

Entanglement temperature for the excitation of SYM theory in the (de)confinement phase

Kazuo Ghoroku^{1,*} and Masafumi Ishihara^{2,†}

¹*Fukuoka Institute of Technology, Wajiro, Higashi-ku Fukuoka 811-0295, Japan*

²*WPI-Advanced Institute for Materials Research (WPI-AIMR), Tohoku University, Sendai 980-8577, Japan*

(Received 26 June 2015; published 14 October 2015)

We study the holographic supersymmetric Yang-Mills (SYM) theory, which is living in a hyperbolic space, in terms of the entanglement entropy. The theory contains a parameter C corresponding to the excitation of the SYM theory, and it controls the dynamical properties of the theory. The entanglement temperature, T_{ent} , is obtained by imposing the thermodynamic law for the relative entanglement entropy and the energy density of the excitation. This temperature is available at any value of the parameter C even in the region where the Hawking temperature disappears. With this new temperature, the dynamical properties of the excited SYM theory are examined in terms of the thermodynamic law. We could find the signatures of phase transitions of the theory.

DOI: [10.1103/PhysRevD.92.085017](https://doi.org/10.1103/PhysRevD.92.085017)

PACS numbers: 12.38.-t, 25.75.Nq, 11.25.Tq, 04.60.-m

I. INTRODUCTION

The holographic approach is a powerful method to study the nonperturbative properties of the strong coupling gauge theories [1–3]. In this context, various attempts have been performed to study the properties of the supersymmetric Yang Mills (SYM) theory in the confinement phase. Recently, the quantum information of strong coupling theory has been studied through the holographic entanglement entropy (S_{EE}), which is very useful to investigate the theory from the thermodynamic viewpoint by supposing the thermodynamic law shown at high temperature [4–15].

As shown in [4,5], S_{EE} is obtained by separating the space to two regions A and its complement \bar{A} as follows:

$$S_{\text{EE}} = \frac{\text{Area}(\gamma_A)}{4G_N^{(5)}}, \quad (1.1)$$

where γ_A denotes the minimal surface whose boundary is defined by ∂A and the surface is extended into the bulk. $G_N^{(5)} = G_N^{(10)}/(\pi^3 R^5)$ denotes the five-dimensional Newton constant reduced from the ten-dimensional one $G_N^{(10)}$. The area is given as

$$\text{Area}(\gamma_A) \equiv S_{\text{Area}}(G^{(0)}, X^{\text{ext}}) = \int_{\gamma_A} d^d \xi \sqrt{g}, \quad (1.2)$$

where the induced metric g_{ab} on γ_A are defined as

$$g = \det(g_{ab}), \quad g_{ab} = G_{MN} \frac{\partial X^M}{\partial \xi^a} \frac{\partial X^N}{\partial \xi^b}. \quad (1.3)$$

The minimal surface γ_A is expressed by the profile X^{ext} , which is embedded in the bulk background defined by $G^{(0)}$. This formulation has been extended to the nonconformal case in terms of the string frame metric by including nontrivial dilaton [15]. Here, we consider the case of the trivial dilaton and the five-dimensional compact space of S^5 . Therefore, the above formula is enough.

At high temperature in the deconfinement phase, it is well known that the entanglement entropy obtained as above approaches the thermal entropy, which satisfies the Bekenstein-Hawking relation, in the limit of large area for the considered system. This fact is convinced in the high temperature SYM theory, which is dual to the AdS₅-Schwarzschild gravity, and the temperature is defined by the Hawking temperature in this case.

On the other hand, in general, the temperature cannot be defined as the Hawking temperature in the confinement phase. S_{EE} is however calculable by using the same formula with (1.1). In other words, the above formula with (1.1) for S_{EE} is useful both in the confinement and the deconfinement phases. In order to see the thermodynamical properties clearly, the calculations are performed for the large area limit of γ_A for the theory which contains a parameter C corresponding to the excitation of the theory from its vacuum state.¹ The energy momentum tensor of this theory is described by this parameter. By the holographic renormalization method [20–22], it has been given in [23]. Using this energy density, the entanglement temperature (T_{ent}) for the excited state is calculated according to the method given in [10]. T_{ent} can be defined even if the state is in the

¹This parameter has been first introduced in the scenario of the brane world [16–19]. However, its role in the present holographic case is different from the case of the brane world.

*gouroku@dontaku.fit.ac.jp

†masafumi@wpi-aimr.tohoku.ac.jp

confinement phase as far as the excitation due to C exists. Therefore the properties of the theory can be investigated thermodynamically by using this new temperature T_{ent} for over all regions of the parameter C .

Our purpose is to investigate the dynamical properties of the excited SYM theories, which could have a rich phase structure, by using the thermodynamic laws represented by the temperature T_{ent} . The merit of using this temperature is that T_{ent} is available in all the regions of the excitation parameter. Through the analysis given here, we could show the important signs of the phase transitions, which have been indicated by performing the nonthermodynamic holographic analysis [23,24]. This indicates that the entanglement temperature T_{ent} is useful to study the dynamics of the excited state thermodynamically even if the Hawking temperature disappears.

The outline of this paper is as follows. In the next section, how T_{ent} is defined is explained. In the Sec. III, the holographic SYM theory with the parameter C , which is mentioned above, is reviewed, and then the energy momentum tensors are given. In the Sec. IV, the entanglement entropy is calculated in our model and discussed in many points through an approximate form. In Sec. V, entanglement temperature T_{ent} is given. Then its meaning and the thermodynamic investigations for the two phase transitions of the theory are discussed. The summary and discussions are given in the final section.

II. ENTANGLEMENT TEMPERATURE

The thermodynamic relation of S_{EE} and the energy density of the system has been related by introducing the modular Hamiltonian (H) as [8]

$$\Delta S_{\text{EE}} = \Delta H, \quad (2.1)$$

$$\rho = e^{-H}, \quad (2.2)$$

where H is defined as above by the density matrix ρ which determines the entanglement entropy as $S_{\text{EE}} = -\text{Tr}\rho \ln \rho$.

In the above, usually, the infinitesimal increasing of S_{EE} and H are given as

$$\begin{aligned} \Delta S_{\text{EE}} &= S_{\text{EE}}(G^{(0)} + \delta G, X^{(0)}) - S_{\text{EE}}(G^{(0)}, X^{(0)}) \\ &= \int d^d \xi \sqrt{g} g^{ij} \delta g_{ij}, \end{aligned} \quad (2.3)$$

for S_{EE} , and for the modular Hamiltonian,

$$\Delta H = \int d^d \xi \sqrt{g^{(0)}} \beta \delta \langle T_{00} \rangle, \quad (2.4)$$

where $G^{(0)}$ denotes a solution of the $(d+1)$ -dimensional bulk gravity which is dual to the corresponding d -dimensional field theory, and $X^{(0)}$ represents the profile X^{ext} of the minimal surface embedded in the bulk

determined by $G^{(0)}$. The energy density $\langle T_{00} \rangle$ of the boundary theory is obtained according to the holographic method for a given $G^{(0)}$ [20,21]. The metric on the boundary is denoted by $g^{(0)} (\neq g)$. Do not confuse $g^{(0)}$ with the induced metric g here. The factor β is introduced as the temperature $\beta = 1/T$. However, in this formulation, it may depend on the coordinates on the boundary and the shape of ∂A as shown in the case of the CFT.

Actually, in the case that $G^{(0)}$ is given by AdS₅, the temperature $1/\beta$ is obtained as follows according to [14]. The modular Hamiltonian for a ball-shaped region with radius l in the Minkowski space is given as [8]

$$H = 2\pi \int d^d x \frac{l^2 - r_d^2}{2l} T^{00}(x), \quad (2.5)$$

where $r_d = \sqrt{(x^1)^2 + \dots + (x^d)^2}$. This implies the temperature defined above as

$$\beta \equiv \frac{1}{T_{\text{ent}}} = 2\pi \frac{l^2 - r_d^2}{2l} \quad (2.6)$$

for this CFT case. Both T_{ent} and T^{00} are dependent on the coordinate on the boundary. Then it is difficult to imagine a thermodynamic picture for the deviation $\delta g_{\mu\nu}$ which is in general a complicated function of coordinates.

