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Classical gauged massless Rarita-Schwinger fields

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We show that, in contrast to known results in the massive case, a minimally gauged massless Rarita–Schwinger field yields a consistent classical theory, with a generalized fermionic gauge invariance realized as a canonical transformation. To simplify the algebra, we study a two-component left chiral reduction of the massless theory. We formulate the classical theory in both Lagrangian and Hamiltonian form for a general non-Abelian gauging and analyze the constraints and the Rarita–Schwinger gauge invariance of the action. An explicit wave front calculation for Abelian gauge fields shows that wavelike modes do not propagate with superluminal velocities. An analysis of Rarita–Schwinger spinor scattering from gauge fields shows that adiabatic decoupling fails in the limit of zero gauge field amplitude, invalidating various "no-go" theorems based on "on-shell" methods that claim to show the impossibility of gauging Rarita–Schwinger fields. Quantization of Rarita–Schwinger fields, using many formulas from this paper, is taken up in the following paper.

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I. INTRODUCTION

A. Motivations and background

Cancellation of gauge anomalies is a basic requirement for constructing grand unified models, and the usual assumption is that anomalies must cancel among spin-$\frac{1}{2}$ fermion fields. However, a 1985 paper of Marcus [1] showed that in principle an $SU(8)$ gauge theory can be constructed with spin-$\frac{3}{2}$ Rarita–Schwinger fermions playing a role in anomaly cancelation, and we have recently constructed [2] a family unification model incorporating this observation. Using gauged spin-$\frac{3}{2}$ fields in a grand unification model raises the question of whether such fields admit a consistent quantum or even classical theory. It is well known, from the papers of Johnson and Sudarshan [3] and Velo and Zwanziger [4] and much subsequent literature (see e.g. Hortacsu [5], Deser and Waldron [6]), that theories of massive gauged Rarita–Schwinger fields have serious problems. Does setting the fermion mass to zero eliminate these difficulties?

The lesson we have learned from the success of the Standard Model is that fundamental fermion masses lead to problems and are to be avoided; all mass is generated by spontaneous symmetry breaking, either through coupling to the Higgs boson or through the formation of chiral symmetry breaking fermion condensates. So from a modern point of view, the Rarita–Schwinger theory with an explicit mass term is suspect. Several hints that the behavior of the massless theory may be satisfactory are already apparent from a study of the zero mass limit of formulas in the Velo–Zwanziger paper. First, in their demonstration of superluminal signaling, the problematic sign change that they find for large $\vec{B}$ fields (Eq. (2.15) of Ref. [4]) is not present when the mass is set to zero. Second, when the mass is zero, the secondary constraint that they derive (Eq. (2.10) of Ref. [4]) appears as a factor in the change in the action under a Rarita–Schwinger gauging $\delta\psi_\mu = D_\mu\epsilon$, with D_μ the usual gauge covariant derivative. [This statement is not in Ref. [4] but is an easy calculation from their Eqs. (2.1)–(2.3), with the D_μ of this paper their $-i\pi_\mu$.] Hence, the constrained action in the massless gauged Rarita–Schwinger theory has a fermionic gauge invariance that is the natural generalization of the fermionic gauge invariance of the massless free Rarita–Schwinger theory. Third, their formula for the anticommutator (Eq. (4.12) of Ref. [4]) in the zero mass case develops an apparent singularity in the limit of vanishing gauge field $\vec{B}$, and so their quantization does not limit to the standard free theory quantization. However, since the massive theory does not have a fermionic gauge invariance, Ref. [4] does not include a gauge-fixing term analogous to that used in the massless case, but gauge fixing is needed to get a consistent quantum theory for a free massless Rarita–Schwinger field. So these observations, following from the equations in Ref. [4], suggest that a study of the massless Rarita–Schwinger field coupled to spin-1 gauge fields is in order.

In a different and more recent setting, massless Rarita–Schwinger fields appear consistently coupled to gravity as the gravitinos of supergravity, as discussed by Das and Freedman [7]. Grisaru, Pendleton, and van Nieuwenhuizen [8] have shown that soft spin-$\frac{3}{2}$ fermions *must* be coupled to gravity as in supergravity, in an analysis based on the free particle external line pole structure of spin-$\frac{3}{2}$ fields that do not have spin-1 gauge couplings. Their result has been extended to gauged spin-$\frac{3}{2}$ fields in various recent "no-go" theorems based on "on-shell" methods [9,10], that again assume a free particle external line pole structure. None of

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these papers has analyzed the gauged Rarita–Schwinger equation to determine the asymptotic field structure. Thus, these papers do not prove that there cannot be a consistent theory of massless, gauged Rarita–Schwinger fields, so again a detailed study of this possibility is warranted.

B. Outline of the paper and summary

With these motivations and background in mind, we embark in this paper on a detailed study of the classical theory of a minimally gauged massless Rarita–Schwinger field. In Sec. II, we give the Lorentz covariant Lagrangian for a gauged four-component Rarita–Schwinger spinor field, derive the source current for the gauge field, and check that it is gauge-covariantly conserved. We also give the Lorentz covariant form of the constraints, of the fermionic gauge transformation, and of the symmetric stress-energy tensor and briefly discuss the generalization to nonflat metrics. Since in the massless case left chiral and right chiral components of the field decouple, in Sec. III, we rewrite the Lagrangian for left chiral components in terms of two-component spinors and Pauli matrices, which simplifies the subsequent analysis. We then give the Euler–Lagrange equations in two-component form and use them to analyze the structure of constraints and the fermionic gauge transformation of the action. In Sec. IV, we introduce canonical momenta for the Rarita–Schwinger field components, which are used to define classical Poisson brackets, and discuss the role of the constraints as generators of gauge transformations under the bracket operation. We show that the constraints group into two sets of four, within each of which there are vanishing Poisson brackets. In Sec. V, we argue that fermionic gauge transformations give a generalized form of gauge invariance, corresponding to the presence of redundant gauge degrees of freedom, by studying the properties of both infinitesimal and general finite gauge transformations. We show that infinitesimal gauge transformations are an invariance of the constrained action functional that governs the influence of Rarita-Schwinger fields on gauge and gravitational fields. We show that finite gauge transformations take the form of generalized auxiliary fields, which lead to an extended action that has an exact invariance under fermionic gauge transformations. In Sec. VI, we specialize to the case of an Abelian gauge field (as in Ref. [4]) and analyze the wave front structure, showing that physical wave modes propagate with luminal velocities; an extension of this discussion, showing that gauge modes are subluminal, is given in Appendix B. In Sec. VII, making a transition to first quantization, we analyze Rarita–Schwinger fermion scattering from an Abelian gauge potential. We show that the asymptotic state structure assumed in "on-shell no-go" theorems is not realized but that a consistent scattering amplitude can be formulated using an analog of the distorted wave Born approimation. In Sec. VIII, we give a brief summary and discussion, and in Appendix A, we summarize our notational conventions and some useful identities. We suggest that the reader skim through Appendix A before going on to Sec. II, since things stated in Appendix A are not repeated in the body of the paper. In the paper that follows this one, we build on our analysis to discuss quantized Rarita–Schwinger fields.

II. LAGRANGIAN AND COVARIANT CURRENT CONSERVATION IN FOUR-COMPONENT FORM

A. Flat spacetime

The action for the massless Rarita–Schwinger theory is

$$\begin{aligned} S(\psi_\mu) &= \frac{1}{2}\int d^4x \bar{\psi}_{\mu\alpha u} R^{\mu\alpha u}, \\ R^{\mu\alpha u} &= i\epsilon^{\mu\eta\nu\rho}(\gamma_5\gamma_\eta)^\alpha{}_\beta (D_\nu \psi_\rho^\beta)^u, \\ (D_\nu\psi_\rho^\beta)^u &\equiv \partial_\nu \psi_\rho^{\beta u} + g A^u_{\nu v}\psi_\rho^{\beta v}, \\ A^u_{\nu v} &= A^A_\nu t^u_{Av}, \end{aligned} \tag{1}$$

with $\psi^{\mu\alpha u} = \psi^{\mu\alpha u}(\vec{x}, t)$ a four-vector four-component spinor, with four-vector index $\mu = 0, \ldots, 3$, spinor index $\alpha = 1, \ldots, 4$, and $SU(n)$ internal symmetry index $u = 1, \ldots, n$, with $SU(n)$ gauge generators $t_A, A = 1, \ldots, n^2 - 1$. Taking u to range from 1 to n means that, for definiteness, we are assuming that the spinors transform according to the fundamental representation of the $SU(n)$ internal symmetry group, but other representations and other compact Lie groups can be accommodated by assigning the internal indices u and A the appropriate range. Note that t_A, $A^u_{\nu v}$, and D_ν all commute with the gamma matrices and the Pauli spin matrices from which the gamma matrices are constructed, and for an Abelian internal symmetry group, the indices u and A are not needed. Using

$$\bar{\psi}_{\mu\alpha u} = \psi^\dagger_{\mu\beta u} i(\gamma^0)^\beta{}_\alpha, \tag{2}$$

together with the adjoint convention $(\chi_1^\dagger\chi_2)^\dagger = \chi_2^\dagger\chi_1$ for Grassmann variables χ_1, χ_2, it is easy to verify that S is self-adjoint.

From here on, we will usually not indicate the spinor indices α, β and internal symmetry indices u, v explicitly, but they are implicit in all formulas. Varying S with respect to the Rarita–Schwinger fields, we get the equations of motion

$$\begin{aligned} \epsilon^{\mu\eta\nu\rho}\partial_\nu\bar{\psi}_\rho\gamma_\eta &= g\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho A^A_\nu t_A\gamma_\eta, \\ \epsilon^{\mu\eta\nu\rho}\gamma_\eta\partial_\nu\psi_\rho &= -g\epsilon^{\mu\eta\nu\rho}\gamma_\eta A^A_\nu t_A\psi_\rho. \end{aligned} \tag{3}$$

Reexpressed in terms of the covariant derivative, these are

$$\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho\overleftarrow{D}_\nu\gamma_\eta = 0,$$
$$\epsilon^{\mu\eta\nu\rho}\gamma_\eta D_\nu\psi_\rho = 0. \tag{4}$$

The $\mu = 0$ component of these equations gives the primary constraints

$$\epsilon^{enr}\bar{\psi}_r\overleftarrow{D}_n\gamma_e = 0,$$
$$\epsilon^{enr}\gamma_e D_n\psi_r = 0, \tag{5}$$

with e, n, r summed from 1 to 3. Contracting the equation of motion for $\bar{\psi}_\rho$ with $g^{-1}\overleftarrow{D}_\mu$ and the equation of motion for ψ_ρ with $g^{-1}D_\mu$, we get the secondary constraints

$$\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho F_{\mu\nu}\gamma_\eta = 0,$$
$$\epsilon^{\mu\eta\nu\rho}\gamma_\eta F_{\mu\nu}\psi_\rho = 0, \tag{6}$$

where we have introduced the gauge field strength

$$F_{\mu\nu} = g^{-1}[D_\mu, D_\nu] = g^{-1}[\overleftarrow{D}_\mu, \overleftarrow{D}_\nu]$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu], \tag{7}$$

which with the adjoint representation index A indicated explicitly reads

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + gf_{ABC}A^B_\mu A^C_\nu. \tag{8}$$

Under a Rarita–Schwinger gauge transformation (with ϵ a four-component spinor), which is a natural gauge field generalization of the fermionic gauge invariance for a free, massless Rarita–Schwinger field discussed in Ref. [11],

$$\psi_\mu \to \psi_\mu + \delta_G\psi_\mu, \qquad \delta_G\psi_\mu \equiv D_\mu\epsilon,$$
$$\bar{\psi}_\mu \to \bar{\psi}_\mu + \delta_G\bar{\psi}_\mu, \qquad \delta_G\bar{\psi}_\mu \equiv \bar{\epsilon}\overleftarrow{D}_\mu, \tag{9}$$

the action of Eq. (1) changes according to

$$\delta_G S(\psi_\mu) = -\frac{1}{4}ig\int d^4x[\bar{\epsilon}\gamma_5(\epsilon^{\mu\eta\nu\rho}\gamma_\eta F_{\mu\nu}\psi_\rho)$$
$$+ (\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho F_{\mu\nu}\gamma_\eta)\gamma_5\epsilon] + O(\bar{\epsilon}\ldots\epsilon). \tag{10}$$

The factors bracketed in parentheses are identical to the secondary constraints of Eq. (6). This equation holds with finite (not necessarily infinitesimal) ϵ and its adjoint $\epsilon^\dagger$; the precise form of the quadratic term is given in Eq. (72) below. We will argue in Sec. V that Eq. (10) implies that, even when coupled to gauge fields, the Rarita–Schwinger theory has a generalized form of fermionic gauge invariance.

Adding the gauge field action

$$S(A^A_\mu) = -\frac{1}{4}\int d^4x F^A_{\mu\nu}F^{A\mu\nu} \tag{11}$$

and varying the sum $S(\psi_\mu) + S(A^A_\mu)$ with respect to the gauge potential, we get the gauge field equation of motion

$$D_\nu F^{A\mu\nu} \equiv \partial_\nu F^{A\mu\nu} + gf_{ABC}A^B_\nu F^{C\mu\nu} = gJ^{A\mu},$$
$$J^{A\mu} = \frac{1}{2}\bar{\psi}_\nu i\epsilon^{\nu\eta\mu\rho}\gamma_5\gamma_\eta t_A\psi_\rho. \tag{12}$$

A straightforward calculation using Eqs. (3) shows that the gauge field source current $J^{A\mu}$ obeys the covariant conservation equation

$$D_\mu J^{A\mu} = \partial_\mu J^{A\mu} + gf_{ABC}A^B_\mu J^{C\mu} = 0, \tag{13}$$

as required for consistency of Eq. (12). So from the Rarita–Schwinger and gauge field actions, we have obtained a formally consistent set of equations of motion.

