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# Non-Abelian electric-magnetic duality with supersymmetry in 4D and 10D

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We present electric-magnetic (Hodge) duality formulation for non-Abelian gauge groups with $N = 1$ supersymmetry in $3 + 1$ (4D) dimensions. Our system consists of three multiplets: (i) A super-Yang-Mills vector multiplet (YMVM) $(A_\mu{}^I, \lambda^I)$, (ii) a dual vector multiplet (DVM) $(B_\mu{}^I, \chi^I)$, and (iii) an unphysical tensor multiplet (TM) $(C_{\mu\nu}{}^I, \rho^I, \varphi^I)$, with the index $^I$ for adjoint representation. The multiplets YMVM and DVM are dual to each other like: $G_{\mu\nu}{}^I = (1/2)\epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}{}^I$. The TM is unphysical, but still plays an important role for establishing the total consistency of the system, based on recently developed tensor-hierarchy formulation. We also apply this technique to non-Abelian electric-magnetic duality in $9 + 1$ (10D) dimensions. The extra bosonic auxiliary field $K_{\mu_1\cdots\mu_6}$ in 10D is shown to play an important role for the closure of supersymmetry on fields.

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## I. INTRODUCTION

It is conjectured that the discrete group $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ is the exact symmetry of the full heterotic string theory [1,2], associated with the target-space duality symmetry $SO(6,22)$ in compactifications to four dimensions (4D). This feature also leads to electric-magnetic (EM) duality in 4D or higher dimensions with Lagrangian formulations [3]. The drawback of nonmanifest Lorentz invariance in [3] was overcome by the manifestly Lorentz-invariant reformulation [4]. The S-duality between the strong and weak string-couplings is also reduced to EM-duality in 4D [5], making D3-branes self-dual [6].

The $SL(2,\mathbb{R})$ symmetry for a vector field was pointed out early in 1980s [7], and is confirmed to be valid, even in the presence of Dirac-Born-Infeld interactions [7,8]. The $N = 1$ and $N = 2$ supersymmetric generalizations have also been accomplished in [9]. Moreover, this duality-symmetry can be generalized to self-duality in even dimensions [10].

In 4D, the EM-duality is $F_{\mu\nu}{}^I = (1/2)\epsilon_{\mu\nu}{}^{\rho\sigma} G_{\rho\sigma}{}^I$, where $G_{\rho\sigma}{}^I$ is the field strength of a new vector field $B_\mu{}^I$ with the adjoint index $I$. However, due to the inconsistency arising for the naÏve definition of the field-strength $G_{\mu\nu}^{(0)I} \equiv 2D_{[\mu}B_{\nu]}{}^I$ for a *non-Abelian* vector $B_\mu{}^I$ [11], such an attempt was again bound to fail in the past. This had been the fate of vector fields with non-Abelian indices, not to mention its supersymmetrization.

This problem was first solved by the work by Samtleben [12] with the purely bosonic EM-duality for non-Abelian YM gauge field with its Hodge-dual field. The essential ingredient is to introduce Chern-Simons-like terms in the $G$-field strength, combined with a new tensor field $C_{\mu\nu\rho}{}^I$ in the adjoint representation. Subsequently, this result was further generalized in terms of "tensor-hierarchy formulations" [13,14].

The next natural step is the supersymmetrization of EM-duality for non-Abelian YM gauge fields. Motivated by this viewpoint, we carry out two objectives in this paper: (i) The $N = 1$ supersymmetrization of the system purely-bosonic EM-duality in 4D [12], and (ii) Its generalization to $N = (1,0)$ YM multiplet in 10D. Even though EM-duality for non-Abelian groups had been known in supergravity, such as $N = 8$ supergravity in 4D with local $SO(8)$, and despite the *purely-bosonic* EM-duality system had been presented as tensor-hierarchy formulation, our new ingredient is the *supersymmetrization* of EM-duality with arbitrary YM groups.

In our formulation in 4D, we introduce the following three multiplets: (i) A super-Yang-Mills vector multiplet (YMVM) which is the conventional vector multiplet, (ii) a dual vector multiplet (DVM) with the field-strength dual to the YM-field-strength, and (iii) a tensor multiplet (TM). The TM plays an important role for the closure of supersymmetry with *no* physical degree of freedom.

The introduction of an extra vector field $B_\mu{}^I$ with the adjoint index in addition to the YM-gauge field $A_\mu{}^I$ is *not* new. In addition to [12], another example is the supersymmetric Jackiw-Pi (JP) model in 3D [15]. The objective of the original JP-model [16] was to improve the parity-odd feature with Chern-Simons (CS) theory in 3D, by introducing an extra vector $B_\mu{}^I$ with the adjoint index. Thus, the introduction of the extra vector $B_\mu{}^I$ is common to our present EM-duality formulation and supersymmetric JP-model [15].

As a by-product of our 4D result, we apply the same mechanism to 10D YM multiplet. The needed field-content

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is the YMVM $(A_\mu{}^I, \lambda^I)$, DVM $(B_{[7]}{}^I, \chi^I)$[1] and auxiliary tensor potential fields $C_{[8]}{}^I$ and $K_{[6]}$. Here the potentials $A_\mu{}^I$ and $B_{[7]}{}^I$ have, respectively, the field-strengths $F_{\mu\nu}{}^I$ and $G_{[8]}{}^I$ dual to each other. The important role played by the extra tensor $K_{[6]}$ is explained both in component and superspace languages.

From a certain viewpoint, our formulation seems just a "trivial" truncation of well-known non-Abelian $N = 1$ systems [14,17,18]. This is because similar structures are found in [14,17,18], after first embedding all fields in super-multiplets and then truncating out all extra fields. Conceptually, that is one way to describe our objective. In practice, however, the most nontrivial process is the realization of such "truncation" consistently with supersymmetry. Whereas the purely-bosonic part of our system had been presented in [12], its supersymmetrization is the most nontrivial part. As we will see also, the necessity of the auxiliary tensor $K_{[6]}$ in the 10D case characterizes our nontrivial formulation.

Our paper is organized as follows: In the next section, we review the tensor-hierarchy formulation [13,14] applied to EM-duality. In Sec. III, we give the $N = 1$ supersymmetrization of *non-Abelian* EM-duality. In Sec. IV, we reformulate our theory in terms of superspace language [19]. We next apply the 4D result to the 10D super YM multiplet in component in Sec. V. In Sec. VI, we present its superspace reformulation. Concluding remarks are given in Sec. VII.

