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Complex Langevin method applied to the 2D $SU(2)$ Yang-Mills theory

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The complex Langevin method in conjunction with the gauge cooling is applied to the two-dimensional lattice $SU(2)$ Yang-Mills theory that is analytically solvable. We obtain strong numerical evidence that at large Langevin time the expectation value of the plaquette variable converges, but to a wrong value when the complex phase of the gauge coupling is large.

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I. INTRODUCTION

As a possible approach to the functional integral with complex measure, such as the one encountered in the finite density QCD [1,2], the complex Langevin method [3–5] has attracted much attention in recent years. This recent interest was triggered mainly by the discovery of sufficient conditions for the convergence of the method to a correct answer [6,7]. Reference [8] is a review on recent developments. Roughly speaking, if the probability distribution of configurations generated by the Langevin dynamics damps sufficiently fast at infinity of configuration space, the statistical average over the configurations is shown to be identical to the integration over the original complex measure. It has been observed that, in systems for which the complex Langevin (CL) method converges to a wrong answer (such as the three-dimensional XY model [9]), this requirement of a sufficiently localized distribution is broken, typically in "imaginary directions" in configuration space.

After the above understanding, a prescription in lattice gauge theory that makes the probability distribution well localized was proposed in Ref. [10]; the prescription is termed "gauge cooling" and it proceeds as follows: The link variables in lattice gauge theory are originally elements of the compact gauge group $SU(N)$. When the (effective) action is complex, however, the corresponding Langevin evolution drives link variables into imaginary directions and link variables become elements of $SL(N,\mathbb{C})$, a *noncompact* gauge group.[1] This evolution tends to make the distribution wide in noncompact directions; in terms of the $SU(N)$ Lie algebra, those noncompact directions are parametrized by imaginary coordinates. At this point, one notes that the definition of a physical observable that is invariant under the original $SU(N)$ gauge transformations can always be tailored so that it is invariant also under the noncompact $SL(N,\mathbb{C})$ gauge transformations. The idea of the gauge cooling is that by applying the $SL(N,\mathbb{C})$ gauge transformations appropriately along the complex Langevin evolution, one squeezes the distribution well localized so that the prerequisite of the convergence theorem [6,7] is fulfiled without changing physical observables.

In a one-dimensional gauge model and in the four-dimensional QCD with heavy quarks, it has been confirmed that the gauge cooling makes the distribution well localized and the complex Langevin method gives rise to correct answers [10]. More recently, this method was applied to the full QCD at finite density [11]. See also Refs. [12–14]. One should note, however, that the Langevin dynamics itself is defined on gauge noninvariant variables (i.e., link variables) and also that the gauge cooling step cannot be regarded as a Langevin evolution that is induced by a *holomorphic* action; the latter is assumed in the convergence theorem [6,7]. Strictly speaking, therefore, the convergence theorem does not apply when the gauge cooling is employed. The method should still be carefully examined in various possible ways.

In the present paper, we apply the complex Langevin method in conjunction with the gauge cooling to the two-dimensional lattice Yang-Mills theory, which can be analytically solved [15–17]. By doing this, we examine the validity of the method. The partition function of the two-dimensional Yang-Mills theory on the lattice is given by

$$\mathcal{Z} = \int \left[\prod_{x,\mu} \mathrm{d}U_{x,\mu}\right] \mathrm{e}^{-S}, \tag{1.1}$$

where $U_{x,\mu}$ are link variables defined on a two-dimensional rectangular lattice, S is the lattice action,[2]

$$S = -\frac{\beta}{2N}\sum_x \mathrm{Tr}[U_{01}(x) + U_{01}(x)^{-1}], \tag{1.2}$$

and the plaquette variable is defined by

$$U_{\mu\nu}(x) = U_{x,\mu} U_{x+\hat{\mu},\nu} U^{-1}_{x+\hat{\nu},\mu} U^{-1}_{x,\nu}. \tag{1.3}$$

For simplicity, we assume that the gauge group is $SU(2)$, that is, $U_{x,\mu} \in SU(2)$ in the original integral (1.1). On the

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[1] We will shortly describe the Langevin evolution of link variables.