We notice that the above deviations of S_{EE} and H are obtained in the linear order of δG . In this case, it has been shown that the relation (2.1) is always satisfied when δG satisfies the linearized five-dimensional Einstein equation under the background $G^{(0)}$ [13].

On the other hand, consider a global excitation as studied in [10],

$$\langle T_{00} \rangle = mR^3 / (4\pi G_N^{(5)}), \quad (2.7)$$

where m denotes a parameter corresponding to the excitation in the vacuum four-dimensional Minkowski space-time. In this case, the bulk metric near the boundary ($z \sim 0$) is given as

$$\begin{aligned} ds^2 &= \left(\frac{R}{z}\right)^2 \left(-\frac{1}{f(z)} dt^2 + f(z) dz^2 + \sum_{i=1}^3 dx^{i2}\right), \\ f(z) &= 1 + mz^4 + \dots \end{aligned} \quad (2.8)$$

Then consider the deviations of S_{EE} and H for small m within the linear order of m (not for the metric deviation δG). In this case, we obtain

$$\begin{aligned} \Delta S_{\text{EE}} &= \partial_m S_{\text{EE}}(G^{(0)}(m), X^{(0)})|_{m=0} \\ &= \beta \int d^d x \sqrt{g^{(0)}} \partial_m \langle T_{00} \rangle|_{m=0}, \end{aligned} \quad (2.9)$$

where $\beta = 1/T_{\text{ent}}$ is written outside of the integration by assuming it as a constant. This assumption seems to be

reasonable since the deviation of (2.7) is independent of the boundary coordinates. This implies that the excitation of the system can be observed uniformly through the global temperature T_{ent} which is independent of the coordinate. We should say however that the entanglement temperature β would generally depend on the choice of the entanglement region. So the above setting of Eq. (2.9) would be restricted to some special cases of the excitation.

Within the linear approximation for the parameter m , it is possible to estimate T_{ent} . For the case of a ball-shaped region with radius l of the Minkowski space, it is given for small l as follows [10],

$$T_{\text{ent}} = \frac{5}{2\pi} \frac{1}{l}. \quad (2.10)$$

For more general cases of global $\langle T_{\mu\nu} \rangle$, a similar evaluation of T_{ent} is obtained for small l [10–12].

For large l , however, it is necessary to give the metric form at large z up to the deep infrared region. Furthermore, we need to obtain the profile function of the minimal surface in the infrared region. Up to now, there has been little research in this direction. Our main purpose is to extend this approach to the infrared region or to the large size area. We define the temperature T_{ent} at any value of the parameter m which expresses the excitation of the system.² In the above CFT example, the excitation is seen from the vacuum, $m = 0$, to the excited state with small m . So T_{ent} can be defined at $m = 0$ as a limit of $m \rightarrow 0$. In our approach, on the other hand, we could define the temperature $T_{\text{ent}}(m)$ at any m by comparing $S_{\text{EE}}(m)$ and $S_{\text{EE}}(m + \delta m)$. As explained below, this is possible since we have a holographic solution in which the parameter m is arbitrary. Thus we can study the thermal properties of the excited state at any value of m .

The definition of the entanglement temperature $T_{\text{ent}}(\alpha)$ is performed by using the relative entropy and the thermodynamic first law as follows. Here the parameter m is generalized to α , and then the energy momentum of the excitation is denoted as $\langle T_{\mu\nu}(\alpha) \rangle$. The details of this formulation are seen in [8,13] for the case of CFT. The relative entropy $S_{\text{EE}}(\rho_1|\rho_0)$ is related to ΔH and ΔS_{EE} as follows [9,10],

$$S(\rho_1|\rho_0) = \Delta H - \Delta S \geq 0, \quad (2.11)$$

for two density matrices ρ_1 and ρ_0 . Consider the case that the density matrices introduced above are characterized by the parameters α as follows:

$$\rho_1 = \rho(\alpha_1), \quad \rho_0 = \rho(\alpha_0), \quad (2.12)$$

where we suppose

²We notice that the parameter m is used here as a symbolic quantity of the excitation.

$$\alpha_1 = \alpha_0 + \delta\alpha. \quad (2.13)$$

In the case of infinitesimally small $\delta\alpha$, we find the following relation,

$$\Delta S_{\text{EE}} = \partial_\alpha S_{\text{EE}} = \Delta H \quad (2.14)$$

at $\alpha = \alpha_0$, namely in the limit of $\delta\alpha = 0$. Further, supposing that the parameter α is global, we can set the following relation,

$$\begin{aligned} \Delta H &= \int d^d \xi \sqrt{g^{(0)}} \beta \delta \langle T_{00} \rangle \\ &= \beta(\alpha) \int d^d \xi \sqrt{g^{(0)}} \partial_\alpha \langle T_{00} \rangle |_{\alpha=\alpha_0}, \end{aligned} \quad (2.15)$$

which has the same form with (2.4). We notice that the second equation of (2.15) is obtained supposing the uniformity of β according to the above equation (2.9). This setting seems to be consistent with our analysis given here.³

Finally, we arrive at the following formula for the entanglement temperature:

$$T_{\text{ent}}^\alpha(\alpha_0) = \left. \frac{\int d^d \xi \sqrt{g^{(0)}} \frac{\partial \langle T_{00} \rangle}{\partial \alpha}}{\frac{\partial S}{\partial \alpha}} \right|_{\alpha=\alpha_0}. \quad (2.16)$$

As mentioned above, notice that this temperature does not depend on the coordinate but does depend on the parameter, α_0 , which determines the excited state of the theory. Furthermore, the above formula (2.16) is available at any α_0 . So it is possible to obtain $T_{\text{ent}}(\alpha)$ as a function of α .

III. GRAVITY DUAL OF EXCITED SYM THEORY

A. Model

The holographic dual to the large N gauge theory embedded in Friedmann-Robertson-Walker (FRW) space-time with two parameters is given as the following form of ten-dimensional metric [23],

$$\begin{aligned} ds_{10}^2 &= \frac{r^2}{R^2} (-\bar{n}^2 dt^2 + \bar{A}^2 a_0^2(t) \gamma_{ij}(x) dx^i dx^j) \\ &\quad + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2. \end{aligned} \quad (3.1)$$

$$\gamma_{ij}(x) = \delta_{ij} \left(1 + k \frac{\bar{r}^2}{4\bar{r}_0^2} \right)^{-2}, \quad \bar{r}^2 = \sum_{i=1}^3 (x^i)^2, \quad (3.2)$$

³For our present case, the ingredients to realize the Eq. (2.15) are considered as the following. The excitation parameter is a global constant and then the energy density of the excitation is also global. Furthermore, a large volume limit of entanglement region is considered here.

where $k = \pm 1$, or 0. The scale parameter of three space is denoted by r_0 . The solution is obtained within the ten-dimensional supergravity of type IIB theory as follows:

$$\bar{A} = \left(\left(1 + \left(\frac{r_0}{r} \right)^2 \right)^2 + \left(\frac{b_0}{r} \right)^4 \right)^{1/2}, \quad (3.3)$$

$$\bar{n} = \frac{(1 + (\frac{r_0}{r})^2)^2 - (\frac{b_0}{r})^4}{\sqrt{\bar{A}}}, \quad (3.4)$$

$$r_0 = \frac{R^2}{2} \sqrt{|\lambda|}, \quad b_0 = \frac{R}{a_0} \left(\frac{CR^2}{4} \right)^{1/4}. \quad (3.5)$$

Two dimensionful parameters, λ and C , are introduced in solving the equation of motion. The parameter C , which is called the ‘‘dark radiation,’’ is introduced as an integration constant. On the other hand, the dark energy λ , which corresponds to the four-dimensional cosmological constant, is introduced by the following relation,

$$\left(\frac{\dot{a}_0}{a_0} \right)^2 + \frac{k}{a_0^2} = \lambda, \quad (3.6)$$

in solving the bulk Einstein equation. We should notice that the above equation is not introduced to solve the four-dimensional Einstein equations with the four-dimensional cosmological constant. Although the value of λ is arbitrary, the above bulk solution is considered for the negative value of $\lambda (= -|\lambda| < 0)$ in order to study the parameter region where the phase transition occurs.