## II. TENSOR-HIERARCHY AND DUALITY

Our field content consists of three multiplets: (i) A YMVM: $(A_\mu{}^I, \lambda^I)$, (ii) a DVM: $(B_\mu{}^I, \chi^I)$, and (iii) a TM: $(C_{\mu\nu}{}^I, \rho^I, \varphi^I)$. The vector fields $A_\mu{}^I$, $B_\mu{}^I$, and the tensor field $C_{\mu\nu}{}^I$ have the following field-strengths defined by [12–14]

$$F_{\mu\nu}{}^I \equiv +2\partial_{[\mu} A_{\nu]}{}^I + m f^{IJK} A_\mu{}^J A_\nu{}^K, \tag{2.1a}$$

$$\begin{aligned} G_{\mu\nu}{}^I &\equiv +2D_{[\mu} B_{\nu]}{}^I + m C_{\mu\nu}{}^I \\ &\equiv +2(\partial_{[\mu} B_{\nu]}{}^I + m f^{IJK} A_{[\mu} B_{\nu]}{}^I) + m C_{\mu\nu}{}^I, \end{aligned} \tag{2.1b}$$

$$H_{\mu\nu\rho}{}^I \equiv = +3D_{[\mu} C_{\nu\rho]}{}^I + 3 f^{IJK} F_{\mu\nu}{}^J B_{\rho]}{}^K. \tag{2.1c}$$

We use $m$ as the YM-gauge coupling constant. These structures with the Chern-Simons (CS) like-terms in $G$ and $H$-field-strengths follow the general pattern in the recently developed tensor-hierarchy formulations [13,14]. Accordingly, the field-strengths $F, G$ and $H$ satisfy their proper Bianchi-identities (BIds):

$$D_{[\mu} F_{\nu\rho]}{}^I \equiv 0, \tag{2.2a}$$

$$D_{[\mu} G_{\nu\rho]}{}^I \equiv +\frac{1}{3} m H_{\mu\nu\rho}{}^I, \tag{2.2b}$$

$$D_{[\mu} H_{\nu\rho\sigma]}{}^I \equiv +\frac{3}{2} f^{IJK} F_{[\mu\nu}{}^J G_{\rho\sigma]}{}^K. \tag{2.2c}$$

The general variation of these field-strengths are given by

$$\delta F_{\mu\nu}{}^I = +2D_{[\mu|}(\delta A_{|\nu]}{}^I), \tag{2.3a}$$

$$\delta G_{\mu\nu}{}^I = +2D_{[\mu|}(\delta B_{|\nu]}{}^I) + m(\tilde{\delta} C_{\mu\nu}{}^I), \tag{2.3b}$$

$$\begin{aligned} \delta H_{\mu\nu\rho}{}^I = &+3D_{[m|}|(\tilde{\delta} C_{|\nu\rho]}{}^I) - 3 f^{IJK} (\delta B_{[\mu|}{}^J) F_{|\nu\rho]}{}^K \\ &+ 3 f^{IJK} (\delta A_{[\mu|}{}^J) G_{|\nu\rho]}{}^K, \end{aligned} \tag{2.3c}$$

$$\tilde{\delta} C_{\mu\nu}{}^I \equiv \delta C_{\mu\nu}{}^I + 2 f^{IJK} (\delta A_{[\mu|}{}^J) B_{|\nu]}{}^I. \tag{2.3d}$$

Since the dual-vector $B_\mu{}^I$ has a space-time index $\mu$, it must have its proper "gauge" transformation: $\delta_U B_\mu{}^I = D_\mu \beta^I$. The tensor $C_{\mu\nu}{}^I$ should also have its tensorial gauge transformation: $\delta_V C_{\mu\nu}{}^I = 2D_{[\mu}\gamma_{\nu]}{}^I$ [13,14]. In total, there are three different (generalized) gauge and tensor transformations $\delta_T, \delta_U$ and $\delta_V$ with the appropriate parameters $\alpha^I, \beta^I$ and $\gamma_\mu{}^I$ [13,14]:

$$\begin{aligned} \delta_T(A_\mu{}^I, B_\mu{}^I, C_{\mu\nu}{}^I) = (&D_\mu \alpha^I, -m f^{IJK} \alpha^J B_\mu{}^K, \\ &- m f^{IJK} \alpha^J C_{\mu\nu}{}^K), \end{aligned} \tag{2.4a}$$

$$\delta_U(A_\mu{}^I, B_\mu{}^I, C_{\mu\nu}{}^I) = (0, +D_\mu \beta^I, +f^{IJK} \beta^J F_{\mu\nu}{}^K), \tag{2.4b}$$

$$\delta_V(A_\mu{}^I, B_\mu{}^I, C_{\mu\nu}{}^I) = (0, -m\gamma_\mu{}^I, +2D_{[\mu}\gamma_{\nu]}{}^I). \tag{2.4c}$$

Using (2.4) in (2.3), we get

$$\delta_T(F_{\mu\nu}{}^I, G_{\mu\nu}{}^I, H_{\mu\nu\rho}{}^I) = -m f^{IJK} \alpha^J (F_{\mu\nu}{}^K, G_{\mu\nu}{}^K, H_{\mu\nu\rho}{}^K), \tag{2.5a}$$

$$\begin{aligned} &\delta_U(F_{\mu\nu}{}^I, G_{\mu\nu}{}^I, H_{\mu\nu\rho}{}^I) = (0,0,0), \\ &\delta_V(F_{\mu\nu}{}^I, G_{\mu\nu}{}^I, H_{\mu\nu\rho}{}^I) = (0,0,0). \end{aligned} \tag{2.5b}$$

In particular, the CS-like terms in the $G$ and $H$-field-strengths play important roles for the $\delta_U$ and $\delta_V$-invariances (2.5b). These results simply follow from the straightforward application of the more general tensor-hierarchy formulation [13,14].

Our crucial starting point is to require the EM-duality between the field-strengths $F$ and $G$[2]:

[1] We use the symbol $_{[n]}$ like $X_{[n]} \equiv X_{\mu_1 \cdots \mu_n}$ to save space for indices.

[2] We use the symbol $\overset{*}{=}$ for an equality related to a duality, or a more general constraint related to consistency with duality. Similarly, we use the symbol $\overset{\bullet}{=}$ for a field equation.

$$G_{\mu\nu}{}^{I} \overset{*}{=} + \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}{}^{I}. \tag{2.6}$$

Note that the right-hand side of the $H$-BI (2.2c) vanishes upon the use of the EM-duality (2.6).

Before the discovery of tensor-hierarchy formulation [13,14], there used to exist inconsistency for EM-duality for *non-Abelian* groups. For example, the gauge noncovariance is one of them. The naïvely-defined field-strength

$$G_{\mu\nu}^{(0)I} \equiv +2D_{[\mu}B_{\nu]}{}^{I} \tag{2.7}$$

is *not* $\delta_\beta$ invariant, because it transforms as

$$\delta_U G_{\mu\nu}^{(0)I} = mf^{IJK}F_{\mu\nu}{}^{J}\beta^{K} \neq 0. \tag{2.8}$$

The trouble is that this transformation does *not* leave the duality condition (2.6) intact. What is needed is an extra term in $G_{\mu\nu}{}^{I}$ as in (2.1b) that cancels the unwanted term (2.8), yielding $\delta_U G_{\mu\nu}{}^{I} = 0$. In contrast, the *non*-invariance of the naïve field-strength $\delta_U G_{\mu\nu}^{(0)I} \neq 0$ used to present an obstruction to establish the EM-duality: $G_{\mu\nu}^{(0)I} \overset{*}{=} (1/2)\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}{}^{I}$.

## III. SUPERSYMMETRIC EM-DUALITY

The next step is to supersymmetrize the duality condition (2.6). Because of the general tensor-hierarchy formulation [13], this process is straightforward. As has been mentioned, the TM in our system is *unphysical*, namely, all fields $(C_{\mu\nu}{}^{I}, \rho^I, \varphi^I)$ have *no* physical degree of freedom.