[2] Throughout this paper, $N = 2$.

other hand, when the gauge coupling β is complex, the corresponding Langevin equation [Eq. (2.1) below] evolves link variables as elements of $SL(2,\mathbb{C})$. Thus, the distinction between $U_{x,\mu}^{\dagger}$ and $U_{x,\mu}^{-1}$ becomes very important in the complex Langevin dynamics. For the convergence theorem in Refs. [6,7] to apply, the action S that generates the drift force in the Langevin equation and physical observables must be a holomorphic function of dynamical variables; our above definitions (1.2)–(1.3) that entirely use $U_{x,\mu}^{-1}$ not $U_{x,\mu}^{\dagger}$ are chosen by this criterion. Note also that the plaquette action (1.2) is invariant under the $SL(2,\mathbb{C})$ lattice gauge transformations [such as the one in Eq. (2.4)].

We consider the expectation value of the plaquette variable:

$$\langle \mathrm{Tr}[U_{01}(x)]\rangle = \frac{1}{\mathcal{Z}}\int\left[\prod_{x,\mu}\mathrm{d}U_{x,\mu}\right]\mathrm{e}^{-S}\mathrm{Tr}[U_{01}(x)]. \tag{1.4}$$

Even if the gauge coupling β is complex, this can be exactly computed by the character expansion [15–17]. Under periodic boundary conditions, one yields

$$\langle \mathrm{Tr}[U_{01}(x)]\rangle = -\frac{N}{V}\frac{\partial}{\partial\beta}\ln\mathcal{Z}, \qquad \mathcal{Z} = \sum_{n=1}^{\infty}\left[\frac{2}{\beta}I_n(\beta)\right]^V, \tag{1.5}$$

where $I_n(x)$ denotes the modified Bessel function of the first kind and V is the number of lattice points.

II. COMPLEX LANGEVIN METHOD AND THE GAUGE COOLING

The following procedures are basically identical to the ones adopted for the four-dimensional lattice QCD in Ref. [11] for example, although our two-dimensional pure-gauge system is much simpler.

For the link variable, the Langevin equation with a discretized Langevin time t with the time step ϵ is defined by

$$U_{x,\mu}(t+\epsilon) = \exp\left[i\sum_a\lambda_a(\sqrt{\epsilon}\eta_{a,x,\mu}(t) - \epsilon D_{a,x,\mu}S)\right]U_{x,\mu}(t), \tag{2.1}$$

where λ_a $(a = 1,\ 2,\ 3)$ are Pauli matrices, $\eta_{a,x,\mu}(t)$ are Gaussian real random numbers of the variant

$$\langle\eta_{a,x,\mu}(t)\eta_{b,y,\nu}(t')\rangle = 2\delta_{ab}\delta_{xy}\delta_{\mu\nu}\delta_{tt'}, \tag{2.2}$$

and $D_{a,x,\mu}S$ is the drift force generated by the action S in Eq. (1.2); the derivative with respect to the link variable is given by

$$D_{a,x,\mu}f(U) = \partial_\xi f(\mathrm{e}^{i\xi\lambda_a}U_{x,\mu})|_{\xi=0}. \tag{2.3}$$

When the gauge coupling β is complex, the drift force becomes complex and the Langevin evolution evolves link variables as elements of $SL(2,\mathbb{C})$.

