dirac bracket structure will then obey Eq. (26) without requiring the assumption of a time-independent $\vec{A}$ and $\vec{D}$.

With $\vec{L} = \vec{D}$, we find for the Dirac bracket of $\Psi_i(\vec{x})$ with $\Psi_j^\dagger(\vec{y})$

$$[\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_D = -2i\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)\delta^3(\vec{x}-\vec{y})$$

$$+ 4\vec{D}_{\vec{x}i}\mathcal{F}(\vec{x}-\vec{y})\vec{D}_{\vec{y}j} - 2D_{\vec{x}i}\mathcal{G}(\vec{x}-\vec{y})(i\vec{D}_{\vec{y}} - \vec{D}_{\vec{y}} \times \vec{\sigma})_j$$

$$+ 2(i\vec{D}_{\vec{x}} - \vec{\sigma} \times \vec{D}_{\vec{x}})_i\mathcal{H}(\vec{x}-\vec{y})\vec{D}_{\vec{y}j} - (i\vec{D}_{\vec{x}} - \vec{\sigma} \times \vec{D}_{\vec{x}})_i\mathcal{I}(\vec{x}-\vec{y})(i\vec{D}_{\vec{y}} - \vec{D}_{\vec{y}} \times \vec{\sigma})_j, \quad (27)$$

which gives the generalization of Eq. (23) to the case when a covariant gauge-fixing constraint is imposed. This equation will be further analyzed in the next section.

III. QUANTIZATION OF THE ANTICOMMUTATOR DERIVED FROM THE DIRAC BRACKET AND POSITIVITY IN THE COVARIANT RADIATION GAUGE

Given the Dirac bracket, the next step is to quantize, by multiplying all Dirac brackets by i and then reinterpreting them as anticommutators or commutators of operators. In the case considered here, this can be done in a constructive way, as follows. First, let us replace the set of $2n$-component column vector constraints ϕ_a and $2n$-component row vector constraints χ_a by the set of $4n$ scalars given by their individual matrix elements. Moreover, since the χ_a are the adjoints of the ϕ_a, we can take linear combinations to make all of these scalars self-adjoint. Labeling the set of self-adjoint scalar

constraints by Φ_a, the Dirac bracket construction for the bracket of F with G reads

$$[F, G]_D = [F, G]_C - \sum_a \sum_b [F, \Phi_a]_C T_{ab}^{-1}[\Phi_b, G]_C,$$

$$T_{ab} = [\Phi_a, \Phi_b]_C, \quad (28)$$

with the matrix T real.

We now observe that, since the Φ_a are all *linear* in the scalar components of $\vec{\Psi}$ and $\vec{\Psi}^\dagger$, if we make the replacement $i[,]_C \to \{,\}_C$, with $\{,\}$ the anticommutatior, and replace all Grassmann variables $\vec{\Psi}$ and $\vec{\Psi}^\dagger$ with operator variables having the standard canonical anticommutators, then, since there is no other operator structure, the same real matrix T_{ab} will be obtained. Moreover, if F and G are both linear in the scalar components of $\vec{\Psi}$ and $\vec{\Psi}^\dagger$, the Grassmann bracket $i[F, G]_C$ formed from the scalar components of F and G will agree with the canonical

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anticommutator $i\{F,G\}_C$ formed from the corresponding operator scalar components and will be a c-number. Thus, for linear F and G, we can define a "Dirac anticommutator" $\{F,G\}_D$ by

$$\{F,G\}_D =\{F,G\}_C - \sum_a \sum_b \{F,\Phi_a\}_C T_{ab}^{-1}\{\Phi_b,G\}_C,$$

$$T_{ab} =\{\Phi_a,\Phi_b\}_C. \qquad (29)$$

When one or both of F and G are bilinear, the Grassmann bracket $i[F,G]_C$ formed from the scalar components of F and G will agree with the canonical commutator formed from the corresponding operator scalar components, and we can define a "Dirac commutator" by a formula analogous to Eq. (29) in which each anticommutator with at least one bilinear argument is replaced by a commutator. In this way, we get a mapping of classical brackets into quantum anticommutators and commutators, that inherits the algebraic properties of the Dirac bracket, including the

chain rule, with the Jacobi identities for odd and even Grassmann variables mapping to the corresponding anti-commutator and commutator Jacobi identities.

To complete this correspondence, we must show that the Dirac anticommutator of $\Psi_i^{\alpha u}$ and $\Psi_j^{\dagger\beta v}$ (with $\alpha = 1,2, \beta = 1,2$ the spin indices; $u = 1,...,n, v = 1,...,n$ the internal symmetry indices; and $i = 1,2,3, j = 1,2,3$ the spatial vector indices) has the expected positivity properties of an operator anticommutator, by showing that for an arbitrary set of complex functions $A_i^{\alpha u}(\vec{x})$ we have

$$\int d^3 x d^3 y A_i^{\alpha u}(\vec{x}) A_j^{*\beta v}(\vec{y})\{\Psi_i^{\alpha u}(\vec{x}), \Psi_j^{\dagger\beta v}(\vec{y})\}_D \geq 0. \qquad (30)$$