To be more specific, the $N = 1$ supersymmetry transformation rule for our multiplets YMVM, DVM and TM is

$$\delta_Q A_\mu{}^{I} = +(\bar{\epsilon}\gamma_\mu\lambda^I), \tag{3.1a}$$

$$\delta_Q \lambda^I = +\frac{1}{2}(\gamma^{\mu\nu}\epsilon)F_{\mu\nu}{}^{I} + im(\gamma_5\epsilon)\varphi^I, \tag{3.1b}$$

$$\delta_Q B_\mu{}^{I} = +i(\bar{\epsilon}\gamma_5\gamma_\mu\chi^I), \tag{3.1c}$$

$$\delta_Q \chi^I = +\frac{i}{2}(\gamma_5\gamma^{\mu\nu}\epsilon)G_{\mu\nu}{}^{I} - im(\gamma_5\epsilon)\varphi^I \tag{3.1d}$$

$$\delta_Q C_{\mu\nu}{}^{I} = +i(\bar{\epsilon}\gamma_5\gamma_{\mu\nu}\rho^I) - 2f^{IJK}(\bar{\epsilon}\gamma_{[\mu|}\lambda^J)B_{|\nu]}{}^{K}, \tag{3.1e}$$

$$\delta_Q \rho^I = -\frac{i}{6}(\gamma_5\gamma^{\mu\nu\rho}\epsilon)H_{\mu\nu\rho}{}^{I} + i(\gamma_5\gamma^\mu\epsilon)D_\mu\varphi^I + \frac{1}{2}f^{IJK}(\gamma_\mu\epsilon)(\bar{\lambda}^J\gamma^\mu\chi^K), \tag{3.1f}$$

$$\delta_Q \varphi^I = +i(\bar{\epsilon}\gamma_5\rho^I). \tag{3.1g}$$

Accordingly, by the use of (2.3) we can get

$$\delta_Q F_{\mu\nu}{}^{I} = -2(\bar{\epsilon}\gamma_{[\mu}D_{\nu]}\lambda^I), \qquad \delta_Q G_{\mu\nu}{}^{I} = -2i(\bar{\epsilon}\gamma_5\gamma_{[\mu}D_{\nu]}\chi^I) + im(\bar{\epsilon}\gamma_5\gamma_{\mu\nu}\rho^I), \tag{3.2a}$$

$$\delta_Q H_{\mu\nu\rho}{}^{I} = +3i(\bar{\epsilon}\gamma_5\gamma_{[\mu\nu}D_{\rho]}\rho^I) + 3f^{IJK}(\bar{\epsilon}\gamma_{[\mu|}\lambda^J)G_{|\nu\rho]}{}^{K} - 3if^{IJK}(\bar{\epsilon}\gamma_5\gamma_{[\mu}\chi^J)F_{\nu\rho]}{}^{K}. \tag{3.2b}$$

The definitions for the $F, G$ and $H$-field-strengths are exactly the same as in (2.1).

Our supersymmetric completion of the duality (2.6) reads as

$$G_{\mu\nu}{}^{I} \overset{*}{=} + \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}{}^{I}, \tag{3.3a}$$

$$\lambda^I \overset{*}{=} -\chi^I, \qquad \not{D}\lambda^I \doteq 0, \qquad \not{D}\chi^I \doteq 0, \tag{3.3b}$$

$$\rho^I \overset{*}{=} 0, \qquad \varphi^I \overset{*}{=} 0, \tag{3.3c}$$

$$H_{\mu\nu\rho}{}^{I} \overset{*}{=} -\frac{i}{2}f^{IJK}(\bar{\lambda}^J\gamma_5\gamma_{\mu\nu\rho}\lambda^K). \tag{3.3d}$$

Some remarks are in order: First, the last two equations in (3.3b) are actually field equations, but they are still indirectly related to the EM-duality by supersymmetry. Second, the first equation in (3.3b) implies that the two fermions $\lambda$ and $\chi$ coincide up to a sign. Third, (3.3c) is needed, so that the TM is *not* physical. Fourth, the condition on $H$ is nontrivial, because if we simply put $H_{\mu\nu\rho}{}^{I} \overset{*}{=} 0$, then its supersymmetric transformation generates *non*vanishing terms on-shell due to (3.2b). Even though the first term in (3.2b) vanishes due to (3.3c), the additional two terms $\approx(\bar{\epsilon}\gamma_5\gamma\lambda) \wedge G$ and $(\bar{\epsilon}\gamma\chi) \wedge F$ remain. Even though the latter is *approximately* equivalent to the former because of (3.3a) and (3.3b), they do *not exactly* cancel each other. It is the variation of the right-hand side of (3.3d) that cancels these two terms: $\delta_Q[H_{\mu\nu\rho}{}^{I} + (i/2)f^{IJK}(\bar{\lambda}^J\gamma_5\gamma_{\mu\nu\rho}\lambda^K)] \overset{*}{=} 0$.

Fifth, all other equations in (3.3) are consistent with supersymmetry. This confirms the total on-shell consistency with supersymmetry.

Sixth, the closure of supersymmetry works as follows:

$$[\delta_{Q_1}, \delta_{Q_2}] = \delta_{P_3} + \delta_{T_3} + \delta_{U_3} + \delta_{V_3},$$
$$\xi_3^\mu = +2(\bar\epsilon_1\gamma^\mu\epsilon_2), \qquad \alpha_3^I = -\xi_3^\mu A_\mu{}^I, \qquad \beta_3^I = -\xi_3^\mu B_\mu{}^I, \qquad \gamma_{3\mu}{}^I = -\xi_3^\nu C_{\nu\mu}{}^I - \xi_{3\mu}\varphi^I, \tag{3.4}$$

where $\delta_P$ is the translation operation. The transformations $\delta_P, \delta_T, \delta_U$ and $\delta_V$, respectively have the parameters $\xi^\mu, \alpha^I, \beta^I$ and $\gamma_\mu{}^I$. The subscript $_3$ on these parameters is to show that they are produced out of the commutator $[\delta_{Q_1}, \delta_{Q_2}]$.

Seventh, other commutators among $\delta_U$ and $\delta_Q$ or $\delta_V$ and $\delta_Q$ are the following:

$$[\delta_Q, \delta_U] = \delta_V, \qquad \gamma_\mu{}^I \equiv -f^{IJK}(\bar\epsilon\gamma_\mu\lambda^J)\beta^K, \tag{3.5a}$$

$$[\delta_{T_1}, \delta_{T_2}] = \delta_{T_3}, \qquad \alpha_3^I \equiv -f^{IJK}\alpha_1^J\alpha_2^K, \tag{3.5b}$$

$$[\delta_Q, \delta_T] = [\delta_Q, \delta_V] = [\delta_T, \delta_U] = [\delta_T, \delta_V] = [\delta_U, \delta_V] = [\delta_{U_1}, \delta_{U_2}] = [\delta_{V_1}, \delta_{V_2}] = [\delta_{K_1}, \delta_{K_2}]$$
$$= [\delta_T, \delta_K] = [\delta_U, \delta_K] = [\delta_V, \delta_K] = 0. \tag{3.5c}$$

Eighth, the degrees of freedom (DOF) in our system are counted as follows: The TM is off-shell without auxiliary fields. However, since it is *unphysical* with DOF are $0 + 0$ on-shell. Both of our YMVM and DVM are *on-shell*, namely, there is *no* $D$-type auxiliary fields. So the total DOF of these two multiplets are $2(2+2)$ on-shell. However, due to the supersymmetric duality (3.3a) and (3.3b), the total DOF are reduced to $2(2+2)/2 = 2+2$.