The above complex Langevin dynamics tends to make the probability distribution function of link variables wide in noncompact directions of $SL(2,\mathbb{C})$. To squeeze the distribution well localized without changing gauge invariant quantities, we apply the following $SL(2,\mathbb{C})$ gauge transformation (this step is the gauge cooling)

$$U_{x,\mu} \to U'_{x,\mu} = V_x U_{x,\mu} V_{x+\hat{\mu}}^{-1}, \tag{2.4}$$

where

$$V_x = \mathrm{e}^{-\epsilon\alpha f_a^x\lambda_a},$$
$$f_a^x = 2\mathrm{Tr}\left[\lambda_a\sum_\mu(U_{x,\mu}U_{x,\mu}^{\dagger} - U_{x-\hat{\mu},\mu}^{\dagger}U_{x-\hat{\mu},\mu})\right], \tag{2.5}$$

and $\alpha > 0$ is a real parameter. The distance defined by [18],

$$d = \frac{1}{V}\sum_{x,\mu}\frac{1}{N}\mathrm{Tr}(U_{x,\mu}U_{x,\mu}^{\dagger} - \mathbb{1}) \geq 0, \tag{2.6}$$

measures how a $SL(2,\mathbb{C})$ gauge field is far away from the subspace of $SU(2)$ gauge fields. It is then straightforward to see that for a sufficiently small ϵ, the gauge cooling (2.4) decreases or does not change the distance d. Note that f_a^x in Eq. (2.5) is not a holomorphic function of link variables and thus the step (2.4) cannot be regarded as a part of the complex Langevin dynamics in which the drift force is generated by a holomorphic action; this fact prevents us from applying the convergence theorem [6,7] to the above procedures.

III. RESULT OF NUMERICAL SIMULATIONS

We numerically solved the Langevin equation with the discretized Langevin time, Eq. (2.1), on a $V = 4^2$ lattice. Periodic boundary conditions are imposed. The maximal size of the time step ϵ we adopted was 0.001 and, when the drift force becomes large, we further reduced ϵ "adaptively" according to the prescription in Ref. [19].

As the parameter α in the gauge cooling (2.5), we tried both $\alpha = 1$ and the adaptive choice (see Ref. [8])

$$\alpha_{\mathrm{ad}} = \frac{1}{D}, \qquad D \equiv \frac{1}{V}\sum_{a,x}|f_a^x| + 1. \tag{3.1}$$

Our numerical results did not show any notable difference in these two choices and we present the results with the latter choice in what follows.

To determine an appropriate rate of the gauge cooling (2.4) along the Langevin evolution, we observed the time evolution of the distance d (2.6) by changing the number of the gauge cooling steps per one Langevin update (2.1).

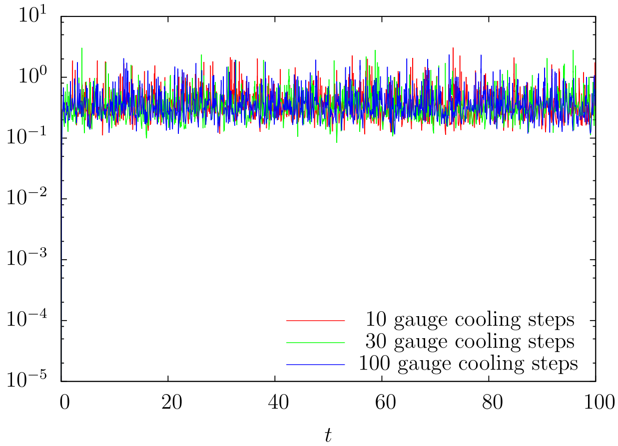

FIG. 1 (color online). Evolution of the distance d (2.6) for $\beta = 0.4 + 2.0i$ with various numbers of the gauge cooling steps per one Langevin update (2.1), 10, 30, and 100.