Here we comment on the time dependence of the scale factor $a_0(t)$. Its time dependent form is given by solving (3.6). In our analysis, we consider the case of very small time derivative of $a_0(t)$ for simplicity. For the sake of the justification of our assumption for $a_0(t)$, we should say that the solution of constant a_0 is supposed here as $a_0 = 1/\sqrt{|\lambda|}$, which is allowed for negative constant λ when we take $k = -1$.

Then the theory is considered in the three-dimensional hyperbolic space. In this case, for a fixed $|\lambda|$, the theory shows two phase transitions with increasing C [25,26]. At small C , the theory is in the confinement and broken chiral symmetry phase [(A)]. With increasing C , deconfinement and broken chiral symmetry phase [(B)] appears. Finally deconfinement and restoration of the chiral symmetry phase [(C)] is realized. In terms of (r_0, b_0) , these phases are assigned as

$$\begin{aligned} & \text{(A)} \quad b_0 < r_0, \quad \text{(B)} \quad r_0 < b_0 < 1.31r_0, \\ & \text{(C)} \quad 1.31r_0 < b_0. \end{aligned} \quad (3.7)$$

The transition from (B) to (C) has been discussed in [26]. Then we expect that the dynamical properties of each phase of the theory would be observed also as thermodynamic

properties in terms of the entanglement temperature which is defined by using the excitation corresponding to C .

B. Energy momentum tensor and meaning of C

For later convenience, we show the four-dimensional stress tensor of the dual field theory for the present model. It has been given in [23], according to the holographic renormalization method [20,21] based on the Fefferman-Graham framework [20–22]. We obtain the following results,

$$\langle T_{\mu\nu} \rangle = \langle \tilde{T}_{\mu\nu}^{(0)} \rangle + \frac{4R^3}{16\pi G_N^{(5)}} \left\{ \frac{3\lambda^2}{16} (1, -g_{(0)ij}) \right\}. \quad (3.8)$$

$$\langle \tilde{T}_{\mu\nu}^{(0)} \rangle = \frac{4R^3}{16\pi G_N^{(5)}} \frac{\tilde{c}_0}{R^4} (3, g_{(0)ij}), \quad (3.9)$$

where $G_N^{(5)} = 8\pi^3 \alpha'^4 g_s / R^5$, $R^4 = 4\pi N \alpha'^2 g_s$ and $g_{(0)ij}$ denotes the three-dimensional metric on the boundary. The first part, $\langle \tilde{T}_{\mu\nu}^{(0)} \rangle$, comes from the conformal YM fields given in [27]. The second term corresponding to the loop corrections of the YM fields leads to the conformal anomaly as follows:

$$\langle T_{\mu}^{\mu} \rangle = -\frac{3\lambda^2}{8\pi^2} N^2. \quad (3.10)$$

Next, we notice the holographic meaning of the dark radiation C . The situation is different from the case of the brane cosmology. Its meaning is clearly understood at $|\lambda| = 0$ or $r_0 = 0$, where we find the AdS-Schwarzschild metric. Then the Hawking temperature is found as

$$T_H|_{r_0=0} \equiv T_H^{(0)} = \frac{\sqrt{2}b_0}{\pi R^2}. \quad (3.11)$$

The energy density is given as

$$\rho = \langle T_{00} \rangle = \frac{3N^2}{8\pi^2} T_H^{(0)4}, \quad (3.12)$$

which represents the Stefan-Boltzmann law of the radiation. This implies that C corresponds to the thermal radiation of SYM fields in the four-dimensional Minkowski space-time.

In the present case, however, we are considering the SYM theory in the four-dimensional curved space-time which is characterized by λ . Then the relations (3.11)–(3.12) are modified by the curvature. As a result, the meaning of C might be changed. In the deconfinement phases, (B)–(C) with $|\lambda| \neq 0$, C still represents the thermal SYM fields. However the formula (3.11) is modified as

$$T_H = \frac{\sqrt{2}b_0}{\pi R^2} \sqrt{1 - (r_0/b_0)^2}. \quad (3.13)$$

In the confinement phase (A), on the other hand, the temperature T_H disappears even if $C \neq 0$. Then we expect that the glueball-like matter may be represented by C . We could find a hint for this expectation by studying the entanglement entropy in these different phases by changing the value of C .

IV. ENTANGLEMENT ENTROPY

In order to estimate T_{ent} , we first examine the entanglement entropy for a sphere with radius p_0 .

A. Minimal surface

For the case of the present holographic theory, from (3.1), the spatial part of the bulk metric is rewritten as

$$ds_{\text{space}}^2 = \frac{1}{R^2} \left(r^2 + 2r_0^2 + \frac{r_c^4}{r^2} \right) ds_{\text{FRW}_3}^2 + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_2^2, \quad (4.1)$$

where

$$ds_{\text{FRW}_3}^2 = a_0^2(t) \gamma^2 (dp^2 + p^2 d\Omega_2^2), \quad (4.2)$$

$$p = \frac{\bar{r}}{r_0}, \quad \gamma = 1/(1 - p^2/4), \quad (4.3)$$

and r_c is defined as

$$r_c \equiv (b_0^4 + r_0^4)^{1/4}. \quad (4.4)$$

We used this coordinate since it is useful to study the large scale region by the finite radial coordinate p . In the Appendix, we give a small comment of this coordinate.

As shown below, the point $r = r_c$ is called the domain wall since the profile of the minimal surface cannot penetrate this point to the infrared region. Namely the solution is restricted to the region $r_c < r < \infty$.

Here, for convenience, we change the variable r to z as

$$z = r_c^2/r, \quad (4.5)$$

so it will be restricted to $0 < z < r_c$. In this case, the spatial part of the bulk metric (4.1) is rewritten as

$$ds_{\text{space}}^2 = \frac{1}{R^2} \left(z^2 + 2r_0^2 + \frac{r_c^4}{z^2} \right) ds_{\text{FRW}_3}^2 + \frac{R^2}{z^2} dz^2 + R^2 d\Omega_2^2. \quad (4.6)$$

We consider an entangling surface at $z = 0$ as a ball with the radius p_0 . Here the profile of the minimal surface in the bulk is set by $p(z)$ which is determined later. Then the area of the minimal surface with this boundary, as shown in the Fig. 1, is given by [25]

$$\frac{S_{\text{Area}}}{4\pi} = \int_0^{z(p=0)} dz \mathcal{L}(z), \quad (4.7)$$

where

$$\mathcal{L}(z) \equiv p(z)^2 B \sqrt{B p'(z)^2 + \frac{R^2}{z^2}}, \quad (4.8)$$

and

$$B \equiv \frac{a_0^2 \gamma^2}{R^2} \left(z^2 + \frac{r_c^4}{z^2} + 2r_0^2 \right). \quad (4.9)$$

By solving the variational equation, which is obtained from (4.7), we can get the profile $p(z)$ of the minimal surface. The numerical solutions for confinement phase ($b_0 < r_0$) and deconfinement phase ($r_0 \leq b_0$) are shown in Fig. 2 where p_0 denotes the ball radius,

$$p_0 \equiv p(z=0) \leq 2. \quad (4.10)$$

The upper bound comes from its definition.