This situation is very similar to the duality-symmetric 11D supergravity [20]. Namely, in [20] we use both the 4th rank field-strength $F_{\mu\nu\rho\sigma}$ and its Hodge dual $G_{\mu_1\cdots\mu_7}$ simultaneously. Originally, there are $2\binom{9}{3} = 2\cdot 84 = 168$ on-shell DOF, but due to the duality relation $F_{[4]} = (1/7!)\epsilon_{[4]}{}^{[7]}G_{[7]}$, the total DOF are reduced again to 84, balancing the usual $128 + 128$ on-shell DOF in 11D supergravity [21].

## IV. SUPERSPACE REFORMULATION

Once we have established the component formulation of our system, it is rather straightforward to translate it into superspace [19]. Our superfield-strengths are $F_{AB}{}^I, G_{AB}{}^I$ and $H_{ABC}{}^I$,[3] defined by

$$F_{AB}{}^I \equiv +E_{[A}A_{B)}{}^I - T_{AB}{}^C A_C{}^I + mf^{IJK}A_A{}^J A_B{}^K, \tag{4.1a}$$

$$G_{AB}{}^I \equiv +\nabla_{[A}B_{B)}{}^I - T_{AB}{}^C B_C{}^I + mC_{AB}{}^I, \tag{4.1b}$$

$$H_{ABC}{}^I \equiv +\frac{1}{2}\nabla_{[A}C_{BC)}{}^I - \frac{1}{2}T_{[AB|}{}^D C_{D|C)}{}^I + \frac{1}{2}f^{IJK}F_{[AB}{}^J B_{C)}{}^K, \tag{4.1c}$$

where $E_A \equiv E_A{}^M\partial_M$, while $\nabla_A$ is the YM-gauge covariant derivative: $\nabla_A \equiv E_A{}^M\partial_M + A_M{}^I\tau^I$ with the YM group generators $\tau^I$. These field-strengths satisfy their respective BIds:

$$+\frac{1}{2}\nabla_{[A}F_{BC)}{}^I - \frac{1}{2}T_{[AB|}{}^D F_{D|C)}{}^I \equiv 0, \tag{4.2a}$$

$$+\frac{1}{2}\nabla_{[A}G_{BC)}{}^I - \frac{1}{2}T_{[AB|}{}^D G_{D|C)}{}^I - mH_{ABC}{}^I \equiv 0, \tag{4.2b}$$

$$+\frac{1}{6}\nabla_{[A}H_{BCD)}{}^I - \frac{1}{4}T_{[AB|}{}^E H_{E|CD)}{}^I - \frac{1}{4}f^{IJK}F_{[AB|}{}^J G_{|CD)}{}^K \equiv 0. \tag{4.2c}$$

Equations (4.1) and (4.2) are nothing but our component results (2.1) and (2.2) recasted into superspace [19].

Our superspace constraints at engineering dimensions $0 \le d \le 1$[4] are

$$T_{\alpha\beta}{}^c = +2(\gamma^c)_{\alpha\beta}, \qquad H_{\alpha\beta c}{}^I = +2(\gamma_c)_{\alpha\beta}\varphi^I, \tag{4.3a}$$

$$F_{\alpha b}{}^I = -(\gamma_b\lambda^I)_\alpha, \qquad G_{\alpha b}{}^I = -i(\gamma_5\gamma_b\chi^I)_\alpha,$$
$$H_{\alpha bc}{}^I = -i(\gamma_5\gamma_{bc}\rho^I)_\alpha, \tag{4.3b}$$

$$\nabla_\alpha\lambda_\beta{}^I = +\frac{1}{2}(\gamma^{cd})_{\alpha\beta}F_{cd}{}^I - im(\gamma_5)_{\alpha\beta}\varphi^I, \tag{4.3c}$$

$$\nabla_\alpha\chi_\beta{}^I = +\frac{i}{2}(\gamma_5\gamma^{cd})_{\alpha\beta}G_{cd}{}^I + im(\gamma_5)_{\alpha\beta}\varphi^I, \tag{4.3d}$$

[3] In superspace, we use the local coordinate indices $A \equiv (a,\alpha), B \equiv (b,\beta), \ldots$ for the bosonic (or fermionic) coordinates $a, b, \ldots = 0,1,2,3$ (or $\alpha,\beta,\ldots = 1,2,3,4$). The (anti)symmetrization in superspace is such as $M_{[AB)} \equiv M_{AB} - (-)^{AB}M_{BA}$. The YM-covariant derivative $D_\mu$ in component language is now $\nabla_a$. For curved coordinates, we use $_{M,N,\ldots}$.

[4] The engineering dimension for our bosonic (or fermionic) fundamental field is $d = 0$ (or $d = 1/2$).

$$\nabla_\alpha \rho_\beta{}^I = -\frac{i}{6}(\gamma_5\gamma^{cde})_{\alpha\beta}H_{cde}{}^I - i(\gamma_5\gamma^c)_{\alpha\beta}\nabla_c\varphi^I + \frac{1}{2}f^{IJK}(\gamma^c)_{\alpha\beta}(\bar\lambda^J\gamma_c\lambda^K), \tag{4.3e}$$

$$\nabla_\alpha\varphi^I = -i(\gamma_5\rho^I)_\alpha. \tag{4.3f}$$

Other independent components, such as $H_{\alpha\beta\gamma}{}^I$ are all zero.

The constraints at $d = 3/2$ are equivalent to (3.2):

$$\nabla_\alpha F_{bc}{}^I = (\gamma_{[b}\nabla_{c]}\lambda^I)_\alpha,$$
$$\nabla_\alpha G_{bc}{}^I = i(\gamma_5\gamma_{[b}\nabla_{c]}\chi)_\alpha - im(\gamma_5\gamma_{bc}\rho^I)_\alpha, \tag{4.4a}$$

$$\nabla_\alpha H_{bcd}{}^I = -\frac{i}{2}(\gamma_5\gamma_{[bc}\nabla_{d]}\rho^I)_\alpha - \frac{1}{2}f^{IJK}(\gamma_{[b}\lambda^J)_\alpha G_{cd]}{}^K + \frac{i}{2}f^{IJK}(\gamma_{[b}\chi^J)_\alpha F_{cd]}{}^K. \tag{4.4b}$$

Our duality-related equations in (3.3) are reexpressed as

$$G_{ab}{}^I \overset{*}{=} +\frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}{}^I, \tag{4.5a}$$

$$\lambda_\alpha{}^I \overset{*}{=} -\chi_\alpha{}^I, \qquad (\nabla\!\!\!/\lambda^I)_\alpha \doteq 0, \qquad (\nabla\!\!\!/\chi^I)_\alpha \doteq 0, \tag{4.5b}$$

$$\rho_\alpha{}^I \overset{*}{=} 0, \qquad \varphi^I \overset{*}{=} 0, \tag{4.5c}$$

$$H_{abc}{}^I \overset{*}{=} -\frac{i}{2}f^{IJK}(\bar\lambda^J\gamma_5\gamma_{abc}\lambda^K), \qquad \tilde H_\mu{}^I \overset{*}{=} +\frac{1}{2}f^{IJK}(\bar\lambda^J\gamma_\mu\lambda^K). \tag{4.5d}$$

It is not too difficult to confirm the mutual consistency of these equations. For example, a spinorial derivative $\nabla_\alpha$ on (4.5a) is shown to vanish:

$$\nabla_\alpha\left(G_{ab}{}^I - \frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}{}^I\right) \overset{*}{=} (\gamma_{[a}\nabla_{b]}\lambda^I)_\alpha + (\gamma_{ab}{}^c\nabla_c\lambda^I)_\alpha \overset{*}{=} (\gamma_{ab}\nabla\!\!\!/\lambda^I)_\alpha \doteq 0, \tag{4.6}$$

by the use of (4.5b) and (4.5c).