We demonstrate this in several steps, in the covariant radiation gauge. First, we examine the conditions for positivity of the canonical anticommutator and Poisson bracket,

$$\int d^3 x d^3 y A_i^{\alpha u}(\vec{x}) A_j^{*\beta v}(\vec{y})\{\Psi_i^{\alpha u}(\vec{x}), \Psi_j^{\dagger\beta v}(\vec{y})\}_C = \int d^3 x d^3 y A_i^{\alpha u}(\vec{x}) A_j^{*\beta v}(\vec{y}) i[\Psi_i^{\alpha u}(\vec{x}), \Psi_j^{\dagger\beta v}(\vec{y})]_C. \qquad (31)$$

From $\Psi_j^{\dagger\beta v} = iP_j^{\beta v} - \epsilon_{jkl}P_k^{\delta v}\sigma_l^{\delta\beta}$, we find that

$$[\Psi_i^{\alpha u}(\vec{x}), \Psi_j^{\dagger\beta v}(\vec{y})]_C = - i(\delta_{ij}\delta^{\alpha\beta} + i\epsilon_{jik}\sigma_k^{\alpha\beta})\delta^{uv}\delta^3(\vec{x}-\vec{y})$$

$$= - i(\sigma_j\sigma_i)^{\alpha\beta}\delta^{uv}\delta^3(\vec{x}-\vec{y}) = -2i\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)^{\alpha\beta}\delta^{uv}\delta^3(\vec{x}-\vec{y}). \qquad (32)$$

Multiplying by $i/2$ and writing $A_i^{\alpha u} = R_i^{\alpha u} + iI_i^{\alpha u}, i = 1,2,3, \alpha = 1,2, u = 1,...,n$, with R and I real, the right-hand side of Eq. (31) evaluates to [we suppress the internal symmetry index u from here on, so $(R_i^{\alpha})^2$ means $\sum_{u=1}^n (R_i^{\alpha u})^2$, etc.]

$$\sum_{i=1}^3 \sum_{\alpha=1}^2 ((R_i^{\alpha})^2 + (I_i^{\alpha})^2) - \frac{1}{2}((R_2^1 - I_1^1 + I_3^2)^2 + (R_1^1 + I_2^1 - R_3^2)^2 + (R_2^2 + I_1^2 + I_3^1)^2 + (R_1^2 - I_2^2 + R_3^1)^2). \qquad (33)$$

If all three components $A_i^{\alpha}, i = 1,...,3$ are present, the expression in Eq. (33) is *not* positive semidefinite. But when only two of the three components are present, as a result of application of a constraint, then each of the four squared terms on the right-hand side of Eq. (33) contains only two terms, and so the expression in Eq. (33) is positive semidefinite by virtue of the inequality

$$X^2 + Y^2 - \frac{1}{2}(X \pm Y)^2 = \frac{1}{2}(X \mp Y)^2 \geq 0. \qquad (34)$$

Another way of seeing this, noted by Velo and Zwanziger [3] and Allcock and Hall [5], is that, because $\sum_{i=1}^3 \sigma_i\sigma_i = 3$, the expression $W_{ij} = \delta_{ij} - \frac{1}{2}\sigma_i\sigma_j$ is not a

projector. But when one component of $\vec{\sigma}$, say σ_3, is replaced by 0, so that one has $\sum_{i=1}^3 \sigma_i\sigma_i = \sum_{i=1}^2 \sigma_i\sigma_i = 2$, then

$$\sum_l W_{il}W_{lj} = \delta_{ij} - 2\frac{1}{2}\sigma_i\sigma_j + \frac{1}{4}\sigma_i\sum_{l=1}^2 \sigma_l\sigma_l\sigma_j$$

$$= \delta_{ij} - \frac{1}{2}\sigma_i\sigma_j = W_{ij}, \qquad (35)$$

and W_{ij} is a projector and hence is positive semidefinite. So we anticipate that proving positivity will require the projection of Eq. (32) into a subspace obeying at least one constraint on $\vec{\Psi}$.

The next step is to use the property that the Dirac bracket of linear quantities F and G reduces to the canonical

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bracket of their projections into the subspace obeying the constraints, when (as is the case here) all constraints are second class, that is they all appear in the Dirac bracket [6]. Referring to Eq. (28), let us define

$$\tilde{F} = F - \sum_a \sum_b [F, \Phi_a]_C T_{ab}^{-1} \Phi_b,$$

$$\tilde{G} = G - \sum_a \sum_b [G, \Phi_a]_C T_{ab}^{-1} \Phi_b, \qquad (36)$$

so that

$$[\tilde{F}, \Phi_c]_C = [F, \Phi_c]_C - \sum_a \sum_b [F, \Phi_a]_C T_{ab}^{-1} [\Phi_b, \Phi_c]_C$$

$$= [F, \Phi_c]_C - \sum_a \sum_b [F, \Phi_a]_C T_{ab}^{-1} T_{bc}$$

$$= [F, \Phi_c]_C - \sum_a [F, \Phi_a]_C \delta_{ac} = 0, \qquad (37)$$

and similarly for $\tilde{G}$. As a result of this relation, which holds when the canonical brackets are simply numbers (as in the case here where Φ_c and F, G are linear), together with symmetry of the canonical bracket $[\tilde{G}, \Phi_c]_C = [\Phi_c, \tilde{G}]_C$, we see that

$$[F, G]_D = [\tilde{F}, \tilde{G}]_C. \qquad (38)$$

These properties of Eqs. (36)–(38) carry over when we replace Grassmann numbers with operators and classical brackets with anticommutators, since in the linear case all anticommutators of linear quantities are c-numbers that commute with the operators and since the anticommutator is symmetric. Thus, we have

$$\{\Psi_i^\alpha(\vec{x}), \Psi_j^{\dagger\beta}(\vec{y})\}_D = \{\tilde{\Psi}_i^\alpha(\vec{x}), \tilde{\Psi}_j^{\dagger\beta}(\vec{y})\}_C. \qquad (39)$$