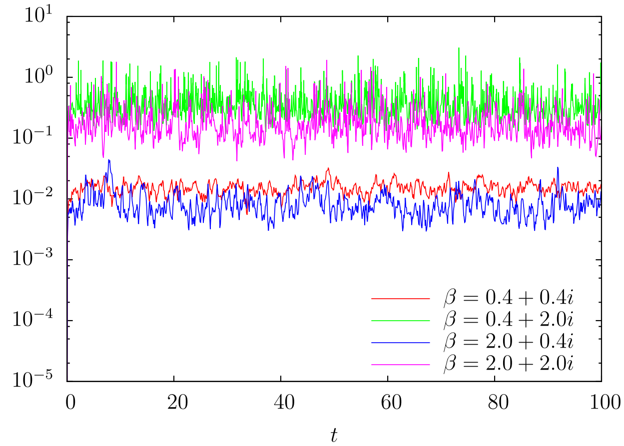

FIG. 2 (color online). Evolution of the distance d (2.6) for various complex gauge couplings, $\beta = 0.4 + 0.4i$, $0.4 + 2.0i$, $2.0 + 0.4i$, and $2.0 + 2.0i$. The number of the gauge cooling steps per one Langevin update (2.1) is 10.

In Fig. 1, for $\beta = 0.4 + 2.0i$, we plotted the evolution of the distance d as the function of the Langevin time t by changing the number of the gauge cooling steps per one Langevin update as 10, 30, and 100.[3] Since this plot shows the evolution including the Langevin stochastic dynamics, the distance d does not necessarily decrease. Since we do not see much difference for those three choices, we adopted ten gauge cooling steps per one Langevin update. With this choice, the evolution of the distance d for various complex gauge couplings, $\beta = 0.4 + 0.4i$, $0.4 + 2.0i$, $2.0 + 0.4i$, and $2.0 + 2.0i$, looks as depicted in Fig. 2. It appears that the gauge cooling is working perfectly for those complex gauge couplings, suppressing the evolution to noncompact imaginary directions.

Now we turn to the computation of the expectation value of the plaquette, Eq. (1.4), by the complex Langevin method. Starting from a configuration of random $SU(2)$ matrices, we discarded configurations until the Langevin time $t = 11$ for thermalization. Then 1000 configurations separated by $\Delta t = 1$ from $t = 11$ to $t = 1010$ are used to compute the expectation value. For typical values of the complex gauge coupling, we confirmed that the plaquette values between configurations separated by $\Delta t = 1$ practically have no autocorrelation. In Figs. 3 and 4, we plotted the real and imaginary parts of the expectation value (1.4) obtained by the CL method. The error bars are statistical ones. The horizontal axis is the the complex phase θ of the gauge coupling[4] with the modulus 1.5:

$$\beta = 1.5e^{i\theta}, \qquad 0 \leq \theta \leq \pi/2. \tag{3.2}$$

[3]We define the Langevin time t such that it does not elapse during the gauge cooling steps.

[4]When the gauge group is $SU(2)$, the partition function (1.1) is invariant under $\beta \to -\beta$. Because of this invariance and the complex conjugation, it is sufficient to consider the range of the phase, $0 \leq \theta \leq \pi/2$.

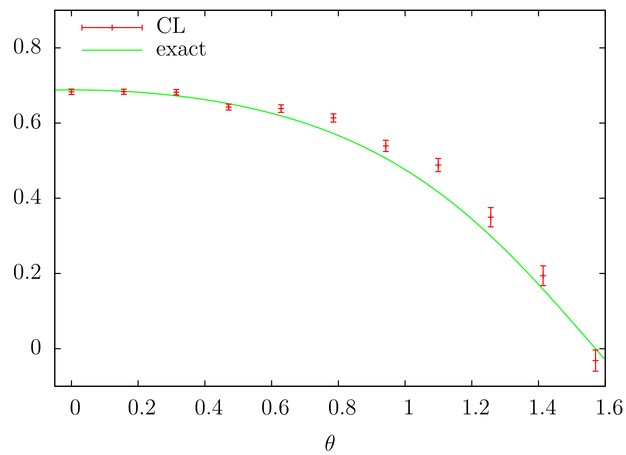

FIG. 3 (color online). Real part of the expectation value (1.4) obtained by the CL method and the exact value given by Eq. (1.5). The horizontal axis is the complex phase θ of the gauge coupling in Eq. (3.2).