## V. 10D APPLICATION

As we have promised, we apply our supersymmetrization technique in 4D to 10D super YM system. Our field content is the YMVM $(A_\mu{}^I, \lambda^I)$, DVM $(B_{[7]}{}^I, \chi^I)$, and auxiliary bosonic tensor fields $C_{[8]}{}^I$ and $K_{[6]}$. Here the fermions $\lambda^I$ and $\chi^I$ are both Majorana-Weyl spinors with the positive chirality, as in the conventional super YM theory in 10D. Compared with the previous 4D case, the tensor $K_{[6]}$ is new, without any adjoint index. The important role played by this tensor will be clarified after (5.7c) below.

The $N = (1, 0)$ supersymmetry transformation rule is

$$\delta_Q A_\mu{}^I = +(\bar\epsilon\gamma_\mu\lambda^I), \tag{5.1a}$$

$$\delta_Q\lambda^I = +\frac{1}{2}(\gamma^{\mu\nu}\epsilon)F_{\mu\nu}{}^I, \tag{5.1b}$$

$$\delta_Q B_{\mu_1\cdots\mu_7}{}^I = +(\bar\epsilon\gamma_{\mu_1\cdots\mu_7}\chi^I) + 7K_{[\mu_1\cdots\mu_6|}(\bar\epsilon\gamma_{|\mu_7]}\lambda^I), \tag{5.1c}$$

$$\delta_Q\chi^I = -\frac{1}{8!}(\gamma^{[8]}\epsilon)G_{[8]}{}^I, \tag{5.1d}$$

$$\delta_Q C_{\mu_1\cdots\mu_8}{}^I = -8f^{IJK}(\bar\epsilon\gamma_{\mu_1}\lambda^J)B_{\mu_2\cdots\mu_8]}{}^K, \tag{5.1e}$$

$$\delta_Q K_{\mu_1\cdots\mu_6} = 0, \tag{5.1f}$$

where $\gamma_{11}\epsilon = +\epsilon$. The field-strengths $F, G, H$ and $L$, respectively, of the potentials $A, B, C$ and $K$ are defined by

$$F_{\mu\nu}{}^I \equiv +2\partial_{[\mu}A_{\nu]}{}^I + mf^{IJK}A_\mu{}^J A_\nu{}^K, \tag{5.2a}$$

$$G_{\mu_1\cdots\mu_8}{}^I \equiv +8\partial_{[\mu_1}B_{\mu_2\cdots\mu_8]}{}^I + mC_{\mu_1\cdots\mu_8}{}^I - 28K_{[\mu_1\cdots\mu_6}F_{\mu_7\mu_8]}{}^I, \tag{5.2b}$$

$$H_{\mu_1\cdots\mu_9}{}^I \equiv +9D_{[\mu_1}C_{\mu_2\cdots\mu_9]}{}^I + 36f^{IJK}F_{[\mu_1\mu_2}{}^J B_{\mu_3\cdots\mu_9]}{}^K, \tag{5.2c}$$

$$L_{\mu_1\cdots\mu_7} \equiv +7\partial_{[\mu_1}K_{\mu_2\cdots\mu_7]}. \tag{5.2d}$$

These field-strengths satisfy the BIds

$$D_{[\mu}F_{\nu\rho]}{}^I \equiv 0, \tag{5.3a}$$

$$D_{[\mu_1}G_{\mu_2\cdots\mu_9]}{}^I \equiv +\frac{1}{9}mH_{\mu_1\cdots\mu_9}{}^I - 4L_{[\mu_1\cdots\mu_7}F_{\mu_8\mu_9]}{}^I, \tag{5.3b}$$

$$D_{[\mu_1}H_{\mu_2\cdots\mu_{10}]}{}^I \equiv +\frac{9}{2}f^{IJK}F_{[\mu_1\mu_2}{}^J G_{\mu_3\cdots\mu_{10}]}{}^K, \tag{5.3c}$$

$$\partial_{[\mu_1}L_{\mu_2\cdots\mu_8]} \equiv 0. \tag{5.3d}$$

The arbitrary variations of these field-strengths are

$$\delta F_{\mu\nu}{}^I = +2D_{[\mu}(\delta A_{\nu]}{}^I), \tag{5.4a}$$

$$\delta G_{\mu_1\cdots\mu_8}{}^I = +8D_{[\mu_1}(\tilde\delta B_{\mu_2\cdots\mu_8]}{}^I) - 8(\delta A_{[\mu_1}{}^I)L_{\mu_2\cdots\mu_8]} + m(\tilde\delta C_{\mu_1\cdots\mu_8}) - 28(\delta K_{[\mu_1\cdots\mu_4})F_{\mu_7\mu_8]}{}^I, \tag{5.4b}$$

$$\delta H_{\mu_1\cdots\mu_9}{}^I = +9D_{[\mu_1}(\tilde\delta C_{\mu_2\cdots\mu_9]}{}^I) - 36f^{IJK}(\tilde\delta B_{[\mu_1\cdots\mu_7}{}^J)F_{\mu_8\mu_9]}{}^K + 9f^{IJK}(\tilde\delta A_{[\mu_1}{}^J)G_{\mu_2\cdots\mu_9]}{}^K, \tag{5.4c}$$

$$\delta L_{\mu_1\cdots\mu_7}{}^I = +7\partial_{[\mu_1}(\delta K_{\mu_1\cdots\mu_7]}{}^I), \tag{5.4d}$$

$$\tilde{\delta} B_{\mu_1\cdots\mu_7}{}^I \equiv \delta B_{\mu_1\cdots\mu_7}{}^I - 7(\delta A_{[\mu_1|}{}^I) K_{|\mu_2\cdots\mu_7]}, \tag{5.4e}$$

$$\tilde{\delta} C_{\mu_1\cdots\mu_8}{}^I \equiv \delta C_{\mu_1\cdots\mu_8}{}^I + 8 f^{IJK} (\delta A_{[\mu_1}{}^J) B_{\mu_2\cdots\mu_8]}{}^K. \tag{5.4f}$$

There are four different gauge transformations $\delta_T, \delta_U, \delta_V$ and $\delta_K$:

$$\delta_T A_\mu{}^I = D_\mu \alpha^I, \qquad \delta_T (B_{[7]}{}^I, C_{[8]}{}^I, K_{[6]}) = -m f^{IJK} \alpha^J (B_{[7]}{}^K, C_{[8]}{}^K, 0), \tag{5.5a}$$

$$\delta_U B_{\mu_1\cdots\mu_7}{}^I = +7 D_{[\mu_1} \beta_{\mu_2\cdots\mu_7]}, \qquad \delta_U C_{\mu_1\cdots\mu_8}{}^I = -28 f^{IJK} F_{[\mu_1\mu_2}{}^J \beta_{\mu_3\cdots\mu_8]}{}^K, \tag{5.5b}$$

$$\delta_V B_{[7]}{}^I = -m \gamma_{[7]}{}^I, \qquad \delta_V C_{\mu_1\cdots\mu_8}{}^I = +8 D_{[\mu_1} \gamma_{\mu_2\cdots\mu_8]}{}^I, \tag{5.5c}$$