In addition to the gauge field source current, there is an additional current J^μ that obeys an ordinary conservation equation,

$$J^\mu = \frac{1}{2}\bar{\psi}_\nu\epsilon^{\nu\eta\mu\rho}\gamma_5\gamma_\eta\psi_\rho,$$
$$\partial_\mu J^\mu = 0. \tag{14}$$

In the massive spinor case, Velo and Zwanziger [4] argue that the analogous current, within the constraint subspace of Eq. (5), should have a positive time component. In the massless case, we see no reason for this requirement, since Eq. (14) is the fermion number current and its time component, giving the fermion number density, can have either sign. However, we shall use parts of the positivity argument of Ref. [4] later on in discussing positivity of the Dirac bracket anticommutator.

The symmetric stress-energy tensor for the free massless Rarita–Schwinger has been computed by Das [12] (see also Allcock and Hall [13]). Changing ordinary derivatives to gauge covariant derivatives, Das's formula becomes

$$T^{\sigma\tau}_{\rm RS} = -\frac{i}{4}\epsilon^{\lambda\mu\nu\rho}[\bar{\psi}_\lambda\gamma_5(\gamma^\tau\delta^\sigma_\mu + \gamma^\sigma\delta^\tau_\mu)D_\nu\psi_\rho$$
$$+ \frac{1}{4}\partial_\alpha(\bar{\psi}_\lambda\gamma_5\gamma_\mu([\gamma^\alpha, \gamma^\sigma]\delta^\tau_\nu + [\gamma^\alpha, \gamma^\tau]\delta^\sigma_\nu)\psi_\rho)]. \tag{15}$$

[This formula can be made manifestly self-adjoint by replacing D_ν by $\frac{1}{2}(D_\nu - \overleftarrow{D}_\nu)$, but this is not needed to verify stress-energy tensor conservation.] Adding the gauge field stress-energy tensor,

$$T^{\sigma\tau}_{\rm gauge} = -\frac{1}{4}\eta^{\sigma\tau}F^A_{\lambda\mu}F^{A\lambda\mu} + F^{A\sigma}_\lambda F^{A\lambda\tau}, \tag{16}$$

a lengthy calculation, using Eq. (13) together with identities and alternative forms of the equations of motion given in Appendix A, shows that the total tensor is conserved,

$$\partial_\sigma(T^{\sigma\tau}_{\rm RS} + T^{\sigma\tau}_{\rm gauge}) = 0. \tag{17}$$

B. Generalization to general $g_{\mu\nu}$

The generalization of the Rarita–Schwinger action to curved spacetime has been reviewed by Deser and Waldron

[6]. In Eq. (1), d^4x is replaced by the invariant volume element $d^4x(-g)^{1/2}$, and the covariant derivative D_ν becomes the curved spacetime covariant derivative

$$D_\nu \psi_\rho = \partial_\nu \psi_\rho - \Gamma^\beta_{\nu\rho}\psi_\beta + \frac{1}{4}\omega_{\nu mn}\gamma^{mn}\psi_\rho + gA_\nu \psi_\rho, \quad (18)$$

with $\Gamma^\beta_{\nu\rho}$ and $\omega_{\nu mn}$ the affine and spin connections. The Rarita–Schwinger equation of Eq. (4) and the primary constraint of Eq. (5) have the same form as before, in terms of the extended covariant derivative D_ν. The secondary constraint of Eq. (6) now reads

$$\begin{aligned} &\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho[\overleftarrow{D}_\mu, \overleftarrow{D}_\nu]\gamma_\eta = 0, \\ &\epsilon^{\mu\eta\nu\rho}\gamma_\eta[D_\mu, D_\nu]\psi_\rho = 0, \end{aligned} \quad (19)$$

with $\overleftarrow{D}_\nu$ defined by the adjoint of D_ν. The commutator of covariant derivatives is now given by [6]

$$[D_\mu, D_\nu]\psi_\rho = -R^\sigma_{\mu\nu\rho}\psi_\sigma + \frac{1}{4}R_{\mu\nu mn}\gamma^{mn}\psi_\rho + gF_{\mu\nu}\psi_\rho, \quad (20)$$

with $R^\sigma_{\mu\nu\rho}$ and $R_{\mu\nu mn}$ components of the Riemann curvature tensor, and as in flat spacetime involves only ψ_ρ and not its time or space derivatives. In terms of the extended covariant derivative, the fermionic gauge transformation is still given by Eq. (9), and under this gauge transformation, the change in the action is now given by

$$\begin{aligned} \delta_G S(\psi_\mu) = -\frac{1}{4}i\int d^4x[&\bar{\epsilon}\gamma_5(\epsilon^{\mu\eta\nu\rho}\gamma_\eta[D_\mu, D_\nu]\psi_\rho) \\ &+ (\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho[\overleftarrow{D}_\mu, \overleftarrow{D}_\nu]\gamma_\eta)\gamma_5\epsilon] + O(\bar{\epsilon}\ldots\epsilon), \end{aligned} \quad (21)$$

with the factors bracketed in parentheses now identical to the secondary constraints of Eq. (19) (and again with ϵ and $\epsilon^\dagger$ finite). The arguments to be given in Sec. V then imply that in the presence of both gravitation and gauge fields the Rarita–Schwinger theory has a generalized form of fermionic gauge invariance. Having established this curved spacetime generalization, we will continue in the remainder of this and the following paper to work in flat spacetime, but we expect everything done in what follows to have a curved spacetime generalization when the covariant derivative is suitably extended.

III. LAGRANGIAN ANALYSIS FOR LEFT CHIRAL SPINORS IN TWO-COMPONENT FORM

Although we could continue with the four-component formalism to study constraints, the Hamiltonian formalism, and quantization, it will be more convenient to first reduce the four-component equation to decoupled equations for left and right chiral components of ψ^α_μ (with α the spinor index and with the internal symmetry index implicit). Since these are related by symmetry, we can then focus our analysis on the two-component equations for the left chiral component, which is the component conventionally used in formulating grand unified models (see, e.g., Ref. [2]).

We convert the action of Eq. (1) to two-component form for the left chiral components of ψ^α_μ, using the Dirac matrices given in Eqs. (A2) and (A4). Defining the two-component four-vector spinor Ψ^α_μ and its adjoint $\Psi^\dagger_{\mu\alpha}$ by

$$\begin{aligned} &P_L\psi^\alpha_\mu = \begin{pmatrix} \Psi^\alpha_\mu \\ 0 \end{pmatrix}, \qquad \mu = 0,1,2,3, \qquad \alpha = 1,2, \\ &\psi^\dagger_{\mu\alpha}P_L = (\,\Psi^\dagger_{\mu\alpha} \quad 0\,), \end{aligned} \quad (22)$$

the action decomposes into uncoupled left and right chiral parts. The left chiral part, with spinor indices α suppressed, is given by

$$\begin{aligned} S(\Psi_\mu) = \frac{1}{2}\int d^4x[&-\Psi^\dagger_0\vec{\sigma}\cdot\vec{D}\times\vec{\Psi} + \vec{\Psi}^\dagger\cdot\vec{\sigma}\times\vec{D}\Psi_0 \\ &+ \vec{\Psi}^\dagger\cdot\vec{D}\times\vec{\Psi} - \vec{\Psi}^\dagger\cdot\vec{\sigma}\times D_0\vec{\Psi}]. \end{aligned} \quad (23)$$

Varying with respect to $\vec{\Psi}^\dagger$, we get the Euler–Lagrange equation

$$0 = \vec{V} \equiv \vec{\sigma}\times\vec{D}\Psi_0 + \vec{D}\times\vec{\Psi} - \vec{\sigma}\times D_0\vec{\Psi}, \quad (24)$$

while varying with respect to $\Psi^\dagger_0$, we get the primary constraint [given in four-component form in Eq. (5)]

$$0 = V_0 \equiv \chi \equiv \vec{\sigma}\cdot\vec{D}\times\vec{\Psi}. \quad (25)$$

(The abbreviation $V_0 \equiv \chi$ conforms to the notation of Ref. [4].) A second primary constraint follows from the fact that the action has no dependence on $d\Psi^\dagger_0/dt$, which implies that the momentum conjugate to $\Psi^\dagger_0$ vanishes identically,

$$P_{\Psi^\dagger_0} = 0. \quad (26)$$

Contracting $\vec{V}$ with $\vec{\sigma}$ and with $g^{-1}\vec{D}$, and using the covariant derivative relations of Eq. (A14), we get, respectively,

$$\begin{aligned} \vec{\sigma}\cdot\vec{V} &= 2i\theta + \chi, \\ g^{-1}\vec{D}\cdot\vec{V} &= i\omega + g^{-1}D_0\chi, \end{aligned} \quad (27)$$

with

$$\begin{aligned} \theta &\equiv \vec{\sigma}\cdot\vec{D}\Psi_0 - D_0\vec{\sigma}\cdot\vec{\Psi}, \\ \omega &\equiv \vec{\sigma}\cdot\vec{B}\Psi_0 - (\vec{B} + \vec{\sigma}\times\vec{E})\cdot\vec{\Psi}. \end{aligned} \quad (28)$$

Since the Euler–Lagrange equations imply that $\vec{V}$ and χ vanish for all times, we learn that θ and ω vanish also for all times. Since θ involves a time derivative, its vanishing is just one component of the equation of motion for Ψ_μ. But ω involves no time derivatives, so it is a secondary constraint that relates Ψ_0 to $\vec{\Psi}$ [given in four-component form in Eq. (6)]. For each of the above equations, there is a corresponding relation for the adjoint quantity.

The equation of motion $\vec{V}=0$ can be written in a simpler form by using the identities of Eqs. (A10) and (A11) as follows. Using Eq. (A10) to simplify $0=\vec{\sigma}\times\vec{V}-i\vec{V}$, we get an equation for $D_0\vec{\Psi}$,

$$D_0\vec{\Psi}=\vec{D}\Psi_0+\frac{1}{2}[-\vec{\sigma}\times(\vec{D}\times\vec{\Psi})+i\vec{D}\times\vec{\Psi}]. \tag{29}$$

A further simplification can be achieved by incorporating the primary constraint $\chi=0$, through applying Eq. (A11) to $\vec{A}=\vec{D}\times\vec{\Psi}$,

$$0=\vec{\sigma}\chi=\vec{\sigma}\,\vec{\sigma}\cdot(\vec{D}\times\vec{\Psi})=\vec{D}\times\vec{\Psi}-i\vec{\sigma}\times(\vec{D}\times\vec{\Psi}). \tag{30}$$

Using this to replace the first term in square brackets in Eq. (29), we get the alternative form of the equation of motion, valid when the constraint $\chi=0$ is satisfied,

$$D_0\vec{\Psi}=\vec{D}\Psi_0+i\vec{D}\times\vec{\Psi}. \tag{31}$$

Writing the gauge field interaction terms in Eq. (23) in the form

$$S_{\rm int}(\Psi_\mu)=\frac{g}{2}\int d^4x(A_0^BJ^{B0}+\vec{A}^B\cdot\vec{J}^B), \tag{32}$$

we find the left chiral contribution to the current of Eq. (12) in the form

$$J^{A0}=-\vec{\Psi}^\dagger t_A\cdot\vec{\sigma}\times\vec{\Psi},$$
$$\vec{J}^A=\Psi_0^\dagger t_A\vec{\sigma}\times\vec{\Psi}+\vec{\Psi}^\dagger\times\vec{\sigma}t_A\Psi_0-\vec{\Psi}^\dagger\times t_A\vec{\Psi}. \tag{33}$$

Replacing t_A by $-i$, we find the corresponding singlet current in the form

$$J^0=i\vec{\Psi}^\dagger\cdot\vec{\sigma}\times\vec{\Psi},$$
$$\vec{J}=-i(\Psi_0^\dagger\vec{\sigma}\times\vec{\Psi}+\vec{\Psi}^\dagger\times\vec{\sigma}\Psi_0-\vec{\Psi}^\dagger\times\vec{\Psi}). \tag{34}$$

For the energy integral computed from the left chiral part of the the stress-energy tensor of Eq. (15), we find

$$H=-\int d^3xT^{00}_{RS}=-\frac{1}{2}\int d^3x\vec{\Psi}^\dagger\cdot\vec{D}\times\vec{\Psi}. \tag{35}$$

To conclude this section, we verify that the action of Eq. (23) has a fermionic gauge invariance on the constraint surface $\omega=0,\omega^\dagger=0$, as already seen in covariant form following Eq. (9). Letting ϵ be a general space and time dependent two-component spinor, we introduce the fermionic gauge changes

$$\vec{\Psi}\to\vec{\Psi}+\delta_G\vec{\Psi},\qquad \delta_G\vec{\Psi}\equiv\vec{D}\epsilon,$$
$$\Psi_0\to\Psi_0+\delta_G\Psi_0,\qquad \delta_G\Psi_0\equiv D_0\epsilon \tag{36}$$

and their adjoints, which are the left chiral form of the gauge change of Eq. (9). Substituting this into Eq. (23), integrating by parts where needed, and using Eqs. (A14) to simplify commutators of covariant derivatives, we find that Eq. (10) takes the two-component spinor form

$$\delta_GS(\Psi_\mu)=\frac{1}{2}ig\int d^4x(\omega^\dagger\epsilon-\epsilon^\dagger\omega)+O(\epsilon^\dagger\ldots\epsilon), \tag{37}$$

with the quadratic term given in Eq. (70) below. Hence, the action on the constraint surface $\omega=\omega^\dagger=0$ has a fermionic gauge invariance. Another gauge invariant, on the constraint surface $\chi=\chi^\dagger=0$, is the fermion number, given by the space integral of the time component of the singlet current of Eq. (34), $\int d^3xJ^0$, which has the gauge variation

$$\delta_G\int d^3xJ^0=\int d^3x[-i(\epsilon^\dagger\chi+\chi^\dagger\epsilon)+g\epsilon^\dagger\vec{\sigma}\cdot\vec{B}\epsilon]. \tag{38}$$

Again, these equations hold for ϵ and its adjoint $\epsilon^\dagger$ finite.