The solid line curves are exact values given by Eq. (1.5). We see that the complex Langevin method reproduces the real part fairly well, while it clearly fails to converge to the correct value of the imaginary part when the complex phase of the gauge coupling is large.

The gradation plot in Fig. 5 shows the relative error

$$\frac{|\langle \mathrm{Tr}[U_{01}(x)]\rangle_{\mathrm{CL}} - \langle \mathrm{Tr}[U_{01}(x)]\rangle_{\mathrm{exact}}|}{|\langle \mathrm{Tr}[U_{01}(x)]\rangle_{\mathrm{exact}}|} \tag{3.3}$$

on the first quadrant of the complex β plane.[5] The (quadrant) circle in the figure is Eq. (3.2) along which Figs. 3 and 4 are

[5]The block around the origin $\beta = 0$ is omitted from the plot because $\langle \mathrm{Tr}[U_{01}(x)]\rangle \sim 0$ for $\beta \sim 0$.

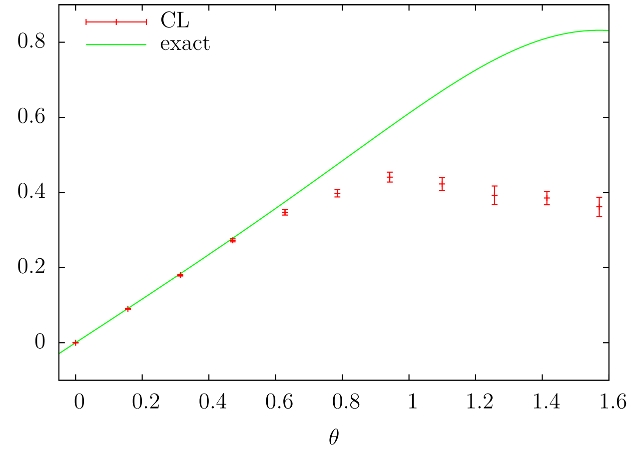

FIG. 4 (color online). Imaginary part of the expectation value (1.4) obtained by the CL method and the exact value given by Eq. (1.5). The horizontal axis is the complex phase θ of the gauge coupling in Eq. (3.2).

plotted. Clearly, the relative error of the complex Langevin method becomes large when the complex phase of the gauge coupling becomes large. Four black crosses in the figure indicate complex gauge couplings we used in Fig. 2; the behavior in Fig. 2 thus suggests that the gauge cooling is correctly working for the region of the complex gauge coupling shown in Fig. 5. Nevertheless, the complex Langevin method shows large deviation from the correct value as in Fig. 4. This is the main result of the present paper.

A similar failure of the complex Langevin method for large complex β has been observed; see Fig. 8 of Ref. [20]. This result of Ref. [20] is, however, for a one-dimensional integral (not a gauge theory) and the validity of the gauge cooling, which is our main issue in this paper, is not relevant to this result of Ref. [20].

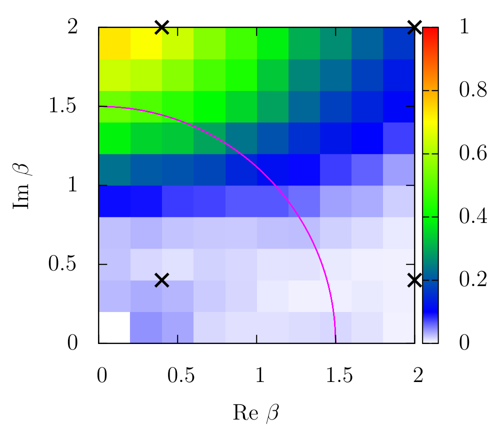

FIG. 5 (color online). Gradation plot of the relative error (3.3) on the complex β (the gauge coupling) plane. The block around the origin $\beta = 0$ is omitted from the plot. The quadrant is Eq. (3.2) along which Figs. 3 and 4 are plotted. Four black crosses indicate complex gauge couplings we used in Fig. 2.