$$\delta_K B_{\mu_1\cdots\mu_7}{}^I = 21 \kappa_{[\mu_1\cdots\mu_5} F_{\mu_6\mu_7]}{}^I, \qquad \delta_K K_{\mu_1\cdots\mu_6} = +6 \partial_{[\mu_1} \kappa_{\mu_2\cdots\mu_6]}. \tag{5.5d}$$

for the potentials $A, B, C$ and $K$, respectively. All other fields not given above are *invariant*, e.g., $\delta_U A_\mu{}^I = 0$, or $\delta_V K_{[6]} = 0$. Under each of $\delta_U, \delta_V$ and $\delta_K$-transformations, there are only two fields transforming. Note that $B_{[7]}{}^I$ also transforms under $\delta_K$. Using (5.4), we can prove the covariance and invariance of our field-strengths:

$$\delta_T (F_\mu{}^I, G_{[8]}{}^I, H_{[9]}{}^I, L_{[7]}) = -m f^{IJK} \alpha^J (F_\mu{}^K, G_{[8]}{}^K, H_{[9]}{}^K, 0),$$
$$\delta_U (F_\mu{}^I, G_{[8]}{}^I, H_{[9]}{}^I, L_{[7]}) = (0,0,0,0), \qquad \delta_U (F_\mu{}^I, G_{[8]}{}^I, H_{[9]}{}^I, L_{[7]}) = (0,0,0,0), \tag{5.6a}$$

$$\delta_K (F_\mu{}^I, G_{[8]}{}^I, H_{[9]}{}^I, L_{[7]}) = (0,0,0,0). \tag{5.6b}$$

The closure of supersymmetry is

$$[\delta_{Q_1}, \delta_{Q_2}] = \delta_{P_3} + \delta_{T_3} + \delta_{U_3} + \delta_{V_3} + \delta_{K_3}, \tag{5.7a}$$

$$\xi_3^\mu \equiv +2(\bar{\epsilon}_1 \gamma^\mu \epsilon_2), \qquad \alpha_3^I \equiv -\xi_3^\mu A_\mu{}^I, \qquad \beta_{3\mu_1\cdots\mu_6}{}^I \equiv -\xi_3^\mu B_{\nu\mu_1\cdots\mu_6}{}^I, \tag{5.7b}$$

$$\gamma_{3\mu_1\cdots\mu_7}{}^I \equiv -\xi_3^\nu C_{\nu\mu_1\cdots\mu_7}{}^I, \qquad \kappa_{3\mu_1\cdots\mu_5} \equiv +2(\bar{\epsilon}_1 \gamma_{\mu_1\cdots\mu_5} \epsilon_2) - \xi_3^\nu K_{\nu\mu_1\cdots\mu_5}. \tag{5.7c}$$

The closures on $B_{[7]}{}^I$ and $C_{[8]}{}^I$ need special care. In $[\delta_{Q_1}, \delta_{Q_2}] B_{[7]}{}^I$, there arises a term $42(\bar{\epsilon}_1 \gamma_{[\mu_1\cdots\mu_5|} \epsilon_2) F_{|\mu_6\mu_7]}{}^I$. Usually, such a term poses a problem, because a $\gamma_{[5]}$-term is *not* acceptable in a supersymmetry-commutator. Even though its leading gradient-term $84(\bar{\epsilon}_1 \gamma_{[\mu_1\cdots\mu_5|} \epsilon_2) \partial_{|\mu_6|} A_{|\mu_7]}{}^I$ may be absorbed into $\delta_U B_{[7]}{}^I$, the *non-Abelian* term $42m(\bar{\epsilon}_1 \gamma_{[\mu_1\cdots\mu_5|} \epsilon_2) f^{IJK} A_{|\mu_6}{}^J A_{\mu_7]}{}^K$ *cannot* be interpreted as a part of $\delta_U B_{[7]}{}^I$. However, in our system, this problematic term can be interpreted as a $\delta_K$-transformation as $\delta_K B_{\mu_1\cdots\mu_7}{}^I = 21 \kappa_{[\mu_1\cdots\mu_5|} F_{|\mu_6\mu_7]}{}^I$ as in (5.5d) and (5.7c). This justifies the necessity of the new gauge symmetry $\delta_K$ for the new field $K_{[6]}$. Note also that in the previous 4D case, the analog of the $K_{[6]}$-field was *not* needed, because there was *no* higher-rank gamma-term in $[\delta_{Q_1}, \delta_{Q_2}] B_{\mu\nu}{}^I$, such as $42m(\bar{\epsilon}_1 \gamma_{[\mu_1\cdots\mu_5|} \epsilon_2) f^{IJK} A_{|\mu_6}{}^J A_{\mu_7]}{}^K$. This is the very reason why we need $K_{[6]}$ in 10D with its associated symmetry $\delta_K$. The necessity of $K_{[6]}$ is also reflected in superspace language [19] in the next section.

Some readers may still wonder what is the real role played by the tensor $K_{[6]}$. Such a question seems legitimate, because the field strength $L_{[7]}$ is zero, so $K_{[6]}$ is unphysical, and completely gauged away. This question is answered as follows: If $K_{[6]}$ *were* gauged away, and its gauge transformation $\delta_K$ *were* no longer available, the aforementioned unwanted term $42m(\bar{\epsilon}_1 \gamma_{[\mu_1\cdots\mu_5|} \epsilon_2) f^{IJK} A_{|\mu_6}{}^J A_{\mu_7]}{}^K$ in $[\delta_{Q_1}, \delta_{Q_2}] B_{[7]}{}^I$ *would not* be absorbed into any gauge transformation, and thus the supersymmetry closure *would be* inconsistent. So, $K_{[6]}$ should *not* be entirely gauged away, maintaining supersymmetry closure. The nontrivial transformation $\delta_K B_{[7]} \neq 0$ is also closely related to this fact. In other words, if we *gauged away* $K_{[6]}$, the $\delta_K$-gauge freedom *would be* lost, and supersymmetry *would not* close. This is a typical example showing that even *nonphysical* fields are playing important roles for the closure of supersymmetry.

As for $[\delta_{Q_1}, \delta_{Q_2}] C_{[8]}{}^I$, there arise three sorts of terms: $FB$, $\lambda^2$ and $K\lambda^2$-terms. The $\lambda^2$-terms need a special $\gamma$-matrix identities

$$(\gamma_{[\mu_1\cdots\mu_4|\nu})_{\alpha\beta}(\gamma_{|\mu_5\cdots\mu_8]}{}^{\nu})_{\gamma\delta} = 0,$$
$$(\gamma_{[\mu_1\cdots\mu_6|}{}^{[3]})_{\alpha\beta}(\gamma_{|\mu_7\mu_8][3]})_{\gamma\delta} = 0. \qquad (5.8)$$

Here all spinorial indices are for the negative chirality, contracted with positive chiral spinors, such as $\epsilon_1^\alpha, \epsilon_2^\beta$ or $\lambda^{\gamma I}$. Equation (5.8) excludes possible $\gamma^{[5]}$ and $\gamma^{[9]}$-terms in the commutator.