However, neither the equation of motion, the constraints χ and ω, the non-Abelian "charge" $\int d^3xJ^{B0}$, nor the integrated Hamiltonian H is gauge invariant in the interacting case. Using δ_G to denote gauge variations, we have

$$\delta_G\vec{V}=-ig(\vec{B}+\vec{\sigma}\times\vec{E})\epsilon,$$
$$\delta_G\theta=-ig\vec{\sigma}\cdot\vec{E}\epsilon,$$
$$\delta_G\chi=-ig\vec{\sigma}\cdot\vec{B}\epsilon,$$
$$\delta_G\omega=\vec{\sigma}\cdot\vec{B}D_0\epsilon-(\vec{B}+\vec{\sigma}\times\vec{E})\cdot\vec{D}\epsilon,$$
$$\delta_G\int d^3xJ^{B0}=g\int d^3x(\epsilon^\dagger[\vec{A},t_B]\cdot\vec{\sigma}\times\vec{\Psi}+\vec{\Psi}^\dagger\times\vec{\sigma}\cdot[t_B,\vec{A}]\epsilon),$$
$$\delta_GH=\frac{1}{2}ig\int d^3x(\vec{\Psi}^\dagger\cdot\vec{B}\epsilon-\epsilon^\dagger\vec{B}\cdot\vec{\Psi}). \tag{39}$$

The *only* global fermionic gauge invariants are the action integral and the fermion number integral, in both flat and curved spacetimes.

These results have an interpretation in terms of the distinction between a gauge transformation, customarily defined as an invariance of the physical state of the system, and a canonical transformation. The usual gauge transformations in gauge field theories and general relativity are invariances of the action without the imposition of a constraint and consequently are invariances of the field equations and the Hamiltonian. Such gauge transformations are a special case of canonical transformations, but the converse is not true; canonical transformations in general alter the action, the field equations, and the Hamiltonian.

We will see in Sec. IV that the fermionic gauge transformations of Eq. (36) are always canonical transformations, which reduce to gauge transformations of the customary type only when the external gauge fields vanish. However, by virtue of the Jacobi identity for the Poisson bracket, canonical transformations preserve inner properties of the theory. As an example, that will be needed in our further discussion of generalized fermionic gauge invariance in Sec. V, we verify that the secondary constraint following from the gauge-varied equation of motion $\vec{V}$ and primary constraint $V_0 = \chi$ agrees with the gauge variation of the original secondary constraint ω. From Eq. (27), we have

$$\vec{D}\cdot\vec{V} - D_0\chi = ig\omega. \tag{40}$$

Preservation of inner properties under the fermionic gauge transformation means that we should find that

$$\vec{D}\cdot\delta_G\vec{V} - D_0\delta_G\chi = ig\delta_G\omega. \tag{41}$$

Substituting Eqs. (39) into the left-hand side of Eq. (41) gives

$$\begin{aligned} &ig[D_0\vec{\sigma}\cdot\vec{B}\epsilon - \vec{D}\cdot(\vec{B}+\vec{\sigma}\times\vec{E})\epsilon)] \\ &\quad = ig[\vec{\sigma}\cdot\vec{B}D_0\epsilon - (\vec{B}+\vec{\sigma}\times\vec{E})\cdot\vec{D}\epsilon + C\epsilon], \end{aligned} \tag{42}$$

with the commutator remainder C given by

$$\begin{aligned} C &= \vec{\sigma}\cdot[D_0\vec{B} - \vec{B}D_0 + \vec{D}\times\vec{E} + \vec{E}\times\vec{D}] - (\vec{D}\cdot\vec{B} - \vec{B}\cdot\vec{D}) \\ &= 0, \end{aligned} \tag{43}$$

which vanishes by virtue of the gauge field Bianchi identity.

In Sec. V, we will discuss in more detail why the fermionic gauge transformation, because it leaves the constrained action invariant, corresponds to an unwanted redundancy in the time evolution. To break the gauge invariance, we can introduce an additional constraint, in the form

$$f(\vec{\Psi}) = 0, \tag{44}$$

with f a scalar function of its argument. This constraint, together with the χ constraint, leaves one independent two-component spinor of the original three in $\vec{\Psi}$, corresponding to the physical massless Rarita–Schwinger modes propagating in the gauge field background. We will limit ourselves to considering linear constraints of the general form

$$f = \vec{L}\cdot\vec{\Psi}, \tag{45}$$

and the choice $\vec{L} = \vec{D}$, a gauge covariant radiation gauge analog, plays a special role in our analysis. By not specializing $\vec{L}$ in our formulas, we can also examine the consequences of omitting a gauge fixing condition, corresponding to taking $\vec{L} = 0$.

We proceed to examine the gauge covariant radiation gauge condition in more detail. We note that, since

$$\vec{\sigma}\cdot\vec{D}\,\vec{\sigma}\cdot\vec{\Psi} = \vec{D}\cdot\vec{\Psi} + i\chi, \tag{46}$$

the primary constraint $\chi = 0$ implies that

$$\vec{\sigma}\cdot\vec{D}\,\vec{\sigma}\cdot\vec{\Psi} = \vec{D}\cdot\vec{\Psi}. \tag{47}$$

Hence, when $\vec{\sigma}\cdot\vec{D}$ is invertible, which is expected in a perturbation expansion in the gauge coupling g, the covariant radiation gauge constraint $\vec{D}\cdot\vec{\Psi} = 0$ implies that

$$\vec{\sigma}\cdot\vec{\Psi} = 0. \tag{48}$$

Conversely, Eqs. (46) and (47) show that $\vec{D}\cdot\vec{\Psi} = 0$ and $\vec{\sigma}\cdot\vec{\Psi} = 0$ together imply the primary constraint $\chi = 0$, and also $\vec{\sigma}\cdot\vec{\Psi} = 0$ and $\chi = 0$ imply $\vec{D}\cdot\vec{\Psi} = 0$.

We next note that on a given initial time slice the covariant radiation gauge is attainable. Under the gauge transformation of Eq. (36), we see that

$$\vec{D}\cdot\vec{\Psi} \to \vec{D}\cdot\vec{\Psi} + (\vec{D})^2\epsilon. \tag{49}$$

Hence, when $(\vec{D})^2$ is invertible, which we expect to be true in a perturbative sense, then we can invert $(\vec{D})^2\epsilon = -\vec{D}\cdot\vec{\Psi}$, to find a gauge function ϵ that brings a general $\vec{\Psi}$ to the covariant radiation gauge. Since

$$(\vec{\sigma}\cdot\vec{D})^2 = (\vec{D})^2 + g\vec{\sigma}\cdot\vec{B}, \tag{50}$$

the conditions for $\vec{\sigma}\cdot\vec{D}$ to be invertible and for $(\vec{D})^2$ to be invertible, are related. For generic non-Abelian gauge fields, both of these operators should be invertible, but there will be isolated gauge field configurations for which $\vec{\sigma}\cdot\vec{D}$ has zeros.

However, although the covariant radiation gauge can be imposed on any time slice, it is not preserved by the equation of motion for $\vec{\Psi}$. To see this, let us consider the simplified case in which the gauge potential is specialized to $A_0 = 0$ and $\partial_0\vec{A} = 0$, so that only a static $\vec{B}$ field is present. From Eq. (31), we have

$$\begin{aligned} \partial_0(\vec{D}\cdot\vec{\Psi}) &= (\vec{D})^2\Psi_0 + g\vec{B}\cdot\vec{\Psi} = [(\vec{D})^2 + g\vec{\sigma}\cdot\vec{B}]\Psi_0 \\ &= (\vec{\sigma}\cdot\vec{D})^2\Psi_0. \end{aligned} \tag{51}$$

So $\partial_0(\vec{D}\cdot\vec{\Psi}) = 0$ implies $\Psi_0 = 0$, but this is one constraint too many. Hence, at each infinitesimal time step, we must make a further infinitesimal fermionic gauge transformation to maintain the covariant radiation gauge condition, as further discussed in Sec. V B below. Only in the absence of gauge fields can we simultaneously impose the constraints $\vec{\nabla}\cdot\vec{\psi}=0$, $\vec{\sigma}\cdot\vec{\psi}=0$, and $\psi_0=0$, as used in the discussion of Ref. [11] for the free Rarita–Schwinger case.

IV. CANONICAL MOMENTA, CLASSICAL BRACKETS, AND GAUGE GENERATORS

We next introduce the canonical momentum conjugate to $\vec{\Psi}$, defined by

$$\vec{P}=\frac{\partial^L S}{\partial(\partial_0\vec{\Psi})}=\frac{1}{2}\vec{\Psi}^\dagger\times\vec{\sigma}, \tag{52}$$

which can be solved for $\vec{\Psi}^\dagger$ using the final line of Eq. (A11),

$$\vec{\Psi}^\dagger=i\vec{P}-\vec{P}\times\vec{\sigma}. \tag{53}$$

We will use Eq. (53) when computing classical brackets involving $\vec{\Psi}^\dagger$ using the formula of Eq. (A17). Equation (52) can be written as an explicit matrix relation for the six components of $\vec{P}$ and $\vec{\Psi}^\dagger$,

$$\begin{pmatrix} P_1^\uparrow \\ P_1^\downarrow \\ P_2^\uparrow \\ P_2^\downarrow \\ P_3^\uparrow \\ P_3^\downarrow \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & -i \\ 0 & 0 & 0 & -1 & i & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & i & 0 & -1 & 0 & 0 \\ -i & 0 & -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1^{\dagger\uparrow} \\ \Psi_1^{\dagger\downarrow} \\ \Psi_2^{\dagger\uparrow} \\ \Psi_2^{\dagger\downarrow} \\ \Psi_3^{\dagger\uparrow} \\ \Psi_3^{\dagger\downarrow} \end{pmatrix}, \tag{54}$$

showing that they are related by an anti-self-adjoint matrix with the determinant $-1/16$.

The four constraints introduced in Sec. III are

$$\begin{aligned}\phi_1&=P_{\Psi_0^\dagger},\\ \phi_2&=(\vec{\sigma}\cdot\vec{B})^{-1}\omega=\Psi_0-(\vec{\sigma}\cdot\vec{B})^{-1}(\vec{B}+\vec{\sigma}\times\vec{E})\cdot\vec{\Psi},\\ \phi_3&=\chi=\vec{\sigma}\cdot\vec{D}\times\vec{\Psi},\\ \phi_4&=\vec{L}\cdot\vec{\Psi}.\end{aligned} \tag{55}$$

In writing these, we are assuming that $\vec{\sigma}\cdot\vec{B}$ is invertible in the non-Abelian case. We are writing the gauge-fixing condition as a general linear gauge-fixing constraint $\vec{L}\cdot\vec{\Psi}$ so as to keep track of which terms in the final answers arise from gauge fixing, which is not evident if we specialize by replacing $\vec{L}$ by $\vec{D}$ at this stage. The constraints of Eq. (55), including the gauge-fixing constraint ϕ_4, are all first class in the Dirac classification, since they have vanishing mutual classical brackets. This is a consequence of the fact that, starting with a constraint depending on $\vec{\Psi}$ but not on $\vec{\Psi}^\dagger$ and taking an arbitrary number of time derivatives, one still has a constraint depending only on $\vec{\Psi}$.

To preserve the adjoint properties of the Rarita–Schwinger equation, for each of these four constraints, we must impose a corresponding adjoint constraint. Using Eq. (53) to express $\vec{\Psi}^\dagger$ in terms of $\vec{P}$, we write these as

$$\begin{aligned}\chi_1&=(P_{\Psi_0^\dagger})^\dagger=-P_{\Psi_0},\\ \chi_2&=\omega^\dagger(\vec{\sigma}\cdot\vec{B})^{-1}=\Psi_0^\dagger\\ &\quad-\vec{P}\cdot[i(\vec{B}+\vec{\sigma}\times\vec{E})-\vec{\sigma}\times(\vec{B}+\vec{\sigma}\times\vec{E})](\vec{\sigma}\cdot\vec{B})^{-1},\\ \chi_3&=\chi^\dagger=2\vec{P}\cdot\overleftarrow{D},\\ \chi_4&=\vec{\Psi}^\dagger\cdot\overleftarrow{L}=\vec{P}\cdot(i\overleftarrow{L}-\vec{\sigma}\times\overleftarrow{L}).\end{aligned} \tag{56}$$

(The reason for the minus sign in the definition $P_{\Psi_0^\dagger}=-P_{\Psi_0}^\dagger$ will be given in Sec. II of the following paper where we discuss the Hamiltonian form of the equations.) The constraints ϕ_a are implicitly $2n$-component column vectors, and the adjoint constraints χ_a are implicitly $2n$-component row vectors, with $2n$ arising from the product of a factor of 2 for the two implicit spinor indices and a factor of n for the n implicit $SU(n)$ internal symmetry indices.

When $\vec{L}=\vec{D}$, we see that ϕ_4 becomes $\phi_4=\vec{D}\cdot\vec{\Psi}$, and χ_4 becomes $\chi_4=i\vec{P}\cdot\overleftarrow{D}-\vec{P}\cdot\vec{\sigma}\times\overleftarrow{D}=(i/2)\chi_3-\vec{P}\cdot\vec{\sigma}\times\overleftarrow{D}$. So a special feature of covariant radiation gauge, which will be exploited later, is that the constraints ϕ_3,ϕ_4 are contractions of $\vec{\sigma}\times\vec{D}$ and $\vec{D}$ with $\vec{\Psi}$, and the constraints χ_3,χ_4 are contractions of linear combinations of the duals $\overleftarrow{D}$ and $\vec{\sigma}\times\overleftarrow{D}$ with $\vec{P}$. That is, in the covariant radiation gauge, the constraint spaces selected by χ_3,χ_4 and ϕ_3,ϕ_4 are duals of one another.