To further study the properties of $\tilde{\Psi}_i(\vec{x})$ and $\tilde{\Psi}_j^\dagger(\vec{y})$ (with spinor indices suppressed), let us now return to our original labeling of the constraints by ϕ_a and χ_a as in Eq. (14), so that we have in the Dirac bracket formalism

$$\tilde{\Psi}_i(\vec{x}) = \Psi_i(\vec{x}) - \sum_a \sum_b [\Psi_i(\vec{x}), \chi_a]_C M_{ab}^{-1} \phi_b \qquad (40)$$

and a similar equation (with the roles of ϕ_a and χ_a interchanged) for $\tilde{\Psi}_j^\dagger(\vec{y})$, with a, b summed from 3 to 4. We now note two important properties of this equation. The first is that it is invariant under the replacement of the constraints χ_a by any linear combination $\chi_a' = \chi_b K_{ba}$, with the matrix K nonsingular, since the factors K and K^{-1} cancel between χ_a' and $M_{ab}'^{-1}$. (More generally, the Dirac bracket is invariant under the replacement of the constraints by any nonsingular linear combination of the constraints, reflecting the fact that the Dirac bracket is a projector onto

the subspace obeying the constraints, and this subspace is invariant under the replacement of the constraints by any nonsingular linear combination of the constraints.) The second is that, if we act on $\tilde{\Psi}_i(\vec{x})$ with either $D_{\bar{x}i}$ or $(\vec{\sigma} \times D_{\bar{x}})_i$, we get zero. For example, recalling that in the covariant radiation gauge $D_{\bar{x}i} \Psi_i(\vec{x}) = \phi_4(\vec{x})$, we have (with spatial variable labels $\vec{x}$ suppressed)

$$D_i \tilde{\Psi}_i = \phi_4 - \sum_a \sum_b [\phi_4, \chi_a]_C M_{ab}^{-1} \phi_b$$

$$= \phi_4 - \sum_a \sum_b M_{4a} M_{ab}^{-1} \phi_b = \phi_4 - \sum_b \delta_{4b} \phi_b = 0 \qquad (41)$$

and similarly for $(\vec{\sigma} \times D_{\bar{x}})_i$, with ϕ_4 replaced by ϕ_3.

Let us now write $\tilde{\Psi}_i(\vec{x})$ as a projector $R_{ij}(\vec{x}, \vec{y})$ acting on $\Psi_j(\vec{y})$, giving after an integration by parts on $\vec{y}$

$$\tilde{\Psi}_i(\vec{x}) = \int d^3y R_{ij}(\vec{x}, \vec{y}) \Psi_j(\vec{y}),$$

$$R_{ij}(\vec{x}, \vec{y}) = \delta_{ij} \delta^3(\vec{x} - \vec{y})$$

$$+ \sum_a \sum_b \int d^3z [\Psi_i(\vec{x}), \chi_a(\vec{z})]_C M_{ab}^{-1}(\vec{z}, \vec{y}) \bar{\eta}_{bj}(\vec{y}), \qquad (42)$$

with

$$\bar{\eta}_{3j}(\vec{y}) = (\vec{\sigma} \times \overleftarrow{D}_{\bar{y}})_j, \qquad \bar{\eta}_{4j}(\vec{y}) = \overleftarrow{D}_{\bar{y}j}. \qquad (43)$$

By virtue of Eq. (41) and its analog for $\vec{\sigma} \times \vec{D}$, we have

$$D_{\bar{x}i} R_{ij}(\vec{x}, \vec{y}) = 0,$$

$$(\vec{\sigma} \times \vec{D}_{\bar{x}})_i R_{ij}(\vec{x}, \vec{y}) = 0. \qquad (44)$$

Since

$$\vec{\sigma} \cdot \vec{D}_{\bar{x}} \sigma_i R_{ij}(\vec{x}, \vec{y}) = D_{\bar{x}i} R_{ij}(\vec{x}, \vec{y}) + i(\vec{\sigma} \times \vec{D}_{\bar{x}})_i R_{ij}(\vec{x}, \vec{y}), \qquad (45)$$

then assuming that $\vec{\sigma} \cdot \vec{D}$ is invertible, Eqs. (44) also imply that

$$\sigma_i R_{ij}(\vec{x}, \vec{y}) = 0. \qquad (46)$$

Next, let us focus on the bracket $[\Psi_i(\vec{x}), \chi_a(\vec{z})]_C$ appearing as the first factor inside the sum. Setting $\vec{L} = \vec{D}$ in Eq. (18), we have

$$[\vec{\Psi}(\vec{x}), \chi_3(\vec{z})]_C = 2\vec{D}_{\bar{x}} \delta^3(\vec{x} - \vec{z}),$$

$$[\vec{\Psi}(\vec{x}), \chi_4(\vec{z})]_C = (i\vec{D}_{\bar{x}} - \vec{\sigma} \times \vec{D}_{\bar{x}}) \delta^3(\vec{x} - \vec{z}). \qquad (47)$$