Other nonvanishing commutators among $\delta_Q, \delta_T, \delta_U, \delta_K$ are

$$[\delta_Q, \delta_U] = \delta_{V_3}, \qquad \gamma_{3\mu_1\cdots\mu_7}{}^I \equiv -7 f^{IJK}(\bar\epsilon\gamma_{[\mu_1}\lambda^J)\beta_{\mu_2\cdots\mu_7]}{}^K, \qquad (5.9a)$$

$$[\delta_Q, \delta_K] = \delta_{U_3}, \qquad \beta_{3\mu_1\cdots\mu_6}{}^I \equiv -6\kappa_{[\mu_1\cdots\mu_5}(\bar\epsilon\gamma_{\mu_6]}\lambda^I). \qquad (5.9b)$$

Our supersymmetric EM-duality relationships are now

$$F_{\mu\nu}{}^I \overset{*}{=} +\frac{1}{8!}\epsilon_{\mu\nu}{}^{[8]}G_{[8]}{}^I \equiv \tilde G_{\mu\nu}{}^I, \qquad (5.10a)$$

$$H_{[9]}{}^I \overset{*}{=} -\frac{1}{2} f^{IJK}(\bar\lambda^J\gamma_{[9]}\lambda^K), \qquad (5.10b)$$

$$\lambda^I \overset{*}{=} -\chi^I, \qquad (5.10c)$$

$$\not{D}\lambda^I \overset{\bullet}{=} 0, \qquad \not{D}\chi^I \overset{\bullet}{=} 0, \qquad (5.10d)$$

$$L_{[7]} \overset{\bullet}{=} 0. \qquad (5.10e)$$

One difference compared with the previous 4D case is the new tensor $L_{[7]}$ needed for the supersymmetry-closure of the system. This will be mentioned in the next section.

## VI. 10D SUPERSPACE REFORMULATION

As reconfirmation and for future applications, we reformulate the 10D result in superspace [19]. Our superfield-strengths are defined by

$$F_{AB}{}^I \equiv +E_{[A}A_{B)}{}^I - T_{AB}{}^C A_C{}^I + m f^{IJK} A_A{}^J A_B{}^K, \qquad (6.1a)$$

$$G_{A_1\cdots A_8}{}^I \equiv +\frac{1}{7!}\nabla_{[A_1}B_{A_2\cdots A_8)}{}^I - \frac{1}{6!\cdot 2}T_{[A_1A_2|}{}^C B_{C|A_3\cdots A_8)}{}^I + mC_{A_1\cdots A_8}{}^I - \frac{1}{6!\cdot 2}K_{[A_1\cdots A_6}F_{A_7A_8)}{}^I, \qquad (6.1b)$$

$$H_{A_1\cdots B_9}{}^I \equiv +\frac{1}{8!}\nabla_{[A_1}C_{A_2\cdots A_9)}{}^I - \frac{1}{7!\cdot 2}T_{[A_1A_2|}{}^C C_{C|A_3\cdots A_9)}{}^I - \frac{1}{7!\cdot 2} f^{IJK} F_{[A_1A_2}{}^J B_{A_3\cdots A_9)}{}^K, \qquad (6.1c)$$

$$L_{A_1\cdots A_7} \equiv +\frac{1}{6!}E_{[A_1}K_{A_2\cdots A_7)} - \frac{1}{5!\cdot 2}T_{[A_1A_2|}{}^B K_{B|A_3\cdots A_7)}. \qquad (6.1d)$$

In particular, the $KF$-term in (6.1b) is the superspace generalization of (5.2b) in component language.

These field-strengths satisfy the superspace BIds

$$\frac{1}{2}\nabla_{[A}F_{BC)}{}^I - \frac{1}{2}T_{[AB|}{}^D F_{D|C)}{}^I \equiv 0, \qquad (6.2a)$$

$$\frac{1}{8!}\nabla_{[A_1}G_{A_2\cdots A_9)}{}^I - \frac{1}{7!\cdot 2}T_{[A_1A_2|}{}^B G_{B|A_3\cdots A_9)}{}^I + \frac{1}{7!\cdot 2}L_{[A_1\cdots A_7}F_{A_8A_9)}{}^I - mH_{A_1\cdots A_9}{}^I \equiv 0, \qquad (6.2b)$$

$$\frac{1}{9!}\nabla_{[A_1}H_{A_2\cdots A_{10})}{}^I - \frac{1}{8!\cdot 2}T_{[A_1A_2|}{}^B H_{B|A_3\cdots A_{10})}{}^I - \frac{1}{8!\cdot 2} f^{IJK}F_{[A_1A_2}{}^J G_{A_3\cdots A_{10})}{}^K \equiv 0, \qquad (6.2c)$$

$$\frac{1}{7!}\nabla_{[A_1}L_{A_2\cdots A_8)} - \frac{1}{6!\cdot 2}T_{[A_1A_2|}{}^B L_{B|A_3\cdots A_8)} \equiv 0. \qquad (6.2d)$$

These are respectively referred to as $(ABC)_F, (A_1\cdots A_9)_G, (A_1\cdots A_{10})_H$ and $(A_1\cdots A_8)_L$-BIds. Here $L_{A_1\cdots A_7}$ plays an important role, as will be clarified shortly.

The superspace constraints at engineering dimensions $0 \le d \le 1$ are

$$T_{\alpha\beta}{}^{c} = +2(\gamma^{c})_{\alpha\beta}, \qquad L_{\alpha\beta c_1\cdots c_5} = +2(\gamma_{c_1\cdots 5})_{\alpha\beta}, \tag{6.3a}$$

$$F_{\alpha b}{}^{I} = +(\gamma_b)_{\alpha\beta}\lambda^{\beta I} \equiv -(\gamma_b \lambda^I)_\alpha, \qquad G_{\alpha b_1\cdots b_7}{}^{I} = +(\gamma_{b_1\cdots b_7})_{\alpha\beta}\chi^{\beta I} \equiv -(\gamma_{b_1\cdots b_7}\chi^I)_\alpha, \tag{6.3b}$$

$$\nabla_\alpha \lambda^{\beta I} = +\frac{1}{2}(\gamma^{cd})_\alpha{}^\beta F_{cd}{}^I, \qquad \nabla_\alpha \chi^{\beta I} = +\frac{1}{8!}(\gamma^{cd})_\alpha{}^\beta G_{[8]}{}^I. \tag{6.3c}$$

Here the upper (or lower) spinorial indices ${}^{\alpha,\beta,\cdots}$ (or ${}_{\alpha,\beta,\ldots}$) are for the positive (or negative) chiralities. We also use the collective indices $\underline{\alpha} \equiv ({}_\alpha, {}^\alpha), \underline{\beta} \equiv ({}_\beta, {}^\beta), \ldots$. Due to the mixed chirality $C_{\alpha\dot\beta}$ or $C^{\alpha\dot\beta}$ for the charge-conjugation matrices in 10D, the upper (or lower) indices are equivalent to *dotted* indices: $X^\alpha = C^{\alpha\dot\beta}X_{\dot\beta}$ (or $X_\alpha = -C_{\alpha\dot\beta}X^{\dot\beta}$). However, we avoid to use the dotted ones. All other independent components, such as $T_\alpha{}^{\beta c}, G_{\underline{\alpha}\underline{\beta}c_1\cdots c_6}{}^I, H_{\underline{\alpha}\underline{\beta}c_1\cdots c_7}{}^I$, etc. are zero.