From these numerical results, we can see that the profile function $p(z)$ approaches the rectangle form, namely the bottom line ($z = \text{const}$) and the side lines ($p = \text{const}$) which are given as

$$z = z_b, \quad p = p_0, \quad (4.11)$$

respectively, where p_0 approaches to the upper limit $p_0 = 2.0$. This behavior is proved as follows. The above Eq. (4.7) is rewritten as

$$\frac{S_{\text{Area}}}{4\pi} = \int_0^{p_0} dp \mathcal{L}(p), \quad (4.12)$$

where

$$\mathcal{L}(p) \equiv p^2 B \sqrt{B + \frac{R^2 \dot{z}^2(p)}{z^2(p)}}, \quad \dot{z}(p) = \frac{\partial z(p)}{\partial p}, \quad (4.13)$$

and B is the same form as the one given by (4.9).

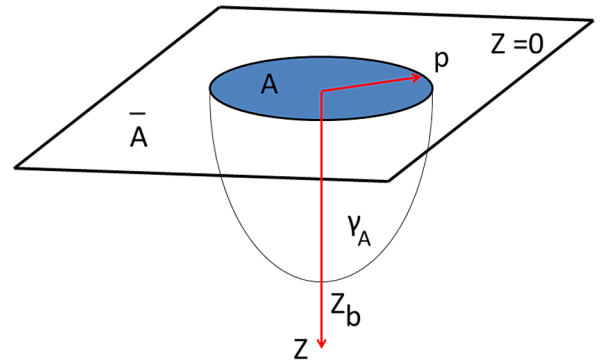


FIG. 1 (color online). The minimal surface γ_A is shown schematically.

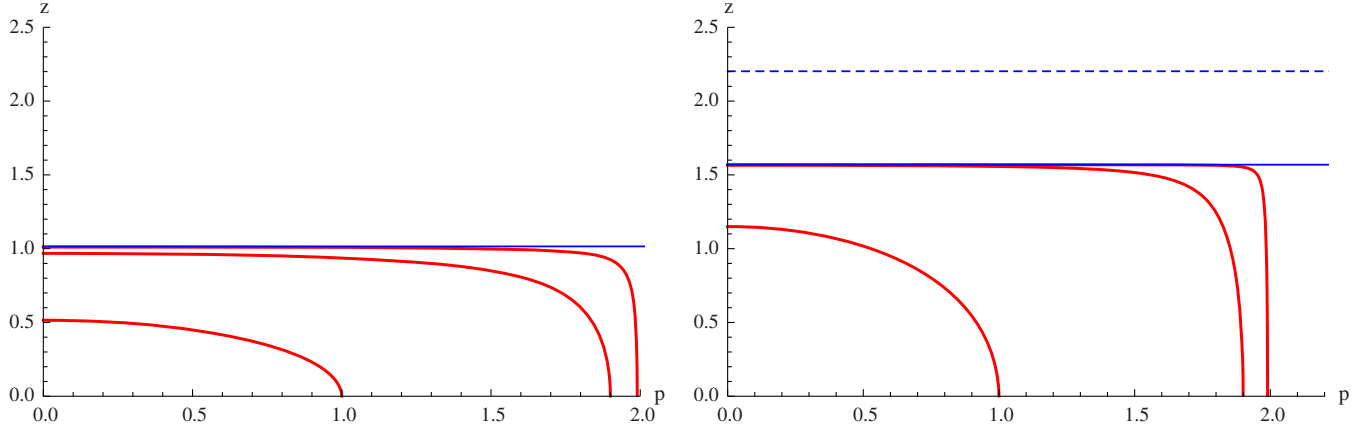


FIG. 2 (color online). Left phase: embedded solutions for $p(z)$ for $p_0 = 0.5, 1.9$ and 1.99 with $r_0 = R = 1, a_0 = 0.5, b_0 = 0.5 (< 1)$. The blue line is the domain wall $r_c = 1.02$. Right phase: Embedded solutions for $p(z)$ for $p_0 = 0.5, 1.9$ and $p_0 = 1.99$ with $r_0 = R = 1, a_0 = 0.5, b_0 = 1.5 (> 1)$. The blue line is the domain wall $r_c = 1.57$ and the dashed blue line is the event horizon $z_H = 2.20$.

In this case, the configuration for the minimal surface is obtained by solving the equation of motion for $z(p)$ which is derived from the above action as

$$\frac{\partial}{\partial z(p)} \left(p^2 B \sqrt{B + \frac{R^2}{z^2} \dot{z}^2(p)} \right) - \frac{\partial}{\partial p} \left(\frac{p^2 B \frac{R^2}{z^2} \dot{z}(p)}{\sqrt{B + \frac{R^2}{z^2} \dot{z}^2(p)}} \right) = 0. \quad (4.14)$$

Now, we concentrate on near the top of the minimal surface, namely near $p = 0$. Here, the following relations

$$\dot{z}(0) = 0, \quad \text{and} \quad \ddot{z}(0) < 0 \quad (4.15)$$

should be satisfied. From Eq. (4.14), we find the following equation,

$$\frac{a_0^2 \gamma^2}{R^2} z_b \left(1 - \frac{r_c^4}{z_b^4} \right) = \frac{R^2}{3z_b^2} \ddot{z}(0), \quad (4.16)$$

where $\dot{z}(0) = 0$ is imposed. Then we find

$$z_b \leq r_c = (b_0^4 + r_0^4)^{1/4} \quad (4.17)$$

to satisfy the second condition of (4.15). This implies that the upper bound of z is given by r_c , which is called here the ‘‘domain wall’’.

We notice the relation of the positions of the domain wall r_c and the horizon r_H . The latter appears only in the deconfinement phase. By setting $z_H \equiv r_c^2/r_H$, we find

$$z_H^4 - r_c^4 = 2b_0^2 r_0^2 \left(\frac{z_H}{r_c} \right)^4 \geq 0. \quad (4.18)$$

This implies that the domain wall r_c is smaller than the horizon z_H . Then the minimal surface could not reach the

horizon even if it appears. This fact implies that the form of the minimal surface is always connected. This point is important in this analysis.

B. Entanglement entropy in the infrared limit

In [15], it has been pointed out that the configuration of the minimal surface changes from the connected one to the disconnected one in the confinement phase. In our case, we find no such topological change of the minimal surface configuration. The minimal surface has the connected form for all regions of p_0 though the theory is in the confinement phase for $b_0/r_0 < 1$. This fact is not contradicted by the statement of [15] since both the bulk geometry and the shape of the divided region in our case are different from those studied in [15].

In order to estimate the entanglement entropy by an approximate formula, it is considered in the infrared limit of $p_0 \rightarrow 2$. In this limit, the minimal surface is estimated by substituting the obtained profile function $z(p)$ in Eq. (4.12). At the limit of $p_0 \rightarrow 2$, as shown above, the profile is approximated by the rectangle form (4.11). Thus S_{Area} of (4.12) can be approximated as

$$\frac{S_{\text{Area}}}{4\pi} = \int_{0, z=z_b, \partial z/\partial p=0}^{p_0} dp p^2 B^{3/2} + \int_{0, p=p_0, \partial p/\partial z=0}^{z_b} dz p^2 B \frac{R^2}{z} \quad (4.19)$$

$$= \int_0^{p_0} dp p^2 \left(\frac{a_0^2 \gamma^2}{R^2} f(z_b) \right)^{3/2} + \int_0^{z_b} dz p_0^2 a_0^2 \gamma^2 (p_0) g(z) \quad (4.20)$$

where

$$f(z_b) = z_b^2 + \frac{r_c^4}{z_b^2} + 2r_0^2, \quad (4.21)$$

$$g(z) = z + \frac{r_c^4}{z^3} + 2\frac{r_0^2}{z}. \quad (4.22)$$

Here we notice that the first term is dominant for $p_0 \rightarrow 2$ since it increases with the volume of A . On the other hand, the second term increases with the surface of A . We could see that the first term has its minimum at $z_b = r_c$ from the form of $f(z_b)$ given above. Then we could understand that the value r_c corresponds to the domain wall for the minimal surface.

Another point to be noticed is that r_c is larger than the horizon r_H in the deconfinement phase. Then the minimal surface bounded at $r = \infty$ could not touch r_H and there appears no disconnected surface as mentioned above.