Using the invariance of $\tilde{\Psi}_i$, or equivalently of R_{ij}, under the replacement of χ_3, χ_4 by any nondegenerate linear combination of χ_3, χ_4, let us choose the new combinations so that

$$[\vec{\Psi}(\vec{x}), \chi_3(\vec{z})]_C = (\vec{\sigma} \times \vec{D}_{\vec{x}})\delta^3(\vec{x} - \vec{z}) = \vec{\eta}_3(\vec{x})\delta^3(\vec{x} - \vec{z}),$$
$$[\vec{\Psi}(\vec{x}), \chi_4(\vec{z})]_C = \vec{D}_{\vec{x}}\delta^3(\vec{x} - \vec{y}) = \vec{\eta}_4(\vec{x})\delta^3(\vec{x} - \vec{z}). \quad (48)$$

Substituting this into Eq. (42), we get the symmetric expression

$$R_{ij}(\vec{x}, \vec{y}) = \delta_{ij}\delta^3(\vec{x} - \vec{y})$$
$$+ \sum_a \sum_b \int d^3 z \vec{\eta}_{ai}(\vec{x}) M_{ab}^{-1}(\vec{x}, \vec{y}) \vec{\eta}_{bj}(\vec{y}). \quad (49)$$

$$\int d^3 x \int d^3 y A_i^\alpha(\vec{x}) A_j^{*\beta}(\vec{y}) \{\Psi_i^\alpha(\vec{x}), \Psi_j^{\dagger\beta}(\vec{y})\}_D$$
$$= \int d^3 x \int d^3 y A_i^\alpha(\vec{x}) A_j^{*\beta}(\vec{y}) \{\tilde{\Psi}_i^\alpha(\vec{x}), \tilde{\Psi}_j^{\dagger\beta}(\vec{y})\}_C$$
$$= \int d^3 x \int d^3 y A_i^\alpha(\vec{x}) A_j^{*\beta}(\vec{y}) \int d^3 z \int d^3 w R_{il}^{\alpha\gamma}(\vec{x}, \vec{z}) \{\Psi_l^\gamma(\vec{z}), \Psi_m^{\dagger\delta}(\vec{w})\}_C R_{jm}^{*\beta\delta}(\vec{y}, \vec{w})$$
$$= \int d^3 x \int d^3 y A_i^\alpha(\vec{x}) A_j^{*\beta}(\vec{y}) \int d^3 z \int d^3 w R_{il}^{\alpha\gamma}(\vec{x}, \vec{z}) 2\left(\delta_{lm}\delta^{\gamma\delta} - \frac{1}{2}\sigma_l^{\gamma\epsilon}\sigma_m^{*\delta\epsilon}\right)\delta^3(\vec{z} - \vec{w}) R_{jm}^{*\beta\delta}(\vec{y}, \vec{w})$$
$$= 2\int d^3 z \left[\int d^3 x A_i^\alpha(\vec{x}) R_{il}^{\alpha\gamma}(\vec{x}, \vec{z})\right]\left[\int d^3 y A_j^\beta(\vec{y}) R_{jl}^{\beta\gamma}(\vec{y}, \vec{z})\right]^*, \quad (51)$$

which is positive semidefinite.

We conclude that the anticommutator of $\vec{\Psi}$ with $\vec{\Psi}^\dagger$ is manifestly positive semidefinite in the covariant radiation gauge. The duality of the $\phi_{3,4}$ and $\chi_{3,4}$ constraints in this gauge is essential to reaching this conclusion; if gauge fixing were omitted, or if another gauge were chosen, this symmetry would not be present, and we could not deduce positivity in a similar fashion.

IV. ALTERNATIVE LAGRANGIAN AND HAMILTONIAN FOR THE $\vec{\Psi}$ EQUATION IN THE COVARIANT RADIATION GAUGE

Up to this point, we have worked with the original action of Eq. (I-23) and the canonical momentum derived from it. We give here another approach, based on setting up an action for the $\vec{\Psi}$ equation of motion from which Ψ_0 has been eliminated by the secondary constraint,

$$D_0\vec{\Psi} = \vec{D}\,\vec{R}\cdot\vec{\Psi} + i\vec{D}\times\vec{\Psi},$$
$$\vec{R} = (\vec{\sigma}\cdot\vec{B})^{-1}(\vec{B} + \vec{\sigma}\times\vec{E}), \quad (52)$$

which holds when the primary constraint $\chi = 0$ is obeyed. Consider the self-adjoint action

By virtue of this symmetry, the projector R_{ij} is annihilated by the constraints $\bar{D}_{\bar{y}j}$ and $(\vec{\sigma} \times \bar{D}_{\bar{y}})_j$ acting from the right, which in turn implies that in addition to Eq. (46) we also have

$$R_{ij}(\vec{x}, \vec{y})\sigma_j = 0. \quad (50)$$

An explicit construction of $R_{ij}(\vec{x}, \vec{y})$ and verification of Eqs. (46) and (50) are given in the Appendix.