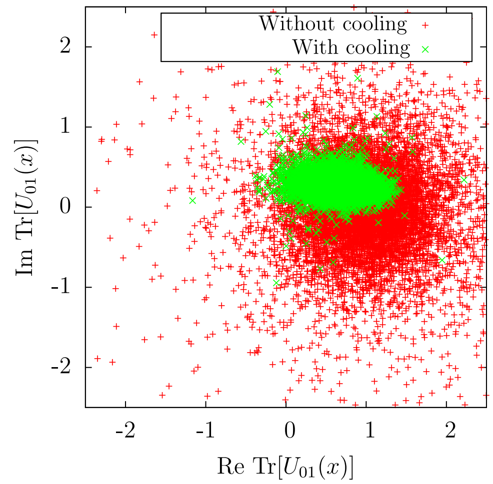

FIG. 6 (color online). Distribution of the plaquette variable averaged over the lattice volume. $\beta = 1.5e^{i(0.3\pi/2)}$ ($\theta = 0.3\pi/2$).

It is of interest how configurations generated by the Langevin dynamics distribute in configuration space. To give some idea on this point, in Figs. 6 and 7, we present scatter plots of the plaquette variable (averaged over the lattice volume) for each configuration. Both cases, with and without the gauge cooling, are shown. Figure 6 is for $\beta = 1.5e^{i(0.3\pi/2)}$ (i.e., $\theta = 0.3\pi/2$) and corresponds to points in Figs. 3 and 4 with a relatively small complex phase. Figure 7 is, on the other hand, for $\beta = 1.5e^{i(0.7\pi/2)}$ (i.e., $\theta = 0.7\pi/2$) and corresponds to points in Figs. 3 and 4 with a large complex phase and with large deviation. Although there is a tendency when the complex phase of the gauge coupling is large for the distribution to become somewhat wider even after the gauge cooling, it is not clear from the scatter plot in Fig. 4 alone whether the distribution is so poorly localized as to break the prerequisite of the convergence theorem [6,7]. More detailed study is needed on this point.

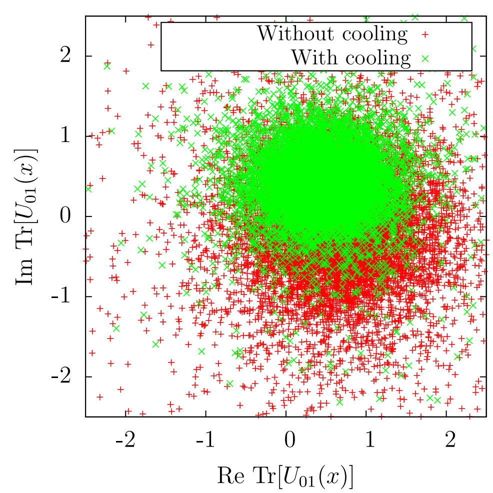

FIG. 7 (color online). Distribution of the plaquette variable averaged over the lattice volume. $\beta = 1.5e^{i(0.7\pi/2)}$ ($\theta = 0.7\pi/2$).

IV. CONCLUSION

In the present paper, we applied the complex Langevin method in conjunction with the gauge cooling to the two-dimensional lattice $SU(2)$ Yang-Mills theory. Our intention was to examine the validity of the method by using this analytically solvable model. Somewhat unexpectedly, as shown in Figs. 4 and 5, we obtained strong numerical evidence that the method fails to converge to the correct value when the complex phase of the gauge coupling is large. As we emphasized in the introduction, the convergence proof of Refs. [6,7] does not necessarily apply when the gauge cooling is employed; thus there is no contradiction even if the method leads to a wrong answer. Nevertheless, it is not yet clear what causes the failure for the gauge coupling with a large complex phase. To find the resolution of the problem we found in the present study, first we have to pin down what the real source of the failure is. For this, consideration on the basis of another approach to the functional integral with complex measure, the Lefschetz thimble [21–25], might provide useful insight. See also Ref. [26] for suggestive observations.

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