The superspace constraints at $d = 3/2$ are

$$\nabla_\alpha F_{bc}{}^I = +(\gamma_{[b}\nabla_{c]}\lambda^I)_\alpha, \qquad \nabla_\alpha G_{b_1\cdots b_8}{}^I = +\frac{1}{7!}(\gamma_{[b_1\cdots b_7}\nabla_{b_8]}\chi^I)_\alpha, \tag{6.4a}$$

$$\nabla_\alpha H_{b_1\cdots b_9}{}^I = -\frac{1}{2}f^{IJK}(\gamma^{cd}\gamma_{b_1\cdots b_9}\lambda^J)_\alpha F_{cd}{}^K. \tag{6.4b}$$

Our supersymmetric EM-duality relations are parallel to the component case (5.10):

$$F_{ab}{}^I \overset{*}{=} +\frac{1}{8!}\epsilon_{ab}{}^{[8]}G_{[8]}{}^I \equiv \tilde G_{ab}{}^I, \tag{6.5a}$$

$$G_{[8]}{}^I \overset{*}{=} -\frac{1}{2}\epsilon_{[8]}{}^{ab}F_{ab}{}^I \equiv -\tilde F_{[8]}{}^I, \tag{6.5b}$$

$$H_{[9]}{}^I \overset{*}{=} -\frac{1}{2}f^{IJK}(\bar\lambda^J\gamma_{[9]}\lambda^K), \tag{6.5c}$$

$$\lambda^{\alpha I} \overset{*}{=} -\chi^{\alpha I}, \tag{6.5d}$$

$$(\nabla\!\!\!\!/\,\lambda^I)_\alpha \overset{\bullet}{=} 0, \qquad (\nabla\!\!\!\!/\,\chi^I)_\alpha \overset{\bullet}{=} 0, \tag{6.5e}$$

$$L_{a_1\cdots a_7} \overset{\bullet}{=} 0. \tag{6.5f}$$

The satisfaction of the BIds (6.2) needs special care, in particular, the role played by the superfield-strength $L_{A_1\cdots A_7}$. For example, if the $LF$-term in (6.2b) *did not* exist in the $(\alpha\beta\gamma d_1\cdots d_6)_G$-BId at $d = 1/2$, then a term proportional to $(\gamma_{[d_1|})_{(\alpha\beta|}(\gamma_{|d_2\cdots d_6]}\chi^I)_{|\gamma)}$ *would* be left over. This term is canceled by the like-term arising from the $LF$-term in the G-BId (6.2b). Similarly at $d = 1$, the $(\alpha\beta c_1\cdots c_7)_G$-BId, which is equivalent to the closure $[\delta_{Q_1}, \delta_{Q_2}]B_{[7]}{}^I$ in component language, works as follows: If there *were no* $LF$-term in this BId, then there *would remain* a term $-(1/6!)(\gamma_{[c_1c_2|}{}^{[3]})_{\alpha\beta}G_{[3]|c_3\cdots c_7]}{}^I \overset{*}{=} -(1/120)(\gamma_{[c_1\cdots c_5|})_{\alpha\beta}\ F_{|c_6c_7]}{}^I$, upon the use of the duality (5.10a). However, this term is exactly canceled by the like-term arising from $(1/240)L_{\alpha\beta[c_1\cdots c_5|}F_{|c_6c_7]}{}^I$. We have thus confirmed the significance of the $K_{[6]}$-field both in component and superspace languages. The significance of the $\delta_K$ for the closure of supersymmetry in component is reflected into the necessity of the $LF$-term in $G$-Bianchi identity (6.2b) in superspace.

For BIds at $d = 1/2$, the following $\gamma$-matrix relationships are crucial:

$$(\gamma_e)_{(\alpha\beta}(\gamma^{ef_1\cdots f_4})_{\gamma\delta)} = 0, \tag{6.6a}$$

$$(\gamma_{[a|}{}^{[4]})_{\alpha\beta}(\gamma_{|b][4]})_{\gamma\delta} = 0, \tag{6.6b}$$

$$(\gamma^{[e_1})_{(\alpha\beta}(\gamma^{e_2\cdots e_6]})_{\beta\delta)} = 0, \tag{6.6c}$$

in addition to (5.8). All of these can be easily confirmed by the use of more fundamental relationships, such as

$$\delta_{(\alpha}{}^\gamma\delta_{\beta)}{}^\delta = -\frac{1}{8}(\gamma_e)_{\alpha\beta}(\gamma^e)^{\gamma\delta} - \frac{1}{1920}(\gamma_{[5]})_{\alpha\beta}(\gamma^{[5]})^{\gamma\delta}. \tag{6.7}$$

As in the 4D case in Sec. III, we can confirm the internal consistency of supersymmetric EM-duality in (6.5). A typical example is the spinorial derivative $\nabla_\alpha$ acting on (6.5a) or (6.5c), yielding zero by the use of other duality-related equations in (6.5). These are parallel to the component case, so that we do not give details.

## VII. CONCLUDING REMARKS

In this paper, we have accomplished the $N = 1$ supersymmetrization of the EM-duality relationship (2.6) for *non-Abelian* gauge groups in 4D. The original EM-duality (2.6) is supersymmetrized to the equations in (3.3).

Subsequently, we have also established the EM-duality (5.10) for $N = (1,0)$ non-Abelian supersymmetric system in 10D.

The total system in 4D is simple with only three multiplets: a YMVM, a DVM and a *nonphysical* TM. Yet the TM plays a very crucial role for avoiding the conventional problem with non-Abelian EM-duality based on tensor-hierarchy [12–14]. Even though EM-duality for $SO(8)$ group with $N = 8$ *local* supersymmetry [22] had been known for a long time, our system is simple only with *global* supersymmetry. Our formulations became possible, thanks to the recently developed tensor-hierarchy formulation [12–14].

We have confirmed the total consistency both in component and superspace languages [19] both in 4D and 10D, as well. The existence of the extra tensors, such as $C_{\mu\nu}{}^I$ in 4D or $C_{[8]}{}^I$ and $K_{[6]}$ in 10D is to maintain the total consistency of the system. In particular, the field-strengths $G$ and $H$ contain CS-like terms, guaranteeing consistency. This aspect is also the result of tensor-hierarchy formulation [12–14].

The validity of the particular $KF$-type CS-term in the $G$-field strength (5.2b), and the $LF$-term in the $G$-Bianchi identity (5.3b) in component language is reconfirmed as (6.1b) and (6.2b) in superspace. The necessity of the potential $K_{[6]}$ or its field strength $L_{[7]}$ is confirmed both in component and superspace languages. It is the sophisticated combination of tensor-hierarchy formalism [13,14] and the special role played by $K_{[6]}$ and $L_{[7]}$ that make our EM-duality possible in 10D.

In our paper, we have dealt with the manifestly-Lorentz-covariant EM-duality, such as $F_{\mu\nu}{}^I \overset{*}{=} +(1/8!)\epsilon_{\mu\nu}{}^{[8]} G_{[8]}{}^I$ in 10D, instead of *nonmanifest* Lorentz covariance as in [3]. Even though our system lacks a Lagrangian formulation, it still maintains *manifest* Lorentz-covariance at the field-equation level.

As some readers may have noticed, (3.3d) indicates that the dual field-strength $\tilde{H}_\mu{}^I$ equals the YM-current vector: $\tilde{H}_\mu{}^I \overset{*}{=} J_\mu{}^I$. The divergence of the left-hand side of this relationship vanishes by the EM-duality (2.6) *via* the $H$-BId (2.2c), while the vanishing of the right-hand side is the usual current conservation. In other words, the new relationship like (3.3d) relates the current $J_\mu{}^I$ directly to field-strength $\tilde{H}_\mu{}^I$ *without* involving derivatives of the latter.

We believe our present result may well be important for generating other and new supersymmetric consistent theories of non-Abelian vectors and tensors associated with general EM-dualities, in diverse space-time dimensions.


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