We can now compute the classical brackets of the constraints. We see that the brackets of the ϕs and χs vanish among themselves,

$$\begin{aligned}[][\phi_a,\phi_b]_C&=0,\\ [\chi_a,\chi_b]_C&=0,\\ a,b&=1,\ldots,4.\end{aligned} \tag{57}$$

On the other hand, the brackets of the ϕs with the χs give a nontrivial matrix of brackets M, which has a nonvanishing determinant,

$$\begin{aligned}M_{ab}(\vec{x},\vec{y})&\equiv[\phi_a(\vec{x}),\chi_b(\vec{y})]_C\neq0,\\ \det M&\neq0.\end{aligned} \tag{58}$$

Thus, in terms of the Dirac classification, the original first class constraints ϕ_a have become second class, not from adding new constraints that follow from differentiation with respect to time or from imposing gauge-fixing conditions but rather from adjoining the adjoint set of constraints. This is a feature of the Rarita–Schwinger constrained fermion system that has no analog in the familiar constrained boson systems such as gauge fields.

Evaluating the brackets shows that M has the general form

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & \mathcal{U} & \mathcal{S} & \mathcal{T} \\ 0 & \mathcal{V} & \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{W} & \mathcal{C} & \mathcal{D} \end{pmatrix}, \tag{59}$$

where in the $SU(n)$ gauge field case each entry in M is a $2n \times 2n$ matrix (corresponding to the fact that ϕ_a is implicitly a $2n$-component column vector and χ_b is implicitly a $2n$-component row vector). Evaluating $\det M$ by a cofactor expansion with respect to the elements of the two unit matrices ± 1, we see that the submatrices $\mathcal{U}, \mathcal{S}, \mathcal{T}, \mathcal{V}, \mathcal{W}$ do not contribute, and we have

$$\det M = \det N$$
$$N = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{60}$$

So we need to only evaluate the brackets $M_{33} = \mathcal{A}$, $M_{34} = \mathcal{B}$, $M_{43} = \mathcal{C}$, $M_{44} = \mathcal{D}$, giving

$$\begin{aligned} \mathcal{A} &= -2ig\vec{\sigma} \cdot \vec{B}(\vec{x})\delta^3(\vec{x} - \vec{y}), \\ \mathcal{B} &= -2\vec{D}_{\vec{x}} \cdot \vec{L}_{\vec{x}}\delta^3(\vec{x} - \vec{y}), \\ \mathcal{C} &= 2\vec{L}_{\vec{x}} \cdot \vec{D}_{\vec{x}}\delta^3(\vec{x} - \vec{y}), \\ \mathcal{D} &= (i(\vec{L}_{\vec{x}})^2 + \vec{\sigma} \cdot (\vec{L}_{\vec{x}} \times \vec{L}_{\vec{x}}))\delta^3(\vec{x} - \vec{y}). \end{aligned} \tag{61}$$

When $\vec{L} = \vec{D}$, these become

$$\begin{aligned} \mathcal{A} &= -2ig\vec{\sigma} \cdot \vec{B}(\vec{x})\delta^3(\vec{x} - \vec{y}), \\ \mathcal{B} &= -2(\vec{D}_{\vec{x}})^2\delta^3(\vec{x} - \vec{y}), \\ \mathcal{C} &= 2(\vec{D}_{\vec{x}})^2\delta^3(\vec{x} - \vec{y}), \\ \mathcal{D} &= i((\vec{D}_{\vec{x}})^2 - g\vec{\sigma} \cdot \vec{B}(\vec{x}))\delta^3(\vec{x} - \vec{y}). \end{aligned} \tag{62}$$

Reflecting the fact that the ϕ_a and χ_a are adjoints of one another, together with the fact that the matrix relating $\vec{\Psi}^\dagger$ to $\vec{P}$ is anti-self-adjoint [see Eq. (54)], these matrix elements obey the adjoint relations

$$M_{ab}(\vec{x}, \vec{y})^\dagger = -M_{ba}(\vec{y}, \vec{x}). \tag{63}$$

Applications of these bracket and determinant calculations will be made in the subsequent paper, where we discuss quantization by both the Dirac bracket formalism and by the Feynman path integral.

To conclude this section, we note that the constraints $\chi, \chi^\dagger, P_{\Psi_0}, P_{\Psi_0^\dagger}$ play the role of gauge transformation generators. For example, we have (with common time argument t suppressed)

$$\begin{aligned} \left[\int d^3x \frac{1}{2}\chi^\dagger(\vec{x})\epsilon(\vec{x}), \vec{\Psi}(\vec{y})\right]_C &= \vec{D}_{\vec{y}}\epsilon(\vec{y}), \\ \left[-\int d^3x P_{\Psi_0}(\vec{x}) D_0\epsilon(\vec{x}), \Psi_0(\vec{y})\right]_C &= D_{0\vec{y}}\epsilon(\vec{y}). \end{aligned} \tag{64}$$

So the fermionic gauge transformation is a canonical transformation. This is also evident from the fact that since Eq. (36) is just a shift in the fermionic variables $\vec{\Psi}$ and Ψ_0 by the quantities $\vec{D}\epsilon$ and $D_0\epsilon$, which have no dependence on the fermionic variables, this shift leaves the canonical brackets $[\vec{\Psi}_i, \vec{P}_j]_c$, $[\Psi_0, P_{\Psi_0}]_c$, etc., unchanged.

V. GENERALIZED GAUGE INVARIANCE OF THE RARITA–SCHWINGER ACTION

We turn now to a justification of our claim that the fermionic gauge transformation introduced in Eqs. (9) and (36) is a generalized form of gauge invariance, which corresponds to redundant degrees of freedom and which leaves essential attributes of the physics of gauged Rarita–Schwinger fields invariant. In the most familiar gauge invariant theories, such as Abelian or non-Abelian gauge fields, the Lagrangian density is invariant under a gauge transformation on the fields. These theories exhibit what one could term “strong” gauge invariance. In a weaker form of gauge invariance, which occurs for the free Rarita–Schwinger equation, the Lagrangian density changes by a total derivative under a gauge transformation of the fields, and so only the action is gauge invariant. Characteristic features of this case have been studied by Das [12]. We argue in this section that there is a still weaker form of gauge invariance, obeyed by the massless Rarita–Schwinger equation with Abelian or non-Abelian gauging, in which, under a gauge transformation, the Lagrangian changes by a total derivative plus terms which vanish when initial value constraints are obeyed.

We divide our argument that the transformation of Eqs. (9) and (36) is a generalized form of a gauge invariance into two parts, first considering infinitesimal transformations and then considering general finite transformations.

A. Infinitesimal gauge transformations

In his seminal analysis of constrained systems, Dirac [14] classifies as “first class” constraints the maximal set of constraints that have vanishing mutual Poisson brackets and notes that "Each of them thus leads to an arbitrary function of the time in the general solution of the equations of motion with given initial conditions." Elaborating on this, he notes that "Different solutions of the equations of motion, obtained by different choices of the arbitrary functions of the time with given initial conditions, should be looked upon as all corresponding to the same physical state of motion, described in various way (sic) by different choices of some mathematical variables that are not of physical significance (e.g. by different choices of the gauge in electrodynamics or of the co-ordinate system in a relativistic theory.)"

These remarks suggest that gauge invariance, in its most general form, corresponds to an arbitrariness in the time evolution of a system, in the sense that the future evolution of the system is not uniquely determined by the initial conditions and the Euler–Lagrange equations following from the action principle. Under this generalized definition, the Rarita–Schwinger equation with coupling to gauge fields has a fermionic gauge invariance. To see this, we note the Euler–Lagrange equations yield equations of two types. The first are the time evolution equations contained in Eq. (3), that determine the field variables at a later time $t+\Delta t$ from those initially given at time t. The second are the primary and secondary constraints of Eqs. (5) and (6), which constrain the initial field values at time t. If we make the gauge transformation of Eq. (9) at time t, with infinitesimal gauge parameter ϵ (with $\epsilon^\dagger$ its adjoint), we see that the action at time t changes, to first order in ϵ, according to Eq. (10). So assuming that the initial data at time t obey both the primary and secondary constraints, then when the constraints at time t are applied, the change in the action is $O((\epsilon)^2)$. After this gauge transformation, we have seen in Eq. (39) that the Euler–Lagrange equations $\vec{V}$, the primary constraint χ, and the secondary constraint ω are all changed at order ϵ, but because the gauge transformation is a canonical transformation that preserves inner properties, we have also seen that the altered secondary constraint is the one implied by the altered $\vec{V}$ and χ, with an error of at most $(\epsilon)^2$. Hence, after the gauge transformation, we still have consistent equations of motion and initial conditions, which can serve as a starting point for time evolution. However, by making the gauge infinitesimal gauge transformation, we have introduced an arbitrariness into the evolved solution. To get a unique time evolution path from the initial data at time t using the action principle, one must impose a gauge-fixing condition, that selects one member out of the equivalence class of equal action field configurations.

In the gauged Rarita–Schwinger theory, only the constrained action and constrained fermion number, in both flat and curved spacetimes, are invariant to first order under infinitesimal fermionic gauge transformations. This has an important physical significance. Consider a set of Rarita–Schwinger fields that, as envisaged in the model of Ref. [2], are permanently bound into condensates. The only way to see that these fields are present is through their gravitational fields, through their gauge field polarizabilities, and possibly also through their influence on overall fermion number counting. The constrained action is the functional of the metric and the gauge fields that determines the influence of the Rarita–Schwinger fields on the metric and the gauge fields, respectively, so the fact that the constrained action is invariant under infinitesimal fermionic gauge transformations means that the physical effects induced by confined Rarita–Schwinger fields are similarly invariant. (This statement is not contradicted by the fermionic gauge noninvariance of the energy integral and the gauge field source currents, since these are calculated by varying the unconstrained action and do not take into account the fact that the constraints that enter into the constrained action are themselves nontrivial functions of the spacetime metric and the gauge fields.)

The fermionic gauge invariance of the constrained action functional of the metric and the gauge fields then allows us to impose a gauge-fixing constraint, making the time evolution determined by the action principle unique. Gauge fixing eliminates the redundancy of gauge degrees of freedom and so is a convenience in checking the correct helicity counting for the Rarita–Schwinger fields but is not needed for this purpose. In the following paper, where we turn to quantization, gauge fixing is needed to get an invertible constraint matrix in the weak field limit, and when covariant radiation gauge fixing is used, one finds manifestly positive semidefinite anticommutation relations for the quantized Rarita–Schwinger fields.

B. Finite gauge transformations: Auxiliary field and the extended action

Since the transformations of Eqs. (9) and (36) are linear in the Rarita-Schwinger field, the relations of Eq. (39) give the most general form of the transformed equations of motion and constraints. Thus, letting Λ denote a finite fermionic gauge transformation, the general form of the equations of motion and constraints are

$$\begin{aligned} 0 &= \vec{V}(\Lambda) = \vec{\sigma}\times\vec{D}\Psi_0 + \vec{D}\times\vec{\Psi} - \vec{\sigma}\times D_0\vec{\Psi} \\ &\quad - ig(\vec{B}+\vec{\sigma}\times\vec{E})\Lambda, \\ 0 &= \chi(\Lambda) = \vec{\sigma}\cdot\vec{D}\times\vec{\Psi} - ig\vec{\sigma}\cdot\vec{B}\Lambda, \\ 0 &= \omega(\Lambda) = \vec{\sigma}\cdot\vec{B}(\Psi_0 + D_0\Lambda) \\ &\quad - (\vec{B}+\vec{\sigma}\times\vec{E})\cdot(\vec{\Psi}+\vec{D}\Lambda). \end{aligned} \tag{65}$$

Under the gauge shifts of Eq. (36), Λ is augmented to $\Lambda+\epsilon$, or, equivalently, under the extended gauge transformation that includes a shift of Λ,

$$\Psi_0 \to \Psi_0 + D_0\epsilon, \qquad \vec{\Psi} \to \vec{\Psi} + \vec{D}\epsilon, \qquad \Lambda \to \Lambda - \epsilon, \tag{66}$$

the formulas of Eq. (65) are left invariant. By using Eq. (43), one can verify that

$$\vec{D}\cdot\vec{V}(\Lambda) - D_0\,\chi(\Lambda) = ig\omega(\Lambda). \tag{67}$$

From Eqs. (65), one deduces alternative forms of the $\vec{\Psi}$ equation of motion, subject to the constraint $\chi(\Lambda) = 0$,

$$\begin{aligned} D_0\vec{\Psi} &= \vec{D}\Psi_0 + i\vec{D}\times\vec{\Psi} + g(\vec{B} - i\vec{E})\Lambda, \\ 0 &= \theta(\Lambda) \equiv \vec{\sigma}\cdot\vec{D}\Psi_0 - D_0\vec{\sigma}\cdot\vec{\Psi} - ig\vec{\sigma}\cdot\vec{E}\Lambda. \end{aligned} \tag{68}$$

From the first of these, one finds

$$D_0\vec{D}\cdot\vec{\Psi} = (\vec{D})^2\Psi_0 + g(\vec{B} + i\vec{E})\cdot\vec{\Psi} + g\vec{D}\cdot((\vec{B} - i\vec{E})\Lambda), \tag{69}$$

which gives a condition on the gauge shift Λ for the covariant radiation gauge condition $\vec{D}\cdot\vec{\Psi} = 0$ to be maintained in time.

We can now write down an action corresponding to the generalized equations of motion and constraints. It is

$$\begin{aligned} S(\Lambda) = \frac{1}{2}\int d^4x[&-\Psi_0^\dagger\vec{\sigma}\cdot\vec{D}\times\vec{\Psi} \\ &+ \vec{\Psi}^\dagger\cdot(\vec{\sigma}\times\vec{D}\Psi_0 + \vec{D}\times\vec{\Psi} - \vec{\sigma}\times D_0\vec{\Psi}) \\ &- ig\vec{\Psi}^\dagger\cdot(\vec{B} + \vec{\sigma}\times\vec{E})\cdot\Lambda + ig\Lambda^\dagger(\vec{B} + \vec{\sigma}\times\vec{E})\cdot\vec{\Psi} \\ &+ ig\Psi_0^\dagger\vec{\sigma}\cdot\vec{B}\Lambda - ig\Lambda^\dagger\vec{\sigma}\cdot\vec{B}\Psi_0 \\ &+ ig\Lambda^\dagger(\vec{B} + \vec{\sigma}\times\vec{E})\cdot\vec{D}\Lambda - ig\Lambda^\dagger\vec{\sigma}\cdot\vec{B}D_0\Lambda]. \end{aligned} \tag{70}$$

One can check that the final line of this action is self-adjoint, by using Eq. (43), and one can also verify that this action is invariant under the transformation of Eq. (66), including quadratic terms in ϵ, without using the constraints following from the equations of motion. The extended action of Eq. (70) gives the most general form of the gauged Rarita-Schwinger action, in which Λ plays the role of an auxiliary field that restores exact fermionic gauge invariance.