We estimate the area of the minimal surface S_{Area} by separating into two parts as follows,

$$\frac{S_{\text{Area}}}{4\pi} = I_{\text{bottom}} + I_{\text{side}}, \quad (4.23)$$

$$I_{\text{bottom}} = \int_0^{p_0} dp p^2 \left(\frac{a_0^2 \gamma^2}{R^2} f(z_b) \right)^{3/2}, \quad (4.24)$$

$$I_{\text{side}} = \int_0^{z_b} dz p_0^2 a_0^2 \gamma^2(p_0) g(z). \quad (4.25)$$

The two parts, I_{bottom} and I_{side} , are corresponding to the parts of $z = z_b$ and $p = p_0$ respectively. The first term is given as

$$I_{\text{bottom}} = \frac{\bar{V}_{(3)}}{4\pi R^3} 2\sqrt{2}(1+h)^{3/2}, \quad (4.26)$$

$$\bar{V}_{(3)} = \frac{\pi R^6}{2} \int_0^{p_0} dp p^2 \gamma^3(p), \quad (4.27)$$

$$h = \sqrt{1 + \frac{b_0^4}{r_0^4}} = \sqrt{1 + \frac{4C}{R^2}}, \quad (4.28)$$

where we used

$$a_0 = \frac{1}{\sqrt{|\lambda|}} = \frac{R^2}{2r_0}. \quad (4.29)$$

It is noticed that I_{bottom} ⁴ does not contain the parameter r_0 or λ . Then the first term is expressed by C only. This point is important as seen below.

⁴Notice that, at the limit of $p_0 = 2$, the three volume is divergent as

$$\frac{2\bar{V}_{(3)}}{\pi R^6} = \int_0^{p_0} \gamma^3 p^2 dp = \frac{1}{2} a_0^3 \left(\frac{4p_0(4+p_0^2)}{(p_0^2-4)^2} + \log \frac{2-p_0}{2+p_0} \right).$$

As for the second term I_{side} , it can be evaluated by introducing ultraviolet cutoff ϵ as

$$I_{\text{side}} = a_0^2 p_0^2 \gamma^2(p_0) \left(\frac{z_b^2}{2} - \frac{r_c^4}{2z_b^2} + 2r_0^2 \ln \frac{z_b}{r_c} - \frac{\epsilon^2}{2} + \frac{r_c^4}{2\epsilon^2} - 2r_0^2 \ln \frac{\epsilon}{r_c} \right). \quad (4.30)$$

Here we take the limit of $\epsilon \rightarrow 0$ by subtracting two divergent terms and we get

$$I_{\text{side}} = a_0^2 p_0^2 \gamma^2(p_0) \left(\frac{z_b^2}{2} - \frac{r_c^4}{2z_b^2} + 2r_0^2 \ln \frac{z_b}{r_c} \right) + F_s. \quad (4.31)$$

This last term F_s denotes an ambiguity of the subtraction. This is usually determined by appropriate boundary conditions or the renormalization conditions.

Our purpose is to see the change of the entanglement entropy when the excitation C increases, so we take the following boundary condition:

$$\left. \frac{S_{\text{Area}}}{4\pi} \right|_{z_b=r_c, C=0} = 0. \quad (4.32)$$

Then we find

$$F_s = 8 \frac{\bar{V}_{(3)}}{4\pi R^3}, \quad (4.33)$$

and by setting $z_b = r_c$, S_{Area} is given as

$$\frac{S_{\text{Area}}}{4\pi} = 8 \frac{\bar{V}_{(3)}}{4\pi R^3} \left(\left(\frac{1+h}{2} \right)^{3/2} - 1 \right). \quad (4.34)$$

As for this result, we notice the following points:

- (i) The result (4.34) indicates that the minimal surface S_{Area} is independent of r_0 . This implies that the entanglement temperature is determined by changing b_0 since the entanglement entropy is controlled only by b_0 . The change of λ is related to the change of the mass of the excited state and the vacuum energy as seen below.
- (ii) At large C (or equivalently at large b_0), we find

$$S_{\text{EE}} = \frac{S_{\text{Area}}/V_{(3)}}{4G_N^{(5)}} = \frac{\pi^2}{2} N^2 T_H^3. \quad (4.35)$$

This indicates the entanglement entropy at large scale limit satisfies the thermodynamic relation,

$$\frac{\partial U}{\partial S_{\text{EE}}} = T \quad (4.36)$$

where $T = T_H$ and

$$U = \langle T_{00} \rangle = \frac{3\pi^3 R^3}{16G_N^{(5)}} T_H^4. \quad (4.37)$$

The above formula for $\langle T_{00} \rangle$ is obtained at large C [23].

- (iii) The resultant form of the minimal surface is determined only by the first term of (4.23), which represents the bottom part of the surface. In other words, the entropy of the excitations due to the dark radiation is given by the area at the bottom.

C. About the logarithmic divergent term

We comment on the divergent terms in S_{Area} . They are found in I_{side} as

$$I_{\text{side}}|_{\text{div}} = a_0^2 p_0^2 \gamma^2(p_0) \left(\frac{r_c^4}{2\epsilon^2} - 2r_0^2 \ln \frac{\epsilon}{r_c} \right). \quad (4.38)$$

However, this is not equivalent to the one given in [25]. The coefficient of the logarithmic divergent term is slightly different from the above formula (4.38). This point is improved by adding a correction term in I_{side} , which is roughly approximated. In getting (4.38), we have approximated as

$$p^2 a_0^2 \gamma^2(p) = p_0^2 a_0^2 \gamma^2(p_0). \quad (4.39)$$

Then the integration with respect to z is performed. This procedure corresponds to adopting the approximation of $p(z) = p_0$. On the other hand, $g(z)$ has a term proportional to $1/z^3$ and we must retain the terms up to z^2 in (4.39) in order to see the logarithmic divergent terms.

Near $z = 0$, the asymptotic solution can be obtained as [25]

$$p = p_0 + p_2 z^2 + p_4 z^4 + p_{4L} z^4 \log z \cdots, \quad (4.40)$$

where p_4 is an arbitrary constant. p_2 and p_{4L} are determined as

$$\begin{aligned} p_2 &= -\frac{(1 - (p_0^2/4)^2)R^3}{2a_0^2 p_0 r_c^4}, \\ p_{4L} &= -\frac{(1 - (p_0^2/4)^2)R^8 \dot{a}_0^2}{4a_0^4 p_0 r_c^8}. \end{aligned} \quad (4.41)$$

Then, instead of (4.25), we obtain

$$I_{\text{side}-2} = \int_0^{z_b} dz p_0^2 a_0^2 \gamma^2(p_0) (1 + s_2 z^2) g(z), \quad (4.42)$$

$$s_2 = \frac{2(1 + p_0^2/4)}{p_0(1 - p_0^2/4)} p_2. \quad (4.43)$$

In this case we have the divergent term as

$$\begin{aligned} I_{\text{side}-2}|_{\text{div}} &= a_0^2 p_0^2 \gamma^2(p_0) \left(\frac{r_c^4}{2\epsilon^2} + \left(\frac{(1 + p_0^2/4)^2}{2a_0^2 p_0^2} R^3 - 2r_0^2 \right) \ln \frac{\epsilon}{r_c} \right). \end{aligned} \quad (4.44)$$

Then we could find the familiar formula,

$$\frac{S_{\text{Area}}}{4G_N^{(5)}} = N^2 \ln \epsilon + \cdots. \quad (4.45)$$

So we could see the correct form of logarithmic divergence contribution by using the higher order term of $p(z)$ with respect to z . However, this is useful in the region of small z , and we should be careful about this expansion to large $z \sim z_b$ with large b_0 . So hereafter we adopt the formula (4.25) in this discussion.