Returning now to Eqs. (30) and (39), writing $\tilde{\Psi}_i^\alpha$ and $\tilde{\Psi}_j^{\dagger\beta}$ in terms of projectors acting on Ψ_i^α and $\Psi_j^{\dagger\beta}$, we have (using $\sigma_m^{\epsilon\delta} = \sigma_m^{*\delta\epsilon}$ and continuing to suppress internal symmetry indices u, v, which are contracted in the same pattern as the spatial vector and spin indices)

$$\hat{S} = \int d^3 x \hat{L} = \frac{i}{2}\int d^4 x \vec{\Psi}^\dagger \cdot (D_0\vec{\Psi} - i\vec{D}\times\vec{\Psi}$$
$$- \vec{D}\,\vec{R}\cdot\vec{\Psi} - \vec{R}^\dagger\vec{D}\cdot\vec{\Psi}). \quad (53)$$

Varying with respect to $\vec{\Psi}^\dagger$, and imposing two constraints, (i) the primary constraint $\chi = \vec{\sigma}\cdot\vec{D}\times\vec{\Psi} = 0$ and (ii) the gauge fixing constraint $\vec{D}\cdot\vec{\Psi} = 0$, we get the equation of motion of Eq. (52). For the canonical momentum conjugate to $\vec{\Psi}$, we find

$$\vec{P} = \frac{\partial^L \hat{S}}{\partial(\partial_0\vec{\Psi})} = -\frac{i}{2}\vec{\Psi}^\dagger, \quad (54)$$

which implies that

$$\vec{\Psi}^\dagger = 2i\vec{P}. \quad (55)$$

For the Hamiltonian corresponding to the new action, we find (again for simplicity taking $A_0 = 0$ and integrating the middle term by parts)

$$\hat{H} = \int d^3x \partial_0 \vec{\bar{\Psi}} \cdot \vec{P} - \hat{L}$$

$$= \frac{1}{2} \int d^3x \vec{\bar{\Psi}}^\dagger \cdot (-\vec{D} \times \vec{\Psi} - i\vec{D}\,\bar{R} \cdot \vec{\Psi} + i\bar{R}^\dagger \vec{D} \cdot \vec{\Psi})$$

$$= -\frac{1}{2} \int d^3x \vec{\bar{\Psi}}^\dagger \cdot \vec{D} \times \vec{\Psi}, \tag{56}$$

where in going from the second to the final line we have used the constraint $\vec{D} \cdot \vec{\Psi} = 0$ and its adjoint $\vec{\bar{\Psi}}^\dagger \cdot \vec{D} = 0$. The Hamiltonian $\hat{H}$ is again the energy integral calculated from the left chiral part of the stress-energy tensor and expressed in terms of the canonical momentum is

$$\hat{H} = -i \int d^3x \vec{P} \cdot \vec{D} \times \vec{\Psi}. \tag{57}$$

From here on, the argument parallels that of Secs. II and III but is simpler. For the canonical bracket of $\Psi_i(\vec{x})$ with $\Psi_j(\vec{y})$, we have

$$[\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_C = [\Psi_i(\vec{x}), 2iP_j(\vec{y})]_C = -2i\delta_{ij}\delta^3(\vec{x} - \vec{y}), \tag{58}$$

and so multiplying by i to convert to a canonical anti-commutator, we get

$$\{\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})\}_C = 2\delta_{ij}\delta^3(\vec{x} - \vec{y}), \tag{59}$$

which is positive semidefinite. The complete set of constraints is

$$\phi_3 = \chi = \vec{\sigma} \cdot \vec{D} \times \vec{\Psi},$$

$$\phi_4 = \vec{D} \cdot \vec{\Psi},$$

$$\chi_3 = \chi^\dagger = -\vec{\bar{\Psi}}^\dagger \times \bar{D} \cdot \vec{\sigma} = 2i\vec{P} \cdot \vec{\sigma} \times \bar{D},$$

$$\chi_4 = \vec{\bar{\Psi}}^\dagger \cdot \bar{D} = 2i\vec{P} \cdot \bar{D}. \tag{60}$$

The constraints ϕ_3, ϕ_4 are identical to ϕ_1, ϕ_2 of Eq. (A1), while the constraints χ_3, χ_4 are χ_1, χ_2 of Eq. (A1) up to an invertible linear transformation (just the interchange of the χ constraints and division by $2i$). Thus, the projector $R_{ij}(\vec{x}, \vec{y})$ is the same as that calculated in the Appendix, and the Dirac anticommutator given by Eq. (39) is positive semidefinite by Eq. (51), this time without using the fact that R_{ij} is projected to zero by σ_i and σ_j.

V. LORENTZ COVARIANCE OF THE COVARIANT RADIATION GAUGE AND LORENTZ INVARIANCE OF THE DIRAC BRACKET

We study next the behavior of the covariant radiation gauge and the Dirac bracket under Lorentz boosts. The Rarita–Schwinger field ψ_μ^α and its left-handed chiral projection Ψ_μ^α both have a four-vector index μ and a spinor index α. Under an infinitesimal Lorentz transformation, the transformations acting on these two types of indices are additive and so can be considered separately. The spinor indices are transformed as in the usual spin-$\frac{1}{2}$ Dirac equation by a matrix constructed from the Dirac gamma matrices, which commutes with D_μ. Hence, the spinor index transformation leaves the covariant radiation gauge condition $\vec{D} \cdot \vec{\Psi}$ invariant.