Varying this action with respect to $\vec{\Psi}^\dagger$ gives the generalized equation of motion $\vec{V}(\Lambda) = 0$, while varying it with respect to $\Psi_0^\dagger$ gives the generalized primary constraint $\chi(\Lambda) = 0$. Since these hold for all times, Eq. (67) then shows that they imply the generalized secondary constraint $\omega(\Lambda) = 0$. Varying this action with respect to $\Lambda^\dagger$ gives just the secondary constraint $\omega(\Lambda) = 0$ as the equation of motion for Λ. This shows that Λ is not an independent dynamical variable but rather is a Lagrange multiplier for the secondary constraint, which plays the role of a generalized auxiliary field. This further supports our argument that the gauge transformation of Eq. (36) corresponds to a generalized gauge invariance, and that the gauge degrees of freedom are redundant degrees of freedom.

Making the shift $\epsilon = -\Lambda$ reduces Λ to zero, so that action of Eq. (70) reduces to the original action of Eq. (23). Conversely, this shows that Eq. (70) is just Eq. (23) with the substitutions $\Psi_0 \to \Psi_0 + D_0\Lambda$ and $\vec{\Psi} \to \vec{\Psi} + \vec{D}\Lambda$, that is

$$\begin{aligned} S(\Lambda) = \frac{1}{2}\int d^4x[&-(\Psi_0^\dagger + \Lambda^\dagger\overleftarrow{D}_0)\vec{\sigma}\cdot\vec{D}\times(\vec{\Psi} + \vec{D}\Lambda) \\ &+ (\vec{\Psi}^\dagger + \Lambda^\dagger\overleftarrow{\vec{D}})\cdot(\vec{\sigma}\times\vec{D}(\Psi_0 + D_0\Lambda) \\ &+ \vec{D}\times(\vec{\Psi} + \vec{D}\Lambda) - \vec{\sigma}\times D_0(\vec{\Psi} + \vec{D}\Lambda))], \end{aligned} \tag{71}$$

which makes manifest the invariance of $S(\Lambda)$ under the shift transformation of Eq. (66). The simplicity of this way of constructing the extended action is a reflection of the fact that the fermionic gauge group is simply an Abelian group under addition of gauge functions. If we now define $\Psi_0' = \Psi_0 + D_0\Lambda$ and $\vec{\Psi}' = \vec{\Psi} + \vec{D}\Lambda$ and fix the choice of Λ by imposing a gauge-fixing condition, such as the gauge covariant radiation gauge, then we see that as a function of the primed, gauge-fixed variables, the generalized action $S(\Lambda)$ takes the same form that the original action of Eq. (23) took as a function of the original variables.

The above analysis in terms of two-component, left chiral spinors can also be carried out in the original four-component formalism. Making the substitution $\psi_\mu \to \psi_\mu + D_\mu\Lambda$ in Eq. (1) gives after some algebra using Eq. (7) the four-component form of the extended action functional of the Rarita-Schwingerfield ψ_ρ and the auxiliary field Λ,

$$\begin{aligned} S(\Lambda) = \frac{i}{2}\int d^4x\epsilon^{\mu\eta\nu\rho}\Big[&\bar{\psi}_\mu\gamma_5\gamma_\eta D_\nu\psi_\rho \\ &+ \frac{g}{2}(-\bar{\Lambda}\gamma_5\gamma_\eta F_{\mu\nu}\psi_\rho + \bar{\psi}_\mu\gamma_5\gamma_\eta F_{\nu\rho}\Lambda \\ &- \bar{\Lambda}\gamma_5\gamma_\eta F_{\nu\rho}D_\mu\Lambda)\Big], \end{aligned} \tag{72}$$

which is self-adjoint by virtue of the Bianchi identity

$$\epsilon^{\mu\eta\nu\rho}[D_\mu, F_{\nu\rho}] = 0. \tag{73}$$

Varying Eq. (72) with respect to $\bar{\psi}_\mu$ gives the generalized Euler–Lagrange equations (which include the generalized primary constraint)

$$\epsilon^{\mu\eta\nu\rho}\Big(D_\nu\psi_\rho + \frac{g}{2}F_{\nu\rho}\Lambda\Big) = 0, \tag{74}$$

while applying $g^{-1}D_\mu$ to this and using Eq. (73) gives the generalized secondary constraint

$$\epsilon^{\mu\eta\nu\rho}F_{\mu\nu}(\psi_\rho + D_\rho\Lambda) = 0. \tag{75}$$

Varying Eq. (72) with respect to $\bar{\Lambda}$ gives just the generalized secondary constraint of Eq. (75), again showing that Λ is a Lagrange multiplier for the secondary constraint, which acts as an auxiliary field, and thus corresponds to a redundant degree of freedom, not a physical degree of freedom.

VI. PROPAGATION OF A RARITA–SCHWINGER FIELD IN AN EXTERNAL ABELIAN GAUGE FIELD: ABSENCE OF SUPERLUMINAL PROPAGATION

We specialize now to the case of a Rarita–Schwinger spinor propagating in an external Abelian gauge field, as studied by Velo and Zwanziger [4]. For an Abelian gauge field,

$$\frac{1}{\vec{\sigma}\cdot\vec{B}} = \frac{\vec{\sigma}\cdot\vec{B}}{(\vec{B})^2}, \tag{76}$$

and so $\vec{\sigma}\cdot\vec{B}$ is invertible as long as $(\vec{B})^2 > 0$, which we assume. Provided the Lorentz invariant expression $(\vec{B})^2 - (\vec{E})^2$ is positive, $(\vec{B})^2$ will be positive in any Lorentz frame. In discussing undamped wave propagation, we will not use the inequality $(\vec{B})^2 - (\vec{E}) > 0$, but in treating damped longitudinal mode propagation in Appendix B, we will assume that $(\vec{E})^2/(\vec{B})^2$ is small, as motivated by the fact that when $(\vec{E})^2$ is of order $(\vec{B})^2$ the vacuum is highly unstable against pair creation. (Strictly speaking, the vacuum is stable against pair production only when $\vec{E}\cdot\vec{B} = 0$ and $(\vec{B})^2 - (\vec{E})^2 > 0$, that is, when there is a Lorentz frame in which the Abelian field has vanishing $\vec{E}$ [15].)

Given that $(\vec{B})^2 > 0$, we can solve the constraint $\omega = 0$ of Eq. (28) for Ψ_0, giving

$$\Psi_0 = \frac{\vec{Q}\cdot\vec{\Psi}}{(\vec{B})^2}, \tag{77}$$

where we have defined

$$\vec{Q} \equiv \vec{\sigma}\cdot\vec{B}(\vec{B} + \vec{\sigma}\times\vec{E}) = \vec{B}\times\vec{E} + \vec{B}\,\vec{\sigma}\cdot(\vec{B} + i\vec{E}) - i\vec{B}\cdot\vec{E}\,\vec{\sigma}. \tag{78}$$

Substituting the solution for Ψ_0 into Eq. (31), we get an equation of motion for $\vec{\Psi}$ by itself,

$$D_0\vec{\Psi} = \vec{D}\frac{\vec{Q}\cdot\vec{\Psi}}{(\vec{B})^2} + i\vec{D}\times\vec{\Psi}. \tag{79}$$

To determine the wave propagation velocity in the neighborhood of a spacetime point $x_* = (t_*, \vec{x}_*)$, we need to calculate the equation for the wave fronts, or characteristics, at that point. Writing the first-order Eq. (79) in the form

$$\partial_0\vec{\Psi} = \vec{\nabla}\frac{\vec{Q}_*\cdot\vec{\Psi}}{(\vec{B}_*)^2} + i\vec{\nabla}\times\vec{\Psi} + \vec{\Delta}[\vec{\Psi}, x_*, x], \tag{80}$$

with $\vec{B}_*$ and $\vec{Q}_*$ the values of the respective quantities at x_*, we see that $\vec{\Delta}[\vec{\Psi}, x_*, x]$ involves no first derivatives of $\vec{\Psi}$ at x_*, and so is not needed [16,17] for determining the wave fronts of Eq. (31). The reason is that, when taking an infinitesimal line integral of Eq. (80), according to

$$\lim_{\delta\to 0}\int_{-\delta}^{\delta} d\ell[\partial_0\vec{\Psi} = \ldots], \tag{81}$$

discontinuities across wave fronts contribute through the first derivative terms, but when the external fields are smooth, the term $\vec{\Delta}[\vec{\Psi}, x_*, x]$ makes a vanishing contribution as $\delta \to 0$. Dropping $\vec{\Delta}$ and multiplying through by $(\vec{B}_*)^2$, we get the equation determining the wave fronts in the form

$$(\vec{B}_*)^2\partial_0\vec{\Psi} = \vec{\nabla}\vec{Q}_*\cdot\vec{\Psi} + i(\vec{B}_*)^2\vec{\nabla}\times\vec{\Psi}. \tag{82}$$

By similar reasoning, the constraint χ can be simplified, for purposes of determining the wave fronts, by replacing $\vec{D}$ by $\vec{\nabla}$, giving

$$0 = \vec{\sigma}\cdot\vec{\nabla}\times\vec{\Psi}. \tag{83}$$

Since these are now linear equations with constant coefficients, the solutions are plane waves, and without loss of generality, we can take the negative $z = x_3$ axis as the direction of wave propagation. So making the Ansatz

$$\vec{\Psi} = \vec{C}\exp(i\Omega t + iKz), \tag{84}$$

Eq. (82) for the wave fronts or characteristics takes the form

$$0 = \vec{F} \equiv (\vec{B}_*)^2\Omega\vec{C} - K\hat{z}\vec{Q}_*\cdot\vec{C} - i(\vec{B}_*)^2K\hat{z}\times\vec{C}, \tag{85}$$

with $\hat{z}$ a unit vector along the z axis, and the constraint Eq. (83) becomes an admissability condition on $\vec{C}$,

$$0 = \vec{\sigma}\cdot\hat{z}\times\vec{C}. \tag{86}$$

Writing F_m as a matrix times C_n (and dropping the subscripts $*$, which are implicit from here on), we have

$$F_m = N_{mn}C_n,$$
$$N_{mn} = (\vec{B})^2\Omega\delta_{mn} - K\delta_{m3}Q_n - i(\vec{B})^2K\epsilon_{m3n}. \tag{87}$$

The equation for the characteristics is now

$$\det(N) = 0, \tag{88}$$

since this is the condition for Eq. (85) to have a solution with nonzero $\vec{C}$. However, since the evaluation of the determinant shows that it factorizes into blocks that determine $C_{1,2}$ and a block that determines C_3, a simpler way to proceed is to work directly from the equations $F_m = 0$, which decouple in a corresponding way. Calculating from Eq. (85), we find

$$0 = F_1^{\uparrow,\downarrow} = (\vec{B})^2(\Omega C_1^{\uparrow,\downarrow} + iKC_2^{\uparrow,\downarrow}),$$
$$0 = F_2^{\uparrow,\downarrow} = (\vec{B})^2(\Omega C_2^{\uparrow,\downarrow} - iKC_1^{\uparrow,\downarrow}),$$
$$0 = F_3^{\uparrow,\downarrow} = (\vec{B})^2\Omega C_3^{\uparrow,\downarrow} - K(\vec{Q}\cdot\vec{C})^{\uparrow,\downarrow}, \tag{89}$$

where $\uparrow, \downarrow$ indicate the up and down spinor components, labeled in Eq. (22) by $\alpha = 1, 2$. Similarly, the constraint Eq. (86) becomes $0 = -\sigma_1 C_2 + \sigma_2 C_1$, that is

$$C_2^{\uparrow} = iC_1^{\uparrow},$$
$$C_2^{\downarrow} = -iC_1^{\downarrow}, \tag{90}$$

with no corresponding condition on $C_3^{\uparrow,\downarrow}$. The first two lines of Eq. (89) together with Eq. (90) have the solution

$$C_1^{\uparrow} = C, \qquad C_2^{\uparrow} = iC, \qquad \Omega = K,$$
$$C_1^{\downarrow} = C, \qquad C_2^{\downarrow} = -iC, \qquad \Omega = -K, \tag{91}$$

with C arbitrary, corresponding to waves with a velocity of magnitude $|\Omega/K| = 1$. Thus, the modes with $C_{1,2} \neq 0$ are exactly luminal. Because general background gauge fields are a nonisotropic medium, these modes have nonzero longitudinal components given by solving the third line of Eq. (89),

$$C_3 = K((\vec{B})^2\Omega - KQ_3)^{-1}(Q_1C_1 + Q_2C_2). \tag{92}$$

The effect on the characteristics of a gauge change $\vec{\Psi} \to \vec{\Psi} + \vec{D}\epsilon$, $\epsilon = E\exp(i\Omega t + iKz)f(t, z)$, where f has a unit slope discontinuity along the z axis at x_*, is to shift $C_3^{\uparrow,\downarrow} \to C_3^{\uparrow,\downarrow} + E^{\uparrow,\downarrow}$, and thus $C_3^{\uparrow,\downarrow}$ are gauge degrees of freedom. In Appendix B, we continue this discussion and show that the longitudinal gauge mode with $C_1 = C_2 = 0, C_3 \neq 0$ also does not propagate superluminally, although in general it is subluminal.