V. EXCITATION AND ENTANGLEMENT TEMPERATURE

A. Entanglement temperature

We calculate the entanglement temperature for the excitation in the SYM theory given above. The global temperature T_{ent} is calculated according to the formula (2.16) by replacing the parameter α with b_0 . Then the meaning and the role of T_{ent} are investigated for our model in the confinement phase as well as in the deconfinement phase. In the latter case, as pointed above, the entanglement temperature T_{ent} approaches the Hawking temperature T_H at large b_0 and infrared limit, namely for the large scale minimal surface [10]. We could also see this behavior in our model.

On the other hand, in the confinement phase, T_H disappears and then T_{ent} would be used to measure the energy of the system due to the excitation of the SYM fields in the form of the glueballs. We therefore expect that the mass of the glueball will be related to T_{ent} in the confinement phase in some way.

In the ultraviolet region, namely at small p_0 , we expect the following behavior as given for the deconfining theory in [10],

$$T_{\text{ent}} = \frac{c_0}{p_0}, \quad (5.1)$$

where c_0 is a calculable number. The reason why this is expected is that the dynamical properties in the infrared region would not affect the quantities at short range physics. In fact we could see it numerically. Here our purpose is to examine the properties at the infrared limit of $p_0 \sim 2$ by using an approximate and simple form of the minimal surface given in our analysis.

In our model, there are two parameters, b_0 and r_0 , as the candidates for the above α , which are used to define the

relative entropy, in (2.16). Both b_0 and r_0 may be considered as such parameters. However, as seen from Eq. (4.34), the entanglement entropy is expressed as a function of $x^4 (= b_0^4/r_0^4)$ which is rewritten as

$$x^4 = \left(\frac{b_0}{r_0}\right)^4 = 4 \frac{C}{R^2}, \quad (5.2)$$

where C represents the excitation of the SYM fields. This means that the entanglement temperature T_{ent} , which should reflect the excitation of the system, is determined by the parameter C only. In other words, it is controlled only by the parameter b_0 . Thus, according to (2.16), we have the entanglement temperature T_{ent} as follows:

$$T_{\text{ent}}^{b_0}(b_0, r_0) \equiv T_{\text{ent}}^{b_0} = \frac{\int d^d \xi \sqrt{g^{(0)}} \frac{\partial \langle T_{00} \rangle}{\partial b_0}}{\frac{\partial S}{\partial b_0}}. \quad (5.3)$$

In the limit of $p_0 \rightarrow 2.0$, the maximum of p_0 , we obtain the following result,

$$T_{\text{ent}}^{b_0} = \frac{\sqrt{2}r_0}{\pi R^2} \frac{h}{\sqrt{1+h}}, \quad (5.4)$$

where we used (4.34).

In order to make clear the difference between T_{ent} and T_H , we compare them. Here and in the followings, $T_{\text{ent}}^{b_0}$ and the Hawking temperature $T_H(r_0)$, which is given in (3.13), are denoted simply as T_{ent} and T_H respectively. They are shown in Fig. 3, which shows that $T_H < T_{\text{ent}}$ and T_{ent} approaches T_H from the above at large b_0 . The ratio of the two temperatures is expressed as

$$T_H/T_{\text{ent}} = \frac{1}{h} \sqrt{(x^2 - 1)(1 + h)} < 1. \quad (5.5)$$

Then, at high temperature, we can use both temperatures to examine the thermodynamic properties of the excited system.

On the other hand, while T_H cannot be defined in the small $b_0 (< r_0)$ region of the confinement phase (A), T_{ent}

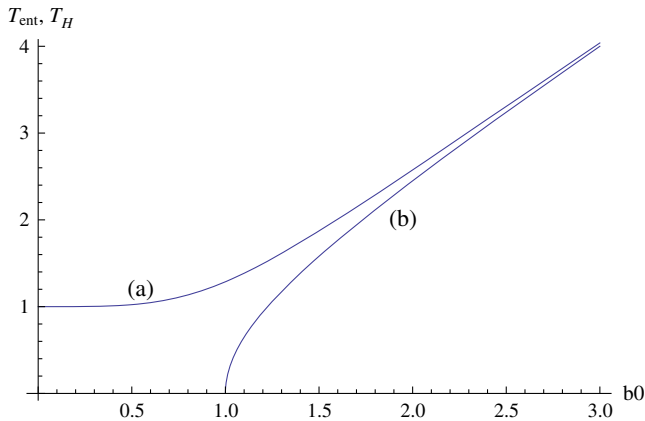


FIG. 3 (color online). (b) T_H and (a) T_{ent} are shown as the function of b_0/r_0 for $r_0 = 1$.

survives in this region and can be defined in all regions where the excitation due to C exists. So we use T_{ent} instead of T_H to see the thermo-dynamical properties in the whole region of C . In other words, it would be possible to extend the thermodynamic viewpoint by using the new temperature T_{ent} . From this viewpoint, our model is studied thermodynamically in terms of T_{ent} as follows.

B. Thermodynamic properties in terms of T_{ent}

In order to see the thermodynamic properties of the excitation, we consider the quantities given by the following equations,

$$E(b_0)/T_{\text{ent}}^4 = \frac{3\pi^2}{8} N^2 f_U(x), \quad f_U(x) = \frac{(1+h)^2}{h^4} (h^2 - 1),$$

$$h(x) = \sqrt{1+x^4}, \quad (5.6)$$

$$S_{\text{EE}}/T_{\text{ent}}^3 = \frac{\pi^2}{2} N^2 f_S(x),$$

$$f_S(x) = \frac{(1+h)^{3/2}}{h^3} ((1+h)^{3/2} - 2\sqrt{2}), \quad (5.7)$$

where T_{ent} is given by (5.4) and

$$E(b_0) \equiv \langle T_{00} \rangle - \langle T_{00} \rangle|_{b_0=0}. \quad (5.8)$$

We investigate the above quantities, f_U and f_S , along the value of b_0 .

At large b_0 :

We notice that the usual thermodynamic relations, (4.35) and (4.37), are obtained at large b_0 (or small r_0), where we find $f_U \rightarrow 1$, $f_S \rightarrow 1$, and $T_{\text{ent}} \rightarrow T_H$. Then, at large b_0 , these two relations approach the one obtained at high temperature deconfining phase. When b_0 decreases, f_U and f_S deviate from one. This is the reflection of the interaction of the dynamical freedom of the excited fields. Then we could see the dynamical properties of the excited fields through this deviation.

Near the region of $b_0 = 0$:

In the region of $b_0 (< r_0)$, the theory is in the confinement phase and the excitation is expected to be the color singlet, namely the glueball. We consider the limit of $b_0 = 0$, where we have the lowest entanglement temperature, which is given as follows:

$$T_{\text{ent}}^{(0)} = \frac{r_0}{\pi R^2}. \quad (5.9)$$

It should be noticed that this is positive and finite in spite of the absence of the excited matter of the system since $b_0 = 0$. On the other hand, this temperature is related to the energy and entropy at small b_0 as follows,

$$E(b_0) \simeq \frac{3\pi^2}{2} N^2 x^4 T_{\text{ent}}^{(0)4}, \quad (5.10)$$

$$S_{EE} \approx \frac{3\pi^2}{2} N^2 x^4 T_{\text{ent}}^{(0)3}, \quad (5.11)$$

where the terms are retained up to $O(x^4)$. These equations implies (i) that the dynamical degrees of freedom (DOF) of the excitation due to b_0 decrease to zero like x^4 . (ii) Secondly, the necessary energy to excite one DOF from the ground state of $b_0 = 0$ is $T_{\text{ent}}^{(0)}$. So there is an energy gap to make the lowest excitation.

This fact is the reflection of the confinement and the existence of the glueball which has the lowest mass. Actually, the glueball mass m_g for $J^{PC} = 2^{++}$ state has been given as follows [25]:

$$m_g^2 = 4(n+1)(n+4) \frac{r_0^2}{R^4} = |\lambda|(n+1)(n+4), \quad (5.12)$$

$$n = 0, 1, 2, \dots$$

From this, the lowest glueball mass is found as $m_g^{(0)} = 4r_0/R^2$. The relation between the mass m_g and $T_{\text{ent}}^{(0)}$ is therefore rewritten as

$$T_{\text{ent}}^{(0)} = \frac{m_g^{(0)}}{4\pi}. \quad (5.13)$$

This implies that we need a small but finite energy to excite the vacuum to the lowest excited state with glueballs of the lowest mass. This is independent of C , and it is determined by the parameter r_0 or λ .