This leaves the transformation on the vector index to be considered, and this is a direct analog of the Lorentz transformation of the radiation gauge in quantum electrodynamics [7]. Since the radiation gauge condition is invariant under spatial rotations, we only have to consider a Lorentz boost,

$$\vec{x} \to \vec{x}' = \vec{x} + \vec{v}t,$$

$$x^0 = t \to t' = t + \vec{v} \cdot \vec{x}. \tag{61}$$

Under this boost, the field $\vec{\bar{\Psi}}$ transforms as

$$\vec{\bar{\Psi}} \to \vec{\bar{\Psi}}' = \vec{\bar{\Psi}} + \vec{v}\Psi^0. \tag{62}$$

For an observer in the boosted frame, the covariant radiation gauge would be $\vec{D}_{\vec{x}'} \cdot \vec{\bar{\Psi}}' = 0$, with $\vec{D}_{\vec{x}'} = \vec{\nabla}_{\vec{x}'} + g\vec{A}'$, where $\vec{A}' = \vec{A} + O(\vec{v})$. Applying this to $\vec{\bar{\Psi}}'(\vec{x}', t')$ and using the covariant radiation gauge condition in the initial frame, we get

$$\vec{D}_{\vec{x}'} \cdot \vec{\bar{\Psi}}' = v_j\Sigma_j(\vec{x}, t), \tag{63}$$

with $\Sigma_j(\vec{x}, t)$ a local polynomial in $\vec{\bar{\Psi}}, \Psi_0$ and the gauge fields, where we have dropped primes on the right-hand side since there is an explicit factor of $\vec{v}$. So in the boosted frame, $\vec{\bar{\Psi}}'$ does not obey the covariant radiation gauge condition, but this can be restored by making a gauge transformation

$$\vec{\bar{\Psi}}' \to \vec{\bar{\Psi}}' - \vec{D}(\vec{D}^2)^{-1}v_j\Sigma_j(\vec{x}, t). \tag{64}$$

Hence, the covariant radiation gauge condition is Lorentz boost covariant, although not Lorentz boost invariant.

Referring now to Eq. (A10), we note that the covariant radiation gauge Dirac bracket and the anticommutation relations are invariant under infinitesimal Rarita–Schwinger gauge transformations, such as that of Eq. (64), up to a remainder that is quadratic in the gauge parameter. Hence, the covariant radiation gauge Dirac bracket and the anticommutation relations following from it are Lorentz invariant, since a finite Lorentz transformation can be built up from a series of infinitesimal ones.

VI. PATH INTEGRAL QUANTIZATION

An alternative method of quantization to the Dirac bracket approach is setting up a Feynman path integral. Again, we will specialize to the case where the external gauge potentials, and hence $\vec{D}$, are time independent, since the simplest discussions of path integrals for constrained systems assume time-independent constraints. As noted above, this assumption can be dropped when the gauge field is quantized along with the Rarita–Schwinger field, leading to a more complex system of constraints and constraint brackets.

When the constraints are time independent, the classical brackets of Eqs. (I-57) and (I-58) have the form needed to apply the Faddeev–Popov [8] method for path integral quantization. (This has been applied in the free Rarita–Schwinger case by Das and Freedman [9] and by Senjanović [10].) The general formula of Ref. [8] for the in to out S matrix element (up to a constant proportionality factor) reads

$$\langle \text{out}|S|\text{in}\rangle \propto \int \exp(iS(q,p)) \prod_t d\mu(q(t),p(t)),$$

$$d\mu(q,p) = \prod_a \delta(\chi_a)\delta(\phi_a)(\det[\phi_a,\chi_b])^\xi \prod_i dp_i dq_i, \quad (65)$$

where $\xi = 1$ when all canonical variables are bosonic, and $\xi = -1$ in our case in which all canonical variables are fermionic, or Grassmann odd. In applying this formula, we note that, since the action S of Eq. (I-23) and the bracket matrix M of Eqs. (I-59)–(I-62) are independent of P_{Ψ_0} and $P_{\Psi_0^\dagger}$, we can immediately integrate out the delta functions in these two constraints. Also, since the canonical momentum $\vec{P}$ is related to $\vec{\Psi}^\dagger$ by the constant numerical transformation of Eq. (I-54), we can take $\vec{\Psi}^\dagger$ as the integration variable instead of $\vec{P}$, up to an overall proportionality constant. So we have the formula, after an integration by parts in the second term,

$$\langle \text{out}|S|\text{in}\rangle \propto \int \exp\left(i\frac{1}{2}\int d^4x[-\Psi_0^\dagger\vec{\sigma}\cdot\vec{D}\times\vec{\Psi} - \vec{\Psi}^\dagger\cdot\vec{\sigma}\times\vec{D}\Psi_0 + \vec{\Psi}^\dagger\cdot\vec{D}\times\vec{\Psi} - \vec{\Psi}^\dagger\cdot\vec{\sigma}\times D_0\vec{\Psi}]\right)$$

$$\times \prod_{t,\vec{x}} d\mu(\Psi_0,\Psi_0^\dagger,\vec{\Psi},\vec{\Psi}^\dagger)$$

$$= \int \exp\left(i\frac{1}{2}\int d^4x[-\Psi_0^\dagger\chi - \chi^\dagger\Psi_0 + \vec{\Psi}^\dagger\cdot\vec{D}\times\vec{\Psi} - \vec{\Psi}^\dagger\cdot\vec{\sigma}\times D_0\vec{\Psi}]\right)$$

$$\times \prod_{t,\vec{x}} d\mu(\Psi_0,\Psi_0^\dagger,\vec{\Psi},\vec{\Psi}^\dagger). \quad (66)$$

Here

$$d\mu(\Psi_0,\Psi_0^\dagger,\vec{\Psi},\vec{\Psi}^\dagger) = \left(\prod_{c=2}^4 \delta(\chi_c)\delta(\phi_c)\right)(\det[\phi_a,\chi_b])^{-1} d\Psi_0 d\Psi_0^\dagger d\vec{\Psi} d\vec{\Psi}^\dagger, \quad (67)$$

with $d\Psi_0$ and $d\Psi_0^\dagger$ each a product over the spinor components and $d\vec{\Psi}$ and $d\vec{\Psi}^\dagger$ each a product over the spinor-vector components.