VII. FAILURE OF ADIABATIC DECOUPLING AND INAPPLICABILITY OF THE S-MATRIX NO-GO THEOREMS

We show in this section that various no-go theorems that claim to rule out the gauging of higher-spin theories do not apply to the gauged Rarita–Schwinger field. The reason is that there is a failure of adiabatic decoupling, arising from the fact that the ω secondary constraint is homogeneous in the gauge fields. For a recent paper on no-go theorems, see Ref. [10], which has extensive references to the earlier literature. In our analysis here, we shall refer specifically to the paper of Porrati [9], which uses so-called on-shell methods to give limits on massless high-spin particles.

The analysis of Porrati assumes that "the general helicity-conserving matrix element of a $U(1)$ current between on-shell spin s states is $\langle v, p+q|J_\mu|u, p\rangle\ldots$," where u and v are free-space spinors that obey the massless Dirac equation. Porrati assumes that the matrix element is bilinear in u and v, and "otherwise depends only on the momenta." We shall see in the following subsections that this assumed form is not realized in the gauged Rarita–Schwinger theory, where, because of the failure of adiabatic decoupling, the matrix element in question also depends on the $U(1)$ gauge field polarization through the dual field strength $\hat{F}_{\eta\nu} = \frac{1}{2}\epsilon_{\eta\nu\lambda\sigma}F^{\lambda\sigma}$. In fact, the initial and final Rarita–Schwinger spinors both must have a $\hat{F}_{\eta\nu}$ dependence in order to obey the secondary constraint of Eq. (6), and so the matrix element has the more complicated form $\langle v, p+q, \hat{F}_{\eta\nu}|J_\mu|u, p, \hat{F}_{\eta\nu}\rangle$.

We show in Sec. VII A that the initial and final Rarita–Schwinger spinors in the limit of a zero gauge field amplitude are equal to free-space spinors u, v of the form assumed by Porrati, plus a fermionic gauge transformation that depends explicitly on the photon field strength $\hat{F}_{\eta\nu}$. This structure arises from the homogeneous form of the secondary constraint and corresponds to an intrinsically nonperturbative aspect of the gauged Rarita–Schwinger equation. As another reflection of this, we show in Sec. VII B that one cannot set up a covariant Lippmann–Schwinger equation [18] for the Rarita–Scwhinger wave function, and so the matrix element that enters into the no-go theorems does not admit a Born approximation. In Sec. VII C, we show that a matrix element that has all the required invariances can be formulated using an analog of the distorted wave Born approximation, in which the initial and final Rarita–Schwinger states have an explicit dependence on the photon polarizations.

A. Zero amplitude limit of the $\vec{\Psi}$ equation: Retained memory of the gauge field

As in Sec. VI, let us consider a Rarita–Schwinger field propagating in an external Abelian gauge field. For convenience, we assume that the ratio $|\vec{E}(\vec{x})|/|\vec{B}(\vec{x})| \equiv r(\vec{x})$ is bounded from above. In the limit as the vector potential amplitude $\vec{A}$ is scaled to zero, Eqs. (77) and (78) become

$$\Psi_0(\vec{x}) = \vec{R}(\vec{x})\cdot\vec{\Psi}(\vec{x}),$$
$$\vec{R}(\vec{x}) = \vec{\sigma}\cdot\hat{B}(\vec{x})(\hat{B}(\vec{x}) + r(\vec{x})\vec{\sigma}\times\hat{E}(\vec{x})), \tag{93}$$

with $\hat{B} = \vec{B}/|\vec{B}|$ and $\hat{E} = \vec{E}/|\vec{E}|$ unit vectors along the $\vec{E}$ and $\vec{B}$ fields. When the external field is a propagating plane wave with the wave vector direction $\hat{q}$, the unit vectors $\hat{q}$, $\hat{B}$, and $\hat{E}$ form an orthonormal set of constant unit vectors, and $|\vec{r}(\vec{x})| = 1$. We see that, because the secondary constraint of Eq. (6) is homogeneous in the field strengths, the relation between Ψ_0 and $\vec{\Psi}$ retains a memory of the gauge field orientations, and thus of the photon polarization, even in the limit as the field amplitude approaches zero.

In the zero amplitude limit, $D_0 = \partial_0$ and $\vec{D} = \vec{\nabla}$, so substituting Eq. (93) into Eq. (79), the zero amplitude limit for the equation of motion for $\vec{\Psi}$ becomes

$$\partial_0 \vec{\Psi} = \vec{\nabla}\vec{R}\cdot\vec{\Psi} + i\vec{\nabla}\times\vec{\Psi}, \tag{94}$$

with the primary constraint now $\vec{\sigma}\cdot\vec{\nabla}\times\vec{\Psi} = 0$. Hence, through $\vec{R}$, the $\vec{\Psi}$ equation of motion retains a memory of the external fields in the limit of zero amplitude; that is, adiabatic decoupling has failed. Let us now consider the situation in which the Rarita–Schwinger field and the external gauge fields are plane waves, so that $\vec{R}$ is a constant and $\vec{\Psi}$ has the form

$$\vec{\Psi} = \vec{C}e^{i(\Omega t + \vec{k}\cdot\vec{x})}. \tag{95}$$

Making the fermionic gauge transformation

$$\vec{\Psi} \to \vec{\Psi}' = \vec{\Psi} + \vec{\nabla}\epsilon,$$
$$\epsilon = Ee^{i(\Omega t + \vec{k}\cdot\vec{x})}, \tag{96}$$

$\vec{\Psi}'$ still obeys the zero amplitude primary constraint since $\vec{\sigma}\cdot\vec{\nabla}\times\vec{\nabla}\epsilon = 0$. Then the gauge choice

$$E = i\frac{\vec{R}\cdot\vec{C}}{\vec{R}\cdot\vec{k}} \tag{97}$$

reduces Eq. (94) to the free-space form

$$\partial_0 \vec{\Psi}' = i\vec{\nabla}\times\vec{\Psi}'. \tag{98}$$

Thus, a Rarita–Schwinger plane wave in a zero amplitude gauge field plane wave background is equal to a free-space solution plus a gauge term that has a memory of the photon polarizations.

B. Breakdown of the Lipmann–Schwinger equation: No Born approximation to scattering

Let us now examine what happens if one tries to set up a covariant Lippmann–Schwinger equation, so as to generate a Born perturbation series for the Rarita–Schwinger wave function in an external gauge field. Let us start from the Rarita–Schwinger equation in the form [see Eq. (A6)]

$$\gamma^{\eta\nu\rho}D_\nu\psi_\rho = 0. \tag{99}$$

Splitting D_ν into ∂_ν and gA_ν, this equation takes the form

$$\gamma^{\eta\nu\rho}\partial_\nu\psi_\rho = -\gamma^{\eta\nu\rho}gA_\nu\psi_\rho. \tag{100}$$

Let us now try to solve this equation as a perturbation series around a free-space solution by writing

$$\psi_\rho(x) = \psi_\rho^{\rm free}(x) + \int d^4y S_{\rho\alpha}(x-y)\gamma^{\alpha\beta\kappa}gA_\beta(y)\psi_\kappa(y), \tag{101}$$

where $\psi_\rho^{\rm free}$ obeys the free-space Rarita–Schwinger equation

$$\gamma^{\eta\nu\rho}\partial_\nu\psi_\rho^{\rm free} = 0. \tag{102}$$

If the free-space Rarita–Schwinger Green's function $S_{\rho\alpha}(x-y)$ obeyed

$$\gamma^{\eta\nu\rho}\partial_{x\nu}S_{\rho\alpha}(x-y) = -\delta^\eta_\alpha\delta^4(x-y), \tag{103}$$

then Eq. (101) would reproduce Eq. (100). But in fact the free-space Green's function cannot obey Eq. (103), because $\partial_{x\eta}\gamma^{\eta\nu\rho}\partial_{x\nu}S_{\rho\alpha}(x-y) = 0$; instead it obeys [19]

$$\gamma^{\eta\nu\rho}\partial_{x\nu}S_{\rho\alpha}(x-y) = -\delta^\eta_\alpha\delta^4(x-y) + \partial_{y\alpha}\Omega^\eta(x-y), \tag{104}$$

with Ω necessarily nonvanishing. Integrating $\partial_{y\alpha}$ by parts onto the factor $\gamma^{\alpha\beta\kappa}gA_\beta(y)\psi_\kappa(y)$, one gets

$$\gamma^{\alpha\beta\kappa}gF_{\alpha\beta}(y)\psi_\kappa(y) + \gamma^{\alpha\beta\kappa}gA_\beta(y)\partial_{y\alpha}\psi_\kappa(y). \tag{105}$$

The first term of this expression vanishes by virtue of the secondary constraint, but the second term is nonvanishing because the Rarita–Schwinger equation for the exact wave function $\psi_\kappa(y)$ is

$$\gamma^{\alpha\beta\kappa}D_{y\alpha}\psi_\kappa(y) = 0; \tag{106}$$

that is, it requires the full covariant derivative $D_{y\alpha}$ in place of its free-space restriction $\partial_{y\alpha}$. The conclusion from this analysis is that one cannot set up a covariant Lippmann–Schwinger equation for the gauged Rarita–Schwinger wave function, and thus one cannot develop this wave function into a Born approximation series expansion in powers of the coupling g to the external gauge field.

C. Lorentz covariance and mode counting in on-shell Rarita–Schwinger field-photon scattering: Distorted wave Born approximation analog

We address finally the question [20] of whether one can write down an amplitude for leading-order on-shell scattering of Rarita–Schwinger fields from an external electromagnetic field, which has the requisite relativistic covariance while preserving the correct counting of massless spin-$\frac{3}{2}$ propagation modes. Looking ahead to the quantization, an operator effective action for this scattering process can be inferred from the interaction term in Eq. (1),

$$S_{\rm eff}(\psi_\mu, A_\nu) = \int d^4x \mathcal{L}_{\rm eff}(\psi_\mu, A_\nu),$$
$$\mathcal{L}_{\rm eff}(\psi_\mu, A_\nu) = \frac{1}{2} g\bar{\psi}_\mu(x) i\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta A_\nu(x)\psi_\rho(x), \quad (107)$$

where we have suppressed spinor indices as in the text from Eq. (3) onward. For Abelian external fields A_ν, the covariant derivatives in the equations of motion and constraints are given by

$$D_\nu = \partial_\nu + gA_\nu, \qquad \overleftarrow{D}_\nu = \overleftarrow{\partial}_\nu - gA_\nu. \quad (108)$$

At the outset, we shall assume that $A_\nu(x)$ is of short range and vanishes for $|\vec{x}| > R$ for some radius R. This effective action, the equations of motion of Eqs. (3) and (4), and the primary and secondary constraints following from them, given in Eqs. (5) and (6), are all relativistically covariant and so provide a starting point for calculating a covariant scattering amplitude. Taking the matrix element of Eq. (107) between an incoming Rarita–Schwinger state of four-momentum p and an outgoing Rarita–Schwinger state of four momentum p', we get the corresponding scattering amplitude

$$\mathcal{A}_S = \frac{1}{2} ig \int d^4x \bar{\psi}_\mu(p', x)\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta A_\nu(x)\psi_\rho(p, x), \quad (109)$$

where ψ_ρ and $\bar{\psi}_\mu$ are now wave functions, rather than operators, that obey the Rarita–Schwinger equations of motion in the presence of the external field A_ν.

We now introduce source currents for the gauge potential A_ν and the Rarita–Schwinger wave functions ψ_ρ and $\bar{\psi}_\mu$, and study their conservation properties. The source current to which the gauge potential A_ν couples is defined by writing the scattering amplitude as

$$\mathcal{A}_S = \frac{1}{2} ig \int d^4x A_\nu(x) J^\nu(x),$$
$$J^\nu(x) = \bar{\psi}_\mu(p', x)\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta\psi_\rho(p, x). \quad (110)$$

The source current for the Rarita–Schwinger field $\bar{\psi}_\mu(p', x)$ is defined by writing the scattering amplitude as

$$\mathcal{A}_S = \frac{1}{2} ig \int d^4x \bar{\psi}_\mu(p', x)\mathcal{J}^\mu(p, x),$$
$$\mathcal{J}^\mu(p, x) = \epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta A_\nu(x)\psi_\rho(p, x). \quad (111)$$

Finally, the source current for the Rarita–Schwinger field $\psi_\rho(p, x)$ is defined by writing the scattering amplitude as

$$\mathcal{A}_S = \frac{1}{2} ig \int d^4x \bar{\mathcal{J}}^\rho(p', x)\psi_\rho(p, x),$$
$$\bar{\mathcal{J}}^\rho(p', x) = \bar{\psi}_\mu(p', x)\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta A_\nu(x). \quad (112)$$

We now show that the three currents that we have just defined are conserved. For the source current J^ν for the gauge potential, we have

$$\begin{aligned}\partial_\nu J^\nu &= \bar{\psi}_\mu(p', x)\overleftarrow{D}_\nu\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta\psi_\rho(p, x) \\ &\quad + \bar{\psi}_\mu(p', x)\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta D_\nu\psi_\rho(p, x) \\ &= 0, \end{aligned} \quad (113)$$

where the first and second terms on the right vanish by the Rarita–Schwinger equations for $\bar{\psi}_\mu(p', x)$ and $\psi_\rho(p, x)$, respectively. For the source current $\mathcal{J}^\mu(p, x)$ for the spinor $\bar{\psi}_\mu(p', x)$, we have

$$\begin{aligned}D_\mu\mathcal{J}^\mu(p, x) &= \epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta(\partial_\mu A_\nu(x))\psi_\rho(p, x) \\ &\quad + \epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta A_\nu(x) D_\mu\psi_\rho(p, x) \\ &= 0. \end{aligned} \quad (114)$$

The second term on the right vanishes by the Rarita–Schwinger equation for $\psi_\rho(p, x)$, while the first term on the right can be rewritten as

$$\frac{1}{2}\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta F_{\mu\nu}(x)\psi_\rho(p, x) \quad (115)$$

and vanishes by the secondary constraint of Eq. (6). Finally, for the source current $\bar{\mathcal{J}}^\rho(p', x)$ for the spinor $\psi_\rho(p, x)$, we have

$$\begin{aligned}\bar{\mathcal{J}}^\rho(p', x)\overleftarrow{D}_\rho &= \bar{\psi}_\mu(p', x)\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta(\partial_\rho A_\nu(x)) \\ &\quad + \bar{\psi}_\mu(p', x)\overleftarrow{D}_\rho\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta A_\nu(x) \\ &= 0. \end{aligned} \quad (116)$$

Again, the second term on the right vanishes by the Rarita–Schwinger equation, while the first term on the right vanishes by the secondary constraint of Eq. (6).