Trangent region, $b_0/r_0 \sim 1$:

The interesting point is seen in the deviations from the high temperature limit, (4.35) and (4.37). They are expressed by the functions f_U and f_S , which are shown in Fig. 4 as functions of $x = b_0/r_0$.

From Fig. 4, we can read the following:

- (i) For the range $0 < x < 0.4$, the values of f_U and f_S are almost zero. This is interpreted as the reflection of the confinement since the color degrees of freedom are suppressed in this region probably to the one of $O(N^0)$. On the other hand, they increase rapidly in the region of $0.4 < b_0 < 1.0$ in spite of the fact that the theory is still in the confinement phase ($b_0 < 1$). This result could be related to the fact that the glueball mass is suppressed to smaller value when b_0 approaches $b_0 = r_0$, the critical point of (de)confinement phase transition, as found in [25]. As a result, it would be possible to excite many higher order states of glueballs since their mass spectrum would be given by Eq. (5.12) with a small prefactor.
- (ii) We should notice that this rapid variation of DOF has been also observed in the lattice simulation of $SU(3)$ gauge theory near the (de)confinement transition temperature [28]. In this case, however, the observed phenomenon is interpreted as the crossover. Namely it is not the first order transition. Therefore the maximum point of its increasing rate ($\partial f_U(x)/\partial b_0$)

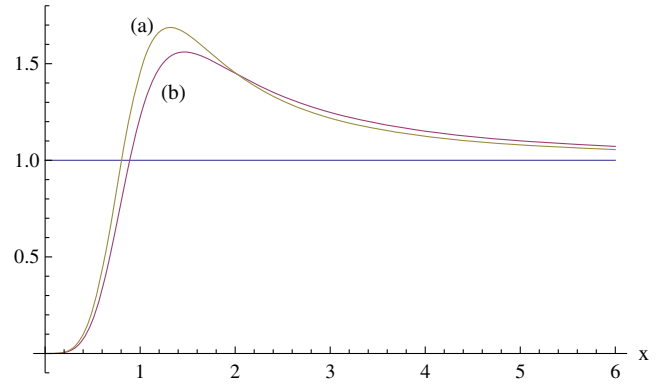


FIG. 4 (color online). (a) $f_U(x)$ and (b) $f_S(x)$ are shown as the functions of $x = b_0/r_0$. With decreasing b_0 , the function f_U (f_S) gradually increases from one to its maximum 1.7(1.6), which is realized at about $x = 1.3$ (1.4). Then both f_U and f_S decrease rapidly to zero, which is realized at $b_0 = 0$.

is identified as the crossover point from the confinement to the deconfinement. In this region, glueballs and color degrees of freedom coexist.

In the present case, however, this transition has been observed as the first order one [27], and the thermodynamic property is examined by using the temperature which is defined by the Hawking temperature T_H . On the other hand, we are now considering the extended thermodynamics in terms of the entanglement temperature T_{ent} .

We could say that the order of the phase transition, which is observed in the thermodynamics defined by T_{ent} , would be made milder than the one given in the analysis in terms of the temperature T_H . Further, the critical point given by the maximum of $\partial f_U(x)/\partial b_0$ is slightly smaller than the actual transition point $b_0/r_0 = 1$.

- (ii) Another point to be noticed is found by comparing our f_U with the one given in [28]. Our f_U has a maximum near $b_0/r_0 = 1$. On the other hand, in the case of the $SU(3)$ lattice gauge simulation, the

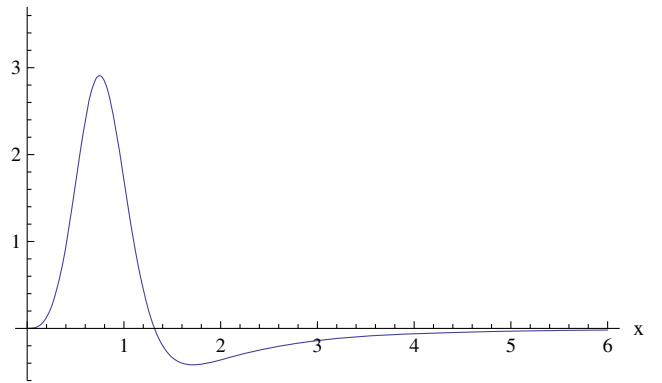


FIG. 5 (color online). (a) $\partial f_U(x)/\partial b_0$ is shown as the functions of $x = b_0/r_0$. We find two extremum points, which would correspond to the two phase transitions of the theory.

corresponding factor f_U increases monotonically without any such a maximum.

As shown in Fig. 5, the existence of the maximum of f_U implies the existence of the minimum point of $\partial f_U(x)/\partial b_0$ also. This point also indicates the point where f_U changes rapidly; then it may correspond to another phase transition point. It might be regarded as the chiral transition point which has been found in our present model used here.

VI. SUMMARY AND DISCUSSION

In terms of entanglement entropy, we have examined the SYM theory, which is living in AdS₄ space-time. The ground state of this theory is in the confinement phase, where we could observe the glueballs as excited modes of the theory. The mass of the glueballs is expressed by the scale r_0 which characterizes AdS₄ curvature.

This theory can be extended to an excited state by adding extra parameter C (or b_0) which is responsible for the excitation of the SYM theory in the background determined by r_0 . This yields its energy density $\langle T_{00} \rangle \propto b_0^4$ in addition to the one of the ground state composed of r_0 . At enough large $b_0 (> r_0)$, this excitation changes the ground state from the confinement phase to the deconfinement phase with a finite temperature.

In the deconfinement phase ($r_0 < b_0$), the temperature of the system is given by the Hawking temperature $T_H(b_0, r_0)$, which depends on b_0 and r_0 . However, $T_H(b_0, r_0)$ disappears in the confinement phase $0 < b_0 \leq r_0$. So, in order to describe all regions of the parameter b_0 as a thermodynamic phenomenon, we here introduced the entanglement temperature T_{ent} , which is available in any phase. It is derived by supposing the thermodynamic relation between the variations of the energy density $\langle T_{00} \rangle$ and the entanglement entropy $S_{\text{EE}}(b_0)$. Thus we could obtain the temperature T_{ent} which is useful at any value of b_0 .

We used the approximate formula for $S_{\text{EE}}(b_0)$, which is evaluated in the limit of large radius of the three-dimensional hyperboloid, p_0 . This approximation is reasonable to see the thermodynamic properties since $S_{\text{EE}}(b_0)$ approaches the Bekenstein-Hawking entropy at large scale. The regularization for $S_{\text{EE}}(b_0)$ is performed by subtracting $S_{\text{EE}}(b_0 = 0)$ since our interest is in the region of $0 \leq b_0$. As for the ultraviolet divergences occurring near the boundary, we could see the expected results in the logarithmic term. We should notice that in the calculation of $S_{\text{EE}}(b_0)$ we could not find the topological change of the minimal surface from the connected to the disconnected one when p_0 increases as found in the case of AdS soliton model [15].

Using the obtained T_{ent} , the two quantities, $\langle T_{00} \rangle/T_{\text{ent}}^4$ and $S_{\text{EE}}/T_{\text{ent}}^3$, which correspond to the effective dynamical degrees of freedom, are considered. Then how these quantities deviate from their high temperature limit is studied for all region of b_0 including the phase transition point of the theory.