As our next step, we can carry out the integrations over Ψ_0 and $\Psi_0^\dagger$, using the delta functions $\delta(\phi_2)$ and $\delta(\chi_2)$. This leaves the formula

$$\langle \text{out}|S|\text{in}\rangle \propto \int \exp\left(i\frac{1}{2}\int d^4x[-\vec{\Psi}^\dagger\cdot(\vec{B}+\vec{\sigma}\times\vec{E})(\vec{\sigma}\cdot\vec{B})^{-1}\chi\right.$$

$$\left. -\chi^\dagger(\vec{\sigma}\cdot\vec{B})^{-1}(\vec{B}+\vec{\sigma}\times\vec{E})\cdot\vec{\Psi} + \vec{\Psi}^\dagger\cdot\vec{D}\times\vec{\Psi} - \vec{\Psi}^\dagger\cdot\vec{\sigma}\times D_0\vec{\Psi}]\right)$$

$$\times \prod_{t,\vec{x}} d\mu(\vec{\Psi},\vec{\Psi}^\dagger), \quad (68)$$

with

$$d\mu(\vec{\Psi},\vec{\Psi}^\dagger) = \left(\prod_{c=3}^4 \delta(\chi_c)\delta(\phi_c)\right)(\det[\phi_a,\chi_b])^{-1} d\vec{\Psi} d\vec{\Psi}^\dagger, \quad (69)$$

so that only the remaining constraints $\phi_{3,4}, \chi_{3,4}$ are used in constructing the determinant $\det[\phi_a,\chi_b]$.

Finally, using the delta functions $\delta(\phi_3) = \delta(\chi)$ and $\delta(\chi_3) = \delta(\chi^\dagger)$ to simplify the exponent, we end up with the elegant formula

$$\langle \text{out}|S|\text{in}\rangle \propto \int \exp\left(i\frac{1}{2}\int d^4x \vec{\Psi}^\dagger \cdot [\vec{D} \times \vec{\Psi} - \vec{\sigma} \times D_0\vec{\Psi}]\right)$$
$$\times \prod_{t,\vec{x}} d\mu(\vec{\Psi}, \vec{\Psi}^\dagger), \tag{70}$$

which as in Dirac bracket quantization, employs as Hamiltonian the energy integral computed in Eq. (I-35) from the stress-energy tensor. In using this formula, the customary procedure [11] would be to put the bracket matrix that is the argument of the determinant back into the exponent by introducing bosonic ghost fields ϕ_G.

VII. CONCLUSION AND DISCUSSION

To conclude, we see that, when a covariant radiation gauge constraint is included, the problems with canonical quantization found in Refs. [2] and [3] are avoided; the Dirac bracket is well-defined in the limit of zero external fields and is positive semidefinite. Thus, our conclusion in (I) that the classical theory of gauged Rarita–Schwinger fields is consistent extends to the quantized theory of gauged Rarita–Schwinger fields as well. As noted in (I), this means that in constructing grand unified theories one can contemplate an anomaly cancellation mechanism in which the gauge anomalies of Rarita–Schwinger fields cancel against those of spin-$\frac{1}{2}$ fields, as first suggested in Ref. [12] and as used in the $SU(8)$ family unification model of Ref. [13].

Some final remarks:

(1) In quantizing, we assumed that the gauge fields $\vec{A}$ are time independent, so that d/dt and $\vec{D}$ commute. As noted, this assumption can be dropped if the gauge fields are treated as dynamical variables, leading to an extension of the bracket structure, involving fermionic brackets as well as bosonic ones. (For a discussion of bosonic vs fermionic constraints, see Ref. [14].)

(2) In demonstrating positivity of the anticommutator in Sec. III (but not in Sec. IV), we used the condition $\vec{\sigma} \cdot \vec{\Psi} = 0$. Deriving this from the covariant radiation gauge condition $\vec{D} \cdot \vec{\Psi} = 0$ assumed the invertibility of $\vec{\sigma} \cdot \vec{D}$, and attainability of the covariant radiation gauge assumed the invertibility of $(\vec{D})^2$. The conditions for invertibility of these two operators remain to be studied. (The open space index theorems of Callias [15] and Weinberg [15] involve $\vec{\sigma} \cdot \vec{D} + i\phi$, with ϕ a scalar field, and so do not give information about the invertibility of $\vec{\sigma} \cdot \vec{D}$.)