Consider now the following three gauge transformations:

$$\begin{aligned} A_\nu(x) &\to A_\nu(x) + \partial_\nu\Lambda, \\ \psi_\rho(p, x) &\to \psi_\rho(p, x) + D_\rho\alpha, \\ \bar{\psi}_\mu(p', x) &\to \bar{\psi}_\mu(p', x) + \bar{\beta}\overleftarrow{D}_\mu, \end{aligned} \quad (117)$$

with α and β independent spinorial gauge parameters. From Eqs. (110)–(112), together with Eqs. (113)–(116), we find that these transformations each leave the amplitude $\mathcal{A}$ invariant,

$$\begin{aligned}\delta_\Lambda \mathcal{A}_S &= \frac{1}{2} ig \int d^4x (\partial_\nu \Lambda) J^\nu(x) \\ &= -\frac{1}{2} ig \int d^4x \Lambda \partial_\nu J^\nu(x) = 0, \\ \delta_\alpha \mathcal{A}_S &= \frac{1}{2} ig \int d^4x \bar{\mathcal{J}}^\rho(p', x) D_\rho \alpha \\ &= -\frac{1}{2} ig \int d^4x \bar{\mathcal{J}}^\rho(p', x) \overleftarrow{D}_\rho \alpha = 0, \\ \delta_\beta \mathcal{A}_S &= \frac{1}{2} ig \int d^4x \bar{\beta} \overleftarrow{D}_\mu \mathcal{J}^\mu(p, x) \\ &= -\frac{1}{2} ig \int d^4x \bar{\beta} D_\mu \mathcal{J}^\mu(p, x) = 0. \end{aligned} \tag{118}$$

This, together with the primary and secondary constraints, implies the correct mode counting for the Rarita–Schwinger wave functions, since the gauge degrees of freedom do not change the amplitude and so are redundant.

We next must specify more precisely the structure of the spinor wave functions entering the formula for $\mathcal{A}_S$. Since the gauge field A_ν is assumed to vanish in the external region $|\vec{x}| > R$, the Rarita–Schwinger wave functions obey free field equations in this region. So for $|\vec{x}| \gg R$, they can be taken asymptotically as plane waves at $t \to \pm\infty$,

$$\begin{aligned}\psi_\mu(p', x) &\sim u_\mu(p') e^{ip'\cdot x}, \qquad t \to +\infty, \\ \psi_\rho(p, x) &\sim u_\rho(p) e^{ip\cdot x}, \qquad t \to -\infty. \end{aligned} \tag{119}$$

With these boundary conditions, the formula for the amplitude takes the final form

$$\mathcal{A} = \frac{1}{2} ig \int d^4x \bar{\psi}_\mu^{(-)}(p', x) \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta A_\nu(x) \psi_\rho^{(+)}(p, x). \tag{120}$$

The out state $(-)$ and in state $(+)$ boundary conditions used here are analogs of the boundary conditions used in the distorted wave Born approximation [21], which the construction of Eq. (120) resembles. Equation (120) then gives an approximation to the matrix element for Rarita–Schwinger scattering by the gauge potential.

Rather than invoking the presence of redundant degrees of freedom to count physical Rarita–Schwinger states, we can follow the usual procedure of imposing a gauge-fixing constraint. To preserve relativistic and gauge covariance, this can be taken as the gauge covariant Lorentz gauge condition

$$\bar{\psi}_\mu(p', x) \overleftarrow{D}^\mu = D^\rho \psi_\rho(p, x) = 0, \tag{121}$$

which is attainable from a generic gauge by the gauge transformation of Eq. (9), provided that $D^\mu D_\mu$ is invertible. In the external region where the gauge field vanishes, one can instead use the condition $\gamma^\rho \psi_\rho = 0$ in place of the secondary constraint together with the gauge condition $\partial^\rho \psi_\rho = 0$, giving the usual covariant degree of freedom counting for the incoming and outgoing Rarita–Schwinger wave functions [22]. Alternatively, if we are not concerned to maintain manifest Lorentz covariance, we can make a gauge transformation in the external region to the gauge $\psi_0 = \vec{\nabla} \cdot \vec{\psi} = 0$ used in Refs. [11,19] to enumerate Rarita–Schwinger degrees of freedom. When a non-Lorentz covariant radiation gauge condition is used, scattering matrix elements depend on a unit timelike vector in addition to the particle momenta, and so the conditions assumed in [9] are not obeyed.

Note that if one were to attempt to construct a Born approximation amplitude, in which the Rarita–Schwinger wave functions in the presence of the gauge field are replaced by plane waves in the interior region where the potential is nonzero, the arguments given above for the compatibility of Lorentz covariance with degree of freedom counting would fail. The reason for this is that the spinor source currents would then no longer be conserved, even to zeroth order in the gauge coupling g, because the free particle plane wave solutions do not obey the secondary constraint of Eq. (6). The nonexistence of a satisfactory Born approximation for Rarita–Schwinger photon scattering agrees with the result obtained in Sec. VII B, that one cannot construct a Lippmann–Schwinger equation for this process. To establish compatibility, we have had to use an analog of the distorted wave Born approximation [21], in which the leading approximation to the amplitude is constructed using interacting rather than free fermion wave functions and does not have a perturbation expansion for small coupling, g.

When the external Abelian potential is a plane wave field which extends to infinity, there is no large $|\vec{x}|$ region where the Rarita–Schwinger solutions reduce to free-space ones. Rather, as shown in Sec. VII A, in the adiabatic decoupling limit of a zero amplitude gauge field, the Rarita–Schwinger solutions become free-space solutions plus gauge terms that remember the photon polarization and which are necessary to enforce the secondary constraint. Thus, one cannot attain the kinematic form assumed in the on-shell no-go theorems. But as shown here, using distorted Born approximation waves, one can write down a consistent covariant scattering amplitude.

VIII. SUMMARY AND REMARKS

To conclude, we see that, unlike the massive case, the massless gauged Rarita–Schwinger equation leads to a consistent classical theory. The theory has the correct counting of propagating nongauge degrees of freedom with no superluminal wave propagation. The theory admits a generalized fermionic gauge transformation, and infinitesimal gauge transformations are an invariance of the constrained flat and curved spacetime actions and of the fermion number. The gauged Rarita–Schwinger equation has a nonperturbative aspect when the secondary constraint ω is eliminated, resulting in a breakdown of adiabatic decoupling, leading to the inapplicability of various S-matrix no-go theorems that claim to forbid gauged massless Rarita–Schwinger fields. The extension of these results to the quantized Rarita–Schwinger theory is given in the following paper, where we show that a consistent quantization by the Dirac bracket and path integral methods is possible, with a manifestly positive semidefinite canonical anticommutator in the covariant radiation gauge. Thus, in the massless case, our analysis eliminates the various objections that have been raised to gauging Rarita–Schwinger fields, showing that non-Abelian gauging of Rarita–Schwinger fields can be contemplated as part of the anomaly cancelation mechanism in constructing grand unified models.

We conclude with several remarks:

(1) We have introduced gauge fixing to make the time evolution of the Rarita–Schwinger fields unique, but the analysis of this paper does not *require* gauge fixing. Specifically, if gauge fixing is not imposed, the correct helicity counting is still obtained because fermionic gauge degrees of freedom are redundant degrees of freedom and are not physical. Gauge fixing makes this redundancy manifest by providing a condition that excludes the gauge degrees of freedom, but in analogy to the case of Maxwell electrodynamics, gauge fixing is not needed to get the correct physical state counting. On the other hand, in the following paper, where we turn to quantization, gauge fixing is needed. This can already be anticipated from the form of the constraint matrix N of Eq. (60), which, when gauge fixing is omitted, reduces to the single element $\mathcal{A} = -2ig\vec{\sigma}\cdot\vec{B}(\vec{x})\delta^3(\vec{x}-\vec{y})$ which is not invertible in the small $\vec{B}$ limit. Inversion of the constraint matrix does not enter into the calculations of this paper but is needed in the following paper both for the Dirac bracket and path integral quantization.

(2) A possible exception to the nonperturbative behavior detailed in Sec. VII is when the $\vec{E}$ and $\vec{B}$ gauge fields are random, since if Eq. (77) is replaced by an average, denoted by AV,

$$\langle\Psi_0\rangle_{\rm AV} \simeq \left\langle\frac{\vec{Q}}{(\vec{B})^2}\right\rangle_{\rm AV}\cdot\langle\vec{\Psi}\rangle_{\rm AV}, \tag{122}$$

it becomes

$$\langle\Psi_0\rangle_{\rm AV} \simeq \frac{1}{3}\vec{\sigma}\cdot\langle\vec{\Psi}\rangle_{\rm AV}, \tag{123}$$

which is compatible with $\langle\Psi_0\rangle_{\rm AV} = \vec{\sigma}\cdot\langle\vec{\Psi}\rangle_{\rm AV} = 0$, the customary free Rarita–Schwinger constraints employed in Refs. [11,19]. This heuristic observation suggests that Rarita–Schwinger fields coupled to quantized gauge fields with a zero background gauge field may have a perturbative $g \to 0$ limit.

(3) In showing in the Abelian case that there is no superluminal propagation, the inversion of $\vec{\sigma}\cdot\vec{B}$ to get Ψ_0 only required $(\vec{B})^2 \neq 0$. In the non-Abelian case, where $\vec{B}$ is itself a matrix, the conditions for invertibility are nontrivial and have yet to be analyzed. We will see in the following paper that this issue is side stepped when the constraints are dealt with by the Dirac bracket or path integral procedures, since these do not require the inversion of $\vec{\sigma}\cdot\vec{B}$ when a gauge constraint is included.

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APPENDIX A: NOTATIONAL CONVENTIONS AND USEFUL IDENTITIES

We follow in general the notational conventions of the book *Supergravity* by Freedman and Van Proeyen [19]. The metric $\eta_{\mu\nu}$ is $(-,+,+,+)$, and the Dirac gamma matrices γ_μ, γ^μ obey the Clifford algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}. \tag{A1}$$

They are given in terms of Pauli matrices σ_j by

$$\gamma_0 = -\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad \gamma_j = \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix},\quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A2}$$

We also note that

$$\epsilon_{0123} = -\epsilon^{0123} = 1, \tag{A3}$$

the left chiral projector P_L is given by

$$P_L = \frac{1}{2}(1+\gamma_5), \tag{A4}$$

and the spinor $\bar{\psi}$ is defined in terms of the adjoint spinor $\psi^\dagger$ by

$$\bar{\psi} = \psi^\dagger i\gamma^0. \tag{A5}$$

As noted in Ref. [19], the Rarita–Schwinger equation of motion can be written in a number of equivalent forms. When ordinary derivatives are replaced by gauge covariant derivatives, these are the vector-spinor equations

$$\begin{aligned} \epsilon^{\mu\eta\nu\rho}\gamma_\eta D_\nu\psi_\rho &= 0, \\ \gamma^{\eta\nu\rho}D_\nu\psi_\rho &= 0, \\ \gamma^\rho(D_\nu\psi_\rho - D_\rho\psi_\nu) &= 0, \\ \gamma^\alpha D_\alpha(D_\sigma\psi_\nu - D_\nu\psi_\sigma) &= \gamma^\rho([D_\rho,D_\sigma]\psi_\nu + [D_\nu,D_\rho]\psi_\sigma \\ &\quad + [D_\sigma,D_\nu]\psi_\rho), \end{aligned} \tag{A6}$$

with only the fourth line, which is quadratic in the covariant derivative, involving more than just a substitution $\partial_\nu \to D_\nu$ in the formulas of Ref. [19]. Using $\gamma_\eta\gamma^{\eta\nu\rho} = 2\gamma^{\nu\rho}$, these also imply the spinor equation $\gamma^{\nu\rho}D_\nu\psi_\rho = 0$. These formulas play a role in verifying stress-energy tensor conservation, as does the identity [23]

$$\begin{aligned} 0 = \epsilon^{\lambda\sigma\mu\nu}(&A_\tau B_\lambda C_\sigma D_\mu E_\nu + A_\nu B_\tau C_\lambda D_\sigma E_\mu + A_\mu B_\nu C_\tau D_\lambda E_\sigma \\ &+ A_\sigma B_\mu C_\nu D_\tau E_\lambda + A_\lambda B_\sigma C_\mu D_\nu E_\tau), \end{aligned} \tag{A7}$$

with $A_\tau, B_\lambda, C_\sigma, D_\mu, E_\nu$ five arbitrary four-vectors. This identity follows from

$$0 = \delta^\alpha_\tau\epsilon^{\lambda\sigma\mu\nu} + \delta^\nu_\tau\epsilon^{\alpha\lambda\sigma\mu} + \delta^\mu_\tau\epsilon^{\nu\alpha\lambda\sigma} + \delta^\sigma_\tau\epsilon^{\mu\nu\alpha\lambda} + \delta^\lambda_\tau\epsilon^{\sigma\mu\nu\alpha}, \tag{A8}$$

which is easily verified by noting that $\lambda, \sigma, \mu, \nu$ must take distinct values from the set 0,1,2,3, and that τ must be equal to one of these values.