We could find that the dynamical degree of freedom reduces to the very small number at small $b_0 \sim 0$. This fact is interpreted as the reflection of the color confinement since the color degree of freedom of the excitation vanishes. Secondly, we notice that there is a lower bound for the temperature T_{ent} , namely $T_{\text{ent}} \geq T_{\text{ent}}^{(0)}$. The lower bound $T_{\text{ent}}^{(0)}$ is related to the glueball mass m_g as $T_{\text{ent}}^{(0)} = m_g/(4\pi)$. These facts indicate that finite minimum energy is necessary for the excitation from the ground state $b_0 = 0$, and the excitation corresponds to the formation of glueballs with the lowest mass. This is also the reflection of the confinement phase at small T_{ent} .

Further, we find from $\langle T_{00} \rangle/T_{\text{ent}}^4$ that the dynamical degrees of freedom increase rapidly near the transition point, from the confinement to the deconfinement phase. This phenomenon is similar to the crossover transition observed in the lattice QCD with $SU(3)$ color symmetry with flavor quarks. So we could understand that the thermodynamic phenomenon with the entanglement temperature would reproduce milder order phase transition than the case of Hawking temperature T_H .

Thirdly, since $\langle T_{00} \rangle/T_{\text{ent}}^4$ has a maximum, it decreases after passing through this maximum and approaches to the expected high temperature limit. So there is a second extremum for the derivative of the deviation of the freedom. This point also could be regarded as the phase transition point as discussed in [28]. In our model, this transition would be interpreted as the chiral restoration point. Therefore, we could say that we can see the expected phase transitions as the thermodynamic phenomenon in terms of the entanglement temperature T_{ent} .

ACKNOWLEDGMENTS

K. G. thanks M. Tachibana, F. Toyoda and I. Maruyama for useful discussion. The work of M. Ishihara is supported by World Premier International Research Center Initiative WPI, MEXT, Japan. The work of M. I. is supported in part by the JSPS Grant-in-Aid for Scientific Research, Grant No. 15K20877.

APPENDIX: COMMENT ON THE BOUNDARY METRIC

We notice the coordinate of the boundary for (3.1). It is given as

$$ds_4^2 = -dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j. \quad (\text{A1})$$

$$\gamma_{ij}(x) = \delta_{ij} \left(1 + k \frac{\bar{r}^2}{4\bar{r}_0^2} \right)^{-2},$$

$$\bar{r}^2 = \sum_{i=1}^3 (x^i)^2, \quad (\text{A2})$$

where $k = \pm 1$, or 0. \bar{r}_0 denotes the scale factor of three space. Here we set $k = -1$ as stated in Sec. III. So the space is opened. Further (A1) is rewritten by (4.2)–(4.3) in terms

of the polar coordinate. Although the radial coordinate p in this metric is restricted as $p < 2$, the volume of the space is infinite and then opened. This point becomes more clear when we rewrite the metric as

$$ds_4^2 = -dt^2 + a_0^2(t) \left(\frac{dq^2}{1+q^2} + q^2 d\Omega_{(2)}^2 \right), \quad (\text{A3})$$

where

$$q = \frac{p}{1 - p^2/4}. \quad (\text{A4})$$

The new radial coordinate is then set in the range of $0 < q < \infty$. For $p \sim 2 (= p_0)$, q approaches to ∞ , then the small change of p near p_0 corresponds to a very large change of q . It is more convenient to use p than q to perform especially the numerical analysis at large scale region. Due to this reason, we used p rather than q .

-
- [1] J. M. Maldacena, The Large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2**, 231 (1998); , The Large N limit of superconformal field theories and supergravity, *Int. J. Theor. Phys.* **38**, 1113 (1999).
- [2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, *Phys. Lett. B* **428**, 105 (1998).
- [3] E. Witten, Anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
- [4] S. Ryu and T. Takayanagi, Holographic Derivation of Entanglement Entropy from AdS/CFT, *Phys. Rev. Lett.* **96**, 181602 (2006).
- [5] S. Ryu and T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, *J. High Energy Phys.* **08** (2006) 045.
- [6] J. M. Maldacena and G. L. Pimentel, Entanglement entropy in de Sitter space, *J. High Energy Phys.* **02** (2013) 038.
- [7] B. Swingle and T. Senthil, Universal crossovers between entanglement entropy and thermal entropy, *Phys. Rev. B* **87**, 045123 (2013).
- [8] H. Casini, M. Huerta, and R. C. Myers, Towards a derivation of holographic entanglement entropy, *J. High Energy Phys.* **05** (2011) 036.
- [9] D. D. Blanco, H. Casini, L.-Y. Hung, and R. C. Myers, Relative entropy and holography, *J. High Energy Phys.* **08** (2013) 060.
- [10] J. Bhattacharya, M. Nozaki, T. Takayanagi, and T. Ugagin, Thermodynamical Property of Entanglement Entropy for Excited States, *Phys. Rev. Lett.* **110**, 091602 (2013).
- [11] M. Alishahiha, D. Allahbakhshi, and A. Naseh, Entanglement thermodynamics, *J. High Energy Phys.* **08** (2013) 102.
- [12] Da-Wei Pang, Entanglement thermodynamics for nonconformal D-branes, *Phys. Rev. D* **88**, 126001 (2013).
- [13] N. Lashkari, M. McDemott, and M. Raamsdonk, Gravitational dynamics from entanglement thermodynamics, *J. High Energy Phys.* **04** (2014) 195.
- [14] G. Wong, I. Klich, L. A. P. Zayas, and D. Vaman, Entanglement temperature and entanglement entropy of excited states, *J. High Energy Phys.* **12** (2013) 020.
- [15] I. R. Klebanov, D. Kutasov, and A. Murugan, Entanglement as a probe of confinement, *Nucl. Phys.* **B796**, 274 (2008).
- [16] P. Binetruy, C. Deffayet, U. Ellwanger, and D. Langlois, Brane cosmological evolution in a bulk with cosmological constant, *Phys. Lett. B* **477**, 285 (2000).
- [17] D. Langlois, Brane cosmological perturbations, *Phys. Rev. D* **62**, 126012 (2000).
- [18] T. Shiromizu, K. Maeda, and M. Sasaki, *Phys. Rev. D* **62**, 024012 (2000).
- [19] M. Sasaki, T. Shiromizu, and K. Maeda, Gravity, stability, and energy conservation on the Randall-Sundrum brane world, *Phys. Rev. D* **62**, 024008 (2000); K. Maeda, S. Mizuno, and T. Torii, Effective gravitational equations on a brane world with induced gravity, *Phys. Rev. D* **68**, 024033 (2003).
- [20] S. de Haro, S. N. Solodukhin, and K. Skenderis, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, *Commun. Math. Phys.* **217**, 595 (2001).
- [21] M. Bianchi, D. Z. Freedman, and K. Skenderis, Holographic renormalization, *Nucl. Phys.* **B631**, 159 (2002).
- [22] C. Fefferman and C. R. Graham, in *Conformal Invariants*, Elie Cartan et les Mathématiques d'aujourd'hui (Astérisque, Paris, 1985), p. 95.
- [23] J. Erdmenger, K. Ghoroku, and R. Meyer, Holographic (de) confinement transitions in cosmological backgrounds, *Phys. Rev. D* **84**, 026004 (2011).
- [24] J. Erdmenger, K. Ghoroku, R. Meyer, and I. Papadimitriou, Holographic cosmological backgrounds, Wilson loop (de) confinement and dilaton singularities, *Fortschr. Phys.* **60**, 991 (2012).
- [25] K. Ghoroku, M. Ishihara, A. Nakamura, and F. Toyota, Glueball instability and thermalization driven by dark radiation, *Phys. Rev. D* **90**, 126011 (2014).
- [26] K. Ghoroku, M. Ishihara, M. Tachibana, and F. Toyota, Chiral symmetry of SYM theory in hyperbolic space at finite temperature, *Phys. Rev. D* **92**, 026011 (2015).
- [27] K. Ghoroku and A. Nakamura, Holographic Friedmann equation and $N = 4$ supersymmetric Yang-Mills theory, *Phys. Rev. D* **87**, 063507 (2013).
- [28] A. Bazavov *et al.*, Equation of state and QCD transition at finite temperature, *Phys. Rev. D* **80**, 014504 (2009).