APPENDIX: CONSTRUCTION OF THE PROJECTOR $R_{ij}(\vec{x}, \vec{y})$

Since there are only two ϕ_a constraints and two χ_a constraints, we index them $a = 1, 2$ rather than $a = 3, 4$ as in the text and use the invariance of $R_{ij}(\vec{x}, \vec{y})$ under changing the linear combination of the χ_a constraints. We start from the constraint set

$$\phi_1 = \vec{\sigma} \times \vec{D} \cdot \vec{\Psi}, \qquad \chi_1 = \vec{P} \cdot \bar{D},$$
$$\phi_2 = \vec{D} \cdot \vec{\Psi}, \qquad \chi_2 = \vec{P} \cdot \vec{\sigma} \times \bar{D}. \tag{A1}$$

For the bracket matrix

$$M_{ab}(\vec{x}, \vec{y}) = [\phi_a(\vec{x}), \chi_b(\vec{y})]_C = \begin{pmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{pmatrix}, \tag{A2}$$

we find the matrix elements

$$\hat{\mathcal{A}} = -ig\vec{\sigma} \cdot \vec{B}\delta^3(\vec{x} - \vec{y}),$$
$$\hat{\mathcal{B}} = (2(\vec{D}_{\vec{x}})^2 + g\vec{\sigma} \cdot \vec{B})\delta^3(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y})(2(\bar{D}_{\vec{y}})^2 + g\vec{\sigma} \cdot \vec{B}),$$
$$\hat{\mathcal{C}} = (\vec{D}_{\vec{x}})^2\delta^3(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y})(\bar{D}_{\vec{y}})^2,$$
$$\hat{\mathcal{D}} = ig\vec{\sigma} \cdot \vec{B}\delta^3(\vec{x} - \vec{y}). \tag{A3}$$

We write the inverse matrix $M^{-1}(\vec{z}, \vec{w})$ as

$$\begin{pmatrix} \hat{\mathcal{F}} & \hat{\mathcal{G}} \\ \hat{\mathcal{H}} & \hat{\mathcal{I}} \end{pmatrix}, \tag{A4}$$

which obeys

$$\begin{pmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{F}} & \hat{\mathcal{G}} \\ \hat{\mathcal{H}} & \hat{\mathcal{I}} \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{F}} & \hat{\mathcal{G}} \\ \hat{\mathcal{H}} & \hat{\mathcal{I}} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(A5)

In terms of the inverse matrix, the projector $R_{ij}(\vec{x}, \vec{w})$ is given by (with internal symmetry indices suppressed)

$$\begin{aligned} R_{ij}(\vec{x}, \vec{w}) = & \delta_{ij}\delta^3(\vec{x} - \vec{w})1 \\ & + D_{\bar{x}i}\hat{\mathcal{F}}(\vec{x} - \vec{w})(\vec{\sigma} \times \bar{D}_{\vec{w}})_j + D_{\bar{x}i}\hat{\mathcal{G}}(\vec{x} - \vec{w})\bar{D}_{\vec{w}j} \\ & + (\vec{\sigma} \times D_{\bar{x}})_i\hat{\mathcal{H}}(\vec{x} - \vec{w})(\vec{\sigma} \times \bar{D}_{\vec{w}})_j \\ & + (\vec{\sigma} \times D_{\bar{x}})_i\hat{\mathcal{I}}(\vec{x} - \vec{w})\bar{D}_{\vec{w}j}. \end{aligned}$$

(A6)

From this expression, we find

$$\begin{aligned} D_{\bar{x}i}R_{ij}(\vec{x}, \vec{w}) = & R_{ij}(\vec{x}, \vec{w})\bar{D}_{\vec{w}j} = (\vec{\sigma} \times D_{\bar{x}})_iR_{ij}(\vec{x}, \vec{w}) \\ & = R_{ij}(\vec{x}, \vec{w})(\vec{\sigma} \times \bar{D}_{\vec{w}})_j = 0. \end{aligned}$$

(A7)

In verifying these, it is not necessary to evaluate the inverse matrix; instead, after contracting on the vector index i or j, one expresses the resulting pre- or postfactor in terms of $\hat{\mathcal{A}}, ..., \hat{\mathcal{D}}$ and then uses the algebraic relations following

from multiplying out the matrices in Eq. (A5). Finally, contracting

$$\begin{aligned} \vec{\sigma} \cdot \vec{D}_{\bar{x}}\sigma_i = & (D_{\bar{x}} + i\vec{\sigma} \times \vec{D}_{\bar{x}})_i, \\ \sigma_j\vec{\sigma} \cdot \bar{D}_{\vec{w}} = & (\bar{D}_{\vec{w}} - i\vec{\sigma} \times \bar{D}_{\vec{w}})_j \end{aligned}$$

(A8)

with $R_{ij}(\vec{x}, \vec{w})$, we conclude that

$$\sigma_iR_{ij}(\vec{x}, \vec{y}) = R_{ij}(\vec{x}, \vec{y})\sigma_j = 0,$$

(A9)

when $\vec{\sigma} \cdot \vec{D}$ is invertible.

As a consequence of Eqs. (42) and (A7), $\tilde{\Psi}_i(\vec{x})$ is invariant under the transformations

$$\begin{aligned} \vec{\tilde{\Psi}} \to & \vec{\tilde{\Psi}} + \vec{D}\epsilon, \\ \vec{\tilde{\Psi}} \to & \vec{\tilde{\Psi}} + \vec{\sigma} \times \vec{D}\epsilon. \end{aligned}$$

(A10)

The first of these implies that the canonical anti-commutation relations are invariant under infinitesimal Rarita–Schwinger gauge transformations starting from the covariant radiation gauge.

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