The fundamental identity for the Pauli matrices is

$$\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c, \tag{A9}$$

with $\epsilon_{123} = 1$ and with the index c summed. We repeatedly use the following two identities that can be derived from Eq. (A9), for a general three-vector $\vec{A}$ that is proportional to a unit matrix in the spinor space and so commutes with $\vec{\sigma}$,

$$\begin{aligned} \vec{\sigma}\times(\vec{\sigma}\times\vec{A}) &= -2\vec{A} + i\vec{\sigma}\times\vec{A}, \\ (\vec{A}\times\vec{\sigma})\times\vec{\sigma} &= -2\vec{A} + i\vec{A}\times\vec{\sigma}. \end{aligned} \tag{A10}$$

Additional useful identities are

$$\begin{aligned} \vec{\sigma}\times\vec{\sigma} &= 2i\vec{\sigma}, \\ \vec{\sigma}\,\vec{\sigma}\cdot\vec{A} &= \vec{A} - i\vec{\sigma}\times\vec{A}, \\ \vec{\sigma}\cdot\vec{A}\,\vec{\sigma} &= \vec{A} + i\vec{\sigma}\times\vec{A}, \\ (\vec{\sigma}\times\vec{A})\cdot\vec{\sigma} &= -2i\vec{\sigma}\cdot\vec{A}, \\ \vec{\sigma}\cdot(\vec{\sigma}\times\vec{A}) &= 2i\vec{\sigma}\cdot\vec{A}, \\ \sigma_a\sigma_b &= 2\left(\delta_{ab} - \frac{1}{2}\sigma_b\sigma_a\right), \\ \vec{B} = i\vec{A} - \vec{A}\times\vec{\sigma} &\leftrightarrow \vec{A} = \frac{1}{2}(\vec{B}\times\vec{\sigma}). \end{aligned} \tag{A11}$$

Gauge field covariant derivatives are

$$D_\mu = \partial_\mu + gA_\mu, \tag{A12}$$

with the gauge potential $A_\mu = A^A_\mu t_A$ and the gauge generators t_A anti-self-adjoint and with the components A^A_μ self-adjoint. The non-Abelian generators t_A obey the compact Lie algebra

$$[t_A, t_B] = f_{ABC}t_C; \tag{A13}$$

in the Abelian case, we replace t_A by $-i$. In writing field strengths $\vec{E}$ and $\vec{B}$, we pull out an additional factor of i to make them self-adjoint, so that we have the identities

$$\begin{aligned} \vec{D}\times\vec{D} &= -ig\vec{B}, \\ [\vec{D}, D_0] &= -ig\vec{E}. \end{aligned} \tag{A14}$$

We will also write a right-acting three-vector covariant derivative as $\vec{D} = \vec{\nabla} + g\vec{A}$ and define a left-acting three-vector covariant derivative as $\overleftarrow{D} = \overleftarrow{\nabla} - g\vec{A}$, so that we have the integration by parts formulas

$$\begin{aligned} \int d^3x A\vec{D}_{\vec{x}}B &= -\int d^3x A\overleftarrow{D}_{\vec{x}}B, \\ \vec{D}_{\vec{x}}\delta^3(\vec{x}-\vec{y}) &= -\delta^3(\vec{x}-\vec{y})\overleftarrow{D}_{\vec{y}}. \end{aligned} \tag{A15}$$

An analogous definition is used for the operators $\vec{L}$ and $\overleftarrow{L}$ which enter the gauge-fixing condition.

At the classical level, variables will be either Grassmann even or odd. Irrespective of the Grassmann parity of monomials A and B, the adjoint operation is defined by [19]

$$(AB)^\dagger = B^\dagger A^\dagger. \tag{A16}$$

For classical brackets, we follow the convention of Henneaux and Teitelboim [24],

$$[F,G]_C = \left(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}\right) + (-)^{\epsilon_F}\left(\frac{\partial^L F}{\partial \theta^\alpha}\frac{\partial^L G}{\partial \pi_\alpha} + \frac{\partial^L F}{\partial \pi_\alpha}\frac{\partial^L G}{\partial \theta^\alpha}\right), \quad (A17)$$

with ϵ_F the Grassmann parity of F, with ∂^L a Grassmann derivative acting from the left, and with q^i, p_i $(\theta^\alpha, \pi_\alpha)$ canonical coordinates and the momenta of even (odd) Grassmann parity. Using the classical bracket, the Dirac bracket is constructed from the constraints as discussed in Sec. II of the following paper. To make the transition to quantum theory, the quantum commutator (anticommutator) is defined to be $i\hbar$ times the corresponding Dirac bracket (with $\hbar = 1$ in our notation). Classical canonical brackets are always denoted, as above, by a subscript C, with a subscript D used for the corresponding Dirac bracket. We use the standard notations $[A,B] = AB - BA$ for the commutator and $\{A,B\} = AB + BA$ for the anticommutator.

To calculate the Dirac bracket, we use block inversion of a matrix. Let

$$M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$
$$M^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \quad (A18)$$

with $A_1, \ldots, A_4$ themselves matrices. Then when A_4 is nonsingular, the blocks $B_1, \ldots, B_4$ of M^{-1} are given by

$$\Delta \equiv A_1 - A_2 A_4^{-1} A_3,$$
$$B_1 = \Delta^{-1},$$
$$B_2 = -\Delta^{-1} A_2 A_4^{-1},$$
$$B_3 = -A_4^{-1} A_3 \Delta^{-1},$$
$$B_4 = A_4^{-1} + A_4^{-1} A_3 \Delta^{-1} A_2 A_4^{-1}. \quad (A19)$$

Even though the blocks are noncommutative, Eqs. (A18) and (A19) give an inverse that obeys $M^{-1}M = MM^{-1} = 1$.

When the constraints ϕ_a and χ_a are combined into an eight element set of constraints $\kappa_a = \phi_a, \kappa_{a+4} = \chi_a$, $a = 1, \ldots, 4$, then the bracket matrix $S_{ab}(\vec{x},\vec{y}) \equiv [\kappa_a(\vec{x}), \kappa_b(\vec{y})]_C$ can be expressed in terms of the matrix $M_{ab}(\vec{x},\vec{y})$ of Eq. (58) as

$$S(\vec{x},\vec{y}) = \begin{pmatrix} 0 & M(\vec{x},\vec{y}) \\ M^T(\vec{y},\vec{x}) & 0 \end{pmatrix}, \quad (A20)$$

where $M^T_{ab}(\vec{x},\vec{y}) = M_{ba}(\vec{x},\vec{y})$ is the matrix transpose. Defining the inverse $M^{-1}(\vec{x},\vec{y})$ by $\int d^3z M^{-1}(\vec{x},\vec{z}) M(\vec{z},\vec{y}) = \int d^3z M(\vec{x},\vec{z}) M^{-1}(\vec{z},\vec{y}) = \delta^3(\vec{x}-\vec{y})$, it is easy to verify that

$$S^{-1}(\vec{x},\vec{y}) = \begin{pmatrix} 0 & M^{T-1}(\vec{y},\vec{x}) \\ M^{-1}(\vec{x},\vec{y}) & 0 \end{pmatrix}. \quad (A21)$$

APPENDIX B: ANALYSIS OF THE RARITA–SCHWINGER FIELD IN AN EXTERNAL ABELIAN GAUGE FIELD: PROPAGATION OF THE LONGITUDINAL GAUGE MODE

We continue here the analysis begun in Sec. V to study propagation of the longitudinal gauge mode. We must now solve for $C_3^{\uparrow,\downarrow}$ starting from Eq. (89) with $C_{1,2} = 0$, so the third line of Eq. (89) simplifies to

$$0 = (\vec{B})^2 \Omega C_3^{\uparrow,\downarrow} - K(Q_3 C_3)^{\uparrow,\downarrow},$$
$$Q_3 = B_1 E_2 - B_2 E_1 + B_3 \vec{\sigma}\cdot(\vec{B} + i\vec{E}) - i\vec{B}\cdot\vec{E}\sigma_3. \quad (B1)$$

Writing this as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} C_3^\uparrow \\ C_3^\downarrow \end{pmatrix}, \quad (B2)$$

we find for the matrix elements

$$U_{11} = (\vec{B})^2\Omega - K[B_1E_2 - B_2E_1 - i(B_1E_1 + B_2E_2) + B_3^2],$$
$$U_{22} = (\vec{B})^2\Omega - K[B_1E_2 - B_2E_1 + i(B_1E_1 + B_2E_2) - B_3^2],$$
$$U_{12} = -KB_3[B_1 + iE_1 - i(B_2 + iE_2)],$$
$$U_{21} = -KB_3[B_1 + iE_1 + i(B_2 + iE_2)]. \quad (B3)$$

The equation $0 = \det(U) = U_{11}U_{22} - U_{12}U_{21}$ reduces, after dividing by an overall factor of $(\vec{B})^2$, to

$$0 = (\vec{B})^2\Omega^2 - 2\Omega K(B_1E_2 - B_2E_1) + K^2(E_1^2 + E_2^2 - B_3^2), \quad (B4)$$

with the solution

$$\frac{\Omega}{K} = \frac{X \pm Y^{1/2}}{(\vec{B})^2},$$
$$X = B_1E_2 - B_2E_1,$$
$$Y = (B_1E_2 - B_2E_1)^2 - (\vec{B})^2(E_1^2 + E_2^2 - B_3^2). \quad (B5)$$

The analysis of the solutions of Eqs. (B4) and (B5) divides into two cases, according to whether the roots of Eq. (B5) are both real or both complex. The roots are both complex if

$$(B_1E_2 - B_2E_1)^2 < (\vec{B})^2(E_1^2 + E_2^2 - B_3^2), \quad (B6)$$

which can be rearranged algebraically to the form

$$[(\vec{B})^2 - (E_1^2 + E_2^2)]B_3^2 < (B_1^2 + B_2^2)(E_1^2 + E_2^2)\cos^2\phi, \tag{B7}$$

where we have written

$$\begin{aligned} B_1E_2 - B_2E_1 &= (B_1^2 + B_2^2)^{1/2}(E_1^2 + E_2^2)^{1/2}\sin\phi, \\ B_1E_1 + B_2E_2 &= (B_1^2 + B_2^2)^{1/2}(E_1^2 + E_2^2)^{1/2}\cos\phi. \end{aligned} \tag{B8}$$

Since the right-hand side of Eq. (B7) is non-negative, when the left-hand side is negative, the inequality is satisfied, and both roots are complex. Hence, a necessary (but not sufficient) condition for both roots to be real is

$$(\vec{B})^2 - (E_1^2 + E_2^2) > 0. \tag{B9}$$

1. Hyperbolic case: Both roots real

When both roots are real, Eq. (B1) describes the hyperbolic case of propagating waves. Introducing the velocity $V = \Omega/K$, Eq. (B4) can be written as

$$0 = (\vec{B})^2V^2 - 2V(B_1E_2 - B_2E_1) + E_1^2 + E_2^2 - B_3^2, \tag{B10}$$

which can be rearranged algebraically to the form

$$\begin{aligned} &[(B_1^2 + B_2^2)^{1/2} - (E_1^2 + E_2^2)^{1/2}]^2 + (\vec{B})^2(V^2 - 1) \\ &\quad = 2(B_1^2 + B_2^2)^{1/2}(E_1^2 + E_2^2)^{1/2}(V\sin\phi - 1). \end{aligned} \tag{B11}$$

Let us now assume that $V^2 > 1$ and show that this leads to a contradiction. When $V^2 > 1$, the left-hand side of Eq. (B11) is non-negative, which implies that $V\sin\phi$ on the right must be non-negative, and so can be replaced by its absolute value. Hence, the right-hand side of Eq. (B11) obeys the inequality

$$\begin{aligned} &2(B_1^2 + B_2^2)^{1/2}(E_1^2 + E_2^2)^{1/2}(V\sin\phi - 1) \\ &\quad = 2(B_1^2 + B_2^2)^{1/2}(E_1^2 + E_2^2)^{1/2}(|V\sin\phi| - 1) \\ &\quad \le 2(\vec{B})^2(|V| - 1), \end{aligned} \tag{B12}$$

where we have used Eq. (B9). But the left-hand side of Eq. (B11) obeys the inequality

$$\begin{aligned} &[(B_1^2 + B_2^2)^{1/2} - (E_1^2 + E_2^2)^{1/2}]^2 + (\vec{B})^2(V^2 - 1) \\ &\quad \ge (\vec{B})^2(|V| + 1)(|V| - 1) > 2(\vec{B})^2(|V| - 1), \end{aligned} \tag{B13}$$

which is a contradiction, since a real number cannot be strictly less than itself. Hence, we must have $V^2 \le 1$, and there is no superluminal propagation.

2. Elliptic case: Both roots complex

When both roots are complex, Eq. (B1) describes the elliptic case in which there are no propagating waves; when a propagating wave enters an elliptic region from a hyperbolic one, it will be damped to zero amplitude. However, in the case of weak damping, one can still define a wave velocity and ask what its magnitude is. When both roots are imaginary, Eq. (B5) takes the form

$$\begin{aligned} \frac{\Omega}{K} &= \frac{X \pm i(-Y)^{1/2}}{(\vec{B})^2}, \\ X &= B_1E_2 - B_2E_1, \\ -Y &= -(B_1E_2 - B_2E_1)^2 + (\vec{B})^2(E_1^2 + E_2^2 - B_3^2). \end{aligned} \tag{B14}$$

Regarding Ω as real and the wave number K as complex, the effective propagation velocity has the magnitude

$$|V_{\rm eff}| = \left|\frac{\Omega}{K_R}\right| = \frac{X^2 - Y}{(\vec{B})^2|X|} = \frac{E_1^2 + E_2^2 - B_3^2}{|B_1E_2 - B_2E_1|}. \tag{B15}$$

The condition for weak damping is $-Y \ll X^2$, which can be rewritten as

$$(\vec{B})^2(E_1^2 + E_2^2 - B_3^2) \ll 2(B_1E_2 - B_2E_1)^2, \tag{B16}$$

and implies

$$|V_{\rm eff}| \ll \frac{2|B_1E_2 - B_2E_1|}{(\vec{B})^2} \le \frac{2|\vec{E}|}{|\vec{B}|}. \tag{B17}$$

Hence, as long as $2|\vec{E}|$ is not much larger than $|\vec{B}|$, which is required by the vacuum stability condition $|\vec{E}| < |\vec{B}|$, the damped wave propagation velocity is subluminal.

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